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Weighted Generalized Fractional Integration by Parts and the Euler–Lagrange Equation

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Abstract: Integration by parts plays a crucial role in mathematical analysis, e.g., during the proof of necessary optimality conditions in the calculus of variations and optimal control. Motivated by this fact, we construct a new, right-weighted generalized fractional derivative in the Riemann–Liouville sense with its associated integral for the recently introduced weighted generalized fractional derivative with Mittag–Leffler kernel. We rewrite these operators equivalently in effective series, proving some interesting properties relating to the left and the right fractional operators. These results permit us to obtain the corresponding integration by parts formula. With the new general formula, we obtain an appropriate weighted Euler–Lagrange equation for dynamic optimization, extending those existing in the literature. We end with the application of an optimization variational problem to the quantum mechanics framework.

Keywords: weighted generalized fractional calculus; integration by parts formula; Euler–Lagrange equation; quantum mechanics; calculus of variations

MSC: 26A33; 49K05

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1. Introduction

In the last decade, fractional calculus played an important role in the theoretical study of dynamical systems by showing significant results in many natural fields and engineering domains [1,2]. For this reason, mathematicians are paying more attention to the generalization of several important formulas in the integral theory of Mathematical Analysis, namely, the Newton–Leibniz formula, the Green formula, and the Gauss and Stokes formulas [3,4]. Some are central tools that enable mathematicians to extend other theories, such as the integration by parts formula, Taylor's formula, the Euler–Lagrange equation, Grönwall's inequality, Lyapunov theorems and LaSalle's invariance principle [5,6].

Often, memory effects are fractionally modeled with Riemann–Liouville and Caputo derivatives [7,8]. However, the fact that the Mittag–Leffler function is a generalization of the exponential function naturally gives rise to new definitions for fractional operators [9,10]. In 2020, Hattaf [11] has proposed a new left-weighted generalized fractional derivative for both Caputo and Riemann–Liouville senses and their associated integral operator, see also [12]. Motivated by their applications in mechanics, where the introduction of the correct operator is needed [8,13], here, we introduce the right-weighted generalized fractional derivative and its associated integral operator, proving their main properties and, in particular, their integration by parts formula.

It is worth emphasizing that integration by parts is of great interest in integral calculus and mathematical analysis. For example, it represents a strong tool to develop the calculus of variations through the so-called Euler–Lagrange equation, which is the central result of dynamic optimization [8]. In recent years, the development of some theoretical practices using fractional derivatives has drawn the attention of several researchers. In 2012, Almeida, Malinowska and Torres [14] reviewed some recent results of fractional variational calculus and discussed the necessary optimality conditions of Euler–Lagrange type for functionals with a Lagrangian containing left and right Caputo derivatives. In 2017, Abdeljawad and Baleanu obtained an adequate integration by parts formula and the corresponding Euler–Lagrange equations using the nonlocal fractional derivative with Mittag–Leffler kernel. In 2019, Abdeljawad et al. [15] developed a fractional integration by parts formula for Riemann–Liouville, Liouville–Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. In 2020, Zine and Torres [16] introduced a stochastic fractional calculus, and obtained a stochastic fractional Euler–Lagrange equation. Motivated by these works, particularly [14–17], and with the help of our weighted generalized fundamental integration by parts formula, we extend the available Euler–Lagrange equations.

The main purpose of our work is to compute a new integration by parts formula for the weighted generalized fractional derivative and to discuss the associated necessary optimality conditions of Euler–Lagrange type. To do this, we organize the paper as follows. In Section 2, we recall some necessary results from the literature. We proceed with Section 3, introducing the right-weighted generalized fractional derivative and its associated integral and studying their well-posedness. Integration by parts is investigated in Section 4, followed by Section 5, where the weighted generalized fractional Euler–Lagrange equation is rigorously proved. We end with Section 6, illustrating the obtained theoretical results with their application in the quantum mechanics framework.

