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On the Group of Absolutely Summable Sequences †

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† Dedicated to María Jesús Chasco.

Abstract: For an abelian topological group G, the sequence group $\ell^1(G)$ of all absolutely summable sequences in G is studied. It is shown that $\ell^1(G)$ is a Pontryagin reflexive group in case G is a reflexive metrizable group or an LCA group. Further, $\ell^1(G)$ has the Schur property if and only if G has it and $\ell^1(G)$ is a Schwartz group if and only if G is linearly topologized.

Keywords: summable sequence; absolutely summable sequence; locally quasi-convex group; Schwartz group; nuclear group; Pontryagin reflexive group; Schur property; LCA groups

MSC: 22A05; 22A10; 22B05; 46A11; 46A20

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1. Preliminaries

1.1. Introduction

It is a well-known result in the theory of locally convex vector spaces that for a metrizable locally convex space (E, τ), the underlying topology τ is the finest locally convex topology giving rise to the dual space $(E, \tau)'$ in all continuous linear forms ([1], p. 263). The idea of a finest compatible topology was generalized in [2] to locally quasi-convex groups. More precisely, for a locally quasi-convex group (G, τ), the topology τ is called the Mackey topology (see [2] for details) if it is the finest among all locally quasi-convex group topologies giving rise to the character group $(G, \tau)^\wedge$. For several years, it was an open question as to whether every metrizable locally quasi-convex group topology is a Mackey topology. The first example giving a negative answer to this question was the group of all null-sequences in the torus $c_0(\mathbb{T}) = \{(z_n) \in \mathbb{T}^{\mathbb{N}} : z_n \to 0\}$ endowed with the topology of uniform convergence. The important observation was that the dual group of $c_0(\mathbb{T})$ is isomorphic to $\mathbb{Z}^{(\mathbb{N})}$; in particular, it is countable. This implies that the weak topology $\sigma(c_0(\mathbb{T}), c_0(\mathbb{T})^\wedge)$ is metrizable and precompact. Because this topology is strictly weaker than the topology of uniform convergence on $c_0(\mathbb{T})$, the metrizable weak topology cannot be the Mackey topology. In [3], this was generalized to $c_0(G)$ where G is a compact connected abelian metrizable group. The main idea was to show that the character group of such a group has a countable dual group. In [4] (Theorem 3.4), an alternative proof for this was given, the structure of the character group of $c_0(G)$ was described, and many properties of these groups have been studied since then (cf. [4–7]).

In [7] (Theorem 1.3), Gabriyelyan proves that for an LCA group G, the following assertions are equivalent: G is totally disconnected iff $c_0(G)$ is a nuclear group iff $c_0(G)$ is a Schwartz group iff $c_0(G)$ respects compactness. Further, in [4] (Theorem 1.2), he generalized the results from [5] and shows that $c_0(G)$ is a reflexive group.

In [5], groups of the form $\ell^p(\mathbb{T}) = \{(z_n) \in \mathbb{T}^{\mathbb{N}} : \sum_{n=1}^{\infty} |1 - z_n|^p < \infty\}$ were investigated and it was shown that for $0 < p < \infty$, $\ell^p(\mathbb{T})$ is a monothetic Polish group which is topologically isomorphic to $\ell^p/\mathbb{Z}^{(\mathbb{N})}$ ([5] Proposition 5/Theorem 1) and $\ell^1(\mathbb{T})$ is reflexive.

Because in the theory of Banach spaces, the sequence space c_0 of (real or complex) null-sequences, the space ℓ^1 of all absolutely summable sequences, and the space ℓ^∞ of bounded sequences play an important role, it is natural to generalize them to the corresponding sequence groups for abelian Hausdorff groups G. This was performed in the case $c_0(G)$ by Gabriyelyan and will now be carried out for the groups $\ell^1(G)$ of absolutely summable sequences (Definition 3).

Alternatively, unconditionally Cauchy sequences and absolutely summable sequences (suitably defined) were studied in the realm of topological vector spaces in order to characterize nuclear vector spaces (cf. ([8], 21.2.1) and ([9], p.73)). This idea was picked up by Domínguez Pérez and Tarieladze in [10,11] in order to characterize nuclear groups (see below).

Our main interest is to find sufficiency conditions for a group G such that $\ell^1(G)$ is reflexive. We prove that a metrizable group G is reflexive if and only if the sequence group $\ell^1(G)$ is reflexive (Corollary 6). Moreover, for every LCA group G, the group $\ell^1(G)$ is reflexive (Theorem 4).

A normed vector space has the Schur property if every sequence which converges in the weak vector space topology is also convergent with respect to the norm. As the vector space ℓ^1 has the Schur property ([12], 27.13), it is natural to ask whether $\ell^1(G)$ also has a similar property. It turns out that for a locally quasi-convex group G, $\ell^1(G)$ has the (analogue of the) Schur property for groups if and only if G has this property (Theorem 6).

In [13], Banaszczyk introduced nuclear groups, a Hausdorff variety of groups which contains all locally convex nuclear vector spaces and all LCA groups. In [14], Schwartz groups were defined, examples were given, and first properties were shown. Because no infinite-dimensional normed space is neither a Schwartz space nor a nuclear vector space, it is not surprising that the hypotheses on a group G such that $\ell^1(G)$ is a Schwartz group or a nuclear group must be rather restrictive. Indeed, we show that for a locally quasi-convex group G, the group $\ell^1(G)$ is a Schwartz group iff $\ell^1(G)$ is a nuclear group iff G is linearly topologized (Theorem 8). This is an analogue of Gabriyelyan's result for $c_0(G)$ as every totally disconnected LCA group is linearly topologized.

The paper is organized as follows:

In Section 1.2, we gather material concerning reflexive groups, and in Section 1.3, we study properties of the Minkowski functional for groups. Section 2 is dedicated to the study of the sequence group $\ell^1(G)$, the focus of the paper. We start in Section 2.1 with the definition and basic properties of the topological group $\ell^1(G)$. We show that, on the one hand, G can be embedded in $\ell^1(G)$ and, on the other, G is a quotient group of $\ell^1(G)$ (Lemma 1). Thus, it is not surprising that G and $\ell^1(G)$ have many properties in common in the sense that G satisfies property P iff $\ell^1(G)$ satisfies P. For example, this holds for cardinal invariants, separation axioms, completeness, and local quasi-convexity. The mapping $G \to \ell^1(G)$ is a covariant functor from the category of abelian topological groups into itself (Lemma 6). Further, the compact subsets of $\ell^1(G)$ are characterized (Proposition 8). In Section 2.2, the dual group of $\ell^1(G)$ is described and it is shown that G is a locally quasi-convex group if and only if $\ell^1(G)$ has this property. Further, sufficiency conditions are established for the continuity of $\alpha_{\ell^1(G)}$, the canonical mapping in the bidual group $G^{\wedge\wedge}$ (see Section 1.2 for a precise definition). In Theorem 2, it is shown that $\alpha_{\ell^1(G)}$ is continuous if G is reflexive and $G^{\wedge}$ is complete with a countable point-separating subgroup. In Section 2.3, the second character group is studied. It is shown that under mild conditions on the group G (e.g., if G is reflexive), $\ell^1(G)^{\wedge\wedge}$ can be canonically identified with $\ell^1(G^{\wedge\wedge})$, from which it follows that $\ell^1(G)$ is reflexive if G is a metrizable reflexive group or an LCA group.

In Section 2.4, we recall first the Schur property for groups (Definition 4) and prove for G locally quasi-convex that $\ell^1(G)$ has the Schur property if and only if G does. In Section 2.5 of this chapter, we recall the definition of Schwartz groups, properties of nuclear groups, and classify locally quasi-convex groups for which $\ell^1(G)$ is a Schwartz group, respectively, a nuclear group.

Finally, in Section 3, we present some open questions related to this article.

1.2. Notation and Preliminaries

Let $\mathbb{N} = \{1, 2, \ldots\}$ denote the natural numbers. For $m \in \mathbb{N}$, put $\underline{m} := \{1, \ldots, m\}$ and denote by $\aleph_0$ the cardinality of $\mathbb{N}$. As usual, $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ denotes the set of integers.

For a topological group G, let $\mathcal{N}_G(0)$ denote the set of all **symmetric** neighborhoods of 0. If the group G is clear from context, the index G will be omitted.

The compact torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is isomorphic to the complex numbers of modulus one. For technical reasons, we prefer the additive notation.

Let G be an abelian Hausdorff group. The set of all continuous characters (i.e., continuous homomorphisms from G into the torus $\mathbb{T}$) is called the **character group** of G, denoted $G^\wedge$. With pointwise addition, $G^\wedge$ is an abelian group; endowed with the compact-open topology, it is an abelian Hausdorff group, allowing us to form the second character group $(G^\wedge)^\wedge =: G^{\wedge\wedge}$. An abelian Hausdorff group G is called **(Pontryagin) reflexive** if the evaluation homomorphism

$$\alpha_G : G \to G^{\wedge\wedge},\ x \mapsto (\alpha_G(x) : \chi \mapsto \chi(x))$$

is a topological isomorphism. The famous Pontryagin–van Kampen duality theorem states that every locally compact abelian group (abbreviated LCA group) is Pontryagin reflexive. It was shown by Smith [15] that every reflexive topological vector space and every Banach space are Pontryagin reflexive groups. The latter result depends deeply on the fact that, in the character group (which can be algebraically identified with the dual space), the compact-open and strong topologies do not agree in general. However, this implies that the real or complex vector spaces c_0, ℓ^1, and ℓ^∞, well-known to be non-reflexive topological vector spaces, are Pontryagin reflexive groups. All other notation and terminology not recalled here can be found in [16] or [17].

Let $\mathbb{T}_+ = \{x + \mathbb{Z} \in \mathbb{T} :\ |x| \leq \frac{1}{4}\}$. For a subset A of G, we call the set $A^\triangleright = \{\chi \in G^\wedge :\ \chi(A) \subseteq \mathbb{T}_+\}$ the **polar** of A, and for a subset $B \subseteq G^\wedge$, we consider $B^\triangleleft = \{x \in G : \chi(x) \in \mathbb{T}_+\ \forall \chi \in B\}$, the **prepolar** of B. A subset A of an abelian topological group G is called **quasi-convex** if, for every $x \in G \setminus A$, there exists a continuous character $\chi \in A^\triangleright$ such that $\chi(x) \notin \mathbb{T}_+$. An abelian topological group G is named **locally quasi-convex** (abbreviated lqc) if there is a neighborhood base at 0 consisting of quasi-convex sets. According to ([13], 2.4), a topological vector space is lqc (as an abelian topological group) if and only if it is locally convex.

A subset B of the character group $G^\wedge$ is called **equicontinuous** if $B \subseteq U^\triangleright$ for a suitable neighborhood $U \in \mathcal{N}_G(0)$. It is well known that the polar of each neighborhood U is a compact subset of $G^\wedge$. The canonical mapping α_G is continuous if and only if every compact subset of $G^\wedge$ is equicontinuous. By a result of Kye ([18]), α_G restricted to every compact subset of G is continuous ([17], 13.4.1). In particular, if G is metrizable (more generally, a k-space), then α_G is continuous.

If G is reflexive, then the sets $\alpha_G^{-1}(U^{\triangleright\triangleright}) = U^{\triangleright\triangleleft}$ form a neighborhood base at 0. Hence, every reflexive group is lqc. The set $U^{\triangleright\triangleleft} =: \mathrm{qc}(U)$ is called the **quasi-convex hull** of U. It is the smallest quasi-convex set containing U.

If a group G is lqc and Hausdorff, then the characters of G separate points; in other words, α_G is injective or, equivalently, G is a maximally almost periodic group (abbreviated MAP group). Further, it is straightforward to prove that if G is an lqc Hausdorff group, then the mapping $\alpha_G^{-1} : \alpha_G(G) \to G,\ \alpha_G(x) \mapsto x$ is continuous.

Thus, in order to prove that G is reflexive, one has to verify that:

- G is an lqc Hausdorff group;
- Every compact subset of $G^\wedge$ is equicontinuous;
- α_G is surjective.

Next, we collect some elementary properties applied later.

Proposition 1. *If G is a second countable Hausdorff group, then $G^\wedge$ is separable.*

Proof. Because G is a second countable regular space, it is separable and metrizable ([16], 4.2.9), in particular, first countable. Thus, $G^\wedge = \bigcup_{n\in\mathbb{N}} U_n^\triangleright$ where (U_n) is a countable neighborhood base at 0. It suffices, therefore, to prove that every $U_n^\triangleright$ is separable. However, on the compact set $U_n^\triangleright$, the compact-open topology coincides with the point-separating topology $\sigma(G^\wedge, D)$ for D, a countable dense subset of G. Thus, each polar $U_n^\triangleright$, whence G is separable. $\square$

Note that the character group of a separable group need not be separable, as $\mathbb{T}^\mathbb{R}$ shows. It is separable by the Pondiczery theorem ([16], 2.3.16), but its discrete character group is uncountable.

