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Note on the Higher-Order Derivatives of the Hyperharmonic Polynomials and the *r*-Stirling Polynomials of the First Kind

José L. Cereceda

Independent Researcher, Collado Villalba, 28400 Madrid, Spain; jl.cereceda@movistar.es

Abstract: In this paper, we focus on the higher-order derivatives of the hyperharmonic polynomials, which are a generalization of the ordinary harmonic numbers. We determine the hyperharmonic polynomials and their successive derivatives in terms of the r-Stirling polynomials of the first kind and show the relationship between the (exponential) complete Bell polynomials and the r-Stirling numbers of the first kind. Furthermore, we provide a new formula for obtaining the generalized Bernoulli polynomials by exploiting their link with the higher-order derivatives of the hyperharmonic polynomials. In addition, we obtain various identities involving the r-Stirling numbers of the first kind, the Bernoulli numbers and polynomials, the Stirling numbers of the first and second kind, and the harmonic numbers.

Keywords: hyperharmonic numbers; hyperharmonic polynomials; generating function; r-Stirling polynomials of the first kind; generalized Bernoulli polynomials; complete Bell polynomials; Stirling numbers

MSC: 11B73; 11B83; 11B68; 05A10; 05A19

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1. Introduction

The n-th harmonic number is the sum of the reciprocals of the first n positive integers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

with $H_0 = 0$. Harmonic numbers play an important role in various branches of mathematics and applications such as number theory, combinatorics, analysis, special functions, and computer science and have been profusely generalized by many authors (see, for instance, [1–14] and references therein). In particular, the importance of the harmonic numbers and their generalizations in the evaluation of the special values of the Riemann zeta function, Hurwitz zeta function, and more generally zeta functions of arithmetical nature (see, for instance, [15–19]) should be stressed.

In this paper, we deal with a generalization of the harmonic numbers known as *hyperharmonic numbers*, which were introduced by Conway and Guy in 1996 [20] (p. 258). Following the notation in [17,21,22], we denote the hyperharmonic numbers by $H_n^{(r)}$. These can be defined recursively as follows.

Definition 1. *For integers $n, r \geq 0$, the n-th hyperharmonic number of order r, $H_n^{(r)}$, is given by*

$$H_n^{(r)} = \begin{cases} 0 & \text{if } n = 0, \\ \frac{1}{n} & \text{if } n > 0 \text{ and } r = 0, \\ \sum_{i=1}^{n} H_i^{(r-1)} & \text{if } n, r \geq 1. \end{cases}$$

Clearly, $H_n^{(1)} = H_n$. Equivalently, the hyperharmonic numbers can be defined by the generating function [23]

$$\sum_{n=0}^{\infty} H_n^{(r)} t^n = -\frac{\ln(1-t)}{(1-t)^r}. \tag{1}$$

Moreover, Conway and Guy [20] (p. 258) provided the following formula

$$H_n^{(r)} = \binom{n+r-1}{r-1}\big(H_{n+r-1} - H_{r-1}\big), \quad r \geq 1, \tag{2}$$

connecting the hyperharmonic numbers with the ordinary harmonic numbers.

Starting with (2), Mező [21] constructed the analytic extension of the hyperharmonic numbers. Specifically, he defined the hyperharmonic function $H_z^{(w)}$ as follows:

$$H_z^{(w)} = \frac{(z)_w}{z\Gamma(w)}\big(\Psi(z+w) - \Psi(w)\big),$$

involving the Pochhammer symbol $(z)_w$, the classical gamma $\Gamma(w)$, and digamma $\Psi(w)$ functions, and where $w, z+w \in \mathbb{C} \setminus \big(\mathbb{Z}^- \cup \{0\}\big)$ (see, e.g., [24] (Chapter 1) for a detailed description of $\Gamma(w)$ and $\Psi(w)$). Furthermore, Mező computed the first derivative of $H_z^{(w)}$ with respect to the variables w and z. Subsequently, Dil [22] presented formulas to calculate special values of $H_z^{(w)}$ and showed a way to obtain higher derivatives of the hyperharmonic function with respect to w and z. In this regard, he proved that [22] (Equation (15))

$$\frac{d^{n+1}}{dw^{n+1}}(w)_z = \Gamma(z+1)\frac{d^n}{dw^n}H_z^{(w)},$$

showing that the higher-order derivatives (with respect to w) of $H_z^{(w)}$ can be expressed in terms of the higher-order derivatives of $(w)_z$. For the special case in which z is a non-negative integer, we have $\Gamma(z+1) = z!$, and thus

$$\frac{d^n}{dw^n}H_z^{(w)} = \frac{1}{z!}\frac{d^{n+1}}{dw^{n+1}}(w)_z = \frac{d^{n+1}}{dw^{n+1}}\binom{w+z-1}{z}, \tag{3}$$

where the general binomial coefficient [25] (Equation (1.2))

$$\binom{w+z-1}{z} = \frac{w(w+1)(w+2)\cdots(w+z-1)}{z!},$$

applies to any complex number w and non-negative integer z. Evaluating the above derivative (3) at $w = 1$ for the simplest case $n = 0$ gives the well-known relation (see, e.g., [2] (Equation (8)))

$$H_z = \frac{d}{dw}\binom{w+z-1}{z}\bigg|_{w=1} = \frac{d}{dw}\binom{w+z}{z}\bigg|_{w=0}.$$

However, the problem of finding closed-form expressions for the higher-order derivatives of $H_z^{(w)}$ considered in [21,22] was left open. In this paper, we completely determine the higher-order derivatives (with respect to x) of the class of *hyperharmonic polynomials* $H_j^{(x)}$, where the subscript j denotes a non-negative integer variable, while the superscript x stands for any arbitrary (real or complex) value. Next, we define the hyperharmonic polynomials $H_j^{(x)}$ through their generating function. Without loss of generality, in what follows, we may restrict x to belong to $\mathbb{R}$.

Definition 2. *For integer $j \geq 0$ and $x \in \mathbb{R}$, the sequence $\{H_0^{(x)}, H_1^{(x)}, H_2^{(x)}, \dots\}$ of the hyperharmonic polynomials $H_j^{(x)} \in \mathbb{Q}[x]$ is determined by the generating function*

$$\sum_{j=0}^{\infty} H_j^{(x)} t^j = -\frac{\ln(1-t)}{(1-t)^x}, \quad |t| < 1, \tag{4}$$

where, by definition, $H_0^{(x)} = 0$.

Note that (4) is just the generating function (1) with r replaced by the continuous variable x, so that $H_j^{(x)}$ reduces to the hyperharmonic number $H_j^{(r)}$ when x is the non-negative integer r. In particular, $H_j^{(1)} = H_j$. (See [5,26–28] for other types of polynomials associated with the harmonic numbers.) The hyperharmonic polynomials admit, among others, the following representations [29]:

$$H_j^{(x)} = \frac{d}{dx}\binom{x+j-1}{j}, \tag{5}$$

$$H_{j+1}^{(x+1)} = \sum_{t=0}^{j} \frac{1}{j+1-t}\binom{x+t}{t}, \tag{6}$$

$$H_{j+1}^{(-x)} = \sum_{t=0}^{j} \frac{(-1)^t}{j+1-t}\binom{x}{t}, \tag{7}$$

and

$$H_{j+1}^{(x)} = \frac{1}{j!}\sum_{t=0}^{j} \begin{bmatrix} j+1 \\ t+1 \end{bmatrix} B_t(x), \tag{8}$$

where $\left[{j \atop t}\right]$ represents the (unsigned) Stirling numbers of the first kind and $B_t(x)$ is the t-th Bernoulli polynomial. For $j \geq 1$, $H_j^{(x)}$ is a polynomial in x of degree $j-1$ with leading coefficient $\frac{1}{(j-1)!}$ and constant term $\frac{1}{j}$. The first few hyperharmonic polynomials are given explicitly by

$$\begin{aligned}
&H_1^{(x)} = 1, \quad H_2^{(x)} = x + \tfrac{1}{2}, \quad H_3^{(x)} = \tfrac{1}{2}x^2 + x + \tfrac{1}{3}, \\
&H_4^{(x)} = \tfrac{1}{6}x^3 + \tfrac{3}{4}x^2 + \tfrac{11}{12}x + \tfrac{1}{4}, \quad H_5^{(x)} = \tfrac{1}{24}x^4 + \tfrac{1}{3}x^3 + \tfrac{7}{8}x^2 + \tfrac{5}{6}x + \tfrac{1}{5}, \\
&H_6^{(x)} = \tfrac{1}{120}x^5 + \tfrac{5}{48}x^4 + \tfrac{17}{36}x^3 + \tfrac{15}{16}x^2 + \tfrac{137}{180}x + \tfrac{1}{6}, \\
&H_7^{(x)} = \tfrac{1}{720}x^6 + \tfrac{1}{40}x^5 + \tfrac{25}{144}x^4 + \tfrac{7}{12}x^3 + \tfrac{29}{30}x^2 + \tfrac{7}{10}x + \tfrac{1}{7}, \\
&H_8^{(x)} = \tfrac{1}{5040}x^7 + \tfrac{7}{1440}x^6 + \tfrac{23}{480}x^5 + \tfrac{35}{144}x^4 + \tfrac{967}{1440}x^3 + \tfrac{469}{480}x^2 + \tfrac{363}{560}x + \tfrac{1}{8}.
\end{aligned}$$

Remark 1. *Notice that (5) immediately implies that $\int_0^1 H_j^{(x)} dx = 1$ for all $j \geq 1$. Furthermore, (5) entails that*

$$\frac{d^i}{dx^i} H_j^{(x)} = \frac{d^{i+1}}{dx^{i+1}}\binom{x+j-1}{j}, \tag{9}$$

in accordance with (3).