2. Preliminaries

In this section, we present some definitions and properties from the fractional calculus literature, which will help us to prove our main results. In the text, $f \in H^1(a,b)$ is a sufficiently smooth function on $[a,b]$ with $a,b \in \mathbb{R}$. In addition, we adopt the following notations:

$$\phi(\alpha) := \frac{1-\alpha}{B(\alpha)}, \quad \psi(\alpha) := \frac{\alpha}{B(\alpha)},$$

where $0 \le \alpha < 1$ and $B(\alpha)$ is a normalization function obeying $B(0) = B(1) = 1$. In the paper, we denote

$$\mu_\alpha := \frac{\alpha}{1-\alpha}.$$

Lemma 1 (See [18])**.** *Let $\alpha > 0$, $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ ($p \ne 1$ and $q \ne 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$\int_a^b f(x)\, {}^{RL}_{a,1}I^\alpha g(x)dx = \int_a^b g(x)\, {}^{RL}I^\alpha_{b,1} f(x)dx,$$

where ${}^{RL}_{a,1}I^\alpha$ is the left standard Riemann–Liouville fractional integral of order α given by

$$ {}^{RL}_{a,1}I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s)ds, \quad x > a, \tag{1}$$

and ${}^{RL}I^\alpha_{b,1}$ is the right standard Riemann–Liouville fractional integral of order α given by

$$ {}^{RL}I^\alpha_{b,1} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s)ds, \quad x < b. \tag{2}$$

Definition 1 (See [11]). *Let $0 \leq \alpha < 1$ and $\beta > 0$. The left-weighted generalized fractional derivative of order α of function f, in the Riemann–Liouville sense, is defined by*

$$ {}^{R}_{a,w}D^{\alpha,\beta} f(x) = \frac{1}{\phi(\alpha)} \frac{1}{w(x)} \frac{d}{dx} \int_a^x (wf)(s) E_\beta \left[-\mu_\alpha (x-s)^\beta \right] ds, \tag{3} $$

where E_β denotes the Mittag–Leffler function of parameter β defined by

$$ E_\beta(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\beta j + 1)}, \quad z \in \mathbb{C}, \tag{4} $$

and $w \in C^1([a,b])$ with $w, w' > 0$. The corresponding fractional integral is defined by

$$ {}_{a,w}I^{\alpha,\beta} f(x) = \phi(\alpha) f(x) + \psi(\alpha)\, {}^{RL}_{a,w}I^\beta f(x), \tag{5} $$

where ${}^{RL}_{a,w}I^\beta$ is the standard weighted Riemann–Liouville fractional integral of order β given by

$$ {}^{RL}_{a,w}I^\beta f(x) = \frac{1}{\Gamma(\beta)} \frac{1}{w(x)} \int_a^x (x-s)^{\beta-1} w(s) f(s) ds, \quad x > a. \tag{6} $$

3. Well-Posedness of the Right-Weighted Fractional Operators

We denote the right-weighted generalized fractional derivative of order α in the Riemann–Liouville sense by ${}^{R}D^{\alpha,\beta}_{b,w}$, and we define this so that the following identity occurs:

$$ Q\left({}^{R}_{a,w}D^{\alpha,\beta} f \right)(x) = \left({}^{R}D^{\alpha,\beta}_{b,w} Qf \right)(x) $$

with Q being the *reflection operator*, that is, $(Qf)(x) = f(a+b-x)$ with function f defined on the interval $[a,b]$.

Definition 2 (right-weighted generalized fractional derivative). *Let $0 \leq \alpha < 1$ and $\beta > 0$. The right-weighted generalized fractional derivative of order α of function f, in the Riemann–Liouville sense, is defined by*

$$ {}^{R}D^{\alpha,\beta}_{b,w} f(x) = \frac{-1}{\phi(\alpha)} \frac{1}{w(x)} \frac{d}{dx} \int_x^b (wf)(s) E_\beta \left[-\mu_\alpha (s-x)^\beta \right] ds, \tag{7} $$

where $w \in C^1([a,b])$ with $w, w' > 0$.

