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Aspects of Differential Calculus Related to Infinite-Dimensional Vector Bundles and Poisson Vector Spaces

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Abstract: We prove various results in infinite-dimensional differential calculus that relate the differentiability properties of functions and associated operator-valued functions (e.g., differentials). The results are applied in two areas: (1) in the theory of infinite-dimensional vector bundles, to construct new bundles from given ones, such as dual bundles, topological tensor products, infinite direct sums, and completions (under suitable hypotheses); (2) in the theory of locally convex Poisson vector spaces, to prove continuity of the Poisson bracket and continuity of passage from a function to the associated Hamiltonian vector field. Topological properties of topological vector spaces are essential for the studies, which allow the hypocontinuity of bilinear mappings to be exploited. Notably, we encounter $k_{\mathbb{R}}$-spaces and locally convex spaces E such that $E \times E$ is a $k_{\mathbb{R}}$-space.

Keywords: vector bundle; dual bundle; direct sum; completion; tensor product; cocycle; smoothness; analyticity; hypocontinuity; k-space; compactly generated space; infinite-dimensional Lie group; Poisson vector space; Poisson bracket; Hamiltonian vector field; group action; multilinear map

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1. Introduction

We study questions of infinite-dimensional differential calculus in the setting of Keller's C^k_c-theory [1] (going back to [2]). Applications to infinite-dimensional vector bundles are given, and also applications in the theory of locally convex Poisson vector spaces.

Differentiability properties of operator-valued maps. Our results are centred around the following basic problem: Consider locally convex spaces X, E and F, an open set $U \subseteq X$ and a map $f\colon U \to L(E,F)_b$ to the space of continuous linear maps, endowed with the topology of uniform convergence on bounded sets. How are the differentiability properties of the operator-valued map f related to those of

$$f^\wedge\colon U \times E \to F, \quad f^\wedge(x,v) := f(x)(v)\,?$$

We show that if $f^\wedge$ is smooth, then also f is smooth (Proposition 1). Conversely, exploiting the hypocontinuity of the bilinear evaluation map

$$L(E,F)_b \times E \to F, \quad (\alpha, v) \mapsto \alpha(v)\,,$$

we find natural hypotheses on E and F ensuring that smoothness of f entails smoothness of $f^\wedge$ (Proposition 2; likewise for compact sets in place of bounded sets). Without extra hypotheses on E and F, this conclusion becomes false, e.g., if $U = X$ is a non-normable, real, locally convex space with dual space $X' := L(X,\mathbb{R})$. Then, $f := \mathrm{id}_{X'}\colon X'_b \to X'_b$ is continuous linear and thus smooth, but $f^\wedge\colon X'_b \times X \to \mathbb{R}$ is the bilinear evaluation map taking (λ, x) to $\lambda(x)$, which is discontinuous for non-normable X (see [3] (p. 2)) and hence not smooth in the sense of Keller's C^∞_c-theory. We also obtain results concerning finite-order differentiability properties, as well as real and complex analyticity. Furthermore, $L(E,F)$ can be replaced with the space $L^k(E_1, \ldots, E_k, F)$ of continuous k-linear maps $E_1 \times \cdots \times E_k \to F$, if $E_1, \ldots, E_k$ are locally convex spaces. (Related questions also play a role in the comparative study of differential calculi [1].) As a very special case of our studies, the differential

$$f' \colon U \to L(E, F)_b$$

is C^{r-2}, for each $r \in \mathbb{N} \cup \{\infty\}$ with $r \geq 2$, locally convex spaces E and F, and C^r-map $f \colon U \to F$ on an open set $U \subseteq E$ (see Corollary 1).

Applications to infinite-dimensional vector bundles. Apparently, mappings of the specific form just described play a vital role in the theory of vector bundles: If F is a locally convex space, M a (not necessarily finite-dimensional) smooth manifold and $(U_i)_{i \in I}$ an open cover of M, then the smooth vector bundles $E \to M$, with fibre F, which are trivial over the sets U_i, can be described by cocycles $g_{ij} \colon U_i \cap U_j \to \mathrm{GL}(F)$ such that $G_{ij} := g_{ij}^{\wedge} \colon (U_i \cap U_j) \times F \to F$, $(x, v) \mapsto g_{ij}(x)(v)$ is smooth (Proposition 3, Remark 7). Then, g_{ij} is smooth as a mapping to the space $L(F)_b := L(F, F)_b$ (see Proposition 1). In various contexts—for example, when trying to construct dual bundles—we are in the opposite situation: we know that each g_{ij} is smooth, and would like to conclude that also the mappings G_{ij} are smooth. Although this is not possible in general (as examples show), our results provide additional conditions ensuring that the conclusion is correct in the specific situation at hand. Notably, we obtain conditions ensuring the existence of a canonical dual bundle (Proposition 13). Without extra conditions, a canonical dual bundle need not exist (Example 2).

Besides dual bundles, we discuss a variety of construction principles of new vector bundles from given ones, including topological tensor products, completions, and finite or infinite direct sums. More generally, given a (finite- or infinite-dimensional) Lie group acting on the base manifold M, we discuss the construction of new equivariant vector bundles from given ones. Most of the constructions require specific hypotheses on the base manifold, the fibre of the bundle, and the Lie group.

As to completions, complementary topics were considered in the literature: Given an infinite-dimensional smooth manifold M, completions of the tangent bundle with respect to a weak Riemannian metric occur in [4] (p. 549), in hypotheses for a so-called *robust* Riemannian manifold.

We mention that multilinear algebra and vector bundle constructions can be performed much more easily in an inequivalent setting of infinite-dimensional calculus, the convenient differential calculus [3]. However, a weak notion of vector bundles is used there, which need not be topological vector bundles. Our discussion of vector bundles intends to pinpoint additional conditions ensuring that the natural construction principles lead to vector bundles in a stronger sense (which are, in particular, topological vector bundles).