At this point, we should remark two previous results that are of particular interest for the present work. For that, it is convenient to introduce the symbol $P(i, j+r, r)$ (with i, j, and r being non-negative integers), which will be defined later in Section 4 (see Definition 4). In essence, $P(i, j+r, r)$ denotes the value of the (exponential) complete Bell polynomial for certain arguments related to the harmonic numbers. The first result that we are interested in was given by Wang [30] (Equation (4.2)) and tells us that (in our notation)

$$i!\sum_{k=0}^{j}\begin{bmatrix}j\\k\end{bmatrix}\binom{k}{i}(r+1)^{k-i}=j!\binom{j+r}{r}P(i,j+r,r)=\frac{d^i}{dx^i}(x)_j\bigg|_{x=r+1}.$$

Let us recall that the r-Stirling numbers of the first kind, $\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]_r$, count the number of permutations of the set $\{1,2,\dots,k\}$ having j disjoint (non-empty) cycles, such that the first r elements belong to distinct cycles [31]. In particular, $\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]_0=\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}k+1\\j+1\end{smallmatrix}\right]_1=\left[\begin{smallmatrix}k+1\\j+1\end{smallmatrix}\right]$, where $\left[\begin{smallmatrix}k\\j\end{smallmatrix}\right]$ is the ordinary Stirling number of the first kind. Actually, as will be shown in Section 2, the leftmost part in the above double identity is equal to $i!$ times the r-Stirling number of the first kind $\left[\begin{smallmatrix}j+r+1\\i+r+1\end{smallmatrix}\right]_{r+1}$. Wang's result can then equivalently be stated as

$$\binom{j+r}{r}P(i,j+r,r)=\frac{i!}{j!}\begin{bmatrix}j+r+1\\i+r+1\end{bmatrix}_{r+1}=\frac{d^i}{dx^i}\binom{x+j+r}{j}\bigg|_{x=0}. \tag{10}$$

In particular, for $r=0$, we have

$$P(i,j,0)=\frac{i!}{j!}\begin{bmatrix}j+1\\i+1\end{bmatrix}=\frac{d^i}{dx^i}\binom{x+j}{j}\bigg|_{x=0},$$

and $P(1,j,0)=H_j$.

Remark 2. *The relation* $\frac{d^i}{dx^i}\binom{x+j+r}{j}\Big|_{x=0}=\binom{j+r}{r}P(i,j+r,r)$ *was rederived (in a somewhat different form) by Wang and Jia in [32] (Theorem 2).*

The second result that we alluded to before appears in Equation (19) of the paper by Kargın et al. [12] and tells us that (in our notation)

$$\frac{d^{i-1}}{dx^{i-1}}H_j^{(x+1)}\bigg|_{x=r}=\binom{j+r}{r}P(i,j+r,r), \tag{11}$$

relating the higher-order derivatives of the hyperharmonic polynomials and the complete Bell polynomials.

Combining the mentioned result of Wang in Equation (10) and that of Kargın et al. in Equation (11), we immediately obtain

$$\frac{d^{i-1}}{dx^{i-1}}H_j^{(x+1)}\bigg|_{x=r}=\frac{i!}{j!}\begin{bmatrix}j+r+1\\i+r+1\end{bmatrix}_{r+1},$$

or, equivalently,

$$\frac{d^{i-1}}{dx^{i-1}}H_j^{(x)}\bigg|_{x=r}=\frac{i!}{j!}\begin{bmatrix}j+r\\i+r\end{bmatrix}_r, \tag{12}$$

which applies to any non-negative integer r.

Remark 3. *When $i=1$,* (12) *reduces to*

$$H_j^{(r)}=\frac{1}{j!}\begin{bmatrix}j+r\\r+1\end{bmatrix}_r, \tag{13}$$

expressing the hyperharmonic numbers in terms of the r-Stirling numbers of the first kind. A combinatorial proof of (13) *was given in [23] (Theorem 2).*

However, although Equation (12) enables us to evaluate the i-th derivative of $H_j^{(x)}$ for the particular case in which x is a non-negative integer, we still lack a general expression for the higher-order derivatives of the hyperharmonic polynomials. One of the principal objectives of the present paper is to provide a thorough account of the hyperharmonic polynomials and their successive derivatives in terms of the r-Stirling polynomials of the first kind. The rest of the paper is organized as follows.

In Section 2, we introduce the r-Stirling polynomials of the first kind $R_{m,i}(x)$ and $\overline{R}_{m,i}(x)$. Based on the properties of these polynomials, in Section 3, we express the higher-order derivatives of the hyperharmonic polynomials, $\frac{d^i}{dx^i}H_{j+1}^{(x)}$ and $\frac{d^i}{dx^i}H_{j+1}^{(x+1)}$, as an explicit polynomial in x of degree $j-i$. As anticipated by Equation (12), such derivatives can in turn be expressed in terms of the r-Stirling numbers of the first kind when they are evaluated at the non-negative integer r. In Section 4, we exhibit the relationship between the (exponential) complete Bell polynomials and the r-Stirling numbers of the first kind. Specifically, we show that Wang's relation

$$\binom{j+r}{r}P(i,j+r,r) = \frac{i!}{j!}\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} \tag{14}$$

arises as a particular case of a theorem due to Kölbig [33] (Theorem). In Section 5, we study the connection (already established in [22] (Proposition 3.13)) between the generalized Bernoulli polynomials and the higher-order derivatives of the hyperharmonic polynomials and provide a new formula for obtaining the generalized Bernoulli polynomials (see Equation (51) below). In Section 6, we consider a series of identities obtained by Spieß [34], Wang [30], and Wuyungaowa [35] involving the numbers $\binom{j+r}{r}P(i,j+r,r)$ and recast them, by means of (14), into a form involving the r-Stirling numbers of the first kind.

In addition, throughout this paper, we obtain various identities involving the r-Stirling numbers of the first kind, the Bernoulli numbers and polynomials, the Stirling numbers of the first and second kind, and the harmonic numbers. As a preliminary example of such identities, let us observe that, from (8) and (13), we quickly obtain

$$\sum_{j=0}^{k}\begin{bmatrix} k+1 \\ j+1 \end{bmatrix} B_j(r) = \frac{1}{k+1}\begin{bmatrix} k+r+1 \\ r+1 \end{bmatrix}_r, \tag{15}$$

for any non-negative integers k and r. Furthermore, the Stirling transform [36] (Appendix A) of (15) yields

$$B_k(r) = (-1)^k\sum_{j=0}^{k}\frac{(-1)^j}{j+1}\begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix}\begin{bmatrix} j+r+1 \\ r+1 \end{bmatrix}_r, \tag{16}$$

and, in particular,

$$B_k = (-1)^k\sum_{j=0}^{k}(-1)^j\frac{j!}{j+1}\begin{Bmatrix} k+1 \\ j+1 \end{Bmatrix},$$

where $B_k = B_k(0)$ are the Bernoulli numbers and $\left\{ k \atop j \right\}$ are the Stirling numbers of the second kind.

2. *r*-Stirling Polynomials of the First Kind

In order to properly define the r-Stirling polynomials of the first kind $R_{m,i}(x)$ and $\overline{R}_{m,i}(x)$, we next formulate the following proposition.

Proposition 1. *For integers $0 \le i \le m$ and $r \ge 0$, we have*

$$\begin{bmatrix} m+r \\ i+r \end{bmatrix}_r = \sum_{j=0}^{m-i}\binom{i+j}{i}\begin{bmatrix} m \\ i+j \end{bmatrix} r^j, \tag{17}$$

and

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} r^j. \tag{18}$$

Proof. We prove only the relation (18), since the proof of relation (17) proceeds in an analogous way. According to [31] (Equation (27)) (see also [37] (p. 224)), $\left[{m+r+1 \atop i+r+1}\right]_{r+1}$ can be expressed as

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{j=i}^{m} \binom{m}{j} \begin{bmatrix} j \\ i \end{bmatrix} (r+1)^{\overline{m-j}},$$

where $r^{\overline{m}} = r(r+1)\dots(r+m-1)$, and $r^{\overline{0}} = 1$. Changing the summation variable from j to t, where $t = m-j$, results in

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{t=0}^{m-i} \binom{m}{t} \begin{bmatrix} m-t \\ i \end{bmatrix} (r+1)^{\overline{t}}.$$

Now, we have

$$(r+1)^{\overline{t}} = \sum_{k=0}^{t} \begin{bmatrix} t \\ k \end{bmatrix} (r+1)^k = \sum_{k=0}^{t} \sum_{s=0}^{k} \binom{k}{s} \begin{bmatrix} t \\ k \end{bmatrix} r^s,$$

and then

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{t=0}^{m-i} \sum_{k=0}^{t} \sum_{s=0}^{k} \binom{m}{t} \begin{bmatrix} m-t \\ i \end{bmatrix} \binom{k}{s} \begin{bmatrix} t \\ k \end{bmatrix} r^s = \sum_{t=0}^{m-i} \sum_{s=0}^{t} \sum_{k=s}^{t} \binom{k}{s} \begin{bmatrix} t \\ k \end{bmatrix} \binom{m}{t} \begin{bmatrix} m-t \\ i \end{bmatrix} r^s.$$

Using the well-known identity (see, e.g., [38] (Equation (6.16))) $\sum_{k=s}^{t} \binom{k}{s}\left[{t \atop k}\right] = \left[{t+1 \atop s+1}\right]$, the preceding equation becomes

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{t=0}^{m-i} \sum_{s=0}^{t} \binom{m}{t} \begin{bmatrix} m-t \\ i \end{bmatrix} \begin{bmatrix} t+1 \\ s+1 \end{bmatrix} r^s = \sum_{j=0}^{m-i} \left(\sum_{t=j}^{m-i} \binom{m}{t} \begin{bmatrix} t+1 \\ j+1 \end{bmatrix} \begin{bmatrix} m-t \\ i \end{bmatrix} \right) r^j, \tag{19}$$

where we have renamed the variable s as j. Invoking the identity (see, e.g., [31] (Equation (52)) and [37] (p. 224))

$$\binom{j+i}{i} \begin{bmatrix} m+r+s \\ j+i+r+s \end{bmatrix}_{r+s} = \sum_{t=j}^{m-i} \binom{m}{t} \begin{bmatrix} t+r \\ j+r \end{bmatrix}_r \begin{bmatrix} m-t+s \\ i+s \end{bmatrix}_s,$$

and specializing to the case in which $r = 1$ and $s = 0$, it follows that

$$\binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} = \sum_{t=j}^{m-i} \binom{m}{t} \begin{bmatrix} t+1 \\ j+1 \end{bmatrix} \begin{bmatrix} m-t \\ i \end{bmatrix}. \tag{20}$$

Subsequently, combining (19) and (20), we obtain (18). □

Remark 4. *From* (17)*, it follows that*

$$\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{k=0}^{j-i} \binom{i+k}{i} \begin{bmatrix} j \\ i+k \end{bmatrix} (r+1)^k = \sum_{k=0}^{j} \binom{k}{i} \begin{bmatrix} j \\ k \end{bmatrix} (r+1)^{k-i},$$

as was noted in the introduction in relation to Wang's identity (10)*. On the other hand, from* (18)*, we can see that* $\left[{m+r+1 \atop i+r+1}\right]_{r+1}$ *can be expressed as a polynomial in r with constant term* $\left[{m+1 \atop i+1}\right]$*. Hence, setting* $r = 0$ *and* $j = 0$ *in* (19) *yields the identity (cf. [31] (Equation (30)))*

$$\begin{bmatrix} m+1 \\ i+1 \end{bmatrix} = \sum_{t=0}^{m-i} t! \binom{m}{t} \begin{bmatrix} m-t \\ i \end{bmatrix}.$$