To properly define the new right-weighted fractional integral, we need to solve the equation ${}^{R}D^{\alpha,\beta}_{b,w} f(x) = u(x)$. We have

$$ {}^{R}D^{\alpha,\beta}_{b,w} f(x) = {}^{R}D^{\alpha,\beta}_{b,w} QQf(x) = Q\, {}^{R}_{a,w}D^{\alpha,\beta} Qf(x) = u(t). $$

Then,

$$ {}^{R}_{a,w}D^{\alpha,\beta} Qf(x) = Qu(x) $$

and thus,

$$ Qf(x) = \phi(\alpha) Qu(x) + \psi(\alpha)\, {}^{RL}_{a,w}I^\beta Qu(x) = \phi(\alpha) Qu(x) + \psi(\alpha)\, Q\, {}^{RL}I^\beta_{b,w} u(x), $$

where ${}^{RL}I^\beta_{b,w}$ is the right-weighted standard Riemann–Liouville fractional integral of order β given by

$$ {}^{RL}I^\beta_{b,w} f(x) = \frac{1}{\Gamma(\beta)} \frac{1}{w(x)} \int_x^b (s-x)^{\beta-1} w(s) f(s) ds, \quad x < b. \tag{8} $$

Applying Q to both sides of (8), we obtain

$$f(t) = \phi(\alpha)u(x) + \psi(\alpha)\, {}^{RL}I^{\beta}_{b,w}u(x).$$

Moreover,

$$\begin{aligned} {}_{a,w}I^{\alpha,\beta}Qf(x) &= \phi(\alpha)Qf(x) + \psi(\alpha)\, {}^{RL}_{a,w}I^{\beta}Qf(x) \\ &= \phi(\alpha)Qf(x) + \psi(\alpha)\, Q^{RL}I^{\beta}_{b,w}f(x) \\ &= Q\big[\phi(\alpha)f(x) + \psi(\alpha)\, {}^{RL}I^{\beta}_{b,w}f(x)\big]. \end{aligned}$$

We are now in the position to introduce the concept of the right-weighted generalized fractional integral.

Definition 3 (right-weighted generalized fractional integral)**.** *Let $0 \le \alpha < 1$ and $\beta > 0$. The right-weighted generalized fractional integral of order α of function f is given by*

$$I^{\alpha,\beta}_{b,w}f(x) = \phi(\alpha)f(x) + \psi(\alpha)\, {}^{RL}I^{\beta}_{b,w}f(x), \tag{9}$$

where $w \in C^1([a,b])$ with $w, w' > 0$.

Our next result provides a series representation to the left- and right-weighted generalized fractional derivatives.

Theorem 1. *Let $0 \le \alpha < 1$ and $\beta > 0$. The left- and right-weighted generalized fractional derivatives of order α of function f can be written, respectively, as*

$${}^{R}_{a,w}D^{\alpha,\beta}f(x) = \frac{1}{\phi(\alpha)} \sum_{j=0}^{\infty} (-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f(x) \tag{10}$$

and

$${}^{R}D^{\alpha,\beta}_{b,w}f(x) = \frac{-1}{\phi(\alpha)} \sum_{j=0}^{\infty} (-\mu_\alpha)^j\, {}^{RL}_{b,w}I^{\beta j}f(x). \tag{11}$$