Proposition 2. *Let G be an abelian MAP group. If $G^\wedge$ endowed with the compact-open topology is separable, then $G^\wedge$ has a countable point-separating subgroup.*

Proof. The weak topology $\sigma(G^\wedge, G)$, induced by the mapping $G^\wedge \to \mathbb{T}^G$, $\chi \mapsto (\chi(x))_{x\in G}$, is coarser than the compact-open topology on $G^\wedge$ and hence also separable. Let $D \leq G^\wedge$ be a countable dense subgroup and let $H = \bigcap_{\chi\in D} \ker(\chi)$. We have to show that $H = \{0\}$ is the trivial subgroup of G. Thus, assume there exists $0 \neq x \in H$. Because G is a MAP group, there exists $\chi \in G^\wedge$ which satisfies $\chi(x) \neq 0_\mathbb{T}$. Because D is dense in $(G^\wedge, \sigma(G^\wedge, G))$, there exists a net $(\chi_\alpha)_{\alpha\in A}$ in D such that $(\chi_\alpha(x))$ converges to $\chi(x)$. Hence, $\chi_\alpha(x) \neq 0_\mathbb{T}$ for some $\alpha \in A$, which shows that D separates the points of G. $\square$

Definition 1 ([19])**.** *A subset A of a topological group G is called* qc**-precompact** *if for every $U \in \mathcal{N}(0)$ there exists a finite subset F of G such that $A \subseteq \mathrm{qc}(F+U)$.*

Proposition 3 ([19], Corollary 3.7)**.** *If G is a locally quasi-convex group, then every* qc*-precompact subset of G is precompact.*

Remark 1 ([20], 6.3.10)**.** *Let C be a compact subset of a reflexive group G, then also $\mathrm{qc}(C)$ is compact.*

Indeed, $\mathrm{qc}(C) = C^{\triangleright\triangleleft} = \alpha_G^{-1}(C^{\triangleright\triangleright})$ holds. Because $C^\triangleright$ is a neighborhood of 0 in $G^\wedge$, its polar $C^{\triangleright\triangleright}$ is a compact subset of $G^{\wedge\wedge}$. Because α_G is a topological isomorphism, $\mathrm{qc}(C) = \alpha_G^{-1}(C^{\triangleright\triangleright})$ is a compact subset of G.

1.3. The Minkowski Functional for Groups

We define an analogue of the Minkowski functional for groups:

Definition 2 ([13], p.8)**.** *Let G be an abelian group and let $S \subseteq G$ be a symmetric subset containing* 0*. Set*

$$\kappa_S : G \to \mathbb{R},\ x \mapsto \begin{cases} 2 & : \quad x \notin S \\ \inf\{\frac{1}{m} : \ kx \in S\ \forall 1 \leq k \leq m\} & : \quad x \in S. \end{cases}$$

We omit an index indicating the group, because κ_S depends only on $S \subseteq G$ and not on the group containing S.

In [13], κ_S was only defined for elements of S. Kaplan defined a generalization of the Minkowski functional slightly differently in [21].

For $n \in \mathbb{N}$, we define $\mathbb{T}_n = \{x + \mathbb{Z} : -\frac{1}{4n} \leq x \leq \frac{1}{4n}\}$ and we put $\mathbb{T}_1 =: \mathbb{T}_+$.

Fact 1. *For $w \in \mathbb{T}$ and $n \in \mathbb{N}$ the following assertions are equivalent:*

(a) $w \in \mathbb{T}_n$;
(b) $kw \in \mathbb{T}_+$ *for all* $1 \leq k \leq n$.

Thus, Fact 1 can be reformulated as follows: $\kappa_{\mathbb{T}_+}(w) \le \frac{1}{n}$ for some $w \in \mathbb{T}$ is equivalent to $w \in \mathbb{T}_n$.

Lemma 1.

(a) *If $A \subseteq B$ are symmetric sets containing 0, then $\kappa_B \le \kappa_A$.*
(b) *Let A and B be symmetric subsets of G and $k \in \mathbb{N}$ such that $0 \in A$ and $\underbrace{A + \ldots + A}_{k \text{ summands}} \subseteq B$. Then, $\kappa_B(x) \le \frac{1}{k}\kappa_A(x)$ holds for all $x \in A$.*
(c) *If A is quasi-convex, then $\kappa_A(x) \le \frac{1}{m}$ for some $m \in \mathbb{N}$ if and only if $\chi(x) \in \mathbb{T}_m$ for all $\chi \in A^{\triangleright}$.*
(d) *If A is a subgroup of G, then $\kappa_A(x) = 0$ if $x \in A$ and $\kappa_A(x) = 2$ for $x \notin A$.*
(e) *If H is a subgroup of G and $A \subseteq G$ is a symmetric set containing $\{0\}$, then $\kappa_A(x) = \kappa_{A\cap H}(x)$ holds for all $x \in H$.*
(f) *If $A_1 \subseteq G_1$ and $A_2 \subseteq G_2$ are symmetric subsets containing the respective neutral elements, then $\kappa_{A_1\times A_2}(x_1, x_2) = \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$ for all $(x_1, x_2) \in G_1 \times G_2$.*

Proof. The proofs of (a) and (b) are straightforward.

(c) Fix $m \in \mathbb{N}$ and $x \in G$ with $\kappa_A(x) \le \frac{1}{m}$. This means, $kx \in A$ for all $1 \le k \le m$. Because A is quasi-convex, $y \in A$ if and only if $\chi(y) \in \mathbb{T}_+$ for all $\chi \in A^{\triangleright}$. Thus, we obtain $k\chi(x) = \chi(kx) \in \mathbb{T}_+$ for all $1 \le k \le m$ and all $\chi \in A^{\triangleright}$. By Fact 1, this is equivalent to $\chi(x) \in \mathbb{T}_m$.

(d) and (e) are trivial.

(f) Fix $m \in \mathbb{N}$. Assume that $\kappa_{A_1\times A_2}(x_1, x_2) \le \frac{1}{m}$. This is equivalent to $kx_1 \in A_1$ and $kx_2 \in A_2$ for all $1 \le k \le m$. Thus, $\kappa_{A_1}(x_1) \le \frac{1}{m}$ and $\kappa_{A_2}(x_2) \le \frac{1}{m}$. This shows that $\kappa_{A_1\times A_2}(x_1, x_2) \ge \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$. Conversely, if $\max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\} \le \frac{1}{m}$, then $k(x_1, x_2) \in A_1 \times A_2$ for all $1 \le k \le m$ and consequently $\kappa_{A_1\times A_2}(x_1, x_2) \le \frac{1}{m}$. This implies $\kappa_{A_1\times A_2}(x_1, x_2) \le \max\{\kappa_{A_1}(x_1), \kappa_{A_2}(x_2)\}$. □

κ_S does, in general, not satisfy the triangle inequality, as the following example shows: Let $A = [-1, 1] \subseteq \mathbb{R}$;
$\kappa_A(\frac{3}{2}) = 2 > 1 + \frac{1}{2} = \kappa_A(1) + \kappa_A(\frac{1}{2})$.

However, we have:

Proposition 4. *If $0 \in A \subseteq G$ is symmetric, then $\kappa_{A+A}(x + y) \le \max\{\kappa_A(x), \kappa_A(y)\} \le \kappa_A(x) + \kappa_A(y)$.*

Proof. It is sufficient to prove the first inequality. If $x \notin A$ or $y \notin A$, the assertion trivially holds. Thus, let us assume that $x, y \in A$. Fix $m \in \mathbb{N}$. If $\kappa_A(x), \kappa_A(y) \le \frac{1}{m}$, then $kx, ky \in A$ for all $1 \le k \le m$ and hence $k(x + y) \in A + A$. This implies $\kappa_{A+A}(x + y) \le \frac{1}{m}$. □

Lemma 2. *If $A \subseteq G$ is quasi-convex, $m \in \mathbb{N}$, and $x, y \in G$ satisfy $\kappa_A(x), \kappa_A(y) \le \frac{1}{2m}$, then $\kappa_A(x + y) \le \frac{1}{m}$.*

Proof. By Lemma 1 (c), $\kappa_A(x), \kappa_A(y) \le \frac{1}{2m}$ is equivalent to $\chi(\{x, y\}) \subseteq \mathbb{T}_{2m}$ for all $\chi \in A^{\triangleright}$. Thus, $\chi(x + y) \in \mathbb{T}_m$ for all $\chi \in A^{\triangleright}$, which is equivalent to $\kappa_A(x + y) \le \frac{1}{m}$. □

Lemma 3. *For $A \subseteq G$ and $\chi \in G^{\wedge}$ and $m \in \mathbb{N}$, the following holds:*

(a) *$\kappa_{A^{\triangleright}}(\chi) = \frac{1}{m}$ if and only if $\chi(A) \subseteq \mathbb{T}_m$ but $\chi(A) \not\subseteq \mathbb{T}_{m+1}$;*
(b) *$\kappa_{A^{\triangleright}}(\chi) = 0$ if and only if $\chi(A) = \{0\}$.*

Proof.

(a) $\kappa_{A^\triangleright}(\chi) = \frac{1}{m}$ is equivalent to $k\chi \in A^\triangleright$ for all $1 \leq k \leq m$ and $(m+1)\chi \notin A^\triangleright$. This means that $k\chi(a) \in \mathbb{T}_+$ for all $1 \leq k \leq m$ and all $a \in A$ and there exists $a_0 \in A$ such that $(m+1)\chi(a_0) \notin \mathbb{T}_+$. The first assertion is equivalent to $\chi(A) \subseteq \mathbb{T}_m$, the second (combined with the first) is equivalent to $\chi(A) \not\subseteq \mathbb{T}_{m+1}$.

(b) The assertions $\kappa_{A^\triangleright}(\chi) = 0$ are equivalent to $k\chi \in A^\triangleright$ and to $k\chi(a) \in \mathbb{T}_+$ for all $a \in A$ and $k \in \mathbb{N}$. The latter is equivalent to $\chi(A) = \{0\}$.

□

Lemma 4. *Let $\varphi : G \to H$ be a homomorphism. Assume that $0 \in A \subseteq G$ and $0 \in B \subseteq H$ are symmetric subsets such that $\varphi(A) \subseteq B$ holds. Then, $\kappa_B \circ \varphi \leq \kappa_A$ follows.*

Proof. Let $x \in G$. WLOG, we may assume that $x \in A$. Assume that $\kappa_A(x) \leq \frac{1}{m}$ for some $m \in \mathbb{N}$. Hence, $kx \in A$ for all $1 \leq k \leq m$ and hence $k\varphi(x) \in B$, which implies $\kappa_B(\varphi(x)) \leq \frac{1}{m}$. □

Lemma 5. *Let G be an abelian topological group and $A \subseteq G$ a symmetric and closed set containing 0. Then, κ_A is lower semicontinuous (i.e., $\kappa_A^{-1}(]y, \infty])$ is open for all $y \in \mathbb{R}$ or, equivalently, $\kappa_A^{-1}([0, y])$ is closed for all $y \geq 0$).*

For any sequence (A_n) of closed symmetric subsets of G containing 0, the mapping $G \to [0, \infty]$, $x \mapsto \sum_{n\in\mathbb{N}} \kappa_{A_n}(x)$ is lower semicontinuous as well.

Proof. For $y < 0$, $\kappa_A^{-1}(]y, \infty[) = G$. Fix $y \geq 0$ and let $x_0 \in G$ satisfy $\kappa_A(x_0) > y$. If $\kappa_A(x_0) = 2$, then $G \setminus A$ is an open neighborhood of x_0 contained in $\kappa_A^{-1}(]y, \infty[)$. Otherwise, $\kappa_A(x_0) = \frac{1}{m}$ for some $m \in \mathbb{N}$. Thus, $(m+1)x_0 \notin A$. For a suitable open neighborhood W of x_0, we have $(m+1)x \notin A$ for all $x \in W$. This implies $\kappa_A(x) \geq \frac{1}{m} > y$ for all $x \in W$ and hence $x_0 \in W \subseteq \kappa_A^{-1}(]y, \infty[)$.

Assume now that (A_n) is a sequence of closed and symmetric sets containing 0. Put $\kappa := \sum_{n\in\mathbb{N}} \kappa_{A_n}$. Fix $y \in \mathbb{R}$. As above, $\kappa^{-1}(]y, \infty[) = G$ in case $y < 0$. Thus, assume now that $y \geq 0$ and let $x_0 \in G$ satisfy $\kappa(x_0) > y$. Then, there is $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} \kappa_{A_n}(x_0) > y$. Let $y_n := \kappa_{A_n}(x_0)$ and $\varepsilon := \frac{1}{N}\left((\sum_{n=1}^{N} y_n) - y\right)$. By what was shown above, there exists an open neighborhood W of x_0 such that $\kappa_{A_n}(x) > y_n - \varepsilon$ for all $1 \leq n \leq N$ and all $x \in W$. Then, $\kappa(x) \geq \sum_{n=1}^{N} \kappa_{A_n}(x) > \sum_{n=1}^{N}(y_n - \varepsilon) = \sum_{n=1}^{N} y_n - N\varepsilon = y$. This shows that κ is lower semicontinuous. □

2. The Group of Absolutely Summable Sequences $\ell^1(G)$

2.1. Basic Properties of $\ell^1(G)$

Definition 3. *Let (G, τ) be an abelian topological group. Denote by*

$$\ell^1(G) = \ell^1(G, \tau) = \{(x_n) \in G^{\mathbb{N}} : \forall U \in \mathcal{N}_G(0) : \sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty\}.$$

The set $\ell^1(G)$ is a group under pointwise addition.

(Indeed, let (x_n), $(y_n) \in \ell^1(G)$. For $U \in \mathcal{N}(0)$, there exists $W \in \mathcal{N}(0)$ such that $W + W \subseteq U$. Then, by Lemma 1 (a) and Proposition 4, $\sum_{n\in\mathbb{N}} \kappa_U(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_{W+W}(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_W(x_n) + \sum_{n\in\mathbb{N}} \kappa_W(y_n) < \infty$ holds.)

