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Randić Index of a Line Graph

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Abstract: The Randić index of a graph G, denoted by $R(G)$, is defined as the sum of $1/\sqrt{d(u)d(v)}$ for all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G. In this note, we show that $R(L(T)) > \frac{n}{4}$ for any tree T of order $n \geq 3$. A number of relevant conjectures are proposed.

Keywords: Randić index; trees; line graphs; claw-free graphs

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1. Introduction

Let $G = (V(G), E(G))$ be a graph. For a vertex $v \in V(G)$, $d_G(v)$ (simply by $d(v)$) denotes the degree of v in G. The symbol $N_G(v)$ presents the set of neighbors of the vertex v. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. In 1975, the Randić index $R(G)$ of a graph G was introduced by Randić [1] as the sum of $\frac{1}{\sqrt{d(u)d(v)}}$ over all edges uv of G, i.e.

$$R(G) = \sum_{uv \in E(G)} 1/\sqrt{d(u)d(v)}.$$

This parameter is quite useful in mathematical chemistry and has been extensively studied, see the monograph [2]. We refer to [3–7] for some recent results. As usual, P_n, C_n and K_n denote the path, the cycle and the complete graphs of order n, respectively. In addition, $K_{m,n}$ represents the complete bipartite graph with m and n vertices in its two parts.

Let us recall two classical results on the Randić index of graphs, which are a lower bound and an upper bound in terms of their orders.

Theorem 1 (Bollobás and Erdős [8])**.** *For a connected graph G of order n, $R(G) \geq \sqrt{n-1}$, with equality, if and only if $G \cong K_{1,n-1}$.*

Theorem 2 (Fajtlowicz [9])**.** *For a graph G of order n, $R(G) \leq \frac{n}{2}$ with equality, if and only if each component of G has order at least two and is regular.*

The line graph of a graph G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$, in which two vertices are adjacent, if and only if they share a common end vertex in G. The relation between Wiener index of a graph and that of its line graph was investigated in [10–13].

Interestingly, for a graph G, $R(L(G))$ is usually large contrast to $R(G)$ (with some exception, P_n for instance). In this note, we investigate the Randić indices of the line graphs of graphs with order given. The following results illustrate that $L(K_n)$ has the maximum Randić index among all line graphs of graphs with order n.

Theorem 3. *For any graph G of order $n \geq 3$, $R(L(G)) \leq \frac{n(n-1)}{4}$, with equality, if and only if $G \cong K_n$.*

Proof. Observe that K_n has the maximum number of edges among all graphs of order n. Thus, the result is an immediate consequence of Theorem 2. $\square$

Our main contribution is to show that $R(L(T)) > \frac{n}{4}$ for any tree T of order $n \geq 3$. A number of relevant conjectures are proposed.

2. Results

We begin with a wider family of graphs than line graphs. A graph G is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. It is well-known that every line graph is claw-free. The following lemma is one of our main tools proving Theorem 5.

Lemma 1. *Let G_1 and G_2 be two disjoint nontrivial connected graphs. If G is a graph obtained from G_1 and G_2 by identifying a vertex $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, then*

$$R(G) = R(G_1) + R(G_2) - (a + b - c),$$

where

$$a = \frac{1}{\sqrt{d_{G_1}(v_1)}} \sum_{x \in N_{G_1}(v_1)} \frac{1}{\sqrt{d_{G_1}(x)}},$$
$$b = \frac{1}{\sqrt{d_{G_2}(v_2)}} \sum_{y \in N_{G_2}(v_2)} \frac{1}{\sqrt{d_{G_2}(y)}},$$
$$c = \frac{1}{\sqrt{(d_{G_1}(v_1) + d_{G_2}(v_2))}} \Big(\sum_{x \in N_{G_1}(v_1)} \frac{1}{\sqrt{d_{G_1}(x)}} + \sum_{y \in N_{G_2}(v_2)} \frac{1}{\sqrt{d_{G_2}(y)}} \Big).$$

Furthermore, if G is claw-free, then $a + b - c < 1$.