Proof. The Mittag–Leffler function $E_\beta(x)$ is an entire series of x. Since the series (4) locally and uniformly converges in the whole complex plane, the left-weighted generalized fractional derivative can be rewritten as

$$\begin{aligned} {}^{R}_{a,w}D^{\alpha,\beta}f(x) &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)}\frac{d}{dx}\int_a^x (wf)(s) \sum_{j=0}^{\infty}(-\mu_\alpha)^j \frac{(x-s)^{\beta j}}{\Gamma(\beta j+1)} ds \\ &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)} \sum_{j=0}^{\infty}(-\mu_\alpha)^j \frac{1}{\Gamma(\beta j+1)}\frac{d}{dx}\int_a^x (wf)(s)(x-s)^{\beta j} ds \\ &= \frac{1}{\phi(\alpha)}\frac{1}{w(x)} \sum_{j=0}^{\infty}(-\mu_\alpha)^j \frac{1}{\Gamma(\beta j)}\int_a^x (wf)(s)(x-s)^{\beta j-1} ds \\ &= \frac{1}{\phi(\alpha)} \sum_{j=0}^{\infty}(-\mu_\alpha)^j \big({}^{RL}_{a,w}I^{\beta j}f(x)\big). \end{aligned}$$

From Definition 2, and using the same steps that were used before, one can easily rewrite the new right-weighted generalized fractional derivative as equality (11). The proof of (10) is similar. □

Theorem 2. *Let $0 \leq \alpha < 1$ and $\beta > 0$. The left- and right-weighted generalized fractional derivative and their associated integrals satisfy the following formulas:*

$$ {}_{a,w}I^{\alpha,\beta}\left({}^{R}_{a,w}D^{\alpha,\beta}f\right)(x) = {}^{R}_{a,w}D^{\alpha,\beta}\left({}_{a,w}I^{\alpha,\beta}f\right)(x) = f(x) \tag{12} $$

and

$$ I^{\alpha,\beta}_{b,w}\left({}^{R}D^{\alpha,\beta}_{b,w}f\right)(x) = {}^{R}D^{\alpha,\beta}_{b,w}\left({}^{\alpha,\beta}I_{b,w}f\right)(x) = -f(x). \tag{13} $$

Proof. We note that

$$ \begin{aligned} {}_{a,w}I^{\alpha,\beta}\left({}^{R}_{a,w}D^{\alpha,\beta}f\right)(x) &= \phi(\alpha)\left({}^{R}_{a,w}D^{\alpha,\beta}f\right)(x) + \psi(\alpha){}^{RL}_{a,w}I^{\beta}\left({}^{R}_{a,w}D^{\alpha,\beta}f\right)(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f(x) + \mu_\alpha\, {}^{RL}_{a,w}I^{\beta}\Big(\sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f\Big)(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f(x) - \sum_{j=0}^{\infty}(-\mu_\alpha)^{j+1}\, {}^{RL}_{a,w}I^{\beta+\beta j}f(x) \\ &= f(t). \end{aligned} $$

Then,

$$ \begin{aligned} {}^{R}_{a,w}D^{\alpha,\beta}\left({}_{a,w}I^{\alpha,\beta}f\right)(x) &= \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}\left({}_{a,w}I^{\alpha,\beta}f\right)(x), \\ &= \frac{1}{\phi(\alpha)}\sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}\left[\phi(\alpha)f(x) + \psi(\alpha)\, {}^{RL}_{a,w}I^{\beta}f(x)\right] \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f(x) + \mu_\alpha\sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j+\beta}f(x) \\ &= \sum_{j=0}^{\infty}(-\mu_\alpha)^j\, {}^{RL}_{a,w}I^{\beta j}f(x) - \sum_{j=0}^{\infty}(-\mu_\alpha)^{j+1}\, {}^{RL}_{a,w}I^{\beta j+\beta}f(x) \\ &= f(x) \end{aligned} $$

and equality (12) holds true. The proof of equality (13) is similar. □

4. Integration by Parts

Our formulas of integration by parts are proved in suitable function spaces.