The work [5] was particularly important for our studies. For an open subset U of a Fréchet space E, smoothness of $f^{\wedge} \colon U \times E^k \to \mathbb{R}$ is deduced from smoothness of $f \colon U \to \Lambda^k(E')_b$ in the proof of [5] (Proposition IV.6). A typical hypocontinuity argument already appears in the proof of [5] (Lemma IV.7). In contrast to the local calculations in charts, the global structure on a dual bundle (and bundles of k-forms) asserted in the first remark of [5] (p. 339) is problematic if Keller's C_c^{∞}-theory is used, without further hypotheses.

Applications related to locally convex Poisson vector spaces. In the wake of works by Odzijewicz and Ratiu on Banach–Poisson vector spaces and Banach–Poisson manifolds [6,7], certain locally convex Poisson vector spaces were introduced [8], which generalise the Lie–Poisson structure on the dual space of a finite-dimensional Lie algebra going back to Kirillov, Kostant and Souriau. By now, the latter spaces can be embedded in a general theory of locally convex Poisson manifolds (see [9]; for generalisations of finite-dimensional Poisson geometry with a different thrust, cf. [10]). Recall that many important examples of bilinear mappings between locally convex topological vector spaces are not continuous, but at least hypocontinuous (cf. [11] for this classical concept). In Sections 12 and 13, we provide the proofs for two fundamental results in the theory of locally convex Poisson vector spaces which are related to hypocontinuity. (These proofs were stated in the preprint version of [8], but not included in the actual publication.) We show that the Poisson bracket associated with a continuous Lie bracket is always continuous (Theorem 1) and that the linear map $C^\infty(E,\mathbb{R}) \to C^\infty(E,E)$ taking a smooth function to the associated Hamiltonian vector field is continuous (Theorem 2). Ideas from [8] and the current article were also taken further in [12] (Section 13).

2. Preliminaries and Notation

We describe our setting of differential calculus and compile useful facts. Either references to the literature are given or a proof; the proofs can be looked up in Appendix A.

Infinite-dimensional calculus. We work in the framework of infinite-dimensional differential calculus known as Keller's C^k_c-theory [1]. Our main reference is [13] (see also [14–17]). If $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we let $\mathbb{D} := \{t \in \mathbb{K} \colon |t| \leq 1\}$ and $\mathbb{D}_\varepsilon := \{t \in \mathbb{K} \colon |t| \leq \varepsilon\}$ for $\varepsilon > 0$. We write $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All topological vector spaces considered in the article are assumed Hausdorff, unless the contrary is stated. For brevity, Hausdorff locally convex topological vector spaces will be called locally convex spaces. As usual, a subset M of a $\mathbb{K}$-vector space is called *balanced* if $tx \in M$ for all $x \in M$ and $t \in \mathbb{D}$. The subset M is called *absolutely convex* if it is both convex and balanced. If $q \colon E \to [0,\infty]$ is a seminorm on a $\mathbb{K}$-vector space E, we write $B^q_\varepsilon(x) := \{y \in E \colon q(y-x) < \varepsilon\}$ for $x \in E$ and $\varepsilon > 0$. We also write $\|x\|_q$ in place of $q(x)$. If E is a locally convex $\mathbb{K}$-vector space, we let E' be the dual space of continuous $\mathbb{K}$-linear functionals $\lambda \colon E \to \mathbb{K}$. We write $M^\circ := \{\lambda \in E' \colon \lambda(M) \subseteq \mathbb{D}\}$ for the polar of a subset $M \subseteq E$. If $\alpha \colon E \to F$ is a continuous $\mathbb{K}$-linear map between locally convex $\mathbb{K}$-vector spaces, we let $\alpha' \colon F' \to E'$, $\lambda \mapsto \lambda \circ \alpha$ be the dual linear map. We say that a mapping $f \colon X \to Y$ between topological spaces is a *topological embedding* if it is a homeomorphism onto its image. We recall:

Definition 1. *Let E and F be locally convex $\mathbb{K}$-vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $U \subseteq E$ be an open subset. A map $f \colon U \to F$ is called $C^0_{\mathbb{K}}$ if it is continuous, in which case we set $d^0 f := f$. Given $x \in U$ and $y \in E$, we define*

$$df(x,y) := (D_y f)(x) := \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t}$$

*if the limit exists (using $t \in \mathbb{K}^\times$ such that $x + ty \in U$). Let $r \in \mathbb{N} \cup \{\infty\}$. We say that a continuous map $f \colon U \to F$ is a $C^r_{\mathbb{K}}$-*map *if the iterated directional derivative*

$$d^k f(x, y_1, \ldots, y_k) := (D_{y_k} \cdots D_{y_1})(f)(x)$$

exists for all $k \in \mathbb{N}$ such that $k \leq r$ and all $(x, y_1, \ldots, y_k) \in U \times E^k$, and if the mappings $d^k f \colon U \times E^k \to F$ so obtained are continuous. Thus, $d^1 f = df$. If $\mathbb{K}$ is understood, we write C^r instead of $C^r_{\mathbb{K}}$. As usual, C^∞-maps are also called smooth.

Remark 1. *For $k \in \mathbb{N}$, it is known that a map $f \colon U \to F$ as before is $C^k_{\mathbb{K}}$ if and only if f is $C^1_{\mathbb{K}}$ and $df \colon U \times E \to F$ is $C^{k-1}_{\mathbb{K}}$ (cf. [13] (Proposition 1.3.10)).*

Remark 2. *If $\mathbb{K} = \mathbb{C}$, it is known that a map $f \colon E \supseteq U \to F$ as before is $C^\infty_{\mathbb{C}}$ if and only if it is* complex analytic *in the sense of [18] (Definition 5.6): f is continuous and for each $x \in U$, there exists a 0-neighbourhood $Y \subseteq E$ such that $x + Y \subseteq U$ and $f(x+y) = \sum_{n=0}^\infty \beta_n(y)$ for all $y \in Y$ as a pointwise limit, where $\beta_n \colon E \to F$ is a continuous homogeneous polynomial over $\mathbb{C}$ of degree n, for each $n \in \mathbb{N}_0$ [13] (Theorem 2.1.12). Furthermore, f is complex analytic if and only if f is $C^\infty_{\mathbb{R}}$ and $df(x,\cdot) \colon E \to F$ is complex linear for all $x \in U$ (see [13] (Theorem 2.1.12)). Complex analytic maps will also be called* $\mathbb{C}$-analytic *or $C^\omega_{\mathbb{C}}$.*