Proof. The first part of the result is obvious. Next we show the second part. For convenience, let $d_i = d_{G_i}(v_i)$ for each $i \in \{1, 2\}$ and $\sum_{x \in N_{G_1}(v_1)} \frac{1}{\sqrt{d_{G_1}(x)}} = \sum_{i=1}^{d_1} \frac{1}{\sqrt{a_i}}$ and $\sum_{x \in N_{G_1}(v_1)} \frac{1}{\sqrt{d_{G_2}(y)}} = \sum_{j=1}^{d_2} \frac{1}{\sqrt{b_j}}$.

Since $d_1 \geq 1$ and $d_2 \geq 1$, we have $\frac{\sqrt{d_1}+\sqrt{d_2}}{\sqrt{d_1+d_2}} > 1$. In addition, since G is claw-free, both $N_{G_1}(v_1)$ and $N_{G_2}(v_2)$ are cliques, implying that $a_i \geq d_1$ for each i and $b_j \geq d_2$ for each j. Thus, we have

$$\begin{aligned}
a + b - c &= \Big(\frac{1}{\sqrt{d_1}} - \frac{1}{\sqrt{d_1 + d_2}}\Big) \sum_{i=1}^{d_1} \frac{1}{\sqrt{a_i}} + \Big(\frac{1}{\sqrt{d_2}} - \frac{1}{\sqrt{d_1 + d_2}}\Big) \sum_{j=1}^{d_2} \frac{1}{\sqrt{b_j}} \\
&\leq \Big(\frac{1}{\sqrt{d_1}} - \frac{1}{\sqrt{d_1 + d_2}}\Big)\sqrt{d_1} + \Big(\frac{1}{\sqrt{d_2}} - \frac{1}{\sqrt{d_1 + d_2}}\Big)\sqrt{d_2} \\
&= 2 - \frac{\sqrt{d_1} + \sqrt{d_2}}{\sqrt{d_1 + d_2}} \\
&< 1.
\end{aligned}$$

The proof is completed. $\square$

We will also use the following result in the proof of our main theorem.

Theorem 4 (Hansen and Vukicević [14]). *Let G be a simple graph. If $d(v) = \delta(G)$, then*

$$R(G) - R(G - v) \geq \frac{1}{2}\sqrt{\frac{\delta(G)}{\Delta(G)}}.$$

By the above theorem, if $\delta(G) > 0$, then $R(G) - R(G - v) > 0$ for any vertex $v \in V(G)$ with $d(v) = \delta(G)$.

Theorem 5. *For any tree T of order $n \geq 3$, $R(L(T)) > \frac{n}{4}$.*

Proof. By induction on n. Observe that $L(P_n) \cong P_{n-1}$and $L(K_{1,n-1}) \cong K_{n-1}$. Moreover, $R(P_n) = \frac{n-3}{2} + \sqrt{2}$ and $R(K_n) = \frac{n}{2}$. A simple computation shows that the result holds for $T \in \{P_n, K_{1,n-1}\}$. So, assume that $n \geq 5$ and T is neither a star nor a path.

Let P be a longest path of T. Label the vertices of P as $v_0, v_1, \ldots, v_t$ consecutively. Clearly, $t \geq 3$. Observe that all neighbors of v_1 except v_2 have degree 1. Let T_{v_1} and T_{v_2} be the two components of $T - v_1v_2$ containing v_1 and v_2, respectively. Let $T_1 = T \setminus E(T_{v_2})$ and $T_2 = T \setminus E(T_{v_1})$. Let $G_i = L(T_i)$ for each $i \in \{1, 2\}$, and $G = L(G)$. Note that G is the graph obtained from G_1 and G_2 by identifying the vertex v_1v_2. By Lemma 1, we have

$$R(G) = R(G_1) + R(G_2) - (a + b - c),$$

where a, b, and c are those defined in the statement of Lemma 1.

By the induction hypothesis, $R(G_2) > \frac{n-d_1}{4}$, where $d_1 = d_T(v_1) - 1$. In addition, $R(G_1) = \frac{d_1+1}{2}$. We consider two cases.