Definition 4 (See [19]). *For $\alpha > 0$, $\beta > 0$ and $1 \leq p \leq \infty$, the following function spaces are defined:*

$$ {}_{a,w}I^{\alpha,\beta}(L_p) := \left\{ f : f = {}_{a,w}I^{\alpha,\beta}(\eta), \eta \in L_p(a,b) \right\} $$

and

$$ I^{\alpha,\beta}_{b,w}(L_p) := \left\{ f : f = I^{\alpha,\beta}_{b,w}(\theta), \theta \in L_p(a,b) \right\}. $$

Theorem 3 (integration by parts without the weighted function). *Let $0 \leq \alpha < 1$, $\beta > 0$, $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$).*

- *If $f \in L_p(a,b)$ and $g \in L_q(a,b)$, then*

$$ \int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx = \int_a^b g(x)(I^{\alpha,\beta}_{b,1}f)(x)dx. \tag{14} $$

- *If* $f \in I_{b,w}^{\alpha,\beta}(L_p)$ *and* $g \in {}_{a,w}I^{\alpha,\beta}(L_q)$*, then*

$$\int_a^b f(x)({}_{a,1}^{R}D^{\alpha,\beta}g)(x)dx = \int_a^b g(x)({}^{R}D_{b,1}^{\alpha,\beta}f)(x)dx. \tag{15}$$

Proof. First, we prove equality (14). Since,

$$\begin{aligned}\int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx &= \int_a^b f(x)\left[\phi(\alpha)g(x) + \psi(\alpha)\,{}_{a,1}^{RL}I^{\beta}g(x)\right] \\ &= \phi(\alpha)\int_a^b f(x)g(x)dx + \psi(\alpha)\int_a^b f(x)\,{}_{a,1}^{RL}I^{\beta}g(x)dx,\end{aligned}$$

it follows from Lemma 1 that

$$\begin{aligned}\int_a^b f(x)({}_{a,1}I^{\alpha,\beta}g)(x)dx &= \phi(\alpha)\int_a^b f(x)g(x)dx + \psi(\alpha)\int_a^b g(x)\,{}^{RL}I_{b,1}^{\beta}f(x)dx \\ &= \int_a^b g(x)\left[\phi(\alpha)f(x) + \psi(\alpha)\,{}^{RL}I_{b,1}^{\beta}f(x)\right] \\ &= \int_a^b g(x)(I_{b,1}^{\alpha,\beta}f)(x)dx.\end{aligned}$$

Now, we prove (15):

$$\begin{aligned}\int_a^b f(x)({}_{a,1}^{R}D^{\alpha,\beta}g)(x)dx &= \int_a^b I_{b,1}^{\alpha,\beta}\theta(x)\left({}_{a,1}^{R}D^{\alpha,\beta}({}_{a,1}I^{\alpha,\beta}\eta)\right)(x)dx \\ &= \int_a^b \eta(x)I_{b,1}^{\alpha,\beta}\theta(x)dx \quad \text{(from Theorem 2)} \\ &= \int_a^b \theta(x)\,{}_{a,1}I^{\alpha,\beta}\eta(x)dx \quad \text{(from equality (14))} \\ &= \int_a^b g(x)({}^{R}D_{b,1}^{\alpha,\beta}f)(x)dx \quad \text{(from Theorem 2)}.\end{aligned}$$

The proof is complete. □

Theorem 4 (weighted generalized integration by parts)**.** *Let* $0 \le \alpha < 1$*,* $\beta > 0$*,* $p \ge 1$*,* $q \ge 1$ *and* $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$ *(*$p \ne 1$ *and* $q \ne 1$ *in the case* $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$*). If* $f \in L_p(a,b)$ *and* $g \in L_q(a,b)$*, then*

$$\int_a^b f(x)({}_{a,w}I^{\alpha,\beta}g)(x)dx = \int_a^b w(x)^2 g(x)\left(I_{b,w}^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx, \tag{16}$$

$$\int_a^b f(x)({}_{a,w}^{R}D^{\alpha,\beta}g)(x)dx = \int_a^b w(x)^2 g(x)\left({}^{R}D_{b,w}^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx. \tag{17}$$