Definition 2. *If $\mathbb{K} = \mathbb{R}$, then a map $f \colon U \to F$ as before is called* real analytic *(or* $\mathbb{R}$-analytic, *or $C^\omega_{\mathbb{R}}$) if it extends to a complex analytic mapping $\widetilde{U} \to F_{\mathbb{C}}$ on some open neighbourhood $\widetilde{U}$ of U in the complexification $E_{\mathbb{C}}$ of E.*

In the following, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, unless the contrary is stated. We use the conventions $\infty + k := \infty - k := \infty$ and $\omega + k := \omega - k := \omega$, for each $k \in \mathbb{N}$. Furthermore, we extend the order on $\mathbb{N}_0$ to an order on $\mathbb{N}_0 \cup \{\infty, \omega\}$ by declaring $n < \infty < \omega$ for each $n \in \mathbb{N}_0$.

Remark 3. *Compositions of composable $C^r_{\mathbb{K}}$-mappings are $C^r_{\mathbb{K}}$-mappings (see Proposition 1.3.4, Remark 2.1.13, and Proposition 2.2.4 in [13]). Thus, $C^r_{\mathbb{K}}$-manifolds modelled on locally convex $\mathbb{K}$-vector spaces can be defined in the usual way (see [13] (Chapter 3) for a detailed exposition). In this article, the word "manifold" (resp., "Lie group") always refers to a manifold (resp., Lie group) modelled on a locally convex space.*

The following basic fact will be used repeatedly.

Lemma 1. *For $k \in \mathbb{N}$, let X, $E_1, \ldots, E_k$, and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq X$ be an open subset and*

$$f\colon U \times E_1 \times \cdots \times E_k \to F$$

be a $C^1_{\mathbb{K}}$-map such that $f^\vee(x) := f(x, \cdot)\colon E_1 \times \cdots \times E_k \to F$ is k-linear, for each $x \in U$. Let $x \in U$ and q be a continuous seminorm on F. Then, there exists a continuous seminorm p on X with $B^p_1(x) \subseteq U$, and continuous seminorms p_j on E_j for $j \in \{1, \ldots, k\}$ such that

$$\|f(y, v_1, \ldots, v_k)\|_q \leq \|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \quad \text{and} \tag{1}$$

$$\|f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \tag{2}$$

for all $y \in B^p_1(x)$ and $(v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k$.

We shall also use the following fact:

Lemma 2. *Let E and F be locally convex $\mathbb{K}$-vector spaces, $k \geq 2$ be an integer and $f\colon U \times E^k \to F$ be a mapping such that $f(x, \cdot)\colon E^k \to F$ is k-linear and symmetric for each $x \in U$. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. If*

$$h\colon U \times E \to F, \quad (x, y) \mapsto f(x, y, \ldots, y)$$

is $C^r_{\mathbb{K}}$, then also f is $C^r_{\mathbb{K}}$. Notably, f is continuous if h is continuous.

***k*-spaces, $k_{\mathbb{R}}$-spaces, k^∞-spaces, and k_ω-spaces.** Recall that a topological space X is said to be *completely regular* if it is Hausdorff and its topology is initial with respect to the set $C(X, \mathbb{R})$ of all continuous real-valued functions on X. Every locally convex space is completely regular, as with every Hausdorff topological group (cf. [19] (Theorem 8.2)). Compare [20,21] for the following.

A topological space X is called a *k-space* if it is Hausdorff and a subset $A \subseteq X$ is closed if and only if $A \cap K$ is closed in K for each compact subset $K \subseteq X$. Every metrisable topological space is a k-space, and every locally compact Hausdorff space. A Hausdorff space X is a k-space if and only if, for each topological space Y, a map $f\colon X \to Y$ is continuous if and only if f is *k-continuous* in the sense that $f|_K$ is continuous for each compact subset $K \subseteq X$. If X is a k-space, then also every subset $M \subseteq X$ which is open or closed in X, when the induced topology is used on M.

A topological space X is called a *$k_{\mathbb{R}}$-space* if it is Hausdorff and a function $f\colon X \to \mathbb{R}$ is continuous if and only if f is k-continuous. Then also a map $f\colon X \to Y$ to a completely regular topological space Y is continuous if and only if it is k-continuous (as the latter condition implies continuity of $g \circ f$ for each $g \in C(Y, \mathbb{R})$). For more information, cf. [22].

Every k-space is a $k_{\mathbb{R}}$-space. The converse is not true: $\mathbb{R}^I$ is known to be a $k_{\mathbb{R}}$-space for each set I (see [22]). If I has cardinality $\geq 2^{\aleph_0}$, then $\mathbb{R}^I$ is not a k-space. (If $\mathbb{R}^I$ was a k-space, then a certain non-discrete subgroup G of $(\mathbb{R}^{\mathbb{R}}, +)$ constructed in [23] would be discrete, which is a contradiction (see [13] (Remark A.6.16 (a)) for more details). Compare also [22].)

The following facts are well known (cf. [22]):

Lemma 3. (a) *If a $k_{\mathbb{R}}$-space X is a direct product $X_1 \times X_2$ of Hausdorff spaces and $X_1 \neq \emptyset$, then X_2 is a $k_{\mathbb{R}}$-space.*
(b) *Every open subset U of a completely regular $k_{\mathbb{R}}$-space X is a $k_{\mathbb{R}}$-space in the induced topology.*

Notably, U is a $k_{\mathbb{R}}$-space for each open subset U of a locally convex space E which is a $k_{\mathbb{R}}$-space. If $E \times E$ is a $k_{\mathbb{R}}$-space, then also E.