The group $\ell^1(G)$ is the **group of all absolutely summable sequences** *in G. The family of sets $(S_U)_{U\in\mathcal{N}(0)}$ where*

$$S_U = \{(x_n) \in \ell^1(G) : \sum_{n\in\mathbb{N}} \kappa_U(x_n) \leq 1\}$$

forms a neighborhood base at 0 *of a group topology on* $\ell^1(G)$.

(Indeed, fix a symmetric neighborhood $U \in \mathcal{N}(0)$ and let $(x_n), (y_n) \in S_U$. Then,

$$\sum_{n\in\mathbb{N}} \kappa_{U+U+U+U}(x_n + y_n) \leq \sum_{n\in\mathbb{N}} \kappa_{U+U}(x_n) + \sum_{n\in\mathbb{N}} \kappa_{U+U}(y_n)$$

$\leq \frac{1}{2}(\sum_{n\in\mathbb{N}} \kappa_U(x_n) + \sum_{n\in\mathbb{N}} \kappa_U(y_n)) \leq 1$ by Proposition 4 and Lemma 1 (b). Thus, the symmetric set S_U satisfies $S_U + S_U \subseteq S_{U+U+U+U}$).

This topology will be denoted $\Sigma_{\ell^1(G)}$.

Further, for $N \in \mathbb{N}$ *and* $U \in \mathcal{N}(0)$, *let*

$$S_{N,U} := \{(x_n) \in \ell^1(G) : \sum_{n\geq N} \kappa_U(x_n) \leq 1\}.$$

Thus, $S_U = S_{1,U}$ *for all* $U \in \mathcal{N}(0)$.

Remark 2. *The direct sum* $G^{(\mathbb{N})}$ *is contained in* $\ell^1(G)$, *while the latter group is a subgroup of* $c_0(G)$, *the group of all null sequences in* G. (The first assertion is trivial. In order to prove the second one, fix $(x_n) \in \ell^1(G)$ and $U \in \mathcal{N}(0)$. Because $\sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that $\kappa_U(x_n) \leq 1$ for all $n \geq n_0$. However, this means that $x_n \in U$ for all $n \geq n_0$. Hence, $x_n \to 0$.)

In case G does not admit any non-trivial convergent sequences, $G^{(\mathbb{N})} = \ell^1(G) = c_0(G)$ holds algebraically. Hrušák, van Mill, Ramos-García, and Shelah [22] proved (under ZFC) that there exists an infinite countably compact group G of exponent 2 which has no non-trivial convergent sequences, whence $\ell^1(G) = G^{(\mathbb{N})}$.

Lemma 6. *If* $\varphi : G \to H$ *is a continuous homomorphism of topological groups, then* $\varphi_\# : \ell^1(G) \to \ell^1(H)$, $(x_n) \mapsto (\varphi(x_n))$ *is a well-defined continuous homomorphism. More precisely, if* $\varphi(U) \subseteq V$ *holds for symmetric neighborhoods* $U \in \mathcal{N}_G(0)$ *and* $V \in \mathcal{N}_H(0)$, *then* $\varphi_\#(S_U) \subseteq S_V$.

Thus, $\mathfrak{F}_1 :$ **ATOP** $\to$ **ATOP**, $G \mapsto \ell^1(G)$ *and* $\varphi \mapsto \varphi_\#$ *defines a covariant functor from the category of all abelian topological groups into itself. In particular, if* φ *is a topological isomorphism, then so is* $\varphi_\#$.

Proof. For $V \in \mathcal{N}_H(0)$, there exists $U \in \mathcal{N}_G(0)$ such that $\varphi(U) \subseteq V$. By Lemma 4, $\kappa_V(\varphi(x)) \leq \kappa_U(x)$ holds for all $x \in G$. Thus, for $(x_n) \in \ell^1(G)$ this gives $\sum_{n\in\mathbb{N}} \kappa_V(\varphi(x_n)) \leq \sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$. This yields that $\varphi_\#$ is well-defined and obviously a homomorphism which satisfies $\varphi_\#(S_U) \subseteq S_V$. Thus, in particular, $\varphi_\#$ is continuous. It is straightforward to check that $(\varphi \circ \psi)_\# = \varphi_\# \circ \psi_\#$ for an appropriate continuous homomorphism $\psi : G_0 \to G$. Now, the assertion follows easily. □

Corollary 1. *Let* G *be a non-necessarily Hausdorff abelian group and denote by* $N = \overline{\{0\}}$ *the core of* G *and by* $\pi : G \to G/N$ *the canonical projection. Then,* $\pi_\# : \ell^1(G) \to \ell^1(G/N)$ *is a projection.*

Proof. By Lemma 6, $\pi_\#$ is continuous, and for a symmetric neighborhood $U \in \mathcal{N}_G(0)$, we have $\pi_\#(S_U) \subseteq S_{\pi(U)}$. Conversely, we are going to show that $\pi_\#(S_{U+U}) \supseteq S_{\pi(U)}$ holds. Therefore, we verify first that $\kappa_{U+U}(x) \leq \kappa_{\pi(U)}(\pi(x))$ holds for all $x \in G$. Thus, assume that $\kappa_{\pi(U)}(\pi(x)) \leq \frac{1}{m}$ for some $m \in \mathbb{N}$. This implies that $k\pi(x) \in \pi(U)$ for all $1 \leq k \leq m$ and hence $kx \in U + N \subseteq U + U$ for all $1 \leq k \leq m$. Thus, $\kappa_{U+U}(x) \leq \frac{1}{m}$. Next, fix $(\pi(x_n)) \in S_{\pi(U)}$. Then, $\sum_{n\in\mathbb{N}} \kappa_{U+U}(x_n) \leq \sum_{n\in\mathbb{N}} \kappa_{\pi(U)}(\pi(x_n)) \leq 1$ follows. Thus, $(x_n) \in S_{U+U}$ and hence $(\pi(x_n)) \in \pi_\#(S_{U+U})$. □

Proposition 5. *Let* G *be an abelian topological group and* F *a finite subset of* $\mathbb{N}$. *Then:*

(a) $\mu_F : G^F \to \ell^1(G)$, $(x_n)_{n\in F} \mapsto (x_n)_{n\in\mathbb{N}}$, *where* $x_n = 0$ *for all* $n \in \mathbb{N} \setminus F$, *is an embedding.*

(b) *$p_F : \ell^1(G) \to G^F,\ (x_n)_{n\in\mathbb{N}} \mapsto (x_n)_{n\in F}$ is a projection.*
(c) *G is Hausdorff if and only if $\ell^1(G)$ is Hausdorff.*
(d) *G is linearly topologized if and only if $\ell^1(G)$ has this property.*

For $F = \{n\}$, we write μ_n and p_n instead of $\mu_{\{n\}}$ and $p_{\{n\}}$.

Proof. We start with the following observation:

For every $U \in \mathcal{N}_G(0)$ and $W \in \mathcal{N}_G(0)$ such that $W \underbrace{+\ldots+}_{|F| \text{ times}} W \subseteq U$ one has

$$\mu_F(W^F) = W^F \times \{0\}^{\mathbb{N}\setminus F} \subseteq S_U.$$

Proof of observation: For $x \in W$, one has $\kappa_U(x) \le \frac{1}{|F|}\kappa_W(x)$ by Lemma 1 (b); hence, $\mu_F(W \times \ldots \times W) \subseteq S_U$, because $\sum_{n\in F} \kappa_U(x_n) \le \sum_{n\in F} \frac{1}{|F|} \cdot \kappa_W(x_n) \le 1$ for all $(x_n)_{n\in F} \in W^F$, as desired.

(a) The observation above implies that μ_F is continuous. In order to show that μ_F is an embedding, observe the following: $\mu_F(G^F) \cap S_W \subseteq \mu_F(W^F)$, because $\mu_F((x_n)_{n\in F}) \in S_W$ if and only if $\sum_{n\in F} \kappa_W(x_n) \le 1$, which implies $x_n \in W$ for all $n \in F$.
(b) Because $p_F(S_U) \subseteq U^F$ for all $U \in \mathcal{N}_G(0)$, the mapping p_F is continuous. In order to show that p_F is open, let U and W be as in the observation. Then, $p_F(S_U) \supseteq p_F(W^F \times \{0\}^{\mathbb{N}\setminus F}) \supseteq W^F$. This shows that p_F is open.
(c) Assume that G is Hausdorff. It is straightforward to prove that $\bigcap\limits_{U\in\mathcal{N}} S_U = \{0\}$. Thus, $\ell^1(G)$ is also a Hausdorff group. Conversely, because $\mu_1 : G \to \ell^1(G)$ is an embedding by item (a), G is Hausdorff provided $\ell^1(G)$ has this property.
(d) Assume that G is linearly topologized. If U is an open subgroup of G, then $\kappa_U = 2 \cdot 1_{G\setminus U}$ where $1_{G\setminus U}$ denotes the indicator function (by Lemma 1 (d)). Thus, $S_U = \{(x_n) \in \ell^1(G) :\ x_n \in U\ \forall n \in \mathbb{N}\} = U^{\mathbb{N}} \cap \ell^1(G)$ is a subgroup. Hence, $\ell^1(G)$ is also linearly topologized.

The converse implication is a consequence of item (a). □

A consequence of item (b) is the continuity of the canonical projections p_n, which immediately implies the following.

Corollary 2. *The canonical mapping $(\ell^1(G), \Sigma_{\ell^1(G)}) \to (G^{\mathbb{N}}, \tau_p)$, where τ_p denotes the product topology, is continuous.*

Proposition 6.

(a) *If H is a subgroup of G and $\iota : H \to G$ denotes the embedding , then $\iota_\# : \ell^1(H) \to \ell^1(G)$ is an embedding. Furthermore, if H is an open, respectively, closed subgroup of G, then $\iota_\#(\ell^1(H))$ is an open, respectively, closed subgroup of $\ell^1(G)$.*
(b) *For abelian topological groups G_1 and G_2, the sequence space $\ell^1(G_1 \times G_2)$ is canonically topologically isomorphic to $\ell^1(G_1) \times \ell^1(G_2)$.*

Proof.

(a) Because for every symmetric neighborhood $U \in \mathcal{N}_G(0)$ the equation $\iota_\#(S_{U\cap H}) = S_U \cap \iota_\#(\ell^1(H))$ holds by Lemma 1 (e), this yields that $\iota_\#$ is an embedding. Further, if H is open, U can be chosen to be contained in H and then $S_U \subseteq \iota_\#(\ell^1(H))$, so $\iota_\#(\ell^1(H)$ is an open subgroup of $\ell^1(G)$. Now, let H be a closed subgroup of G and let $p_n : \ell^1(G) \to G$ denote the projection on the n-th coordinate. Then, $\iota_\#(\ell^1(H)) = \bigcap_{n\in\mathbb{N}} p_n^{-1}(H)$ is closed in $\ell^1(G)$ by Proposition 5 (b).
(b) For $i \in \{1, 2\}$, let $\pi_i : G_1 \times G_2 \to G_i$ be the canonical projection and consider the canonical mapping

$\psi = ((\pi_1)_\# \times (\pi_2)_\#) : \ell^1(G_1 \times G_2) \to \ell^1(G_1) \times \ell^1(G_2),\ ((x_n, y_n)) \mapsto ((x_n), (y_n))$, which is a continuous monomorphism by Lemma 6.

By Lemma 1 (f), we have for $U_i \in \mathcal{N}_{G_i}(0)$ and $x \in G_1, y \in G_2$ $\kappa_{U_1 \times U_2}(x, y) \leq \kappa_{U_1}(x) + \kappa_{U_2}(y)$. This implies that ψ is surjective. In order to prove that ψ is open, we are going to show that $\psi(S_{(U_1+U_1)\times(U_2+U_2)}) \supseteq S_{U_1} \times S_{U_2}$ for $U_i \in \mathcal{N}_{G_i}(0)$. Thus, fix $(x_n) \in S_{U_1}$ and $(y_n) \in S_{U_2}$. Then, by Lemma 1 (b),

$$\sum_{n\in\mathbb{N}} \kappa_{(U_1+U_1)\times(U_2+U_2)}((x_n, y_n)) = \sum_{n\in\mathbb{N}} \max\{\kappa_{U_1+U_1}(x_n), \kappa_{U_2+U_2}(y_n)\} \leq$$

$$\leq \sum_{n\in\mathbb{N}} \kappa_{U_1+U_1}(x_n) + \sum_{n\in\mathbb{N}} \kappa_{U_2+U_2}(y_n) \leq \sum_{n\in\mathbb{N}} \frac{1}{2}\kappa_{U_1}(x_n) + \sum_{n\in\mathbb{N}} \frac{1}{2}\kappa_{U_2}(y_n) \leq 1.$$

This shows that ψ is open and completes the proof. □

Lemma 7. *Let (C_n) be a sequence of complete subsets of a Hausdorff abelian group G and let $((x_n^{(\alpha)})_n)_{\alpha\in A}$ be a Cauchy net in $\ell^1(G)$. Assume that $\{x_n^{(\alpha)} : \alpha \in A\} \subseteq C_n$ for every $n \in \mathbb{N}$. Then, $((x_n^{(\alpha)})_n)_{\alpha\in A}$ is convergent.*

Proof. By Proposition 5 (b), all p_n are continuous, so for every $n \in \mathbb{N}$, the net $(x_n^{(\alpha)})_{\alpha\in A}$ is a Cauchy net in G contained in C_n. Because C_n was assumed to be complete, $x_n = \lim_{\alpha\in A} x_n^{(\alpha)}$ exists for all $n \in \mathbb{N}$. □

Claim: For every $U \in \mathcal{N}(0)$, there exists $\alpha_0 \in A$ such that $\sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha)} - x_n) \leq 1$ for all $\alpha \geq \alpha_0$.