Remark 5. *For integers $k, n \geq 0$ and $m \geq 1$, the hyper-sums of powers of integers $S_k^{(m)}(n)$ are defined recursively by*

$$S_k^{(m)}(n) = \sum_{j=1}^{n} S_k^{(m-1)}(j),$$

with initial conditions $S_k^{(0)}(n) \equiv S_k(n) = 1^k + 2^k + \cdots + n^k$, and $S_k^{(m)}(0) = 0$. In [12] (Equation (27)), Kargın et al. obtained the following formula for $S_k^{(m)}(n)$ (see also the related paper [39])

$$S_k^{(m)}(n) = \frac{1}{m!} \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m+n+1 \\ i+n+1 \end{bmatrix}_{n+1} S_{k+i}(n).$$

When $k = 0$, the hyper-sum $S_k^{(m)}(n)$ is equal to $S_0^{(m)}(n) = \binom{n+m}{m+1}$. Hence, putting $k = 0$ in the last equation gives

$$\sum_{i=0}^{m} (-1)^i \begin{bmatrix} m+n+1 \\ i+n+1 \end{bmatrix}_{n+1} S_i(n) = m! \binom{n+m}{m+1}. \tag{21}$$

In particular, for $n = 1$, we find

$$\sum_{i=0}^{m} (-1)^i \begin{bmatrix} m+2 \\ i+2 \end{bmatrix}_2 = m!.$$

In view of Proposition 1, and following Broder [31] (Equation (56)) and Carlitz [40] (Equation (5.2)), we define the r-Stirling polynomials of the first kind as follows.

Definition 3. *For integers $0 \leq i \leq m$, the r-Stirling polynomials of the first kind $R_{m,i}(x)$ and $\overline{R}_{m,i}(x)$ are defined as*

$$R_{m,i}(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m \\ i+j \end{bmatrix} x^j, \tag{22}$$

and

$$\overline{R}_{m,i}(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} x^j, \tag{23}$$

respectively.

By construction, $R_{m,i}(x)$ and $\overline{R}_{m,i}(x)$ reduce to $R_{m,i}(r) = \begin{bmatrix} m+r \\ i+r \end{bmatrix}_r$ and $\overline{R}_{m,i}(r) = \begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1}$, respectively, when x is a non-negative integer r, so that $\overline{R}_{m,i}(r) = R_{m,i}(r+1)$ for any integer $r \geq 0$. Since $\overline{R}_{m,i}(x) - R_{m,i}(x+1)$ vanishes for infinitely many $r \in \mathbb{N}$, $\overline{R}_{m,i}(x) = R_{m,i}(x+1)$ for an arbitrary x. Hence, we have

$$\sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} x^j = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m \\ i+j \end{bmatrix} (x+1)^j. \tag{24}$$

In particular, when $x = -1$, it follows that

$$\sum_{j=0}^{m-i} (-1)^j \binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} = \sum_{j=0}^{m} (-1)^{j-i} \binom{j}{i} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix} = \begin{bmatrix} m \\ i \end{bmatrix},$$

which corresponds to the identity [38] (Equation (6.18)).

3. Hyperharmonic Polynomials and Their Derivatives

From [31] (Theorem 28), we know that $R_{m,i}(x) = \frac{1}{i!}\frac{d^i}{dx^i}x^{\overline{m}}$. This follows immediately from the horizontal generating function of the Stirling numbers of the first kind, namely,

$$x^{\overline{m}} = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} x^j. \tag{25}$$

Indeed, differentiating i times (with respect to x) both sides of (25), we obtain

$$\frac{1}{i!}\frac{d^i}{dx^i}x^{\overline{m}} = \sum_{j=0}^{m} \binom{j}{i} \begin{bmatrix} m \\ j \end{bmatrix} x^{j-i} = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m \\ i+j \end{bmatrix} x^j.$$

Thus,

$$\frac{d^i}{dx^i}x^{\overline{m}} = i! R_{m,i}(x). \tag{26}$$

Now, we present the following theorem, which shows the explicit expression of the higher-order derivatives of the hyperharmonic polynomials $H_{j+1}^{(x)}$ and $H_{j+1}^{(x+1)}$.

Theorem 1. *For integers $i, j \geq 0$, the i-th derivative of $H_{j+1}^{(x)}$ and $H_{j+1}^{(x+1)}$ with respect to x is given, respectively, by*

$$\frac{d^i}{dx^i}H_{j+1}^{(x)} = \frac{(i+1)!}{(j+1)!} \sum_{t=0}^{j-i} \binom{i+t+1}{i+1} \begin{bmatrix} j+1 \\ i+t+1 \end{bmatrix} x^t, \tag{27}$$

and

$$\frac{d^i}{dx^i}H_{j+1}^{(x+1)} = \frac{(i+1)!}{(j+1)!} \sum_{t=0}^{j-i} \binom{i+t+1}{i+1} \begin{bmatrix} j+2 \\ i+t+2 \end{bmatrix} x^t. \tag{28}$$

Proof. Relation (27) readily follows from (26) and the representation (9) for the i-th derivative of $H_j^{(x)}$. Thus, combining (9) and (26), we obtain

$$\frac{d^i}{dx^i}H_{j+1}^{(x)} = \frac{d^{i+1}}{dx^{i+1}}\binom{x+j}{j+1} = \frac{1}{(j+1)!}\frac{d^{i+1}}{dx^{i+1}}x^{\overline{j+1}} = \frac{(i+1)!}{(j+1)!}R_{j+1,i+1}(x).$$

Now, using the expression for $R_{j+1,i+1}(x)$ that is obtained from (22), we obtain (27). On the other hand, we have

$$\frac{d^i}{dx^i}H_{j+1}^{(x+1)} = \frac{(i+1)!}{(j+1)!}R_{j+1,i+1}(x+1) = \frac{(i+1)!}{(j+1)!}\overline{R}_{j+1,i+1}(x),$$

and, using the expression for $\overline{R}_{j+1,i+1}(x)$ that is obtained from (23), we then obtain (28). □

Letting $i = 0$ in (27) and (28) gives us, respectively, $H_{j+1}^{(x)}$ and $H_{j+1}^{(x+1)}$ as polynomials in x of degree j, namely,

$$H_{j+1}^{(x)} = \frac{R_{j+1,1}(x)}{(j+1)!} = \frac{1}{(j+1)!}\sum_{t=0}^{j}(t+1)\begin{bmatrix} j+1 \\ t+1 \end{bmatrix} x^t,$$

and

$$H_{j+1}^{(x+1)} = \frac{\overline{R}_{j+1,1}(x)}{(j+1)!} = \frac{1}{(j+1)!}\sum_{t=0}^{j}(t+1)\begin{bmatrix} j+2 \\ t+2 \end{bmatrix} x^t.$$

Additionally, setting $x = 1$ and $j \to j-1$ in the first of the two equations above produces the well-known identity $H_j = \frac{1}{j!}\sum_{t=0}^{j} t\left[{j \atop t}\right]$.

The following corollary is a direct consequence of Theorem 1 and Definition 3.

Corollary 1. *For non-negative integer r, we have*

$$\frac{d^i}{dx^i}H_{j+1}^{(x)}\bigg|_{x=r} = \frac{(i+1)!}{(j+1)!}\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_r, \tag{29}$$

and

$$\frac{d^i}{dx^i}H_{j+1}^{(x+1)}\bigg|_{x=r} = \frac{(i+1)!}{(j+1)!}\begin{bmatrix} j+r+2 \\ i+r+2 \end{bmatrix}_{r+1}. \tag{30}$$

Note that, obviously, any of (29) or (30) is equivalent to (12).

Example 1. *As a simple application of Theorem 1, we may use the representation (6) in combination with (28) to obtain*

$$\sum_{t=0}^{j}\frac{1}{j+1-t}\frac{d^i}{dx^i}\binom{x+t}{t} = \frac{(i+1)!}{(j+1)!}\sum_{t=0}^{j-i}\binom{i+t+1}{i+1}\begin{bmatrix} j+2 \\ i+t+2 \end{bmatrix} x^t.$$

Therefore, it follows from (30) that

$$\sum_{t=0}^{j}\frac{1}{j+1-t}\frac{d^i}{dx^i}\binom{x+t}{t}\bigg|_{x=r} = \frac{(i+1)!}{(j+1)!}\begin{bmatrix} j+r+2 \\ i+r+2 \end{bmatrix}_{r+1}.$$

Since $\frac{d^i}{dx^i}\binom{x+t}{t}\Big|_{x=r} = \frac{i!}{t!}\left[{t+r+1 \atop i+r+1}\right]_{r+1}$, this can be expressed as (after replacing r by $r-1$)

$$\sum_{t=0}^{j}\frac{1}{t!\,(j+1-t)}\begin{bmatrix} t+r \\ i+r \end{bmatrix}_r = \frac{i+1}{(j+1)!}\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_r, \tag{31}$$

which holds for any integers $0 \le i \le j$ and $r \ge 0$. In particular, for $r = 0$, we have

$$\sum_{t=0}^{j}\frac{1}{t!\,(j+1-t)}\begin{bmatrix} t \\ i \end{bmatrix} = \frac{i+1}{(j+1)!}\begin{bmatrix} j+1 \\ i+1 \end{bmatrix}. \tag{32}$$

Furthermore, setting $i = r = 1$ in (31) yields the identity

$$\sum_{t=0}^{j}\frac{H_t}{j+1-t} = \frac{2}{(j+1)!}\begin{bmatrix} j+2 \\ 3 \end{bmatrix} = H_{j+1}^2 - H_{j+1}^{[2]},$$

which can also be found in [41] (p. 544), and where the notation $H_j^{[2]}$ means $\sum_{t=1}^{j} 1/t^2$.