Following [8], a topological space X is called a *k^∞-space* if the Cartesian power X^n is a k-space for each $n \in \mathbb{N}$, using the product topology. A Hausdorff space X is called *hemicompact* if $X = \bigcup_{n\in\mathbb{N}} K_n$ for a sequence $K_1 \subseteq K_2 \subseteq \cdots$ of compact subsets $K_n \subseteq X$ such that each compact subset of X is a subset of some K_n. Hemicompact k-spaces are also called *k_ω-spaces*. If X and Y are k_ω-spaces, then the product topology makes $X \times Y$ a k_ω-space. Notably, every k_ω-space is a k^∞-space. See [24,25] for further information. Finite products of metrisable spaces being metrisable, every metrisable topological space is a k^∞-space. Recall that a locally convex space E is said to be a *Silva space* or (DFS)-space if it is the locally convex inductive limit of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of Banach spaces such that each inclusion map $E_n \to E_{n+1}$ is a compact operator. Every Silva space is a k_ω-space (see, e.g., [13] (Proposition B13.13(g))).

Spaces of multilinear maps. Given $k \in \mathbb{N}$, locally convex $\mathbb{K}$-vector spaces $E_1, \ldots, E_k$ and F, and a set $\mathcal{S}$ of bounded subsets of $E_1 \times \cdots \times E_k$, we write $L^k(E_1, \ldots, E_k, F)_{\mathcal{S}}$ or $L^k_{\mathbb{K}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ for the space of continuous k-linear maps $E_1 \times \cdots \times E_k \to F$, endowed with the topology $\mathcal{O}_{\mathcal{S}}$ of uniform convergence on the sets $B \in \mathcal{S}$. Recall that finite intersections of sets of the form

$$\lfloor B, U \rfloor := \{\beta \in L^k(E_1, \ldots, E_k, F) \colon \beta(B) \subseteq U\}$$

yield a basis of 0-neighbourhoods for this (not necessarily Hausdorff) locally convex vector topology, for U ranging through the 0-neighbourhoods in F and B through $\mathcal{S}$. If $\bigcup_{B\in\mathcal{S}} B = E_1 \times \cdots \times E_k$, then $\mathcal{O}_{\mathcal{S}}$ is Hausdorff. If $E_1 = \cdots = E_k$, we abbreviate $L^k(E, F)_{\mathcal{S}} := L^k(E, \ldots, E, F)_{\mathcal{S}}$. If $k = 1$ and $E := E_1$, we abbreviate $L(E, F)_{\mathcal{S}} := L^1(E, F)_{\mathcal{S}}$, $L_{\mathbb{K}}(E, F)_{\mathcal{S}} := L^1_{\mathbb{K}}(E, F)_{\mathcal{S}}$ and $L(E)_{\mathcal{S}} := L(E, E)_{\mathcal{S}}$. We write $\mathrm{GL}(E) = L(E)^\times$ for the group of all automorphisms of the locally convex $\mathbb{K}$-vector space E. If $\mathcal{S}$ is the set of all bounded, compact, and finite subsets of $E_1 \times \cdots \times E_k$, respectively, we shall usually write "b," "c," and "p" in place of $\mathcal{S}$. For example, we shall write $L^k(E_1, \ldots, E_k, F)_b$, $L^k(E_1, \ldots, E_k, F)_c$, and $L^k(E_1, \ldots, E_k, F)_p$.

Remark 4. *Let $E_1, \ldots, E_k$ and F be complex locally convex spaces and $f \colon U \to L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)$ be a map, defined on an open subset U of a real locally convex space. Let $\mathcal{S} := b$ or $\mathcal{S} := c$. Since $L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ is a closed real vector subspace of $L^k_{\mathbb{R}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$, the map f is $C^r_{\mathbb{R}}$ as a map to $L^k_{\mathbb{C}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ if and only if f is $C^r_{\mathbb{R}}$ as a map to $L^k_{\mathbb{R}}(E_1, \ldots, E_k, F)_{\mathcal{S}}$ (see [13] (Lemma 1.3.19 and Exercise 2.2.4)).*

Given a $C^r_{\mathbb{K}}$-map $f \colon E \supseteq U \to F$ as in Definition 1, we define $f^{(0)} := f$ and

$$f^{(j)} \colon U \to L^j_{\mathbb{K}}(E, F), \quad f^{(j)}(x) := (d^j f)^\vee(x) = d^j f(x, \cdot)$$

for $j \in \mathbb{N}$ such that $j \leq r$.

Hypocontinuous multilinear maps. Beyond normed spaces, typical multilinear maps are not continuous, but merely hypocontinuous. Hypocontinuous *bi*linear maps are discussed in many textbooks. An analogous notion of hypocontinuity for multilinear maps (to be described presently) is useful to us. It can be discussed similarly to the bilinear case.

Lemma 4. *For an integer $k \geq 2$, let $\beta \colon E_1 \times \cdots \times E_k \to F$ be a separately continuous k-linear mapping and $j \in \{2, \ldots, k\}$ such that, for each $x \in E_1 \times \cdots \times E_{j-1}$, the map*

$$\beta^\vee(x) := \beta(x, \cdot) \colon E_j \times \cdots \times E_k \to F$$

is continuous. Let $\mathcal{S}$ be a set of bounded subsets of $E_j \times \cdots \times E_k$. Consider the conditions:

(a) *For each $M \in \mathcal{S}$ and each 0-neighbourhood $W \subseteq F$, there exists a 0-neighbourhood $V \subseteq E_1 \times \cdots \times E_{j-1}$ such that $\beta(V \times M) \subseteq W$.*
(b) *The $(j-1)$-linear map $\beta^\vee \colon E_1 \times \cdots \times E_{j-1} \to L^{k-j+1}(E_j, \ldots, E_k, F)_{\mathcal{S}}$ is continuous.*
(c) *$\beta|_{E_1 \times \cdots \times E_{j-1} \times M} \colon E_1 \times \cdots \times E_{j-1} \times M \to F$ is continuous, for each $M \in \mathcal{S}$.*
Then (a) *and* (b) *are equivalent, and* (b) *implies* (c). *If*

$$(\forall M \in \mathcal{S})\ (\exists N \in \mathcal{S})\quad \mathbb{D}M \subseteq N, \tag{3}$$

then (a), (b), *and* (c) *are equivalent.*

Definition 3. *A k-linear map β which satisfies the hypotheses and Condition* (a) *of Lemma 4 is called $\mathcal{S}$-hypocontinuous in its arguments $(j, \ldots, k)$. If $j = k$, we also say that β is $\mathcal{S}$-hypocontinuous in the k-th argument. Analogously, we define $\mathcal{S}$-hypocontinuity of β in the j-th argument, if $j \in \{1, \ldots, k\}$ and a set $\mathcal{S}$ of bounded subsets of E_j are given.*