Proof. Fix $U \in \mathcal{N}(0)$. We choose a closed and symmetric neighborhood $W \in \mathcal{N}(0)$ such that $W + W + W + W \subseteq U$. By assumption, there exists $\alpha_W \in A$ such that $\sum_{n\in\mathbb{N}} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) \leq 1$ holds for all $\alpha, \beta \geq \alpha_W$. Because W is closed and $x_n^{(\alpha)} - x_n^{(\beta)} \in W$ for all $\alpha, \beta \geq \alpha_W$, we obtain $x_n^{(\alpha)} - x_n \in W$ for all $\alpha \geq \alpha_W$. Now, fix $\alpha \geq \alpha_W$ and assume that $\sum_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha)} - x_n) > 1$. Choose a finite subset $F \subseteq \mathbb{N}$ such that $\kappa_U(x_n^{(\alpha)} - x_n) > 0$ for all $n \in F$ and $\sum_{n\in F} \kappa_U(x_n^{(\alpha)} - x_n) > 1$. For $n \in F$, we have $0 < \kappa_U(x_n^{(\alpha)} - x_n) \leq \kappa_W(x_n^{(\alpha)} - x_n) \leq 1$. Thus, choose $m_n \in \mathbb{N}$ such that $\frac{1}{m_n} = \kappa_W(x_n^{(\alpha)} - x_n)$. Because $(x_n^{(\beta)})_{\beta\in A}$ converges to x_n, there exists $\beta \geq \alpha_W$ such that $\kappa_W(x_n^{(\beta)} - x_n) \leq \frac{1}{|F|}$ for all $n \in F$. We obtain

$$\begin{aligned}
1 &< \sum_{n\in F} \kappa_U(x_n^{(\alpha)} - x_n) \\
&\leq \sum_{n\in F} \kappa_{W+W+W+W}(x_n^{(\alpha)} - x_n^{(\beta)} + x_n^{(\beta)} - x_n) \\
&\overset{\text{Proposition 4}}{\leq} \sum_{n\in F} \kappa_{W+W}(x_n^{(\alpha)} - x_n^{(\beta)}) + \sum_{n\in F} \kappa_{W+W}(x_n^{(\beta)} - x_n) \\
&\overset{\text{Lemma 1(b)}}{\leq} \frac{1}{2}\sum_{n\in F} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) + \frac{1}{2}\sum_{n\in F} \kappa_W(x_n^{(\beta)} - x_n) \\
&\leq \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(x_n^{(\alpha)} - x_n^{(\beta)}) + \frac{1}{2}\sum_{n\in F} \frac{1}{|F|} \leq \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

This contradiction proves the Claim with $\alpha_0 = \alpha_W$.

We now show that $(x_n) \in \ell^1(G)$. Fix a symmetric closed neighborhood $U \in \mathcal{N}(0)$. Choose α_0 as in the Claim. We obtain $\sum\limits_{n\in\mathbb{N}} \kappa_{U+U}(x_n) \leq \sum\limits_{n\in\mathbb{N}} \kappa_U(x_n - x_n^{(\alpha_0)}) + \sum\limits_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha_0)}) \leq 1 + \sum\limits_{n\in\mathbb{N}} \kappa_U(x_n^{(\alpha_0)}) < \infty$. Thus, $(x_n) \in \ell^1(G)$. It follows from the Claim that $((x_n^{(\alpha)})_n)_{\alpha\in A}$ converges to $(x_n)_n$. □

Corollary 3. *If G is a Hausdorff complete abelian group, then so is $\ell^1(G)$.*

Proof. Apply Lemma 7 to $C_n = G$ for all $n \in \mathbb{N}$. □

Proposition 7. *$G^{(\mathbb{N})}$ is dense in $\ell^1(G)$.*

Proof. Fix $(x_n) \in \ell^1(G)$ and $U \in \mathcal{N}(0)$. Because $\sum_{n\in\mathbb{N}} \kappa_U(x_n) < \infty$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} \kappa_U(x_n) \leq 1$. This shows that $(x_n) - \mu_{\underline{n_0}}(x_1, \ldots, x_{n_0}) \in S_U$. □

Proposition 8. *Let G be an abelian Hausdorff group. A subset K of $\ell^1(G)$ is compact if and only if the following three conditions hold:*

(a) *K is closed;*
(b) *$p_n(K)$ is compact for every $n \in \mathbb{N}$;*
(c) *For every $U \in \mathcal{N}(0)$, there exists $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$.*

Proof. Assume that $K \subseteq \ell^1(G)$ is compact. Then, obviously, conditions (a) and (b) are satisfied. In order to prove (c), fix $U \in \mathcal{N}(0)$. Because K is totally bounded and $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$ by Proposition 7, there exists a finite subset $F \subseteq G^{(\mathbb{N})}$ such that $K \subseteq F + S_U$. Fix $N_U \in \mathbb{N}$ such that $p_k((y_n)) = 0$ for all $k \geq N_U$ and all $(y_n) \in F$. Fix $(x_n) \in K$. There exists $(y_n) \in F$ such that $(x_n - y_n) \in S_U$. Hence, $\sum_{n\geq N_U} \kappa_U(x_n) = \sum_{n\geq N_U} \kappa_U(x_n - y_n) \leq \sum_{n\geq 1} \kappa_U(x_n - y_n) \leq 1$. This shows that $K \subseteq S_{N_U,U}$.

Conversely, assume that $K \subseteq \ell^1(G)$ satisfies the conditions (a), (b), and (c). By Lemma 7 (with $C_n = p_n(K)$), we conclude that K is complete. In order to prove that K is totally bounded, we fix $U \in \mathcal{N}(0)$. By item (c), there exists $N_U \in \mathbb{N}$ such that $\sum_{n\geq N_U} \kappa_U(x_n) \leq 1$ for all $(x_n) \in K$. Because $K \subseteq \mu_{\underline{N_U}}(\prod_{n=1}^{N_U} p_n(K)) + S_U$ and $\mu_{\underline{N_U}}(\prod_{n=1}^{N_U} p_n(K))$ is compact, K is totally bounded. □

Corollary 4. *Let G be an abelian Hausdorff group. For the density (the minimal cardinality of a dense subset), the following holds: $d(\ell^1(G)) = \max\{\aleph_0, d(G)\}$ in case $d(G) > 1$.*

Proof. Let $D \subseteq G$ be a dense subset of cardinality $d(G)$. Because $\mu_{\underline{n}}$ is an embedding for every $n \in \mathbb{N}$, the closure of $D^{(\mathbb{N})}$ contains the dense set $G^{(\mathbb{N})}$. This shows that $D^{(\mathbb{N})}$ is dense in $\ell^1(G)$ and hence $d(\ell^1(G)) \leq \max\{\aleph_0, d(G)\}$. In case $d(G)$ is infinite, $d(G) = d(\ell^1(G))$, because p_1 maps a dense subset of $\ell^1(G)$ onto a dense subset of G.

Assume now that $1 < d(G) < \infty$. Then, G is a finite discrete group and hence $\ell^1(G) = G^{(\mathbb{N})}$ is a countably infinite discrete group. Hence, $d(\ell^1(G)) = \aleph_0$ in this case. □

Proposition 9. *Let G be an abelian Hausdorff group. For the character χ (the minimal cardinality of a neighborhood base at 0) and the weight w (the minimal cardinality of a base), the following holds:*

(a) *$\chi(G) = \chi(\ell^1(G))$.*
(b) *$w(G) = w(\ell^1(G))$ if $w(G)$ is infinite.*

Proof.

(a) is trivial.
(b) Recall that for every topological group H, one has $w(H) = \chi(H) \cdot d(H)$ (Lemma 5.1.7 in [17]). If $d(G)$ were finite, then G would be a finite discrete group and hence $w(G)$ had to be finite in contradiction to the assumption. Thus, $d(G)$ is infinite. Applying item (a) and Corollary 4, we obtain

$$w(\ell^1(G)) = \chi(\ell^1(G)) \cdot d(\ell^1(G)) = \chi(G) \cdot \max\{\aleph_0, d(G)\} = \chi(G) \cdot d(G) = w(G).$$

□

2.2. The Character Group of $\ell^1(G)$

Proposition 10. *The mapping*

$$(\mu_n^\wedge) : \ell^1(G)^\wedge \longrightarrow G^{\wedge\mathbb{N}},\ \chi \longmapsto (\mu_n^\wedge(\chi)) = (\chi \circ \mu_n)$$

is a continuous injective homomorphism. Thus, algebraically, $\ell^1(G)^\wedge$ can be identified with a subgroup of $G^{\wedge\mathbb{N}}$.

Proof. Because μ_n is continuous for every $n \in \mathbb{N}$ by Proposition 5 (a), so is $(\mu_n^\wedge)$. We are going to show now that $(\mu_n^\wedge)$ is injective: Let $\chi \in \ell^1(G)^\wedge$ and assume that $\mu_n^\wedge(\chi) = \chi \circ \mu_n$ is the trivial character for every $n \in \mathbb{N}$. This implies that χ restricted to the subgroup $G^{(\mathbb{N})}$ is trivial. By Proposition 7, $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$; hence, χ is trivial. □

This result allows us to identify a character $\chi \in \ell^1(G)^\wedge$ with the sequence $(\chi_n) = (\mu_n^\wedge(\chi))_{n\in\mathbb{N}}$.

Next, we are going to describe the structure of the dual group of $\ell^1(G)$.

Proposition 11. *For an abelian topological group, the following assertions hold:*

$$\ell^1(G)^\wedge = \bigcup_{U\in\mathcal{N}(0)} (U^\triangleright)^{\mathbb{N}}.$$

and

$$(S_U)^\triangleright = (U^\triangleright)^{\mathbb{N}}.$$

Proof. A homomorphism $\chi : \ell^1(G) \to \mathbb{T}$ is continuous if and only if χ maps a suitable neighborhood of 0 in $\ell^1(G)$ into $\mathbb{T}_+$ or, equivalently, if χ belongs to the polar of a neighborhood of 0. Hence, $\ell^1(G)^\wedge = \bigcup_{U\in\mathcal{N}_G(0)} (S_U)^\triangleright$.

Next, we are going to describe such a polar $(S_U)^\triangleright$: Fix $\chi = (\chi_n) \in (S_U)^\triangleright$. Because $\mu_n(U) \subseteq S_U$ for all $n \in \mathbb{N}$, we obtain $\chi_n = \mu_n^\wedge(\chi) = \chi \circ \mu_n \in U^\triangleright$. This shows that $(S_U)^\triangleright \subseteq (U^\triangleright)^{\mathbb{N}}$.

Conversely, assume that $\chi = (\chi_n) \in (U^\triangleright)^{\mathbb{N}}$ and fix $(x_n) \in S_U$. Recall that for $\psi \in U^\triangleright$ and $x \in U$ with $\kappa_U(x) \le \frac{1}{m}$, one has $k\psi(x) \in \mathbb{T}_+$ for all $1 \le k \le m$ and hence $\psi(x) \in \mathbb{T}_m$ (Fact 1). We obtain $\chi(x_n) = \sum_{n\in\mathbb{N}} \chi_n(x_n) \in \mathbb{T}_+$, so $\chi \in (S_U)^\triangleright$. □

Proposition 12. *A topological group G is lqc if and only if $\ell^1(G)$ is lqc.*

Proof. Because $\mu_1 : G \to \ell^1(G)$ is an embedding and because subgroups of lqc groups are again lqc, the condition is necessary. Conversely, assume that G is lqc. Fix a quasi-convex neighborhood $U \in \mathcal{N}_G(0)$ and choose $W \in \mathcal{N}_G(0)$ quasi-convex such that $W + W + W \subseteq U$. We are going to prove that $\mathrm{qc}(S_W) \subseteq S_U$. Thus, let $(x_n) \notin S_U$, i.e., $\sum_{n\in\mathbb{N}} \kappa_U(x_n) > 1$. We have to find $\chi = (\chi_n) \in (S_W)^\triangleright = (W^\triangleright)^{\mathbb{N}}$ such that $\chi(x_n) \notin \mathbb{T}_+$. In case there is $n \in \mathbb{N}$ such that $x_n \notin W$, there exists $\chi_n \in W^\triangleright$ such that $\chi_n(x_n) \notin \mathbb{T}_+$. Then, $\chi = p_n^\wedge(\chi_n)$ has the desired property. Assume now that $x_n \in W$ for all $n \in \mathbb{N}$. This implies $\kappa_U(x_n) \le \frac{1}{3}$. Let $N \in \mathbb{N}$ be minimal with the property that $\sum_{n=1}^N \kappa_U(x_n) > 1$, $F = \{n :\ 1 \le n \le N,\ \kappa_U(x_n) > 0\}$, and put $\kappa_U(x_n) = \frac{1}{m_n}$ for $n \in F$, where $m_n \ge 3$ must hold. By the minimality condition, $N \in F$. For $n \in F$, we choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n) = t_n + \mathbb{Z}$ where $\dfrac{1}{4(m_n+1)} < t_n \le \dfrac{1}{4m_n}$ (cf. Lemma 1 (c)). Because $W + W \subseteq U$, we obtain $U^\triangleright + U^\triangleright \subseteq W^\triangleright$ (Fact 1). Thus, $\chi = p_F^\wedge((2\chi_n)_{n\in F}) \in (W^\triangleright)^{\mathbb{N}}$. We obtain: $\chi(x_n) = \sum_{n\in F} 2\chi_n(x_n) = \sum_{n\in F} 2t_n + \mathbb{Z}$ where

$$\frac{1}{4} < \frac{1}{4}\sum_{n\in F} \kappa_U(x_n) = \sum_{n\in F} \frac{1}{4m_n} \le \sum_{n\in F} \frac{1}{2m_n+2} < \sum_{n\in F} 2t_n \le$$

$$\le \sum_{n\in F} \frac{1}{2m_n} = \sum_{n\in F\setminus\{N\}} \frac{1}{2m_n} + \frac{1}{2m_N} \le \frac{1}{2} + \frac{1}{2m_N} \le \frac{2}{3} < \frac{3}{4}$$

because N was chosen to be minimal and hence $\sum_{n\in F\setminus\{N\}} \frac{1}{m_n} \leq 1$; further, because $x_N \in W$, we have $\frac{1}{m_N} = \kappa_U(x_N) \leq \frac{1}{3}$. This shows that $\chi = p_F^\wedge((2\chi_n)_{n\in F})$ has the desired properties. □