Remark 6. *Using (25) and the recurrence relation $\left[{j+1 \atop i+1}\right] = j\left[{j \atop i+1}\right] + \left[{j \atop i}\right]$, it is easy to show that*

$$\frac{d^{i+1}}{dx^{i+1}}x^{\overline{j+1}} = (x+j)\frac{d^{i+1}}{dx^{i+1}}x^{\overline{j}} + (i+1)\frac{d^i}{dx^i}x^{\overline{j}},$$

or, in view of (26),

$$R_{j+1,i+1}(x) = (x+j)R_{j,i+1}(x) + R_{j,i}(x).$$

Hence, setting $x = r$ (where r is a non-negative integer), we obtain the corresponding recurrence for the r-Stirling numbers of the first kind

$$\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_r = (r+j)\begin{bmatrix} j+r \\ i+r+1 \end{bmatrix}_r + \begin{bmatrix} j+r \\ i+r \end{bmatrix}_r. \tag{33}$$

Furthermore, letting $i = 0$ in (33) *and recalling* (13) *leads to the following recurrence for the hyperharmonic numbers:*

$$H_{j+1}^{(r)} = \frac{r+j}{j+1}H_j^{(r)} + \frac{1}{j+1}\binom{j+r-1}{j},$$

which holds for any integers $r, j \geq 0$. In particular, when $r = 1$, we recover the recurrence relation defining the harmonic numbers, namely, $H_{j+1} = H_j + \frac{1}{j+1}$, with $H_0 = 0$.

4. Complete Bell Polynomials and *r*-Stirling Numbers of the First Kind

Let $Y_n(x_1, x_2, \ldots, x_n)$ be the n-th (exponential) complete Bell polynomial defined by $Y_0 = 1$ and (see, e.g., [42] (p. 134))

$$\exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right) = 1 + \sum_{p=1}^{\infty} Y_p(x_1, x_2, \ldots, x_p)\frac{t^p}{p!},$$

and let $H(j,k;r)$ be the function

$$H(j,k;r) = \sum_{t=1}^{j} \frac{1}{(t+r)^k},$$

for integers $r \geq 0$ and $j, k \geq 1$. Following Spieß [34] and Kargın et al. [12], we define the symbol $P(i, j+r, r)$ as follows.

Definition 4. *For non-negative integers i, j, and r, we have*

$$P(i,j+r,r) = \begin{cases} 1, & \text{for } i = 0, \text{ and } j, r \geq 0, \\ P_i\big(H(j,1;r), H(j,2;r), \ldots, H(j,i;r)\big), & \text{for } 1 \leq i \leq j,\ j \geq 1, \text{ and } r \geq 0, \\ 0, & \text{for } 0 \leq j < i,\ i \geq 1, \text{ and } r \geq 0, \end{cases}$$

where the polynomial $P_i(x_1, x_2, \ldots, x_i)$ is defined by

$$P_i(x_1, x_2, \ldots, x_i) = (-1)^i Y_i(-0!x_1, -1!x_2, \ldots, -(i-1)!x_i),$$

or, equivalently,

$$P_i(x_1, x_2, \ldots, x_i) = Y_i(0!x_1, -1!x_2, \ldots, (-1)^{i-1}(i-1)!x_i).$$

The first five polynomials $P_i(x_1, x_2, \ldots, x_i)$ are given by

$$\begin{aligned}
&P_1(x_1) = x_1, \\
&P_2(x_1, x_2) = x_1^2 - x_2, \\
&P_3(x_1, x_2, x_3) = x_1^3 - 3x_1x_2 + 2x_3, \\
&P_4(x_1, x_2, x_3, x_4) = x_1^4 - 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 - 6x_4, \\
&P_5(x_1, x_2, x_3, x_4, x_5) = x_1^5 - 10x_1^3x_2 + 20x_1^2x_3 + 15x_1x_2^2 - 30x_1x_4 - 20x_2x_3 + 24x_5.
\end{aligned}$$

Thus, we have, for example, $P(1, j+r, r) = H(j,1;r)$, $P(2, j+r, r) = \big(H(j,1;r)\big)^2 - H(j,2;r)$, $P(3, j+r, r) = \big(H(j,1;r)\big)^3 - 3H(j,1;r)H(j,2;r) + 2H(j,3;r)$, etc.

Remark 7. *By letting $i = 1$ and $r \to r-1$ in (11), and noting that $P(1, j+r-1, r-1) = H(j,1;r-1)$, we obtain (assuming that $r \geq 1$)*

$$\begin{aligned} H_j^{(x+1)}\Big|_{x=r-1} = H_j^{(r)} &= \binom{j+r-1}{r-1} H(j,1;r-1) \\ &= \binom{j+r-1}{r-1} \sum_{t=1}^{j} \frac{1}{t+r-1} = \binom{j+r-1}{r-1} \big(H_{j+r-1} - H_{r-1}\big), \end{aligned}$$

thus recovering Conway and Guy's formula (2).

With these ingredients at hand, next we show that Wang's relation (14) (which we reproduce here for convenience)

$$\binom{j+r}{r} P(i, j+r, r) = \frac{i!}{j!} \begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1}, \tag{34}$$

is a direct consequence of the following theorem set forth by Kölbig [33] (Theorem).

Theorem 2 (Kölbig, 1994)**.** *Let $\alpha \in \mathbb{R}$ with $\alpha \neq -1, -2, \ldots, -j$, and*

$$H(j,k;\alpha) = \sum_{t=1}^{j} \frac{1}{(t+\alpha)^k},$$

for integers $j, k \geq 1$. We then have

$$P_q\big(H(j,1;\alpha), H(j,2;\alpha), \ldots, H(j,q;\alpha)\big) = \begin{cases} \dfrac{q!}{(1+\alpha)^{\bar{j}}} S(j,q;\alpha), & q \leq j, \\ 0, & q > j, \end{cases}$$

where

$$S(j,q;\alpha) = \sum_{t=q}^{j} \binom{t}{q} \begin{bmatrix} j \\ t \end{bmatrix} (1+\alpha)^{t-q}.$$

Indeed, taking $\alpha = r$ in Kölbig's theorem (with integer $r \geq 0$), we have

$$\begin{aligned} \binom{j+r}{r} P(i, j+r, r) &= \binom{j+r}{r} P_i\big(H(j,1;r), H(j,2;r), \ldots, H(j,i;r)\big) \\ &= \binom{j+r}{r} \frac{i!}{(1+r)^{\bar{j}}} S(j,i;r), \end{aligned}$$

where it is assumed that $i \leq j$, with $i = 1, 2, \ldots$. Furthermore, it turns out that $\binom{j+r}{r} = \frac{1}{j!}(1+r)^{\bar{j}}$; thus,

$$\binom{j+r}{r} P(i, j+r, r) = \frac{i!}{j!} S(j,i;r),$$

where

$$S(j,i;r) = \sum_{t=i}^{j} \binom{t}{i} \begin{bmatrix} j \\ t \end{bmatrix} (1+r)^{t-i}.$$

On the other hand, we have

$$\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = R_{j,i}(r+1) = \sum_{s=0}^{j-i} \binom{i+s}{i} \begin{bmatrix} j \\ i+s \end{bmatrix} (r+1)^s = \sum_{t=i}^{j} \binom{t}{i} \begin{bmatrix} j \\ t \end{bmatrix} (r+1)^{t-i}.$$

Therefore, $S(j,i;r) = \left[\begin{smallmatrix} j+r+1 \\ i+r+1 \end{smallmatrix}\right]_{r+1}$; thus, we obtain (34).

Remark 8. *By setting $r = 0$ in (34), we recover the well-known result (see [42] (Equation (7b)))*

$$\begin{bmatrix} j+1 \\ i+1 \end{bmatrix} = \frac{j!}{i!} P(i,j,0) = \frac{j!}{i!} P_i\left(H_j^{[1]}, H_j^{[2]}, \dots, H_j^{[i]}\right),$$

where the notation $H_j^{[i]}$ means $\sum_{t=1}^{j} 1/t^i$.

Remark 9. *According to [34] (Theorem 16), it turns out that, for $m, r \geq 0$,*

$$\sum_{k=0}^{m} \frac{P(r,k,0)}{k+1} = \frac{P(r+1,m+1,0)}{r+1}.$$

Noting that $P(r,k,0) = \frac{r!}{k!}\left[\begin{smallmatrix} k+1 \\ r+1 \end{smallmatrix}\right]$, the above relation is equivalent to the identity (cf. [38] (Equation (6.21)))

$$\begin{bmatrix} m+1 \\ r+1 \end{bmatrix} = m! \sum_{k=0}^{m} \frac{1}{k!} \begin{bmatrix} k \\ r \end{bmatrix}.$$

Remark 10. *The generating function of the numbers $\binom{j+r}{r} P(i, j+r, r)$ is given by (see, e.g., [30] (Equation (1.6)))*

$$\sum_{j=i}^{\infty} \binom{j+r}{r} P(i, j+r, r) t^j = \frac{(-\ln(1-t))^i}{(1-t)^{r+1}}. \tag{35}$$

Therefore, taking $r \to r-1$ in (34) and using (35), we obtain the exponential generating function of the r-Stirling numbers of the first kind, namely,

$$\sum_{j=i}^{\infty} \begin{bmatrix} j+r \\ i+r \end{bmatrix}_r \frac{t^j}{j!} = \frac{1}{i!} \frac{(-\ln(1-t))^i}{(1-t)^r}.$$

In particular, for $r = 0$ and $i \geq 1$, it follows that

$$(-\ln(1-t))^i = \sum_{j=i}^{\infty} \frac{i!}{j!} \begin{bmatrix} j \\ i \end{bmatrix} t^j = \sum_{j=i}^{\infty} \frac{i}{j} P(i-1, j-1, 0) t^j,$$

in agreement with [34] (Theorem 9). Note that setting $i = 1$ in the last equation yields the Maclaurin series of the natural logarithm

$$\ln(1-t) = -\sum_{j=1}^{\infty} \frac{t^j}{j} = -t - \frac{t^2}{2} - \frac{t^3}{3} - \dots.$$