We are mainly interested in b-, c-, and p-hypocontinuity, viz., in $\mathcal{S}$-hypocontinuity with respect to the set $\mathcal{S}$ of all bounded subsets of $E_j \times \cdots \times E_k$, the set $\mathcal{S}$ of all compact subsets, and the set $\mathcal{S}$ of all finite subsets, respectively. If $\mathcal{S}$ and $\mathcal{T}$ are sets of bounded subsets of $E_j \times \cdots \times E_k$ such that $\mathcal{S} \subseteq \mathcal{T}$ and β is $\mathcal{T}$-hypocontinuous in its variables $(j, \ldots, k)$, then β is also $\mathcal{S}$-hypocontinuous in the latter. The following is obvious from Lemma 4 (c) (as the elements of a convergent sequence, together with its limit, form a compact set):

Lemma 5. *If $\beta \colon E_1 \times \cdots \times E_k \to F$ is c-hypocontinuous in some argument, or in its arguments $(j, \ldots, k)$ for some $j \in \{2, \ldots, k\}$, then β is sequentially continuous.*

In many cases, separately continuous bilinear maps are automatically hypocontinuous. Recall that a subset B of a locally convex space E is a *barrel* if it is closed, absolutely convex, and absorbing. The space E is called *barrelled* if every barrel is a 0-neighbourhood. See Proposition 6 in [11] (Chapter III, §5, no. 3) for the following fact.

Lemma 6. *If $\beta \colon E_1 \times E_2 \to F$ is a separately continuous bilinear map and E_1 is barrelled, then β is $\mathcal{S}$-hypocontinuous in its second argument, with respect to any set $\mathcal{S}$ of bounded subsets of E_2.*

Evaluation maps are paradigmatic examples of hypocontinuous multilinear maps.

Lemma 7. *Let $E_1, \ldots, E_k$ and F be locally convex $\mathbb{K}$-vector spaces and $\mathcal{S}$ be a set of bounded subsets of $E := E_1 \times \cdots \times E_k$ with $\bigcup_{M \in \mathcal{S}} M = E$. Then, the $(k+1)$-linear map*

$$\varepsilon \colon L^k(E_1, \ldots, E_k, F)_{\mathcal{S}} \times E_1 \times \cdots \times E_k \to F, \quad (\beta, x) \mapsto \beta(x)$$

is $\mathcal{S}$-hypocontinuous in its arguments $(2, \ldots, k+1)$. If $k = 1$ and $E = E_1$ is barrelled, then $\varepsilon \colon L(E, F) \times E \to F$ is also hypocontinuous in the first argument, with respect to any locally convex topology $\mathcal{O}$ on $L(E, F)$ which is finer than the topology of pointwise convergence, and any set $\mathcal{T}$ of bounded subsets of $(L(E, F), \mathcal{O})$.

Lemma 8. *Consider locally convex spaces $E_1, \ldots, E_k$ and F with $k \geq 2$ and a k-linear map $\beta \colon E_1 \times \cdots \times E_k \to F$.*

(a) *If β is sequentially continuous, then the composition $\beta \circ f$ is continuous for each continuous function $f \colon X \to E_1 \times \cdots \times E_k$ on a topological space X which is metrisable or satisfies the first axiom of countability.*
(b) *If β is c-hypocontinuous in its arguments $(j, \ldots, k)$ for some $j \in \{2, \ldots, k\}$ and X is a $k_{\mathbb{R}}$-space, then $\beta \circ f$ is continuous for each continuous function $f \colon X \to E_1 \times \cdots \times E_k$.*

Lipschitz differentiable maps. In Section 7, it will be useful to work with certain Lipschitz differentiable maps, instead of C^r-maps. We briefly recall concepts and facts.

Definition 4. *Let E and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ be open and $f\colon U \to F$ be a map. We say that f is* locally Lipschitz continuous *or $LC^0_{\mathbb{K}}$ if it has the following property: For each $x \in U$ and continuous seminorm q on F, there exists a continuous seminorm p on E such that $B^p_1(x) \subseteq U$ and*

$$q(f(z) - f(y)) \leq p(z-y) \quad \text{for all } y, z \in B^p_1(x).$$

Given $r \in \mathbb{N}_0 \cup \{\infty\}$, we say that f is $LC^r_{\mathbb{K}}$ if f is $C^r_{\mathbb{K}}$ and $d^k f\colon U \times E^k \to F$ is $LC^0_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$ such that $k \leq r$.

Every C^1-map is $LC^0_{\mathbb{K}}$ (see, for example, [13] (Exercise 1.5.4)). As a consequence, for each $r \in \mathbb{N} \cup \{\infty\}$, every $C^r_{\mathbb{K}}$-map is $LC^{r-1}_{\mathbb{K}}$. Notably, every smooth map is $LC^\infty_{\mathbb{K}}$. Moreover, a $C^r_{\mathbb{K}}$-map with finite r is $LC^r_{\mathbb{K}}$ if and only if $d^r f$ is $LC^0_{\mathbb{K}}$. The following facts are known, or part of the folklore.

Lemma 9. *For locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$, we have:*

(a) *A map $f\colon E \supseteq U \to \prod_{j\in J} F_j$ to a direct product of locally convex spaces is $LC^r_{\mathbb{K}}$ if and only each component is $LC^r_{\mathbb{K}}$;*
(b) *Compositions of composable $LC^r_{\mathbb{K}}$-maps are $LC^r_{\mathbb{K}}$;*
(c) *Let F be a locally convex space and $F_0 \subseteq F$ be a vector subspace which is closed in F, or sequentially closed. Then, a map $f\colon E \supseteq U \to F_0$ is $FC^r_{\mathbb{K}}$ if and only if it is $FC^r_{\mathbb{K}}$ as a map to F.*
(d) *A map $E \supseteq U \to P$ to a projective limit $P = \varprojlim F_j$ of locally convex spaces is $LC^r_{\mathbb{K}}$ if and only if $p_j \circ f\colon U \to F_j$ is $LC^r_{\mathbb{K}}$ for all $j \in J$, where $p_j\colon P \to F_j$ is the limit map.*

Our concept of local Lipschitz continuity is weaker than the one in [13] (Definition 1.5.4).