Proposition 13. *Let G be a Hausdorff abelian group. Then, $(G^\wedge)^{(\mathbb{N})}$ is dense in $\ell^1(G)^\wedge$.*

Proof. Fix $\chi = (\chi_n) \in \ell^1(G)^\wedge$ and a compact subset K of $\ell^1(G)$. Because χ is continuous, there exists $U \in \mathcal{N}_G(0)$ such that $\chi \in (S_U)^\triangleright = (U^\triangleright)^{\mathbb{N}}$. Choose $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$ (cf. Proposition 8).

For $(x_n) \in S_{N_U,U}$ we have
$(\chi - p^\wedge_{\underline{N_U}}(\chi_1, \ldots, \chi_{N_U}))(x_n) = \sum_{n>N_U} \chi_n(x_n) \in \mathbb{T}_+$ because $\sum_{n>N_U} \kappa_U(x_n) \leq 1$ and $\chi_n \in U^\triangleright$ for all $n \in \mathbb{N}$, so $\chi - p^\wedge_{\underline{N_U}}(\chi_1, \ldots, \chi_{N_U}) \in (S_{N_U,U})^\triangleright \subseteq K^\triangleright$, as desired. □

Next, we are going to study the continuity of $\alpha_{\ell^1(G)}$ and start with the following obvious

Proposition 14. *Let G be a metrizable group. Then, $\alpha_{\ell^1(G)}$ is continuous.*

Proof. Because G is first countable, so is $\ell^1(G)$ by Proposition 9 (a). Hence, $\alpha_{\ell^1(G)}$ is continuous. □

Lemma 8. *Let G be an abelian Hausdorff group. Then, $\alpha_{\ell^1(G)}$ is continuous if and only if for every compact subset $K \subseteq \ell^1(G)^\wedge$ the set $T_K := \overline{\bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)} \subseteq G^\wedge$ is equicontinuous.*

Proof. Recall that for an abelian topological group G, the canonical homomorphism α_G is continuous if and only if every compact subset of $G^\wedge$ is equicontinuous. Thus, $\alpha_{\ell^1(G)}$ is continuous if and only if for every compact subset K of $\ell^1(G)^\wedge$ there exists a neighborhood $U \in \mathcal{N}_G(0)$ such that $K \subseteq (S_U)^\triangleright = (U^\triangleright)^{\mathbb{N}}$. This implies $\mu_m^\wedge(K) \subseteq U^\triangleright$ for all $m \in \mathbb{N}$ and hence $T_K \subseteq U^\triangleright$.

Conversely, assume that for every compact subset $K \subseteq \ell^1(G)^\wedge$ there exists $U \in \mathcal{N}_G(0)$ such that $T_K \subseteq U^\triangleright$. Then, $K \subseteq \prod_{m\in\mathbb{N}} \mu_m^\wedge(K) \subseteq (U^\triangleright)^{\mathbb{N}} = (S_U)^\triangleright$. This shows that K is equicontinuous and hence $\alpha_{\ell^1(G)}$ is continuous. □

For a continuous homomorphism $\psi : H \to G$ between Hausdorff groups, the homomorphism $\psi_\# : \ell^1(H) \to \ell^1(G)$ is continuous and so is its dual homomorphism $(\psi_\#)^\wedge : \ell^1(G)^\wedge \to \ell^1(H)^\wedge$.

Lemma 9. *Let $\psi : H \to G$ be a continuous homomorphism between abelian Hausdorff groups. Then, $(\psi_\#)^\wedge(\chi_n) = (\psi^\wedge(\chi_n))$ holds for all $(\chi_n) \in \ell^1(G)^\wedge$.*

If $K \subseteq \ell^1(G)^\wedge$ is compact and $T_K = \overline{\bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)}$ and $T_{(\psi_\#)^\wedge(K)}$ is the analogous subset of $H^\wedge$ corresponding to the compact set $(\psi_\#)^\wedge(K)$, then $\psi^\wedge(T_K) \subseteq T_{(\psi_\#)^\wedge(K)}$.

Proof. By Lemma 6, the mapping $\psi_\# : \ell^1(H) \to \ell^1(G)$ is a continuous homomorphism. Hence, $(\psi_\#)^\wedge : \ell^1(G)^\wedge \to \ell^1(H)^\wedge$ is a well-defined continuous homomorphism. Fix $(\chi_n) \in \ell^1(G)^\wedge$ and $(h_n) \in \ell^1(H)$. Then, we have $(\psi_\#)^\wedge((\chi_n))(h_n) = (\chi_n)(\psi_\#(h_n)) = (\chi_n)(\psi(h_n)) = \sum_{n\in\mathbb{N}} \chi_n(\psi(h_n)) = \sum_{n\in\mathbb{N}} \psi^\wedge(\chi_n)(h_n) = (\psi^\wedge(\chi_n))(h_n)$. Now, the first assertion follows. This yields

$$\psi^\wedge(T_K) \subseteq \overline{\{\psi^\wedge(\mu_m^\wedge(\chi)) : \chi \in K,\ m \in \mathbb{N}\}}$$

$$= \overline{\{\psi^\wedge(\chi_m) : \chi = (\chi_n) \in K,\ m \in \mathbb{N}\}} = T_{(\psi_\#)^\wedge(K)}.$$

□

Theorem 1. *For every compact abelian group G, the mapping $\alpha_{\ell^1(G)}$ is continuous.*

Proof. Let $K \subseteq \ell^1(G)^\wedge$ be compact and let T_K be as in Lemma 8. Assume that T_K is an infinite subset of $G^\wedge$. Let D be the divisible hull of the discrete group $G^\wedge$ and consider the embedding $G^\wedge \to D$. Let D_0 be a divisible countably infinite subgroup of D such that $T_K \cap D_0$ is infinite. Because D_0 splits, there is a continuous homomorphism $\gamma : G^\wedge \to D_0$ such that $\gamma(T_K)$ is infinite. Because G and D_0 are reflexive groups, we may consider $\gamma = \psi^\wedge$ for a suitable homomorphism $\psi : D_0^\wedge \to G$ (after identifying D_0 with its second dual group). Indeed, let $\gamma^\wedge : D_0^\wedge \to G^{\wedge\wedge}$ be the dual homomorphism and let $\psi = \alpha_G^{-1} \circ \gamma^\wedge : D_0^\wedge \to G$ be the composition of $\gamma^\wedge$ with the topological isomorphism α_G^{-1}. (Observe that G is compact and hence reflexive.) Then, $\psi^\wedge = \gamma^{\wedge\wedge} \circ (\alpha_G^{-1})^\wedge = \gamma^{\wedge\wedge} \circ \alpha_{G^\wedge} : G^\wedge \to D_0^{\wedge\wedge}$ holds, because $(\alpha_G^{-1})^\wedge = \alpha_{G^\wedge}$. Finally, because the discrete group D_0 is reflexive, $\alpha_{D_0}^{-1} \circ \psi^\wedge = \alpha_{D_0}^{-1} \circ \gamma^{\wedge\wedge} \circ \alpha_{G^\wedge} = \gamma$. We are going to identify $D_0^{\wedge\wedge}$ with D_0 via the topological isomorphism $\alpha_{D_0}^{-1}$ and obtain that $\psi^\wedge = \gamma$. Thus, $\psi^\wedge(T_K)$ is an infinite subset of D_0.

By Lemma 9, $\psi^\wedge(T_K)$ is contained in the set $T_{(\psi_\#)^\wedge(K)}$. Because $D_0^\wedge$ is metrizable, $\alpha_{\ell^1(D_0^\wedge)}$ is continuous by Proposition 14. As $(\psi_\#)^\wedge(K)$ is a compact subset of $\ell^1(D_0^\wedge)^\wedge$, the set $T_{(\psi_\#)^\wedge(K)}$ is an equicontinuous and hence compact subset of the discrete group $D_0^{\wedge\wedge}$ by Lemma 8, and hence finite. This contradiction proves that T_K must be finite and hence equicontinuous. Thus, again by Lemma 8, $\alpha_{\ell^1(G)}$ is continuous. □

Lemma 10. *Let (G, τ) be a reflexive group such that $G^\wedge$ has a countable point-separating subgroup. For a compact subset K of $\ell^1(G)^\wedge$ and $m \in \mathbb{N}$, put $T_m = \mu_m^\wedge(K)$ and $T = \bigcup_{m\in\mathbb{N}} T_m$. Then, T is totally bounded.*

Recall that the hypothesis that $G^\wedge$ has a countable point-separating subgroup is fulfilled in case G is second countable or $G^\wedge$ is separable by Propositions 1 and 2.

Proof. Let $D = \{\psi_k : k \in \mathbb{N}\}$ be a countable point-separating subgroup of $G^\wedge$. Because the topology $\sigma(G, D)$ induced by the mapping $G \to \mathbb{T}^D$, $x \mapsto (\psi(x))_{\psi\in D}$ is Hausdorff, we obtain that on every τ-compact subset C of G, the subspace topologies induced by τ and by $\sigma(G, D)$ coincide. Denote by $\mathcal{F}$ the set of all finite subsets of $G^\wedge$ containing 0.

Assume that T is not precompact. By Proposition 3, T is not qc-precompact either, because $G^\wedge$ is lqc. Thus, there exists a compact subset $0 \in C \subseteq G$ such that for every $F \in \mathcal{F}$ we have $T \nsubseteq \mathrm{qc}(F + C^\triangleright)$. Because $\mathrm{qc}(F \cup C^\triangleright) \subseteq \mathrm{qc}(F + C^\triangleright)$, we even have $T \nsubseteq \mathrm{qc}(F \cup C^\triangleright)$ for all $F \in \mathcal{F}$. This is equivalent to

$$\mathrm{qc}(T) \nsubseteq \mathrm{qc}(F \cup C^\triangleright).$$

As $C^\triangleright = \mathrm{qc}(C)^\triangleright$ and because $\mathrm{qc}(C)$ is compact according to Remark 1, we may assume that C is quasi-convex. Hence, we have

$$T^{\triangleleft\triangleright} \overset{(*)}{=} T^{\triangleright\triangleleft} = \mathrm{qc}(T) \nsubseteq \mathrm{qc}(F \cup C^\triangleright) = (F \cup C^\triangleright)^{\triangleright\triangleleft} \overset{(*)}{=} (F \cup C^\triangleright)^{\triangleleft\triangleright} = (F^\triangleleft \cap \underbrace{C^{\triangleright\triangleleft}}_{=\mathrm{qc}(C)=C})^\triangleright = (F^\triangleleft \cap C)^\triangleright$$

The equations marked by $(*)$ hold because G is reflexive, so α_G is surjective. Hence, we have

$$\forall F \in \mathcal{F} \quad T^\triangleleft \nsupseteq F^\triangleleft \cap C. \tag{1}$$

We inductively construct:

(a) A sequence $(\chi^{(n)})_{n\in\mathbb{N}_0}$ in K where $\chi^{(n)} = (\chi_k^{(n)})_{k\in\mathbb{N}}$;
(b) A strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ of natural numbers;
(c) An increasing sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\psi_n \in F_n$ for all $n \in \mathbb{N}$;
(d) A sequence $(x_n)_{n\in\mathbb{N}}$ in C such that for all $n \in \mathbb{N}$
 (i) $x_n \in C \cap F_n^\triangleleft$
 (ii) $\chi_{m_n}^{(n)}(x_n) \notin \mathbb{T}_+$;

(iii) $\chi_m^{(j)}(C \cap F_n^{\triangleleft}) \subseteq \mathbb{T}_2$ for all $m \in \mathbb{N}$ and $0 \leq j < n$.

Choose $\chi^{(0)} \in K$ arbitrarily.

Assume now that for some $n \in \mathbb{N}_0$ $(\chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(n)})$, $m_1 < \ldots < m_n$, $F_1 \subseteq \ldots \subseteq F_n$ and $(x_1, \ldots, x_n)$ have been constructed, satisfying the above-listed properties.