Remark 11. *Using (34) into (21) leads to the relation*

$$\sum_{i=0}^{m} \frac{(-1)^i}{i!} P(i, m+n, n) S_i(n) = \frac{n}{m+1},$$

and, in particular,

$$\sum_{i=0}^{m} \frac{(-1)^i}{i!} P(i, m+1, 1) = \frac{1}{m+1}.$$

We conclude this section by noting that, by virtue of (34), the theorem [12] (Theorem 6) can be reformulated as follows:

$$\sum_{k=l}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \binom{k}{l} B_{k-l}(q) = \frac{l+1}{n+1} \begin{bmatrix} n+q+r \\ l+q+r \end{bmatrix}_{q+r-1}, \tag{36}$$

where $B_{k-l}(q)$ is a Bernoulli polynomial. Equation (36) may also be written as

$$\sum_{k=l}^{n} \begin{bmatrix} n+r+1 \\ k+r+1 \end{bmatrix}_{r+1} \binom{k}{l} B_{k-l}(q) = \frac{l+1}{n+1} \begin{bmatrix} n+q+r+1 \\ l+q+r+1 \end{bmatrix}_{q+r}, \tag{37}$$

or else,

$$\sum_{k=l}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \binom{k}{l} B_{k-l}(q+1) = \frac{l+1}{n+1} \begin{bmatrix} n+q+r+1 \\ l+q+r+1 \end{bmatrix}_{q+r}, \tag{38}$$

for non-negative integers l, q, r, and $n \geq l$. As an example, putting $l = 2$, $q = 0$, and $r = 1$ in (38), we obtain

$$\sum_{k=2}^{n} (-1)^k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} k(k-1) B_{k-2} = \frac{6}{n+1} \begin{bmatrix} n+2 \\ 4 \end{bmatrix} = n!\big(H_{n+1}^3 - 3H_{n+1}H_{n+1}^{[2]} + 2H_{n+1}^{[3]}\big),$$

in accordance with the particular identity found in [12] (p. 8). Moreover, from (37) and (38), we find that

$$\sum_{k=l}^{n} \begin{bmatrix} n+r+1 \\ k+r+1 \end{bmatrix}_{r+1} \binom{k}{l} B_{k-l}(x) = \sum_{k=l}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \binom{k}{l} B_{k-l}(x+1),$$

which holds for an arbitrary x. In particular, for $r = 0$, and renaming the indices $k - l \to j$, $l \to i$, and $n \to m$, the preceding identity becomes

$$\sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} B_j(x) = \sum_{j=0}^{m-i} \binom{i+j}{i} \begin{bmatrix} m \\ i+j \end{bmatrix} B_j(x+1),$$

which may be compared with (24).

Moreover, (36) can be generalized as follows:

$$\sum_{k=l}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \binom{k}{l} B_{k-l}(x) = \frac{l+1}{n+1} \sum_{k=l}^{n} \binom{k+1}{l+1} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (x+r-1)^{k-l}, \tag{39}$$

which holds for an arbitrary x. For $l = 0$, (39) can be compactly written as

$$\sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r B_k(x) = n! H_{n+1}^{(x+r-1)}. \tag{40}$$

Conversely, (39) can be obtained by performing the l-th derivative with respect to x of both sides of (40).

5. Connection with the Generalized Bernoulli Polynomials

For a complex parameter α, the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ are generated by the relation (see, e.g., [43–45])

$$\left(\frac{z}{e^z - 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!}, \quad |z| < 2\pi,$$

where $B_0^{(\alpha)} = 1$. Note that the value $\alpha = 1$ corresponds to the classical Bernoulli polynomials $B_n(x) = B_n^{(1)}(x)$. In the same way, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are the generalized Bernoulli numbers (also known as Bernoulli numbers of order α or Nörlund polynomials), which are rational polynomials in α of degree n. In particular, $B_n = B_n^{(1)}$ are the ordinary Bernoulli numbers.

There is a close relationship between the generalized Bernoulli polynomials and the higher-order derivatives of the hyperharmonic polynomials. This relationship stems from the fact that (see, e.g., [46] (Equation (2.5)) and [22] (Equation (35)))

$$B_\nu^{(m+1)}(x) = (-1)^m \frac{\nu!}{m!} \frac{d^{m-\nu}}{dx^{m-\nu}} (1-x)^{\overline{m}}, \tag{41}$$

where m and ν are non-negative integers. Equivalently to (41), we have

$$B_\nu^{(m+1)}(x+1) = \nu! \frac{d^{m-\nu}}{dx^{m-\nu}} \binom{x}{m}, \tag{42}$$

in accordance with the result found by Gould in [25] (Equation (13.2)). In particular, for $\nu = m$, from (42), we quickly obtain

$$B_m^{(m+1)}(x+1) = m! \binom{x}{m} = (-1)^m \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix} x^j.$$

As indicated in [25] (see the sentence immediately following Equation (13.2) of [25]), a negative exponent in $\frac{d^{m-\nu}}{dx^{m-\nu}} \binom{x}{m}$ has to be interpreted as integrating $\binom{x}{m}$ $\nu - m$ times. Therefore, for example, when $\nu = m+1$, from (42), we have

$$B_{m+1}^{(m+1)}(x+1) = (m+1)! \int \binom{x}{m} dx = B_{m+1}^{(m+1)}(1) + (-1)^m (m+1) \sum_{j=0}^{m} \frac{(-1)^j}{j+1} \begin{bmatrix} m \\ j \end{bmatrix} x^{j+1},$$

where $B_{m+1}^{(m+1)}(1)$ is the integration constant. Likewise, when $\nu = m+2$, from (42) and the preceding equation, we have

$$\begin{aligned} B_{m+2}^{(m+1)}(x+1) &= (m+2) \int B_{m+1}^{(m+1)}(x+1) dx = B_{m+2}^{(m+1)}(1) \\ &\quad + (m+2) B_{m+1}^{(m+1)}(1)\, x + (-1)^m (m+1)(m+2) \sum_{j=0}^{m} \frac{(-1)^j}{(j+1)(j+2)} \begin{bmatrix} m \\ j \end{bmatrix} x^{j+2}, \end{aligned}$$

where $B_{m+2}^{(m+1)}(1)$ is the integration constant. In general, for an arbitrary $m \geq 0$ and $\nu \geq 0$, we have

$$B_{m+\nu}^{(m+1)}(x+1) = \sum_{t=0}^{\nu-1} \binom{m+\nu}{t} B_{m+\nu-t}^{(m+1)}(1)\, x^t + (-1)^m \binom{m+\nu}{\nu} \sum_{j=0}^{m} (-1)^j \binom{j+\nu}{\nu}^{-1} \begin{bmatrix} m \\ j \end{bmatrix} x^{j+\nu}, \tag{43}$$

where it is understood that the first summation on the right hand side of (43) is zero when $\nu = 0$. For the special case in which $m = 0$, (43) yields the Bernoulli polynomials evaluated at $x+1$, namely,

$$B_\nu(x+1) = (-1)^\nu \sum_{t=0}^{\nu} (-1)^t \binom{\nu}{t} B_{\nu-t}\, x^t,$$

with $B_{\nu-t}$ being the Bernoulli numbers.

On the other hand, as can be easily verified, (41) can be expressed in terms of the hyperharmonic polynomials as follows (see [22] (Proposition 3.13))

$$B_\nu^{(m+1)}(1-x) = (-1)^\nu \nu! \frac{d^{m-\nu-1}}{dx^{m-\nu-1}} H_m^{(x)}. \tag{44}$$

(Note the corrected factor $(-1)^\nu$ in (44) instead of the original factor $(-1)^{m+1}$ appearing in [22] (Proposition 3.13).) Furthermore, (44) can in turn be written as

$$\binom{j}{i} B_{j-i}^{(j+1)}(1-x) = (-1)^{j-i} \frac{j!}{i!} \frac{d^{i-1}}{dx^{i-1}} H_j^{(x)}. \tag{45}$$

Remark 12. *Recalling (12), it follows from (45) that*

$$B_{j-i}^{(j+1)}(1-r) = (-1)^{j-i} \binom{j}{i}^{-1} \begin{bmatrix} j+r \\ i+r \end{bmatrix}_r, \tag{46}$$

which holds for any integers $0 \le i \le j$ and $r \ge 0$. The following special cases can be obtained from (46):

- $B_j^{(j+1)}(1-r) = (-1)^j \begin{bmatrix} j+r \\ r \end{bmatrix}_r = (-1)^j r^{\overline{j}}$. *In particular,* $B_j^{(j+1)} = (-1)^j j!$.
- $B_{j-1}^{(j+1)}(1-r) = (-1)^{j-1}(j-1)! H_j^{(r)}, \quad j \ge 1.$
- $B_{j-i}^{(j+1)} = (-1)^{j-i} \binom{j}{i}^{-1} \begin{bmatrix} j+1 \\ i+1 \end{bmatrix}$. *In particular,* $B_j^{(j+2)} = (-1)^j j! H_{j+1}$.
- $B_{j-i}^{(j+1)}(1) = (-1)^{j-i} \binom{j}{i}^{-1} \begin{bmatrix} j \\ i \end{bmatrix}$. *In particular,* $B_j^{(j+2)}(1) = (-1)^j \frac{j!}{j+1}$.

Let us now invoke the so-called *harmonic polynomials* $H_j(x)$, which were introduced by Cheon and El-Mikkawy in [5] (Section 5). They are defined by the generating function

$$\sum_{j=0}^{\infty} H_j(x) t^j = -\frac{\ln(1-t)}{t(1-t)^{1-x}}, \tag{47}$$

where $H_j(0) = H_{j+1}$. The first few harmonic polynomials are given explicitly by

$$\begin{aligned}
&H_0(x) = 1, \quad H_1(x) = -x + \tfrac{3}{2}, \quad H_2(x) = \tfrac{1}{2}x^2 - 2x + \tfrac{11}{6}, \\
&H_3(x) = -\tfrac{1}{6}x^3 + \tfrac{5}{4}x^2 - \tfrac{35}{12}x + \tfrac{25}{12}, \quad H_4(x) = \tfrac{1}{24}x^4 - \tfrac{1}{2}x^3 + \tfrac{17}{8}x^2 - \tfrac{15}{4}x + \tfrac{137}{60}, \\
&H_5(x) = -\tfrac{1}{120}x^5 + \tfrac{7}{48}x^4 - \tfrac{35}{36}x^3 + \tfrac{49}{16}x^2 - \tfrac{203}{45}x + \tfrac{49}{20}, \\
&H_6(x) = \tfrac{1}{720}x^6 - \tfrac{1}{30}x^5 + \tfrac{23}{72}x^4 - \tfrac{14}{9}x^3 + \tfrac{967}{240}x^2 - \tfrac{469}{90}x + \tfrac{363}{140}, \\
&H_7(x) = -\tfrac{1}{5040}x^7 + \tfrac{1}{160}x^6 - \tfrac{13}{160}x^5 + \tfrac{9}{16}x^4 - \tfrac{1069}{480}x^3 + \tfrac{801}{160}x^2 - \tfrac{29531}{5040}x + \tfrac{761}{280}.
\end{aligned}$$