The compact-open C^r-topology. If E and F are locally convex $\mathbb{K}$-vector spaces, $U \subseteq E$ is an open set and $r \in \mathbb{N}_0 \cup \{\infty\}$, then the vector space $C^r_{\mathbb{K}}(U, F)$ of all $C^r_{\mathbb{K}}$-maps $U \to F$ carries a natural topology (the "compact-open C^r-topology"), namely the initial topology with respect to the mappings

$$C^r_{\mathbb{K}}(U, F) \to C(U \times E^j, F)_{c.o.} \quad f \mapsto d^j f$$

for $j \in \mathbb{N}_0$ such that $j \leq r$, where the right-hand side is endowed with the compact-open topology. Then, $C^r_{\mathbb{K}}(U, F)$ is a locally convex $\mathbb{K}$-vector space. If F is a complex locally convex space, then also $C^r_{\mathbb{K}}(U, F)$. See, e.g., [13] (§1.7) for further information, or [26].

3. Differentiability Properties of Operator-Valued Maps

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. In this section, we establish the following proposition.

Proposition 1. *Let $k \in \mathbb{N}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $E_1, \ldots, E_k$ and F be locally convex $\mathbb{L}$-vector spaces, X be a locally convex $\mathbb{K}$-vector space, and $U \subseteq X$ be an open subset. Let $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)$ be a map such that*

$$f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F, \quad f^\wedge(x, v) := f(x)(v) \quad \text{for } x \in U, v \in E_1 \times \cdots \times E_k$$

is $C^r_{\mathbb{K}}$. Then, the following holds:

(a) *f is $C^r_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$.*
(b) *If $r \geq 1$, then f is $C^{r-1}_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$.*

Furthermore,

$$d^j f(x, y_1, \ldots, y_j)(v) = d^j(f^\wedge)((x, v), (y_1, 0), \ldots, (y_j, 0)) \tag{4}$$

for all $j \in \mathbb{N}$ with $j \leq r$ (resp., $j \leq r-1$, in (b)*), all $x \in U$, $v \in E_1 \times \cdots \times E_k$, and $y_1, \ldots, y_j \in X$.*

Corollary 1. *Let E and F be locally convex $\mathbb{K}$-vector spaces and $f\colon U \to F$ be a $C^r_{\mathbb{K}}$-map on an open subset $U \subseteq E$, where $r \in \mathbb{N} \cup \{\infty, \omega\}$. Then, the following holds:*

(a) *The map $f^{(k)}\colon U \to L^k_{\mathbb{K}}(E, F)_c$, $x \mapsto f^{(k)}(x) = d^k f(x, \cdot)$ is $C^{r-k}_{\mathbb{K}}$, for each $k \in \mathbb{N}$ such that $k \leq r$.*

(b) *The map $f^{(k)}\colon U \to L^k_{\mathbb{K}}(E, F)_b$ is $C^{r-k-1}_{\mathbb{K}}$, for each $k \in \mathbb{N}$ such that $k \leq r-1$.*

Furthermore, $d^j(f^{(k)})(x, y_1, \ldots, y_j) = d^{j+k} f(x, \cdot, y_1, \ldots, y_j)$, for all $j \in \mathbb{N}$ with $j + k \leq r$ (resp., $j + k \leq r-1$), all $x \in U$, and $y_1, \ldots, y_j \in E$.

Proof. For each $k \in \mathbb{N}$ such that $k \leq r$, the map $d^k f\colon U \times E^k \to F$ is $C^{r-k}_{\mathbb{K}}$ (see [13] (Remark 1.3.13 and Exercise 2.2.7)), and $f^{(k)}(x) = d^k f(x, \cdot)$ is k-linear for each $x \in U$, by [13] (Proposition 1.3.17). Moreover, $(f^{(k)})^\wedge = d^k f$. Thus, Proposition 1 applies with $f^{(k)}$ in place of f and $r - k$ in place of r. □

Given a topological space X and locally convex space F, we endow the space $C(X, F)$ of continuous F-valued functions on X with the compact-open topology. It is known that this topology coincides with the topology of uniform convergence on compact sets. The next lemma will be useful when we discuss mappings to $L^k(E, F)_c$.

Lemma 10. *Let X, E, and F be locally convex $\mathbb{K}$-vector spaces, $U \subseteq X$ and $W \subseteq E$ be open subsets, and $f\colon U \times W \to F$ be a $C^r_{\mathbb{K}}$-map, with $r \in \mathbb{N}_0 \cup \{\infty\}$. Then, also the map*

$$f^\vee\colon U \to C(W, F), \quad x \mapsto f(x, \cdot)$$

is $C^r_{\mathbb{K}}$. If $\mathbb{K} = \mathbb{R}$ and f admits a complex analytic extension $h\colon \widetilde{U} \times \widetilde{W} \to F_{\mathbb{C}}$ for suitable open neighbourhoods $\widetilde{U}$ of U in $X_{\mathbb{C}}$ and $\widetilde{W}$ of W in $E_{\mathbb{C}}$, then $f^\vee$ is real analytic.