Case 1. $d_1 \geq 2$

Since G is a line graph (so it is claw-free), by Lemma 1, $a + b - c < 1$.

$$\begin{aligned} R(G) &= R(G_1) + R(G_2) - (a + b - c) \\ &> \frac{d_1 + 1}{2} + \frac{n - d_1}{4} - 1 \\ &= \frac{n}{4} + \frac{d_1}{4} - \frac{1}{2} \\ &\geq \frac{n}{4}. \end{aligned}$$

Note that if there exists an edge $uv \in T$ with $d_T(u) \geq 3$ such that all neighbors of u except v have degree 1, then by the argument as in Case 1, we can show that $R(G) > \frac{n}{4}$. So, in what follows, we may assume that $d_T(u) = 2$ for any vertex $u \in V(T)$ with all neighbors but one having degree 1.

Case 2. $d_1 = 1$

Let $d_2 = d_T(v_2) - 1$. We consider two subcases.

Subcase 2.1. $d_2 = 1$

Since $d_G(v_1v_2) = d_1 + d_2 = 2$, we have

$$\begin{aligned} R(G) &= R(G_1) + R(G_2) - (a + b - c) \\ &> 1 + \frac{n - 1}{4} - \left(2 - \frac{\sqrt{d_1} + \sqrt{d_2}}{\sqrt{d_1 + d_2}}\right) \\ &\geq \frac{n - 1}{4} + 1 - (2 - \sqrt{2}) \\ &> \frac{n}{4}. \end{aligned}$$

Subcase 2.2. $d_2 \geq 2$

By the choice of P and the remark before Case 2, the component of $T - v_2v_3$ containing v_2 is a wounded spider, as shown in Figure 1. Denote this component by $W_{r,s}$, where r and s are the numbers of neighbors of v_2 having degrees 1 and 2, respectively.

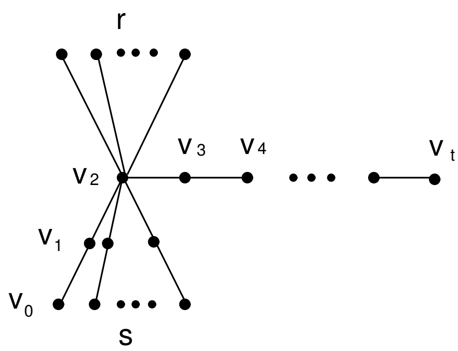

Figure 1. The local structure of T.

Let T_1 be the subtree of T obtained from $W_{r,s}$ by joining v_3 to v_2, and $T_2 = T \setminus E(W_{r,s})$. Moreover, let $G_i = L(T_i)$ for each $i \in \{1, 2\}$, and let $G = L(T)$. Clearly, G is obtained from G_1 and G_2 by identifying the vertex v_2v_3. By the induction hypothesis, $R(G_2) > \frac{n-r-2s}{4}$. One can see that

$$R(G_1) = \frac{s}{\sqrt{r+s+1}} + \frac{\binom{s}{2}}{r+s+1} + \frac{\binom{r+1}{2}}{r+s} + \frac{(r+1)s}{\sqrt{(r+s+1)(r+s)}}. \tag{1}$$

Deleting leaves (minimum degree vertices) of G_1 one-by-one, we end up with K_{r+s+1}. By Theorem 2.2, we have

$$R(G_1) > R(K_{r+s+1}) = \frac{r+s+1}{2}. \tag{2}$$

Subcase 2.2.1. $r \geq 2$

By (2), $R(G_1) > \frac{r+s+1}{2}$. By Lemma 1, we have

$$\begin{aligned} R(G) &> R(G_1) + R(G_2) - 1 \\ &> \frac{r+s+1}{2} + \frac{n-r-2s}{4} - 1 \\ &\geq \frac{n}{4}. \end{aligned}$$

Subcase 2.2.2. $r = 0$

Since $d_2 = r + s$, $s = d_2 \geq 2$. By (1), $R(G_1) = \frac{s}{\sqrt{s+1}} + \frac{s(s-1)}{2(s+1)} + \frac{s}{\sqrt{(s+1)s}} > \frac{s}{2} + 1$ for any $s \geq 2$. By Lemma 1,

$$\begin{aligned} R(G) &= R(G_1) + R(G_2) - (a+b-c) \\ &> \frac{s}{2} + 1 + \frac{n-2s}{4} - 1 \\ &= \frac{n}{4}. \end{aligned}$$