Upon comparing (4) and (47), it is clear that

$$H_j(x) = H_{j+1}^{(1-x)}, \quad j \ge 0. \tag{48}$$

Therefore, by virtue of (48), we can write (44) as

$$B_\nu^{(m+2)}(x) = (-1)^m \nu! \frac{d^{m-\nu}}{dx^{m-\nu}} H_m(x), \tag{49}$$

where now we assume that $\nu \le m$. In particular,

$$B_m^{(m+2)}(x) = (-1)^m m! H_m(x).$$

As with the hyperharmonic polynomials, the harmonic polynomials enjoy different representations [5,29]. Among them, we highlight the following one:

$$H_m(x) = \frac{1}{m!} \sum_{t=0}^{m} (-1)^t \begin{bmatrix} m+1 \\ t+1 \end{bmatrix} B_t(x), \tag{50}$$

expressing the harmonic polynomials in terms of the Stirling numbers of the first kind and the Bernoulli polynomials. Therefore, combining (49) and (50) yields

$$B_\nu^{(m+2)}(x) = (-1)^m \binom{m}{\nu}^{-1} \sum_{t=m-\nu}^{m} (-1)^t \binom{t}{m-v} \begin{bmatrix} m+1 \\ t+1 \end{bmatrix} B_{t+\nu-m}(x). \tag{51}$$

Thus,

$$B_\nu^{(m+2)} = (-1)^m \binom{m}{\nu}^{-1} \sum_{t=m-\nu}^{m} (-1)^t \binom{t}{m-v} \begin{bmatrix} m+1 \\ t+1 \end{bmatrix} B_{t+\nu-m},$$

and

$$B_\nu^{(m+2)}(1) = (-1)^\nu \binom{m}{\nu}^{-1} \sum_{t=m-\nu}^{m} \binom{t}{m-v} \begin{bmatrix} m+1 \\ t+1 \end{bmatrix} B_{t+\nu-m}.$$

It should be emphasized that, by using (51), one can obtain (at least in principle) the explicit expression of the generalized Bernoulli polynomial $B_\nu^{(\alpha)}(x)$ for any $\nu \geq 0$ and arbitrary parameters α and x. For example, setting $\nu = 1, \ldots, 6$ in (51), we obtain in the first place

$$\begin{aligned}
B_1^{(m+2)}(x) &= x - \tfrac{1}{2}(m+2), \\
B_2^{(m+2)}(x) &= x^2 - (m+2)x + \tfrac{1}{12}(m+2)(3m+5), \\
B_3^{(m+2)}(x) &= x^3 - \tfrac{3}{2}(m+2)x^2 + \tfrac{1}{4}(m+2)(3m+5)\,x - \tfrac{1}{8}(m+1)(m+2)^2, \\
B_4^{(m+2)}(x) &= x^4 - 2(m+2)x^3 + \tfrac{1}{2}(m+2)(3m+5)x^2 - \tfrac{1}{2}(m+1)(m+2)^2\,x \\
&\quad + \tfrac{1}{240}(m+2)(15m^3+60m^2+65m+12), \\
B_5^{(m+2)}(x) &= x^5 - \tfrac{5}{2}(m+2)x^4 + \tfrac{5}{6}(m+2)(3m+5)x^3 - \tfrac{5}{4}(m+1)(m+2)^2\,x^2 \\
&\quad + \tfrac{1}{48}(m+2)(15m^3+60m^2+65m+12)x - \tfrac{1}{96}(m+1)(m+2)^2(3m^2+5m-4), \\
B_6^{(m+2)}(x) &= x^6 - 3(m+2)x^5 + \tfrac{5}{4}(m+2)(3m+5)x^4 - \tfrac{5}{2}(m+1)(m+2)^2\,x^3 \\
&\quad + \tfrac{1}{16}(m+2)(15m^3+60m^2+65m+12)x^2 - \tfrac{1}{16}(m+1)(m+2)^2(3m^2+5m-4)x \\
&\quad + \tfrac{1}{4032}(m+2)(63m^5+315m^4+315m^3-539m^2-938m-240).
\end{aligned}$$

Subsequently, making the transformation $m \to \alpha - 2$ in the above expressions for $B_\nu^{(m+2)}(x)$, $\nu = 1, \ldots, 6$, we obtain the corresponding generalized Bernoulli polynomials (cf. [45] (p. 143))

$$\begin{aligned}
B_1^{(\alpha)}(x) &= x - \tfrac{1}{2}\alpha, \\
B_2^{(\alpha)}(x) &= x^2 - \alpha x + \tfrac{1}{12}\alpha(3\alpha-1), \\
B_3^{(\alpha)}(x) &= x^3 - \tfrac{3}{2}\alpha x^2 + \tfrac{1}{4}\alpha(3\alpha-1)x - \tfrac{1}{8}\alpha^2(\alpha-1), \\
B_4^{(\alpha)}(x) &= x^4 - 2\alpha x^3 + \tfrac{1}{2}\alpha(3\alpha-1)x^2 - \tfrac{1}{2}\alpha^2(\alpha-1)x + \tfrac{1}{240}\alpha(15\alpha^3-30\alpha^2+5\alpha+2), \\
B_5^{(\alpha)}(x) &= x^5 - \tfrac{5}{2}\alpha x^4 + \tfrac{5}{6}\alpha(3\alpha-1)x^3 - \tfrac{5}{4}\alpha^2(\alpha-1)x^2 + \tfrac{1}{48}\alpha(15\alpha^3-30\alpha^2+5\alpha+2)x \\
&\quad - \tfrac{1}{96}\alpha^2(\alpha-1)(3\alpha^2-7\alpha-2), \\
B_6^{(\alpha)}(x) &= x^6 - 3\alpha x^5 + \tfrac{5}{4}\alpha(3\alpha-1)x^4 - \tfrac{5}{2}\alpha^2(\alpha-1)x^3 + \tfrac{1}{16}\alpha(15\alpha^3-30\alpha^2+5\alpha+2)x^2 \\
&\quad - \tfrac{1}{16}\alpha^2(\alpha-1)(3\alpha^2-7\alpha-2)x + \tfrac{1}{4032}\alpha(63\alpha^5-315\alpha^4+315\alpha^3+91\alpha^2-42\alpha-16).
\end{aligned}$$

Note that, when $\alpha = 1$, the above generalized polynomials $B_\nu^{(\alpha)}(x)$, where $\nu = 1, \ldots, 6$, become the ordinary Bernoulli polynomials

$$
\begin{aligned}
B_1(x) &= x - \tfrac{1}{2}, \\
B_2(x) &= x^2 - x + \tfrac{1}{6}, \\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}, \\
B_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x, \\
B_6(x) &= x^6 - 3x^5 + \tfrac{5}{2}x^4 - \tfrac{1}{2}x^2 + \tfrac{1}{42}.
\end{aligned}
$$

Furthermore, when $x = 0$, the above generalized polynomials $B_\nu^{(\alpha)}(x)$, where $\nu = 1, \ldots, 6$, reduce to the following Bernoulli numbers of order α:

$$
\begin{aligned}
B_1^{(\alpha)} &= -\tfrac{1}{2}\alpha, \\
B_2^{(\alpha)} &= \tfrac{1}{4}\alpha^2 - \tfrac{1}{12}\alpha, \\
B_3^{(\alpha)} &= -\tfrac{1}{8}\alpha^3 + \tfrac{1}{8}\alpha^2, \\
B_4^{(\alpha)} &= \tfrac{1}{16}\alpha^4 - \tfrac{1}{8}\alpha^3 + \tfrac{1}{48}\alpha^2 + \tfrac{1}{120}\alpha, \\
B_5^{(\alpha)} &= -\tfrac{1}{32}\alpha^5 + \tfrac{5}{48}\alpha^4 - \tfrac{5}{96}\alpha^3 - \tfrac{1}{48}\alpha^2, \\
B_6^{(\alpha)} &= \tfrac{1}{64}\alpha^6 - \tfrac{5}{64}\alpha^5 + \tfrac{5}{64}\alpha^4 + \tfrac{13}{576}\alpha^3 - \tfrac{1}{96}\alpha^2 - \tfrac{1}{252}\alpha.
\end{aligned}
$$

Of course, by setting $\alpha = 1$ in $B_\nu^{(\alpha)}$, $\nu = 1, \ldots, 6$, we recover the ordinary Bernoulli numbers $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, and $B_6 = \frac{1}{42}$.

6. Further Identities Involving the r-Stirling Numbers of the First Kind

In this section, we will provide further identities involving the r-Stirling numbers of the first kind, the Bernoulli numbers and polynomials, the ordinary Stirling numbers of the first and second kind, and the harmonic numbers.

The following proposition exceptionally involves the r-Stirling numbers of the second kind $\left\{ {k+r \atop j+r} \right\}_r$ [31]. It provides a representation of the Bernoulli polynomials $B_k(x)$ evaluated at $x = -r$ (with r being a non-negative integer).

Proposition 2. *For any non-negative integers k, r, we have*

$$
B_k(-r) = (-1)^k \left(\left\{ {k+r \atop r} \right\}_r + \sum_{j=1}^{k} (-1)^{j+1} \frac{(j-1)!}{j+1} \left\{ {k+r \atop j+r} \right\}_r \right), \tag{52}
$$

where it is understood that the summation on the right hand side of (52) is zero when $k = 0$. In particular,

$$
B_k = \delta_{k,0} + (-1)^{k+1} \sum_{j=1}^{k} (-1)^j \frac{(j-1)!}{j+1} \left\{ {k \atop j} \right\}.
$$

Proof. First note that, for $x = -r$, (40) can be written as

$$
\sum_{j=0}^{k} \left[{k+r \atop j+r} \right]_r B_j(-r) = k! H_{k+1}^{(-1)},
$$

whose Stirling transform is given by

$$
B_k(-r) = \sum_{j=0}^{k} (-1)^{k-j} j! \left\{ {k+r \atop j+r} \right\}_r H_{j+1}^{(-1)}.
$$

On the other hand, it can be seen from (7) that

$$H_{j+1}^{(-1)} = \begin{cases} -\dfrac{1}{j(j+1)}, & j \geq 1, \\ 1, & j = 0. \end{cases}$$

Therefore, by combining the last two equations, we arrive at (52). □

Remark 13. *The above representation* (52) *for $B_k(-r)$ complements the formula for $B_k(r)$ given in* (16) *as well as the following formula established in [47] (Theorem 1.1):*

$$B_k(r) = \sum_{j=0}^{k} (-1)^j \frac{j!}{j+1} \left\{ {k+r \atop j+r} \right\}_r.$$

Moreover, as was noted in the introduction, Spieß [34], Wang [30], and Wuyungaowa [35] derived a number of identities involving the numbers $\binom{j+r}{r} P(i, j+r, r)$. By making use of (34), we next restate some of these identities in terms of the r-Stirling numbers of the first kind.