Proof. We first assume that $r \in \mathbb{N}_0$, and proceed by induction. For $r = 0$, the assertion is well known (see, e.g., [13] (Proposition A.6.17)). Now assume that $r \in \mathbb{N}$. Given $x \in U$ and $y \in X$, there exists $\varepsilon > 0$ such that $x + \mathbb{D}^0_\varepsilon y \subseteq U$, where $\mathbb{D}^0_r := \{t \in \mathbb{K}\colon |t| < \varepsilon\}$. Consider

$$g\colon \mathbb{D}^0_\varepsilon \times W \to F, \quad (t, w) \mapsto \begin{cases} \frac{f(x+ty,w) - f(x,w)}{t} & \text{if } t \neq 0; \\ df((x, w), (y, 0)) & \text{if } t = 0. \end{cases}$$

Then, $g(t, w) = \int_0^1 df((x + sty, w), (y, 0))\, ds$, by the Mean Value Theorem. The integrand being continuous, also g is continuous (by the Theorem on Parameter-Dependent Integrals, [13] (Lemma 1.1.11)). Hence, $g^\vee\colon V \to C(W, F)$ is continuous, by induction, and hence

$$\frac{f^\vee(x + ty) - f^\vee(x)}{t} = g^\vee(t) \to g^\vee(0)$$

as $t \to 0$, where $g^\vee(0) = df((x, \cdot), (y, 0)) = k^\vee(x, y)$ with

$$k\colon (U \times E) \times W \to F, \quad (x, y, w) \mapsto df((x, w), (y, 0)).$$

Since k is $C^{r-1}_{\mathbb{K}}$, the map $d(f^\vee) = k^\vee$ is $C^{r-1}_{\mathbb{K}}$, by the inductive hypothesis. Notably, $d(f^\vee)$ is continuous and hence $f^\vee$ is $C^1_{\mathbb{K}}$. Now, $f^\vee$ being $C^1_{\mathbb{K}}$ with $d(f^\vee)$ a $C^{r-1}_{\mathbb{K}}$-map, $f^\vee$ is $C^r_{\mathbb{K}}$.

The case $r = \infty$. If f is $C^\infty_{\mathbb{K}}$, then f is $C^k_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$. Hence, $f^\vee$ is $C^k_{\mathbb{K}}$ for each $k \in \mathbb{N}_0$ (by the case already treated), and thus $f^\vee$ is $C^\infty_{\mathbb{K}}$.

Final assertion. By the $C^\infty_{\mathbb{C}}$-case already treated, the map

$$h^\vee : \widetilde{U} \to C(\widetilde{W}, F_{\mathbb{C}})$$

is $C^\infty_{\mathbb{C}}$. The restriction map

$$\rho\colon C(\widetilde{W}, F_{\mathbb{C}}) \to C(W, F_{\mathbb{C}}), \quad \gamma \mapsto \gamma|_W$$

being continuous $\mathbb{C}$-linear and thus $C^\infty_{\mathbb{C}}$, it follows that the composition

$$\rho \circ h^\vee : \widetilde{U} \to C(W, F_{\mathbb{C}}) = C(W, F)_{\mathbb{C}}$$

is $C^\infty_{\mathbb{C}}$ and thus complex analytic. Since $\rho \circ h^\vee$ extends $f^\vee$, we see that $f^\vee$ is real analytic. □

Proof of Proposition 1. (a) Abbreviate $E := E_1 \times \cdots \times E_k$. Because $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ is a closed $\mathbb{K}$-vector subspace of $C(E, F)$ and carries the induced topology, f will be $C^r_{\mathbb{K}}$ as a map to $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ if we can show that f is $C^r_{\mathbb{K}}$ as a map to $C(E, F)$ (see [13] (Lemma 1.3.19 and Exercise 2.2.4)). Since $f^\wedge$ is $C^r_{\mathbb{K}}$ and $f = (f^\wedge)^\vee$, the latter follows from Lemma 10. This is obvious unless $\mathbb{K} = \mathbb{R}$ and $r = \omega$. In this case, the map $f^\wedge$ admits a $\mathbb{C}$-analytic extension $p\colon Q \to F_{\mathbb{C}}$ to an open neighbourhood Q of $U \times E$ in $X_{\mathbb{C}} \times E_{\mathbb{C}}$. For each $x \in U$, there exists an open, connected neighbourhood U_x of x in $X_{\mathbb{C}}$ and a balanced, open 0-neighbourhood $W_x \subseteq E_{\mathbb{C}}$ such that $U_x \times W_x \subseteq Q$ and $U_x \cap X \subseteq U$. Let $D := \{z \in \mathbb{C} \colon |z| < 1\}$. Then,

$$q\colon U_x \times W_x \times D \to F_{\mathbb{C}}, \quad (y, w, z) \mapsto p(y, zw) - z^k p(y, w)$$

is a $\mathbb{C}$-analytic map which vanishes on $(U_x \times W_x \times D) \cap (X \times E \times \mathbb{R})$. Hence, $q = 0$, by the Identity Theorem (see [13] (Theorem 2.1.16 (c))). Then, $p(y, zw) = z^k p(y, w)$ for all $z \in \mathbb{C}$ such that $|z| \leq 1$, by continuity. This implies that the map

$$g\colon U_x \times E_{\mathbb{C}} \to F_{\mathbb{C}}, \quad (y, w) \mapsto z^k p(y, z^{-1}w) \quad \text{for some } z \in \mathbb{C}^\times \text{ with } z^{-1}w \in W_x$$

is well defined. Since g is $\mathbb{C}$-analytic, the final statement of Lemma 10 applies.

(b) We prove the assertion for $r \in \mathbb{N}$ first; then, also the case $r = \infty$ follows. If $r = 1$, let $x \in U$. Given an open 0-neighbourhood $W \subseteq F$ and bounded subset $B \subseteq E := E_1 \times \cdots \times E_k$, let q be a continuous seminorm on F such that $B^q_1(0) \subseteq W$. By Lemma 1, there exist continuous seminorms p on X and p_j on E_j for $j \in \{1, \ldots, k\}$ such that $B^p_1(x) \subseteq U$ and

$$\|f^\wedge(y, v) - f^\wedge(x, v)\|_q \leq \|y - x\|_p \|v_1\|_{p_1} \cdots \|v_k\|_{p_k}$$

for all $y \in B^p_1(x)$ and all $v = (v_1, \ldots, v_k) \in E_1 \times \cdots \times E_k$. Since B is bounded, we have

$$C := \sup\{\|v_1\|_{p_1} \cdots \|v_k\|_{p_k} \colon v = (v_1, \ldots, v_k) \in B\} \; < \; \infty.$$

Choose $\delta \in \,]0, 1]$ such that $\delta C \leq 1$. For each $y \in B^p_\delta(x)$, we get $\|f^\wedge(y, v) - f^\wedge(x, v)\|_q < \delta C \leq 1$ for each $v \in B$ and thus $f^\wedge(y, v) - f^\wedge(x, v) \in B^q_1(0) \subseteq W$. Hence,

$$f(y) - f(x) \in \lfloor B, W \rfloor \quad \text{for each } y \in B^p_1(x),$$

entailing that f is continuous.