Proof. We have

$$\begin{aligned}\int_a^b f(x)({}_{a,w}I^{\alpha,\beta}g)(x)dx &= \int_a^b w(x)\frac{f(x)}{w(x)}\left({}_{a,w}I^{\alpha,\beta}\left(\frac{gw}{w}\right)\right)(x)dx \\ &= \int_a^b \frac{f(x)}{w(x)}\left({}_{a,1}I^{\alpha,\beta}(gw)\right)(x)dx \\ &= \int_a^b w(x)g(x)\left(I_{b,1}^{\alpha,\beta}\left(\frac{f}{w}\right)\right)(x)dx \text{ (from Theorem 3)} \\ &= \int_a^b g(x)w(x)^2\left(I_{b,w}^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx.\end{aligned}$$

Therefore, equality (16) is true. Similarly, we have

$$\begin{aligned}\int_a^b f(x)({}^{R}_{a,w}D^{\alpha,\beta}g)(x)dx &= \int_a^b w(x)\frac{f(x)}{w(x)}\left({}^{R}_{a,w}D^{\alpha,\beta}\left(\frac{gw}{w}\right)\right)(x)dx \\ &= \int_a^b \frac{f(x)}{w(x)}\left({}^{R}_{a,1}D^{\alpha,\beta}(gw)\right)(x)dx \\ &= \int_a^b w(x)g(x)\left(D^{\alpha,\beta}_{b,1}\left(\frac{f}{w}\right)\right)(x)dx \text{ (from Theorem 3)} \\ &= \int_a^b g(x)w(x)^2\left(D^{\alpha,\beta}_{b,w}\left(\frac{f}{w^2}\right)\right)(x)dx\end{aligned}$$

and equality (17) holds. □

Remark 1. *When $w(t)=1$ and $\alpha=\beta$, then we can obtain from our Theorem 4 the integration by parts formula [17] associated with Atangana–Baleanu derivatives:*

$$\begin{aligned}\int_a^b f(x)\left({}^{AB}_{a}I^{\alpha}g\right)(x)dx &= \int_a^b g(x)\left({}^{AB}I^{\alpha}_{b}f\right)(x)dx, \\ \int_a^b f(x)\left({}^{ABR}_{a}D^{\alpha}g\right)(x)dx &= \int_a^b g(x)\left({}^{ABR}D^{\alpha}_{b}f\right)(x)dx.\end{aligned}$$

From (16) and (17), we obtain the following consequence.

Corollary 1. *Let $0\le\alpha<1$, $\beta>0$, $p\ge1$, $q\ge1$ and $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ ($p\ne1$ and $q\ne1$ in the case $\frac{1}{p}+\frac{1}{q}=1+\alpha$). If $f\in L_p(a,b)$ and $g\in L_q(a,b)$, then*

$$\begin{aligned}\int_a^b f(x)\left(I^{\alpha,\beta}_{b,w}g\right)(x)dx &= \int_a^b w(x)^2 g(x)\left({}_{a,w}I^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx, \\ \int_a^b f(x)\left({}^{R}D^{\alpha,\beta}_{b,w}g\right)(x)dx &= \int_a^b w(x)^2 g(x)\left({}^{R}_{a,w}D^{\alpha,\beta}\left(\frac{f}{w^2}\right)\right)(x)dx.\end{aligned}$$

For a symmetric view of weighted generalized integration by parts, we propose the following corollary of Theorem 4.

Corollary 2. *Let $0\le\alpha<1$, $\beta>0$, $p\ge1$, $q\ge1$ and $\frac{1}{p}+\frac{1}{q}\le 1+\alpha$ ($p\ne1$ and $q\ne1$ in the case $\frac{1}{p}+\frac{1}{q}=1+\alpha$). If $f\in L_p(a,b)$ and $g\in L_q(a,b)$, then*

$$\int_a^b w(x)f(x)\left({}_{a,w}I^{\alpha,\beta}\frac{g}{w}\right)(x)dx = \int_a^b w(x)g(x)\left(I^{\alpha,\beta}_{b,w}\frac{f}{w}\right)(x)dx, \tag{18}$$

$$\int_a^b w(x)f(x)\left({}^{R}_{a,w}D^{\alpha,\beta}\frac{g}{w}\right)(x)dx = \int_a^b w(x)g(x)\left({}^{R}D^{\alpha,\beta}_{b,w}\frac{f}{w}\right)(x)dx. \tag{19}$$