Because $\chi^{(0)}, \chi^{(1)}, \ldots, \chi^{(n)}$ are continuous, there exists $U \in \mathcal{N}_G(0)$ such that $\chi^{(j)} \in (S_{U+U})^{\triangleright}$ for all $0 \leq j \leq n$, which implies

$$\chi_m^{(j)}(U) \subseteq \mathbb{T}_2 \quad \text{for all } m \in \mathbb{N} \text{ and } 0 \leq j \leq n. \tag{2}$$

As a finite union of compact sets, $\bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K)$ is compact. (In case $n = 0$, this set is empty and hence compact.) Because α_G is assumed to be continuous, $\bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K)$ is equicontinuous. Thus, for a suitable neighborhood $W \in \mathcal{N}_G(0)$, we have

$$\psi(x) \in \mathbb{T}_+ \quad \text{for all } x \in W \text{ and } \psi \in \bigcup_{k=1}^{m_n} \mu_k^{\wedge}(K). \tag{3}$$

Because C is compact, the original topology τ coincides with the weak topology $\sigma(G, G^{\wedge})$ on C; hence, there exists a finite subset $F_{n+1} \in \mathcal{F}$ such that

$$0 \in C \cap F_{n+1}^{\triangleleft} \subseteq C \cap U \cap W. \tag{4}$$

WLOG, we may assume that $F_n \cup \{\psi_{n+1}\} \subseteq F_{n+1}$, so that item (c) is fulfilled. Thus, for all $0 \leq j \leq n$ and $m \in \mathbb{N}$, we have $\chi_m^{(j)}(C \cap F_{n+1}^{\triangleleft}) \subseteq \chi_m^{(j)}(U) \subseteq \mathbb{T}_2$ by Equations (2) and (4) (i.e., (d)(iii) is satisfied).

Because by Equation (1) $T^{\triangleleft} \not\supseteq C \cap F_{n+1}^{\triangleleft}$, there exists $x_{n+1} \in C \cap F_{n+1}^{\triangleleft} \setminus T^{\triangleleft}$. This means that there exist $\chi^{(n+1)} \in K$ and $m_{n+1} \in \mathbb{N}$ such that $\mu_{m_{n+1}}^{\wedge}(\chi^{(n+1)})(x_{n+1}) = \chi_{m_{n+1}}^{(n+1)}(x_{n+1}) \notin \mathbb{T}_+$. As $x_{n+1} \in C \cap F_{n+1}^{\triangleleft} \subseteq W$ by Equation (4), the index m_{n+1} must be strictly larger than m_n, because otherwise $\chi_{m_{n+1}}^{(n+1)}(x_{n+1}) \in \mathbb{T}_+$ would follow from Equation (3). Thus, $\chi^{(n+1)}$, x_{n+1} and m_{n+1} satisfy the properties stated in (a), (b), (d)(i), and (d)(ii). This completes the inductive step.

Let $S := \{0\} \cup \{\mu_{m_n}(x_n) : n \in \mathbb{N}\}$. Applying Proposition 8, we are going to show first that S is a compact subset of $\ell^1(G)$. Of course, $p_m(S)$ consists of at most 2 points, because the sequence $(m_n)_{n \in \mathbb{N}}$ is strictly increasing. It can be easily checked that S is closed in the product topology and by Corollary 2 also in the topology $\Sigma_{\ell^1(G)}$.

Fix $U \in \mathcal{N}_G(0)$. We have to show that there exists $N_U \in \mathbb{N}$ such that for all $(y_n) \in S$ $\sum_{n \geq N_U} \kappa_U(y_n) \leq 1$ holds. By the special form of the elements of S, this is equivalent to $\kappa_U(x_n) \leq 1$ for all n such that $m_n \geq N_U$. Because C is compact, there exists a finite subset F of D such that $F^{\triangleleft} \cap C \subseteq U \cap C$. By item (c), there exists $n_0 \in \mathbb{N}$ such that $F \subseteq F_{n_0} \subseteq F_n$ for all $n \geq n_0$. Thus, for $n \geq n_0$, we have $x_n \in F_n^{\triangleleft} \cap C \subseteq F_{n_0}^{\triangleleft} \cap C \subseteq U \cap C \subseteq U$ by item (d)(i) and hence $\kappa_U(x_n) \leq 1$ for all $n \geq n_0$. Now, choose $N_U := 1 + m_{n_0}$. For n, such that $m_n \geq N_U$, we have (because (m_n) is strictly increasing) $n > n_0$ and hence $\kappa_U(x_n) \leq 1$. This shows that S is compact.

Let us prove that

$$\forall k_1, k_2 \in \mathbb{N} \quad k_1 \neq k_2 \implies \chi^{(k_2)} - \chi^{(k_1)} \notin (S+S)^{\triangleright} : \tag{5}$$

WLOG, we may assume that $k_1 < k_2$. Because $\chi_m^{(k_1)}(F_{k_2}^{\triangleleft} \cap C) \subseteq \mathbb{T}_2$ for all $m \in \mathbb{N}$ by item (d)(iii) and $x_{k_2} \in F_{k_2}^{\triangleleft} \cap C$ by item (d)(i) and $\chi_{m_{k_2}}^{(k_2)}(x_{k_2}) \notin \mathbb{T}_+$ by item (d)(ii), this implies

$$(\chi^{(k_2)} - \chi^{(k_1)})(\mu_{m_{k_2}}(x_{k_2})) = \underbrace{\chi_{m_{k_2}}^{(k_2)}(x_{k_2})}_{\notin \mathbb{T}_+} - \underbrace{\chi_{m_{k_2}}^{(k_1)}(x_{k_2})}_{\in \mathbb{T}_2} \notin \mathbb{T}_2. \tag{6}$$

Because $\psi \in (S+S)^{\triangleright}$ if and only if $\psi(S) \subseteq \mathbb{T}_2$, Equation (5) is an immediate consequence of Equation (6).

Because by item (a) $\chi^{(n)} \in K$ for all $n \in \mathbb{N}$, Equation (5) implies that K is not totally bounded. This contradiction implies that T is precompact, whence totally bounded. $\square$

Theorem 2. *Let G be a reflexive group which has the following additional properties:*

1. *$G^\wedge$ has a countable point-separating subgroup.*
2. *$G^\wedge$ is complete.*

Then, $\alpha_{\ell^1(G)}$ is continuous.

Proof. Let K be a compact subset of $\ell^1(G)^\wedge$. By Lemma 10, $T = \bigcup_{m\in\mathbb{N}} \mu_m^\wedge(K)$ is totally bounded. Hence, its closure $T_K = \overline{T}$ is also totally bounded and complete by the assumption that $G^\wedge$ is complete. Thus, T_K is a compact subset of $G^\wedge$. Because α_G is continuous, the compact subset T_K of $G^\wedge$ is equicontinuous. By Lemma 8, the canonical homomorphism $\alpha_{\ell^1(G)}$ is continuous. $\square$

2.3. The Second Character Group of $\ell^1(G)$

In this section, we study the second character group of $\ell^1(G)$ and show that each element $\eta \in \ell^1(G)^{\wedge\wedge}$ can be identified with a sequence (η_n) in $G^{\wedge\wedge}$. Next, we study necessary and sufficient conditions for G such that (η_n) belongs to $\ell^1(G^{\wedge\wedge})$. As a consequence, it is possible to prove the main theorems of this paper, asserting that $\ell^1(G)$ is reflexive if G is metrizable and reflexive or an LCA group.

Proposition 15. *For every abelian topological group G, the mapping*

$$\Psi = (p_n^{\wedge\wedge}) : \ell^1(G)^{\wedge\wedge} \to (G^{\wedge\wedge})^{\mathbb{N}},\ \eta \mapsto (p_n^{\wedge\wedge}(\eta))_n$$

is a continuous monomorphism. For all $(x_n) \in \ell^1(G)$, $\quad \Psi \circ \alpha_{\ell^1(G)}(x_n) = (\alpha_G(x_n))$ holds. If α_G is continuous, then

$$\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#.$$

Proof. It is clear that Ψ is a continuous homomorphism. Fix $\eta \in \ell^1(G)^{\wedge\wedge}$ with $p_n^{\wedge\wedge}(\eta) = 0$ for all $n \in \mathbb{N}$. Then, $\eta \circ p_n^\wedge$ is trivial for all $n \in \mathbb{N}$. Hence, η vanishes on the subgroup $G^{\wedge(\mathbb{N})}$ of $\ell^1(G)^\wedge$, which is dense by Proposition 13. This implies that η is trivial. Because Ψ is a homomorphism, we conclude that Ψ is injective.

Observe that for $(x_n) \in \ell^1(G)$, we have $\Psi(\alpha_{\ell^1(G)}((x_n))) = (p_m^{\wedge\wedge}(\alpha_{\ell^1(G)}((x_n))))_{m\in\mathbb{N}} = (\alpha_{\ell^1(G)}((x_n)) \circ p_m^\wedge)_{m\in\mathbb{N}}$. Further, for $\chi \in G^\wedge$, we have

$$\alpha_{\ell^1(G)}((x_n))(p_m^\wedge(\chi)) = \alpha_{\ell^1(G)}((x_n))(\chi \circ p_m) = (\chi \circ p_m)((x_n)) = \chi(x_m) = \alpha_G(x_m)(\chi).$$

Combining these observations yields $\Psi(\alpha_{\ell^1(G)}((x_n))) = (\alpha_G(x_n))$. If α_G is continuous (and hence $(\alpha_G)_\#$ is well-defined), then $\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#$. $\square$

For the remainder, we identify an element $\eta \in \ell^1(G)^{\wedge\wedge}$ with the sequence $\Psi(\eta) = (\eta_n)$ where $\eta_n = p_n^{\wedge\wedge}(\eta)$.

Proposition 16. *Let G be an abelian Hausdorff group and let Ψ be as in Proposition 15.*

(a) *If α_G is continuous, then $\Psi(\ell^1(G)^{\wedge\wedge}) \subseteq \ell^1(G^{\wedge\wedge})$.*
(b) *If α_G is surjective and G is lqc, then $\Psi(\ell^1(G)^{\wedge\wedge}) \supseteq \ell^1(G^{\wedge\wedge})$.*

In particular, if G is reflexive, then $\Psi(\ell^1(G)^{\wedge\wedge}) = \ell^1(G^{\wedge\wedge})$.

Proof. (a) Assume first that α_G is continuous. Fix $\eta \in \ell^1(G)^{\wedge\wedge}$ and let $\eta_n := p_n^{\wedge\wedge}(\eta)$. Because η is a continuous character of $\ell^1(G)^\wedge$, there exists—by definition of the compact-open topology—a compact subset $K \subseteq \ell^1(G)$ such that $\eta \in K^{\triangleright\triangleright}$. In order to show that $\Psi(\eta) = (\eta_n) \in \ell^1(G^{\wedge\wedge})$, we fix a compact subset C of $G^\wedge$ and wish to prove that $\sum_{n\in\mathbb{N}} \kappa_{C^\triangleright}(\eta_n) < \infty$. Because, by assumption, α_G is continuous, there exists a neighborhood $U \in \mathcal{N}_G(0)$ such that $C \subseteq U^\triangleright$, whence $C^\triangleright \supseteq U^{\triangleright\triangleright}$. Because $K \subseteq \ell^1(G)$ is compact, there exists by Proposition 8 $N_U \in \mathbb{N}$ such that $K \subseteq S_{N_U,U}$. Hence, $\eta \in K^{\triangleright\triangleright} \subseteq (S_{N_U,U})^{\triangleright\triangleright} = \left(\{0\}^{\{1,\dots,N_U-1\}} \times (U^\triangleright)^{\mathbb{N}\setminus\{1,\dots,N_U-1\}}\right)^\triangleright$.

Because $p_n^\wedge(U^\triangleright) \subseteq \{0\}^{\{1,\dots,N_U-1\}} \times (U^\triangleright)^{\mathbb{N}\setminus\{1,\dots,N_U-1\}} =: M$ for all $n \geq N_U$, we obtain $\eta_n(U^\triangleright) = p_n^{\wedge\wedge}(\eta)(U^\triangleright) = \eta(p_n^\wedge(U^\triangleright)) \subseteq \mathbb{T}_+$, which implies that $\eta_n \in U^{\triangleright\triangleright}$ for all $n \geq N_U$. We want to show that

$$\sum_{n\geq N_U} \kappa_{U^{\triangleright\triangleright}}(\eta_n) < 2.$$

Assume that this does not hold and let $\nu \geq N_U$ be minimal with $\displaystyle\sum_{n=N_U}^{\nu} \kappa_{U^{\triangleright\triangleright}}(\eta_n) \geq 2$.