Subcase 2.2.3. $r = 1$

By (1), $R(G_1) = \frac{s}{\sqrt{s+2}} + \frac{s(s-1)}{2(s+2)} + \frac{1}{s+2} + \frac{2s}{\sqrt{(s+2)(s+1)}} \geq \frac{1+2s}{4} + 1 = \frac{r+2s}{4} + 1$. Thus, by Lemma 1,

$$\begin{aligned} R(G) &= R(G_1) + R(G_2) - (a+b-c) \\ &> \frac{1+2s}{4} + 1 + \frac{n-1-2s}{4} - 1 \\ &= \frac{n}{4}. \end{aligned}$$

The proof is completed. □

3. Discussion

In this paper, we show that $R(L(T)) > \frac{n}{4}$ for any tree T of order $n \geq 3$. For a graph G, $S(G)$ denotes the graph obtained from G by inserting exactly one vertex into each edge of G. For a positive even integer n, $S(K_{1,\frac{n}{2}})^-$ denotes the tree obtained from $S(K_{1,\frac{n}{2}})$ by deleting a leaf. Define a function $f(n)$ as

$$f(n) = \begin{cases} \frac{n-3}{4} + \sqrt{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ \frac{n}{4} - \frac{3}{2} + \frac{2}{n} + \sqrt{1 - \frac{2}{n}} + \sqrt{\frac{n}{2}} - \sqrt{\frac{2}{n}}, & \text{if } n \text{ is even .} \end{cases}$$

Indeed,

$$f(n) = \begin{cases} R(L(S(K_{1,\frac{n-1}{2}}))), & \text{if } n \text{ is odd} \\ R(L(S(K_{1,\frac{n}{2}})^-)), & \text{if } n \text{ is even .} \end{cases}$$

We strongly believe that the following conjectures holds.

Conjecture 1. *For any tree T of order $n \geq 3$, $R(L(T)) \geq f(n)$, with equality, if and only if*

$$T \cong \begin{cases} S(K_{1,\frac{n-1}{2}}), & \text{if } n \text{ is odd} \\ S(K_{1,\frac{n}{2}})^-, & \text{if } n \text{ is even .} \end{cases}$$

Conjecture 2. *For any connected graph G of order $n \geq 3$, $R(L(G)) \geq f(n)$, with equality holds, if and only if*

$$T \cong \begin{cases} S(K_{1,\frac{n-1}{2}}), & \text{if } n \text{ is odd} \\ S(K_{1,\frac{n}{2}})^-, & \text{if } n \text{ is even .} \end{cases}$$

Since every line graph is claw-free, we propose a more general conjecture.

Conjecture 3. *For any connected claw-free graph of order $n \geq 2$, $R(G) \geq f(n)$.*

A weaker conjecture than the above is the following one.

Conjecture 4. *For any connected claw-free graph of order $n \geq 2$, $R(G) > \frac{n}{4}$.*

As we have seen before, $L(P_n) \cong P_{n-1}$ for any $n \geq 2$ and $K_{1,3} \cong K_3$. We guess that the following is true.

Conjecture 5. *Let G be a connected graph of order $n \geq 3$. If $\delta(G) \geq 2$, then $R(L(G)) \geq R(G)$, with equality, if and only if $G \cong C_n$.*

The Harmonic index $H(G)$ of a graph G is defined as $H(G) = \sum_{uv \in E(G)} \frac{2}{d(u)+d(v)}$. It is natural that one may consider the same problems for the Harmonic index as we did in this note. Specifically,

Conjecture 6. *$H(L(T)) > \frac{n}{4}$ for any tree T of order $n \geq 3$.*

If the above conjecture is true, it implies the main result of this note, since $R(G) \geq H(G)$ for any graph G.

Recall that for a real number α, the general Randić index of a graph G, denoted by $R_\alpha(G)$, is

$$R_\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha.$$

The sum-connectivity index $\chi(G)$ and the general sum-connectivity index $\chi_\alpha(G)$ were proposed by Zhou and Trinajstić in [15,16] and were defined as

$$\chi(G) = \sum_{uv \in E(G)} (d(u) + d(v))^{-\frac{1}{2}}$$

and

$$\chi_\alpha(G) = \sum_{uv \in E(G)} (d(u) + d(v))^\alpha.$$

It is interesting to consider the general Randić index and the general sum-connectivity index of a line graph for different value of α.

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