5. The Weighted Generalized Fractional Euler–Lagrange Equation

Let us denote by $AC(I\to\mathbb{R})$ the set of absolutely continuous functions X, where $I=[a,b]$, such that the left and right Riemann–Liouville-weighted generalized fractional derivatives of X exist, endowed with the norm

$$\|X\| = \sup_{t\in I}\Big(|\,X(t)\,| + |\,{}^{RL}_{a,w}D^{\alpha,\beta}X(t)\,| + |{}^{RL}\,D^{\alpha,\beta}_{b,w}X(t)\,|\Big).$$

Let $L \in C^1(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and consider the following minimization problem:

$$J[X] = \left(\int_a^b L\left(t, X(t), {}^{RL}D_{a,w}^{\alpha,\beta} X(t), {}^{RL}D_{b,w}^{\alpha,\beta} X(t)\right) dt \right) \longrightarrow \min \tag{20}$$

subject to the boundary conditions

$$X(a) = X_a, \quad X(b) = X_b. \tag{21}$$

Under appropriate general conditions, one can prove that the minimum of $J[\cdot]$ exists [20]. Here, we are interested in showing the usefulness of our Theorem 4 to prove the necessary optimality conditions for problem (20) and (21). With the help of weighted generalized fractional integration by parts, we obtain the following Euler–Lagrange necessary optimality condition for the fundamental weighted generalized fractional problem of the calculus of variations (20) and (21).

Theorem 5 (the weighted generalized fractional Euler–Lagrange equation)**.** *If $L \in C^1(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ and $X \in AC([a,b] \to \mathbb{R})$ is a minimizer of (20) subject to the fixed end points (21); then, X satisfies the following weighted generalized fractional Euler–Lagrange equation:*

$$\partial_2 L + w(t)^2 \, {}^{R}D_{b,w}^{\alpha,\beta}\left(\frac{\partial_3 L}{w(t)^2}\right) + w(t)^2 \, {}^{R}_{a,w}D^{\alpha,\beta}\left(\frac{\partial_4 L}{w(t)^2}\right) = 0,$$

where $\partial_i L$ denotes the partial derivative of the Lagrangian L with respect to its ith argument evaluated at $\left(t, X(t), {}^{RL}D_{a,w}^{\alpha,\beta} X(t), {}^{RL}D_{b,w}^{\alpha,\beta} X(t)\right)$.

Proof. Let $J[X] = \displaystyle\int_a^b L\left(t, X(t), {}^{R}_{a,w}D^{\alpha,\beta} X(t), {}^{R}D_{b,w}^{\alpha,\beta} X(t)\right) dt$ and assume that X^* is the optimal solution of problem (20) and (21). Set

$$X = X^* + \varepsilon \eta,$$

where $\eta, X \in AC([a,b] \to \mathbb{R})$ and ε is a small, real parameter. By linearity of the weighted generalized fractional derivative, we obtain

$${}^{R}_{a,w}D^{\alpha,\beta} X(t) = {}^{R}_{a,w}D^{\alpha,\beta} X^* + \varepsilon\left({}^{R}_{a,w}D^{\alpha,\beta} \eta(t)\right)$$

and

$${}^{R}D_{b,w}^{\alpha,\beta} X(t) = {}^{R}D_{b,w}^{\alpha,\beta} X^* + \varepsilon\left({}^{R}D_{b,w}^{\alpha,\beta} \eta(t)\right).$$