- From [34] (Theorem 10) and (34), we obtain

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \binom{r}{m-k} \left[{k+r \atop i+r} \right]_r = \frac{(-1)^m}{m!} \left[{m \atop i} \right], \tag{53}$$

  which holds for any integers $r, i \geq 0$. When $r = 1$, (53) gives the recurrence $\left[{m+1 \atop i+1} \right] = \left[{m \atop i} \right] + m \left[{m \atop i+1} \right]$. On the other hand, when $i = 1$, we can write (53) in terms of the hyperharmonic numbers $H_k^{(r)}$ as

$$\sum_{k=0}^{m} (-1)^k \binom{r}{m-k} H_k^{(r)} = \frac{(-1)^m}{m}, \quad m \geq 1.$$

- From [34] (Theorem 13) and (34), we obtain

$$\sum_{k=r}^{m+1-s} \frac{r!s!}{k!\,(m+1-k)!} \left[{k \atop r} \right] \left[{m+1-k \atop s} \right] = \frac{(r+s)!}{(m+1)!} \left[{m+1 \atop r+s} \right],$$

  which holds for any integers $r, s \geq 0$. In particular, for $s = 1$, we retrieve the identity in (32).
- From [34] (Theorem 15) and (34), and after some rearrangements, we obtain

$$q^r \sum_{k=1}^{m} \frac{1}{(k+q)!} \left[{k \atop r} \right] = \frac{1}{q!} - \frac{1}{(m+q)!} \sum_{j=1}^{r} q^{j-1} \left[{m+1 \atop j} \right], \tag{54}$$

  which is valid for any integers $1 \leq r \leq m$ and $q \geq 1$. Setting $r = 1, 2$, and 3 in (54) gives

$$\sum_{k=1}^{m} \frac{1}{k(k+1)\cdots(k+q)} = \frac{1}{q \cdot q!} - \frac{1}{q(m+1)\cdots(m+q)},$$
$$\sum_{k=1}^{m} \frac{H_{k-1}}{k(k+1)\cdots(k+q)} = \frac{1}{q^2 \cdot q!} - \frac{1+qH_m}{q^2(m+1)\cdots(m+q)},$$

  and

$$\sum_{k=1}^{m} \frac{H_{k-1}^2 - H_{k-1}^{[2]}}{k(k+1)\cdots(k+q)} = \frac{2}{q^3 \cdot q!} - \frac{2 + 2qH_m + q^2\left(H_m^2 - H_m^{[2]}\right)}{q^3(m+1)\cdots(m+q)},$$

respectively. The above three identities are to be compared with the corresponding Examples 1, 2, and 3 previously obtained in [34] (p. 849). Moreover, it is worth pointing out that, for the case in which $r = m$, (54) yields the horizontal generating function for the Stirling numbers $\left[{m+1 \atop j+1}\right]$, namely,

$$\sum_{j=0}^{m} \begin{bmatrix} m+1 \\ j+1 \end{bmatrix} q^j = (q+1)(q+2)\dots(q+m),$$

which holds for an arbitrary q.

- From [30] (Equation (3.4)) and (34), we obtain

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!} \binom{m}{k} \binom{k+r}{r}^{-1} \begin{bmatrix} k+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \frac{(-1)^i}{m!} \binom{m+r}{r}^{-1} \begin{bmatrix} m \\ i \end{bmatrix}.$$

Setting here $r = i = 1$, we obtain

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{k+1} = -\frac{1}{m(m+1)}, \quad m \geq 1,$$

which may be compared with the more commonly known identity (see, for example, [32] (Equation (3.2))) $\sum_{k=0}^{m}(-1)^k \binom{m}{k} H_k = -\frac{1}{m}$, where $m \geq 1$.

- From [30] (Equation (3.22)) and (34), we obtain

$$\begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = m! \sum_{k=0}^{m} \frac{1}{k!} \binom{m-k+r}{r} \begin{bmatrix} k \\ i \end{bmatrix} = m! \sum_{k=0}^{m} (-1)^{k-i} \frac{1}{k!} \binom{m+r}{k+r} \begin{bmatrix} k \\ i \end{bmatrix}.$$

In particular, for $r = 0$, we have (cf. [30] (Equation (3.25)))

$$\begin{bmatrix} m+1 \\ i+1 \end{bmatrix} = m! \sum_{k=0}^{m} \frac{1}{k!} \begin{bmatrix} k \\ i \end{bmatrix} = m! \sum_{k=0}^{m} (-1)^{k-i} \frac{1}{k!} \binom{m}{k} \begin{bmatrix} k \\ i \end{bmatrix}.$$

- From [30] (Equations (3.32) and (3.33)) and (34), we obtain

$$\sum_{k=0}^{m} \frac{1}{k!} \binom{r+m-k-1}{m-k} \begin{bmatrix} k+s \\ i+s \end{bmatrix}_s = \frac{1}{m!} \begin{bmatrix} m+r+s \\ i+r+s \end{bmatrix}_{r+s}, \tag{55}$$

and

$$\begin{bmatrix} m+r \\ i+r \end{bmatrix}_r = m! \sum_{k=0}^{m} \frac{1}{k!} \binom{r+m-k-2}{m-k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}, \tag{56}$$

respectively, where r and s are non-negative integers. Notice that, from (55) and (56), one quickly obtains

$$\sum_{k=0}^{m} \frac{1}{k!} \binom{r+m-k-1}{m-k} \begin{bmatrix} k+s \\ i+s \end{bmatrix}_s = \sum_{k=0}^{m} \frac{1}{k!} \binom{r+s+m-k-2}{m-k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}.$$

Furthermore, regarding (56), for $r = 2$, it reads

$$\begin{bmatrix} m+2 \\ i+2 \end{bmatrix}_2 = m! \sum_{k=0}^{m} \frac{1}{k!} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix},$$

or, equivalently,

$$P_i\big(H(m,1;1), H(m,2;1), \dots, H(m,i;1)\big) = \frac{i!}{m+1} \sum_{k=0}^{m} \frac{1}{k!} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}.$$

In particular, when $i = 1$, we recover the well-known identity $\sum_{k=0}^{m} H_k = (m+1)H_m - m$.

- From [30] (Equations (4.1) and (4.3)) and (34), we obtain

$$\sum_{k=0}^{m}(-1)^{m-k}\left\{{m \atop k}\right\}\left[{k+r \atop i+r}\right]_r = \binom{m}{i} r^{m-i}, \tag{57}$$

and

$$\sum_{k=i}^{m}(-r)^{k-i}\binom{k}{i}\left[{m+r \atop k+r}\right]_r = \left[{m \atop i}\right], \tag{58}$$

respectively, where i and r are non-negative integers. When $r = 1$, (57) and (58) reduce to

$$\sum_{k=0}^{m}(-1)^{m-k}\left\{{m \atop k}\right\}\left[{k+1 \atop i+1}\right] = \binom{m}{i},$$

and

$$\sum_{k=i}^{m}(-1)^{k-i}\binom{k}{i}\left[{m+1 \atop k+1}\right] = \left[{m \atop i}\right],$$

respectively. Furthermore, when $i = 1$ and $r = 2$, from (57) we obtain

$$\sum_{k=0}^{m}(-1)^{m-k}(k+1)!\left\{{m \atop k}\right\}(H_{k+1} - 1) = m2^{m-1}.$$

On the other hand, putting $i = 1$ in [30] (Equation (4.5)) yields

$$\sum_{k=0}^{m}(-1)^{m-k}(k+1)!\left\{{m \atop k}\right\} = 2^m.$$

Therefore, combining the last two identities, we obtain

$$\sum_{k=0}^{m}(-1)^{m-k}(k+1)!\left\{{m \atop k}\right\}H_{k+1} = (m+2)\,2^{m-1},$$

which may be compared with [30] (Equation (4.10)).

- From [30] (Equation (4.40)) and (34), we obtain

$$\sum_{k=0}^{m}(-1)^{k-i}i!\left\{{k \atop i}\right\}\left[{m+r+1 \atop k+r+1}\right]_{r+1} = m!\binom{r+m-i}{m-i},$$

which holds for any integers $0 \le i \le m$ and $r \ge 0$. For $r = 0$, the above identity becomes

$$\sum_{k=0}^{m}(-1)^{k-i}i!\left\{{k \atop i}\right\}\left[{m+1 \atop k+1}\right] = m!.$$

- As we saw at the beginning of Section 5, the generalized Bernoulli numbers $B_k^{(i)}$ are defined by the generating function

$$\left(\frac{t}{e^t - 1}\right)^i = \sum_{k=0}^{\infty} B_k^{(i)}\frac{t^k}{k!},$$

where $B_k^{(1)} = B_k$ corresponds to the ordinary Bernoulli numbers. In what follows, we restrict the superscript i to be a positive integer $i \ge 1$. Then, from [30] (Equations (5.2)) and (34), we obtain

$$\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r B_k^{(i)} = \binom{m+i}{i}^{-1} \begin{bmatrix} m+i+r \\ i+r \end{bmatrix}_r. \tag{59}$$

In particular, for $i = 1$, we obtain

$$\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r B_k = \frac{1}{m+1} \begin{bmatrix} m+r+1 \\ r+1 \end{bmatrix}_r,$$

which, for $r = 1$, reduces to the well-known identity (see, e.g., [30] (Equation (5.6)))

$$\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m+1 \\ k+1 \end{bmatrix} B_k = m!H_{m+1}.$$

Moreover, substituting the representation of $B_k^{(i)}$ given by Kim et al. [48]

$$B_k^{(i)} = \sum_{j=0}^{k}(-1)^j \binom{i+j}{i}^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix},$$

into (59), we obtain

$$\sum_{k=0}^{m}\sum_{j=0}^{k}(-1)^{k+j} \binom{i+j}{i}^{-1} \begin{bmatrix} i+j \\ i \end{bmatrix} \begin{Bmatrix} k \\ j \end{Bmatrix} \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r = \binom{m+i}{i}^{-1} \begin{bmatrix} m+i+r \\ i+r \end{bmatrix}_r. \tag{60}$$