Let $N = \{n \in \mathbb{N} : \; N_U \leq n \leq \nu \text{ and } \kappa_{U^{\triangleright\triangleright}}(\eta_n) > 0\}$. For $n \in N$, we have $\kappa_{U^{\triangleright\triangleright}}(\eta_n) = \frac{1}{m_n}$ for a suitable natural number m_n, because $\eta_n \in U^{\triangleright\triangleright}$. Next, for $n \in N$, choose $\chi_n \in U^\triangleright$ such that $\eta_n(\chi_n) = t_n + \mathbb{Z}$ where $\dfrac{1}{4(m_n+1)} < t_n \leq \dfrac{1}{4m_n}$. For $k \in \mathbb{N} \setminus N$, put $\chi_k = 0$. Then, $\chi = (\chi_n)_{n\in\mathbb{N}} \in M$ and hence $\eta(\chi) = (\sum_{n\in N} t_n) + \mathbb{Z} \in \mathbb{T}_+$. Further,

$$\frac{1}{4} \leq \sum_{n\in N} \frac{1}{8m_n} \leq \sum_{n\in N} \frac{1}{4(m_n+1)} < \sum_{n\in N} t_n \leq \sum_{n\in N} \frac{1}{4m_n}$$

holds. Because ν was chosen minimal, we conclude that $\displaystyle\sum_{n\in N} \frac{1}{4m_n} = \sum_{n\in N, n\neq\nu} \frac{1}{4m_n} + \frac{1}{4m_\nu} < \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. This yields $\eta(\chi) = \sum_{n\in N} t_n + \mathbb{Z} \notin \mathbb{T}_+$ and gives the desired contradiction.

Now, it easily follows that $\sum_{n\in\mathbb{N}} \kappa_{C^\triangleright}(\eta_n) \leq \sum_{n\in\mathbb{N}} \kappa_{U^{\triangleright\triangleright}}(\eta_n) < \infty$.

(b) Assume now that α_G is surjective and G is lqc. Then, $\alpha_G^{-1} : G^{\wedge\wedge} \to G$ is continuous. We show that the composition

$$\ell^1(G^{\wedge\wedge}) \xrightarrow{(\alpha_G^{-1})_\#} \ell^1(G) \xrightarrow{\alpha_{\ell^1(G)}} \ell^1(G)^{\wedge\wedge} \xrightarrow{\Psi} (G^{\wedge\wedge})^{\mathbb{N}}$$

is the identity on $\ell^1(G^{\wedge\wedge})$. Fix $(\eta_n) \in \ell^1(G^{\wedge\wedge})$. Because α_G is surjective, there is a sequence $(x_n) \in G^{\mathbb{N}}$ such that $\alpha_G(x_n) = \eta_n$ for all $n \in \mathbb{N}$. Applying Proposition 15, we obtain

$$\Psi \circ \alpha_{\ell^1(G)} \circ (\alpha_G^{-1})_\#(\eta_n) = \Psi \circ \alpha_{\ell^1(G)}(x_n) = (\alpha_G(x_n)) = (\eta_n)$$

This shows $\ell^1(G^{\wedge\wedge}) \subseteq \Psi(\ell^1(G)^{\wedge\wedge})$. The final statement is an immediate consequence of (a) and (b). $\square$

Corollary 5. *Let G be a reflexive group. Then, $\alpha_{\ell^1(G)}$ is surjective.*

Proof. By Proposition 15, $\Psi \circ \alpha_{\ell^1(G)} = (\alpha_G)_\#$ holds. Because α_G is a topological isomorphism, Proposition 6 implies that $(\alpha_G)_\# : \ell^1(G) \to \ell^1(G^{\wedge\wedge})$ is a topological isomorphism. Because $\Psi : \ell^1(G)^{\wedge\wedge} \to \ell^1(G^{\wedge\wedge})$ is an isomorphism by Proposition 16, we obtain that $\alpha_{\ell^1(G)} = \Psi^{-1} \circ (\alpha_G)_\#$ is an isomorphism, whence surjective. $\square$

Theorem 3. *Let G be a reflexive group. Then, $\alpha_{\ell^1(G)}$ is an open isomorphism. If*

(a) *G is metrizable or*
(b) *$G^\wedge$ is complete and has a countable point-separating subgroup,*

then $\ell^1(G)$ is reflexive.

Proof. If G is reflexive, then G is an lqc Hausdorff group. According to Propositions 5 (c) and 12, $\ell^1(G)$ is lqc and Hausdorff as well and hence $\alpha_{\ell^1(G)}$ is an open isomorphism by Corollary 5.

It remains to show that $\alpha_{\ell^1(G)}$ is continuous if (a) or (b) holds. In case (a), it is a consequence of Proposition 14. In case (b), it is a consequence of Theorem 2. □

Theorem 4. *For every LCA group G, the group $\ell^1(G)$ is reflexive.*

Proof. Let G be an LCA group. By the structure theorem for LCA groups, G has an open subgroup H topologically isomorphic to $\mathbb{R}^n \times K$ where $n \in \mathbb{N}_0$ and K is a compact abelian group. By Proposition 6 (a), $\ell^1(H)$ can be considered to be an open subgroup of $\ell^1(G)$. Because by Theorem (2.3), in [23], a group is reflexive if and only if it has an open reflexive subgroup, it is sufficient to show that $\ell^1(H)$ is reflexive, or, by Proposition 6 (b), that $\ell^1(\mathbb{R})^n \times \ell^1(K)$ is reflexive. The group $\ell^1(\mathbb{R})$ is reflexive by Theorem 3.

By Corollary 5, $\alpha_{\ell^1(K)}$ is surjective, and by Theorem 1, $\alpha_{\ell^1(K)}$ is continuous. As the group $\ell^1(K)$ is lqc and Hausdorff by Propositions 12 and 5 (c), the assertion follows. □

Recall that a subgroup H of an abelian topological group G is **dually closed** if for every $x \in G \setminus H$ there exists a continuous character $\chi \in G^\wedge$ such that $\chi(H) = \{0\}$ and $\chi(x) \neq 0$. The subgroup H is **dually embedded** if every continuous character of H can be extended to a continuous character of G; in other words, the dual homomorphism of the canonical embedding $\iota : H \to G$ is surjective.

It is straightforward to check that $\mu_1(G)$ is a dually closed and dually embedded subgroup of $\ell^1(G)$ provided that G is an MAP group. We are going to apply the following result of Noble:

Proposition 17 ([24], Theorem 3.1)**.** *Let G be an abelian Hausdorff group such that α_G is an open isomorphism. If H is a dually closed and dually embedded subgroup of G, then also α_H is an open isomorphism.*

Theorem 5. *If $\ell^1(G)$ is Pontryagin reflexive, then so is G.*

Proof. Assume that $\ell^1(G)$ is reflexive. Because μ_1 is an embedding and $\mu_1(G)$ is a dually closed and dually embedded subgroup of $\ell^1(G)$, we obtain from Proposition 17 that α_G is an open isomorphism. Because $p_1 : \ell^1(G) \to G$ is a projection (Proposition 5 (b), Lemma (14.7) in [13]) implies α_G is continuous. □

Corollary 6. *Let G be a metrizable group. Then, G is reflexive if and only if $\ell^1(G)$ is reflexive.*

Proof. If G is a metrizable reflexive group, then $\ell^1(G)$ is reflexive by Theorem 3 (a). If $\ell^1(G)$ is reflexive, then G is reflexive by Theorem 5. □

2.4. The Schur Property of $\ell^1(G)$

A normed space V is said to have the **Schur property** if a sequence (x_n) converges to 0 provided that $(f(x_n))$ converges to 0 for every continuous linear form. In this section, we first recall the definition of the Schur property for MAP groups; afterward, having in mind that $\ell^1(\mathbb{R})$ has the Schur property, we prove that $\ell^1(G)$ has the Schur property for groups if and only if G has this property (Theorem 6).

Definition 4. *For a topological group (G, τ), denote by τ^+ the topology on G induced by $G \to \mathbb{T}^{G^\wedge}$, $x \mapsto (\chi(x))_{\chi \in G^\wedge}$. The topology τ^+ is called **weak topology**.*

The weak topology τ^+ is Hausdorff if and only if the characters of G separate the points. A subset C of G is called **weakly compact** if it is compact with respect to τ^+.

Definition 5. *A MAP group (G, τ) is said to have the* **Schur property** *if every τ^+-convergent sequence converges in τ.*

Theorem 6. *Let G be an lqc Hausdorff group. Then, G has the Schur property if and only if $\ell^1(G)$ has the* **Schur property**.

Proof. Assume first that G has the Schur property. Let $(x^{(m)})_{m\in\mathbb{N}}$ be a weakly convergent sequence in $\ell^1(G)$, where $x^{(m)} = (x_n^{(m)})_{n\in\mathbb{N}}$. WLOG, we may assume that $(x^{(m)})_{m\in\mathbb{N}}$ converges to 0. For $\chi \in G^\wedge$ and $n \in \mathbb{N}$, the sequence $(p_n^\wedge(\chi)(x^{(m)}))_{m\in\mathbb{N}} = (\chi(x_n^{(m)}))_{m\in\mathbb{N}}$ converges to 0 in $\mathbb{T}$. The assumption that G has the Schur property implies that $(x_n^{(m)})_{m\in\mathbb{N}}$ converges in the original topology of G to 0 for every $n \in \mathbb{N}$.

Assume that the sequence $(x^{(m)})_{m\in\mathbb{N}}$ is not convergent in the original topology. This means that there exists a quasi-convex neighborhood $U \in \mathcal{N}_G(0)$ such that for infinitely many $m \in \mathbb{N}$, $x^{(m)} \notin S_U$. After passing to a subsequence, we may assume that $x^{(m)} \notin S_U$ for all $m \in \mathbb{N}$. In order to obtain a contradiction, we are going to inductively construct strictly increasing sequences (m_k), (n_k) and (N_k) of natural numbers and a sequence $(\chi_k) \in (U^\triangleright)^{\mathbb{N}}$ such that

(a) $N_k \leq n_k < N_{k+1}$ for all $k \in \mathbb{N}$;
(b) $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_k)}) \notin \mathbb{T}_3$, and $\sum_{n>n_k} \kappa_U(x_n^{(m_k)}) < \frac{1}{8}$ for all $k \in \mathbb{N}$.

Let $m_1 = 1$. Because $x^{(1)} \notin S_U$, there exists $N_1 \in \mathbb{N}$ minimal such that $\sum_{n=1}^{N_1} \kappa_U(x_n^{(1)}) > 1$. Further, there is $n_1 \geq N_1$ such that $\sum_{n>n_1} \kappa_U(x_n^{(1)}) < \frac{1}{8}$. If for some $1 \leq n \leq N_1$ the element $x_n^{(1)} \notin U$, then we choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n^{(1)}) \notin \mathbb{T}_+$ and for $j \in \{1 \ldots, n_1\} \setminus \{n\}$ we put $\chi_j = 0$. Otherwise, let $F = \{n : 1 \leq n \leq N_1, \kappa_U(x_n^{(1)}) > 0\}$. Fix $n \in F$. Because $x_n^{(1)} \in U$, we have $0 < \kappa_U(x_n^{(1)}) \leq 1$; hence, there exists $l_n \in \mathbb{N}$ such that $\frac{1}{l_n} = \kappa_U(x_n^{(1)})$ for some $l_n \in \mathbb{N}$. The minimality of N_1 implies that $N_1 \in F$ and $\sum_{n\in F} \kappa_U(x_n^{(1)}) = \sum_{n=1}^{N_1} \kappa_U(x_n^{(1)}) \leq 2$. For every $n \in F$, choose $\chi_n \in U^\triangleright$ such that $\chi_n(x_n^{(1)}) = t_n + \mathbb{Z}$ for some $t_n \in]\frac{1}{4(l_n+1)}, \frac{1}{4l_n}]$. Because

$$\frac{1}{8} < \frac{1}{4}\sum_{n\in F} \frac{1}{2l_n} \leq \frac{1}{4}\sum_{n\in F} \frac{1}{l_n+1} < \sum_{n\in F} t_n \leq \frac{1}{4}\sum_{n\in F} \frac{1}{l_n} \leq \frac{1}{2},$$

we obtain $\sum_{n\in F} \chi_n(x_n^{(1)}) \notin \mathbb{T}_3$. For $n \in \{1, \ldots, n_1\} \setminus F$, we put $\chi_j = 0$. Then, conditions (a) and (b) are satisfied for $k = 1$.

Assume now that for some $k \in \mathbb{N}$, $m_1, \ldots, m_k$, $N_1, \ldots, N_k$, $n_1, \ldots, n_k$ and $\chi_1, \ldots, \chi_{n_k} \in U^\triangleright$ have been constructed such that (a) and (b) hold. By the initial observation, $(\kappa_U(x_n^{(m)}))_{m\in\mathbb{N}}$ converges to 0 for every $n \in \mathbb{N}$; hence, there exists $m_{k+1} > m_k$ such that $\sum_{n=1}^{n_k} \kappa_U(x_n^{(m_{k+1})}) < \frac{1}{8}$. Because $x^{(m_{k+1})} \notin S_U$, there exists a minimal $N_{k+1} > n_k$ such that $\sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) > \frac{7}{8}$. We choose $n_{k+1} \geq N_{k+1}$ such that $\sum_{n>n_{k+1}} \kappa_U(x_n^{(m_{k+1})}) < \frac{1}{8}$.

Fix $s \in [-\frac{1}{4}, \frac{1}{4}]$ such that $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_{k+1})}) = s + \mathbb{Z}$. If $s + \mathbb{Z} \notin \mathbb{T}_3$, we define $\chi_j = 0$ for all $n_k < j \leq n_{k+1}$.

Assume now that $s + \mathbb{Z} \in \mathbb{T}_3$ and that for some $n_k < j \leq N_{k+1}$, the element $x_j^{(m_{k+1})} \notin U$. Then, we choose $\chi_j \in U^\triangleright$ such that $\chi_j(x_j^{(m_{k+1})}) \notin \mathbb{T}_+$ and then $\sum_{n=1}^{n_k} \chi_n(x_n^{(m_{k+1})}) + \chi_j(x_j^{(m_{k+1})}) \notin \mathbb{T}_3$. Further, for all $n \in \{n_k + 1, \ldots, n_{k+1}\} \setminus \{j\}$, we put $\chi_n = 0$.