Now, consider the following function:

$$\begin{aligned} J(\varepsilon) = \int_a^b L\Big(t, X^*(t) + \varepsilon\eta(t),\ & {}^{R}_{a,w}D^{\alpha,\beta} X^*(t) + \varepsilon\left({}^{R}_{a,w}D^{\alpha,\beta} \eta(t)\right), \\ & {}^{R}D_{b,w}^{\alpha,\beta} X^*(t) + \varepsilon\left({}^{R}D_{b,w}^{\alpha,\beta} \eta(t)\right)\Big) dt. \end{aligned}$$

Fermat's theorem asserts that $\displaystyle\frac{d}{d\varepsilon} J(\varepsilon)\Big|_{\varepsilon=0} = 0$ and we deduce, by the chain rule, that

$$\int_a^b \left(\partial_2 L \cdot \eta + \partial_3 L \cdot {}^{R}_{a,w}D^{\alpha,\beta} \eta + \partial_4 L \cdot {}^{R}D_{b,w}^{\alpha,\beta} \eta\right) dt = 0.$$

Using Theorem 4 of weighted fractional integration by parts, we obtain that

$$\int_a^b \left(\partial_2 L \cdot \eta + w(t)^2 \cdot \eta \cdot {}^{R}D_{b,w}^{\alpha,\beta}\left(\frac{\partial_3 L}{w(t)^2}\right) + w(t)^2 \cdot \eta \cdot {}^{R}_{a,w}D^{\alpha,\beta}\left(\frac{\partial_4 L}{w(t)^2}\right)\right) dt = 0.$$

The result follows by the fundamental theorem of the calculus of variations. □

6. An Application

Let us consider the weighted generalized fractional variational problem (20) and (21) with

$$L\left(t, X(t), {}^{R}_{a,w} D^{\alpha,\beta} X(t), {}^{R} D^{\alpha,\beta}_{b,w} X(t)\right) = \frac{1}{2}\left(\frac{1}{2} m \mid {}^{R}_{a,w} D^{\alpha,\beta} X(t) \mid^2 + \frac{1}{2} m \mid {}^{R} D^{\alpha,\beta}_{b,w} X(t) \mid^2\right) - V(X(t)),$$

where X is an absolutely continuous function on $[a,b]$ and V maps $C^1(I \to \mathbb{R})$ to $\mathbb{R}$. Note that

$$\frac{1}{2}\left(\frac{1}{2} m \mid {}^{R}_{a,w} D^{\alpha,\beta} X(t) \mid^2 + \frac{1}{2} m \mid {}^{R} D^{\alpha,\beta}_{b,w} X(t) \mid^2\right)$$

can be viewed as a weighted generalized kinetic energy in the quantum mechanics framework. By applying our Theorem 5 to the current variational problem, we obtain that

$$\frac{1}{2} m \left[w(t)^2 \, {}^{R} D^{\alpha,\beta}_{b,w} \left(\frac{{}^{R}_{a,w} D^{\alpha,\beta} X(t)}{w(t)^2} \right) + w(t)^2 \, {}^{R}_{a,w} D^{\alpha,\beta} \left(\frac{{}^{R} D^{\alpha,\beta}_{b,w} X(t)}{w(t)^2} \right) \right] = V'(X(t)), \tag{22}$$

where V' is the derivative of the potential energy of the system. We observe that relation (22) generalizes Newton's dynamical law $m\ddot{X}(t) = V'(X(t))$.

7. Conclusions

In this work, some definitions and properties of a recent class of fractional operators defined by general integral operators, with and without singular kernels, are recalled. A new definition of a right-weighted generalized fractional operator in the Riemann–Liouville sense is then posed, serving as a prerequisite for the establishment of a new weighted generalized integration by parts formula, which shows a duality relation with the existing left weighted generalized fractional operator in the Riemann–Liouville–Hattaf sense [11]. In the context of the fractional calculus of variations, we have investigated weighted generalized Euler–Lagrange equations, which were then used to produce an effective application in the quantum mechanics setting, after a proper definition of kinetic energy.

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