In particular, setting $i = 2$ and $r = 0$ in (60) leads to

$$\sum_{k=0}^{m}\sum_{j=0}^{k}(-1)^{k+j} \frac{j!}{j+2} \begin{Bmatrix} k \\ j \end{Bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} H_{j+1} = \frac{m!}{m+2} H_{m+1}.$$

Likewise, employing the representation of $B_k^{(i)}$ given in [43] (Equation (15)),

$$B_k^{(i)} = \sum_{j=0}^{k}(-1)^j \binom{k+i}{i+j} \binom{i+j-1}{i-1} \binom{k+j}{j}^{-1} \begin{Bmatrix} k+j \\ j \end{Bmatrix},$$

we obtain from (59)

$$\sum_{k=0}^{m}\sum_{j=0}^{k}(-1)^{k+j} \frac{\binom{k+i}{i+j}\binom{i+j-1}{i-1}}{\binom{k+j}{j}} \begin{Bmatrix} k+j \\ j \end{Bmatrix} \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r = \binom{m+i}{i}^{-1} \begin{bmatrix} m+i+r \\ i+r \end{bmatrix}_r. \tag{61}$$

In particular, when $i = 3$ and $r = 0$, (61) implies the relation

$$\sum_{k=0}^{m}\sum_{j=0}^{k}(-1)^{k+j}(j+1)(j+2) \frac{\binom{k+3}{j+3}}{\binom{k+j}{j}} \begin{Bmatrix} k+j \\ j \end{Bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} = \frac{6 \cdot m!}{m+3} \left(H_{m+2}^2 - H_{m+2}^{[2]}\right).$$

- From [30] (Equations (5.3)) and (34), we obtain

$$\sum_{k=0}^{m} \begin{bmatrix} m+r+1 \\ k+r+1 \end{bmatrix}_{r+1} B_k(i+1) = m! \binom{m+i+r+1}{i+r} \left(H_{m+i+r+1} - H_{i+r}\right). \tag{62}$$

Taking $r \to r-1$ and $i \to i-1$ in (62) gives

$$\sum_{k=0}^{m} \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r B_k(i) = m! \binom{m+i+r-1}{i+r-2} \left(H_{m+i+r-1} - H_{i+r-2}\right) = m!H_{m+1}^{(i+r-1)},$$

which is just the identity (40) with x replaced by i.

- In [30] (p. 1508), we find the identity

$$\sum_{k=0}^{m} P(k, m+r+i, r+i) \frac{B_k^{(i)}}{k!} = \binom{r+i}{i} \binom{m+i}{i}^{-1} P(i, m+r+i, r).$$

Using (34), and after some minor manipulations, we can write this as

$$\sum_{k=0}^{m} \begin{bmatrix} m+r+i \\ k+r+i \end{bmatrix}_{r+i} B_k^{(i)} = \binom{m+i}{i}^{-1} \begin{bmatrix} m+i+r \\ i+r \end{bmatrix}_r. \tag{63}$$

In particular, if $r = i = 1$ here, then we have

$$\sum_{k=0}^{m} P_k\big(H(m,1;1), H(m,2;1), \dots, H(m,k;1)\big) \frac{B_k}{k!} = \frac{H_{m+1}}{m+1}.$$

Furthermore, from (59) and (63), we find

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r B_k^{(i)} = \sum_{k=0}^{m} \begin{bmatrix} m+r+i \\ k+r+i \end{bmatrix}_{r+i} B_k^{(i)}.$$

Lastly, it is to be noted that, by setting $i = 1$ in (59) and (63), and comparing the resulting equations with (15), we obtain the double identity

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m+r \\ k+r \end{bmatrix}_r B_k = \sum_{k=0}^{m} \begin{bmatrix} m+r+1 \\ k+r+1 \end{bmatrix}_{r+1} B_k = \sum_{k=0}^{m} \begin{bmatrix} m+1 \\ k+1 \end{bmatrix} B_k(r),$$

which holds for any non-negative integers m and r.

- Combining the recurrence of the numbers $\binom{j+r}{r} P(i, j+r, r)$ appearing in [30] (p. 1505)

$$\binom{j+r}{r} P(i, j+r, r) = \binom{j+r-1}{r-1} P(i, j+r-1, r-1) + \binom{j+r-1}{r} P(i, j+r-1, r)$$

as well as (34), we readily obtain the corresponding recurrence for the r-Stirling numbers of the first kind:

$$\begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \begin{bmatrix} j+r \\ i+r \end{bmatrix}_r + j \begin{bmatrix} j+r \\ i+r+1 \end{bmatrix}_{r+1},$$

which may be compared with (33).

- From [35] (Equation (4)) and (34), we obtain the identity

$$\sum_{j=i}^{m} \frac{1}{m+1-j} \frac{1}{(j-1)!} \begin{bmatrix} j+r \\ i+r \end{bmatrix}_{r+1} = \frac{i}{m!} \begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1},$$

which holds for any integers $1 \le i \le m$ and $r \ge 0$. Note the close resemblance of this identity to (31). In particular, for $r = 0$, we find

$$\sum_{j=i}^{m} \frac{1}{m+1-j} \frac{1}{(j-1)!} \begin{bmatrix} j \\ i \end{bmatrix} = \frac{i}{m!} \begin{bmatrix} m+1 \\ i+1 \end{bmatrix},$$

which may be compared with (32).

7. Conclusions

In this paper, we have unified and generalized some previous, unconnected results obtained in [12,30] (see also [21–23,29]) concerning the higher-order derivatives of the hyperharmonic polynomials and their relationship with the complete Bell polynomials. Specifically, we have fully characterized the hyperharmonic polynomials and their successive derivatives in terms of the r-Stirling polynomials of the first kind. In particular, when evaluated at some non-negative integer r, such derivatives can be expressed in terms of the r-Stirling numbers of the first kind or, equivalently, in terms of the numbers $P(i, j+r, r)$, which, in turn, can be expressed in terms of the complete Bell polynomials. Moreover, by exploiting the link between the generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the higher-order derivatives of the hyperharmonic polynomials, we have derived a new formula that enables us to obtain $B_n^{(\alpha)}(x)$ for any non-negative integer n. Finally, we have provided a number of identities involving the r-Stirling numbers of the first kind, the Bernoulli numbers and polynomials, the ordinary Stirling numbers of both kinds, and the harmonic numbers.

It is to be noted that some of the identities obtained in this paper may be new, including the identity considered in the following proposition.

Proposition 3. *For any integers $0 \le i \le m$ and $r \ge 1$, we have*

$$\sum_{j=0}^{m} \frac{1}{j!} \begin{bmatrix} j+r \\ i+r \end{bmatrix}_r = \frac{(-1)^i}{m! r^{i+1}} \sum_{k=0}^{i} (-r)^k \begin{bmatrix} m+r+1 \\ k+r \end{bmatrix}_r. \tag{64}$$

Proof. We start with the following elementary identity

$$\sum_{j=0}^{m} \binom{x+j+r}{j} = \left(1 + \frac{m}{x+r+1}\right) \binom{x+m+r}{m}.$$

On the other hand, from (10), we have

$$\frac{d^i}{dx^i} \binom{x+j+r}{j} \bigg|_{x=0} = \frac{i!}{j!} \begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1}.$$

Hence, it follows that

$$\sum_{j=0}^{m} \frac{i!}{j!} \begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \frac{d^i}{dx^i} \left(1 + \frac{m}{x+r+1}\right) \binom{x+m+r}{m} \bigg|_{x=0}. \tag{65}$$

Now, applying the Leibniz rule for the i-th derivative of the product of two functions yields

$$\begin{aligned}
\frac{d^i}{dx^i} \left(1 + \frac{m}{x+r+1}\right) \binom{x+m+r}{m} \bigg|_{x=0} &= \sum_{k=0}^{i} \binom{i}{k} \frac{d^{i-k}}{dx^{i-k}} \left(1 + \frac{m}{x+r+1}\right) \bigg|_{x=0} \frac{d^k}{dx^k} \binom{x+m+r}{m} \bigg|_{x=0} \\
&= \left(1 + \frac{m}{r+1}\right) \frac{i!}{m!} \begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} + \sum_{k=0}^{i-1} \binom{i}{k} \frac{d^{i-k}}{dx^{i-k}} \left(1 + \frac{m}{x+r+1}\right) \bigg|_{x=0} \frac{d^k}{dx^k} \binom{x+m+r}{m} \bigg|_{x=0} \\
&= \left(1 + \frac{m}{r+1}\right) \frac{i!}{m!} \begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} + \sum_{k=0}^{i-1} \frac{(-1)^{i-k} i!}{(m-1)!(r+1)^{i-k+1}} \begin{bmatrix} m+r+1 \\ k+r+1 \end{bmatrix}_{r+1}.
\end{aligned} \tag{66}$$

Thus, from (65) and (66), we have

$$\sum_{j=0}^{m} \frac{1}{j!} \begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \left(1 + \frac{m}{r+1}\right) \frac{1}{m!} \begin{bmatrix} m+r+1 \\ i+r+1 \end{bmatrix}_{r+1} + \sum_{k=0}^{i-1} \frac{(-1)^{i-k}}{(m-1)!(r+1)^{i-k+1}} \begin{bmatrix} m+r+1 \\ k+r+1 \end{bmatrix}_{r+1},$$

which can be expressed more compactly as

$$\sum_{j=0}^{m-1} \frac{1}{j!} \begin{bmatrix} j+r+1 \\ i+r+1 \end{bmatrix}_{r+1} = \sum_{k=0}^{i} \frac{(-1)^{i-k}}{(m-1)!(r+1)^{i-k+1}} \begin{bmatrix} m+r+1 \\ k+r+1 \end{bmatrix}_{r+1}.$$

Finally, making $m \to m+1$ and $r \to r-1$ in the last equation, we obtain (64). □

Remark 14. *For $i = 0, 1$, and m, (64) leads to the relations*

$$\sum_{j=0}^{m} \binom{r+j-1}{j} = \binom{r+m}{m},$$

$$\sum_{j=0}^{m} H_j^{(r)} = \frac{1}{m!r} \begin{bmatrix} m+r+1 \\ r+1 \end{bmatrix}_r - \frac{1}{r}\binom{r+m}{m},$$

and

$$r^{m+1} = (-1)^m \sum_{k=0}^{m} (-r)^k \begin{bmatrix} m+r+1 \\ k+r \end{bmatrix}_r,$$

respectively, where r stands for any arbitrary integer $r \geq 1$.

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