Finally, assume that $s + \mathbb{Z} \in \mathbb{T}_3$ and $x_j^{(m_{k+1})} \in U$ for all $n_k < j \leq N_{k+1}$. The minimality of N_{k+1} implies $\sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) \leq \frac{15}{8}$. Let $F = \{n \in \mathbb{N} : n_k < n \leq N_{k+1}$ and $\kappa_U(x_n^{(m_{k+1})}) > 0\}$. For $n \in F$, $\kappa_U(x_n^{(m_{k+1})}) = \frac{1}{l_n}$ for suitable $l_n \in \mathbb{N}$. Hence, there exist $\chi_n \in U^{\triangleright}$ such that $\chi_n(x_n^{(m_{k+1})}) = t_n + \mathbb{Z}$ where $|t_n| \in [\frac{1}{4(l_n+1)}, \frac{1}{4l_n}]$. Because $\sum_{n\in F} \frac{1}{l_n} = \sum_{n=n_k+1}^{N_{k+1}} \kappa_U(x_n^{(m_{k+1})}) \in [\frac{7}{8}, \frac{15}{8}]$, we obtain

$$\frac{1}{12} < \frac{1}{4} \cdot \frac{7}{16} < \frac{1}{4} \sum_{n\in F} \frac{1}{2l_n} \leq \frac{1}{4} \sum_{n\in F} \frac{1}{l_n+1} < \sum_{n\in F} |t_n| \leq \frac{1}{4} \sum_{n\in F} \frac{1}{l_n} \leq \frac{15}{32} < \frac{1}{2}.$$

For $n \in \{n_k+1, \ldots, n_{k+1}\} \setminus F$, we put $\chi_n = 0$.
If $s \in [0, \frac{1}{12}]$, then $\sum_{n=1}^{n_{k+1}} \chi_n(x^{(m_{k+1})}) \notin \mathbb{T}_3$.
If $s \in [-\frac{1}{12}, 0]$, replace χ_n by $-\chi_n$ for $n \in F$ such that $\sum_{n=1}^{n_{k+1}} \chi_n(x^{(m_{k+1})}) \notin \mathbb{T}_3$ holds.

We have constructed the subsequence $(x^{(m_k)})_k$ of $(x^{(m)})_{m\in\mathbb{N}}$ and the character $\chi = (\chi_n) \in (U^{\triangleright})^{\mathbb{N}} \subseteq \ell^1(G)^\wedge$. We obtain

$$\chi(x^{(m_k)}) = \sum_{n=1}^{\infty} \chi_n(x_n^{(m_k)}) = \underbrace{\sum_{n=1}^{n_k} \chi_n(x_n^{(m_k)})}_{\notin \mathbb{T}_3} + \underbrace{\sum_{n>n_k} \chi_n(x_n^{(m_k)})}_{\in \mathbb{T}_8 \subseteq \mathbb{T}_6} \notin \mathbb{T}_6,$$

because $\sum_{n>n_k} \kappa_U(x_n^{(m_k)}) < \frac{1}{8}$ and $\chi_n \in U^{\triangleright}$ for all $n \in \mathbb{N}$ (cf. Lemma 1 (c)). This shows $(\chi(x^{(m_k)}))$ does not converge to 0 and gives the desired contradiction.

Because $\mu_1 : G \to \ell^1(G)$ is an embedding, and the class of groups having the Schur property is closed under taking subgroups, the result follows. $\square$

An MAP group (G, τ) is said to have the **Glicksberg property** if every weakly compact subset is compact.

Every group which has the Glicksberg property also has the Schur property. This definition honors Glicksberg who proved that every LCA group has this property. Because then many other examples of groups having the Glicksberg property were established, for example, it is a consequence of the Eberlein–Šmulian theorem [25] and the Schur theorem [12] that $\ell^1(\mathbb{R})$ has the Glicksberg property. Further, the class of groups having the Glicksberg property is stable under taking subgroups and products. In particular, if for an MAP group G, the sequence group $\ell^1(G)$ has the Glicksberg property, then also G has the Glicksberg property (as G can be embedded in $\ell^1(G)$). However, the converse implication is not clear, see Question 6.

2.5. Schwartz Groups

In this final section, we show that only under very restrictive conditions is the sequence group $\ell^1(G)$ a Schwartz group, a class of groups introduced in [14] generalizing Schwartz topological vector spaces (see ([8], p. 201) for the definition).

Notation 1. *Let G be an abelian group. For a symmetric subset U of G containing 0, one defines*

$$(1/n)U := \{x \in G : \ jx \in U \, \forall \, 1 \leq j \leq n\} = \kappa_U^{-1}([0, \frac{1}{n}]).$$

Observe that if U is a symmetric neighborhood of 0 in a topological group, then also $(1/n)U$ is a neighborhood of 0 for every $n \in \mathbb{N}$.

Definition 6 ([14])**.** *An abelian topological group G is called a* **Schwartz group** *if for every symmetric neighborhood $U \in \mathcal{N}_G(0)$ there exists a symmetric neighborhood $V \in \mathcal{N}_G(0)$ and a sequence (F_n) of finite subsets of G such that $V \subseteq (1/n)U + F_n$ for all $n \in \mathbb{N}$.*

The class of Schwartz groups is closed under taking subgroups, arbitrary products, and Hausdorff quotients ([14], 3.6). Every lqc Schwartz group has the Glicksberg property, in particular, the Schur property ([19]). A topological vector space is a Schwartz space if and only if the additive group is a Schwartz group ([14], 4.2).

Definition 7 (Tarieladze)**.** *A symmetric subset U containing* 0 *of an abelian group G is called a* **GTG-set** *(***G***roup* **T***opology* **G***enerating set), if the sets $((1/n)U)_{n\in\mathbb{N}}$ form a neighborhood base at* 0 *of a not-necessarily Hausdorff group topology. An abelian topological group G is called a* **locally GTG-group** *if it has a neighborhood base at* 0 *consisting of GTG-sets.*

The following two statements follow straightforward from the definitions. For a GTG-set U in G, the intersection $U_\infty := \bigcap_{n\in\mathbb{N}}(1/n)U$ is a subgroup of G. It is a direct consequence of Lemma 1 (c) that every lqc group is locally GTG.

By ([8], 10.4.3), every bounded subset of a Schwartz space is precompact. Hence, a normed space is a Schwartz space if and only if it is finite-dimensional. Thus, $\ell^1(\mathbb{R})$ is not a Schwartz space and hence no Schwartz group either. Conversely, if $\ell^1(G)$ is a Schwartz group, then necessarily G is a Schwartz group, because G embeds in $\ell^1(G)$. However, the example $\mathbb{R} \to \ell^1(\mathbb{R})$ shows that this property is not sufficient.

Theorem 7. *For a locally GTG-group G, the following assertions are equivalent:*

(a) *$\ell^1(G)$ is a Schwartz group.*
(b) *G is linearly topologized.*

Proof. (a) $\Longrightarrow$ (b) Let $\mathcal{N}_0$ be a neighborhood base at $0 \in G$ consisting of GTG-sets. Fix a neighborhood $U \in \mathcal{N}_0$. There is neighborhood $W \in \mathcal{N}_0$ such that $W + W + W + W \subseteq U$. Because $\ell^1(G)$ is a Schwartz group by assumption, there is a sequence $(\widetilde{F}_n)$ of finite subsets in $\ell^1(G)$ and a neighborhood $V \in \mathcal{N}_0$ such that $S_V \subseteq \widetilde{F}_n + (1/n)S_W$. Because $G^{(\mathbb{N})}$ is dense in $\ell^1(G)$ (Proposition 7), there exists for every $n \in \mathbb{N}$ a finite subset $F_n \subseteq G^{(\mathbb{N})}$ such that $\widetilde{F}_n \subseteq F_n + (1/n)S_W$. Thus, we have $S_V \subseteq F_n + (1/n)S_W + (1/n)S_W$. We are going to show that $(1/n)S_W + (1/n)S_W \subseteq (1/n)S_U$. Therefore, we fix $(x_n), (y_n) \in (1/n)S_W$. For $1 \le j \le n$, we obtain by Lemma 1 (a) and (b) and Proposition 4

$$\textstyle\sum_{n\in\mathbb{N}} \kappa_U(j(x_n + y_n)) \le \sum_{n\in\mathbb{N}} \kappa_{W+W+W+W}(jx_n + jy_n) \le$$

$$\le \textstyle\sum_{n\in\mathbb{N}} \kappa_{W+W}(jx_n) + \sum_{n\in\mathbb{N}} \kappa_{W+W}(jy_n) \le \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(jx_n) + \frac{1}{2}\sum_{n\in\mathbb{N}} \kappa_W(jy_n) \le 1.$$

It follows that

$$S_V \subseteq F_n + (1/n)S_U$$

for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Because F_n is a finite subset of $G^{(\mathbb{N})}$, we can choose $N_n \in \mathbb{N}$ such that $p_m(F_n) = \{0\}$ for all $m \ge N_n$. For $m \ge N_n$, $\mu_m(V) \subseteq S_V \subseteq F_n + (1/n)S_U$, or equivalently because $p_m(F_n) = \{0\}$, $\mu_m(V) \subseteq (1/n)S_U$. Thus, for all $x \in V$ and $1 \le j \le n$, we have $j\mu_m(x) \in S_U$ which is equivalent to $\kappa_U(jx) \le 1$ for all $1 \le j \le n$. Thus, $jx \in U$ for all $1 \le j \le n$, which means that $x \in (1/n)U$. We have shown that $V \subseteq (1/n)U$ for all $n \in \mathbb{N}$. This yields $V \subseteq \bigcap_{n\in\mathbb{N}}(1/n)U = U_\infty$. As a consequence, U_∞ is an open subgroup of G. Hence, $(U_\infty)_{U\in\mathcal{N}_0}$ is a neighborhood base at 0 for G consisting of open subgroups. This means that G is linearly topologized.

(b) $\Longrightarrow$ (a) If G is linearly topologized, so is $\ell^1(G)$ by Proposition 5. It is obvious that every linearly topologized group is a Schwartz group, so the assertion follows. □

The class of nuclear groups was introduced by Banaszczyk in [13]. Nuclear groups include all Schwartz groups ([14], 4.3), all LCA groups ([13], 7.10), and all nuclear locally convex vector spaces ([13], 7.4). This class of groups is closed under taking products, subgroups, and Hausdorff quotient groups ([13], 7.5 and 7.6), and every nuclear group is lqc ([13], 8.5). Because every Hausdorff linearly topologized group can be embedded into a product of discrete groups, every linearly topologized Hausdorff group is nuclear.

Theorem 8. *For an lqc Hausdorff group G, the following are equivalent:*

(a) *$\ell^1(G)$ is a nuclear group;*
(b) *$\ell^1(G)$ is a Schwartz group;*
(c) *G is linearly topologized.*

Proof.
(a)$\Longrightarrow$ (b) holds, because every nuclear group is a Schwartz group ([14], 4.3).
(b)$\Longrightarrow$ (c) is a consequence of Theorem 7.
(c)$\Longrightarrow$ (a) If G is linearly topologized, so is $\ell^1(G)$ by Proposition 5. Hence, $\ell^1(G)$ is a nuclear group.
$\square$

3. Open Questions

In this final chapter, we gather some open questions concerning sequence groups.

Question 1. *Characterize those abelian Hausdorff groups for which $\ell^1(G) = G^{(\mathbb{N})}$ holds. In particular, is it possible that $c_0(G) \neq G^{(\mathbb{N})} = \ell^1(G)$?*

A dense subgroup H of an abelian Hausdorff group G is said to **determine** G if the dual homomorphism $\iota^\wedge : G^\wedge \to H^\wedge$ of the natural embedding ι is a topological isomorphism. It was shown in [26] and in ([27], 4.10) that every metrizable abelian group determines its completion.

Question 2. *Assume that H is a dense subgroup of the abelian topological group G which determines G. Does $\ell^1(H)$ determine $\ell^1(G)$?*

It was shown (Theorem 5) that if $\ell^1(G)$ is reflexive, then G must be reflexive. Conversely, if G is reflexive, then $\alpha_{\ell^1(G)}$ is an open isomorphism (Theorem 3). However, we do not know if $\alpha_{\ell^1(G)}$ is continuous.

Question 3. *Let G be an abelian Hausdorff group such that α_G is continuous. Is it true that $\alpha_{\ell^1(G)}$ is continuous?*

Or, a bit weaker,

Question 4. *Let G be a reflexive group. Is it true that $\alpha_{\ell^1(G)}$ is continuous (and hence $\ell^1(G)$ is reflexive)?*

It was shown in [4] that for every LCA group G, the group of null-sequences $c_0(G)$ is reflexive.

Question 5. *Is it true that G is a reflexive group if and only if $c_0(G)$ is reflexive?*

Question 6. *Assume that G has the Glicksberg property. Does $\ell^1(G)$ have the Glicksberg property?*

Question 7. *What can be said about the groups*

$$\ell^p(G) = \{(x_n) \in G^{\mathbb{N}} : \sum_{n\in\mathbb{N}} (\kappa_U(x_n))^p < \infty \ \forall U \in \mathcal{N}_G(0)\}$$

for $1 \leq p < \infty$? In particular, what are the properties of $\ell^2(G)$?

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