Induction step: Now, assume that $r \geq 2$. Given $x \in U$ and $y \in X$, there exists $\varepsilon > 0$ such that $x + \mathbb{D}^0_\varepsilon y \subseteq U$, where $\mathbb{D}^0_r := \{t \in \mathbb{K} \colon |t| < \varepsilon\}$. Consider

$$g\colon \mathbb{D}^0_\varepsilon \times E^k \to F, \quad (t, v) \mapsto \begin{cases} \frac{f^\wedge(x+ty, v) - f^\wedge(x, v)}{t} & \text{if } t \neq 0; \\ d(f^\wedge)((x, v), (y, 0)) & \text{if } t = 0. \end{cases}$$

Then, g is $C^{r-1}_{\mathbb{K}}$ and hence $C^1_{\mathbb{K}}$, as a consequence of [27] (Propositions 7.4 and 7.7). Since $g(t,v)$ is k-linear in v, it follows that $g^\vee\colon U \to L^k(E,F)_b$ is continuous, by induction. As a consequence,

$$\frac{f(x+ty)-f(x)}{t} \;=\; g^\vee(t) \;\to\; g^\vee(0)$$

as $t \to 0$, where $g^\vee(0) = d(f^\wedge)((x,\cdot),(y,0)) = h^\vee(x,y)$ with

$$h\colon (U \times E^k) \times W \to F\,, \quad h((x,y),v) := d(f^\wedge)((x,v),(y,0))\,.$$

Since h is $C^{r-1}_{\mathbb{K}}$ and $h((x,y),v)$ is k-linear in v, the map $df = h^\vee$ is $C^{r-2}_{\mathbb{K}}$, by induction. Hence, df is continuous and thus f is $C^1_{\mathbb{K}}$. Now, f being $C^1_{\mathbb{K}}$ with df a $C^{r-2}_{\mathbb{K}}$-map, f is $C^{r-1}_{\mathbb{K}}$.

The case $\mathbb{K} = \mathbb{R}$, $r = \omega$. By Remark 4, we may assume that $\mathbb{L} = \mathbb{R}$ (the case $\mathbb{L} = \mathbb{C}$ then follows). Given $x \in U$, let $g\colon U_x \times E_{\mathbb{C}} \to F_{\mathbb{C}}$ be as in the proof of (a). Identifying $E_{\mathbb{C}}$ with $(E_1)_{\mathbb{C}} \times \cdots \times (E_k)_{\mathbb{C}}$, the mapping g is complex k-linear in the second variable. Hence $g^\vee\colon U_x \to L^k_{\mathbb{C}}((E_1)_{\mathbb{C}},\ldots,(E_k)_{\mathbb{C}},F_{\mathbb{C}})_b$ is $\mathbb{C}$-analytic, by the $C^\infty_{\mathbb{C}}$-case already discussed. Because the map $\rho\colon L^k_{\mathbb{C}}((E_1)_{\mathbb{C}},\ldots,(E_k)_{\mathbb{C}},F_{\mathbb{C}})_b \to L^k_{\mathbb{R}}(E_1,\ldots,E_k,F_{\mathbb{C}})_b = (L^k_{\mathbb{R}}(E_1,\ldots,E_k,F)_b)_{\mathbb{C}}$, $\alpha \mapsto \alpha|_E$ is continuous $\mathbb{C}$-linear, the composition $\rho \circ g^\vee$ is $\mathbb{C}$-analytic. However, this mapping extends $f|_{U_x \cap X}$. Hence, $f|_{U_x\cap X}$ is real analytic and hence so is f, using that the open sets $U_x \cap X$ form an open cover of U.

Formula for the differentials: Let $j \in \mathbb{N}$ with $j \leq r$, $x \in U$, $v \in E_1 \times \cdots \times E_k$ and $y_1,\ldots,y_j \in X$. Exploiting that $\mathrm{ev}_v\colon L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_c \to F$, $\beta \mapsto \beta(v)$ is continuous and linear, we deduce that

$$\begin{aligned}\mathrm{ev}_v(d^j f(x,y_1,\ldots,y_j)) &= d^j(\mathrm{ev}_v \circ f)(x,y_1,\ldots,y_j) \;=\; d^j(f^\wedge(\cdot,v))(x,y_1,\ldots,y_j)\\ &= d^j(f^\wedge)((x,v),(y_1,0),\ldots,(y_j,0))\end{aligned}$$

for f as a map to $L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_c$. If $j \leq r-1$, the same calculation applies to f as a mapping to $L^k_{\mathbb{L}}(E_1,\ldots,E_k,F)_b$. □

For the special case of (a) when $r = 0$ and X as well as $E_1 = \cdots = E_k$ are metrisable, see already [1] (Lemma 0.1.2).

4. Compositions with Hypocontinuous k-Linear Maps

We study the differentiability properties of compositions of the form $\beta \circ f$, where β is a k-linear map which need not be continuous.

Lemma 11. *Let $k \geq 2$ be an integer, $E_1,\ldots,E_k$, X, and F be locally convex $\mathbb{K}$-vector spaces, $\beta\colon E_1 \times \cdots \times E_k \to F$ be a k-linear map, $r \in \mathbb{N}_0 \cup \{\infty,\omega\}$ and $f\colon U \to E_1 \times \cdots \times E_k =: E$ be a $C^r_{\mathbb{K}}$-map on an open subset $U \subseteq X$. Assume that*

(a) *β is sequentially continuous and X is metrisable; or*

(b) *For some $j \in \{2,\ldots,k\}$, the k-linear map β is c-hypocontinous in its variables $(j,\ldots,k)$. Moreover, $X \times X$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and X is a $k_{\mathbb{R}}$-space, or $(r,\mathbb{K}) = (\infty,\mathbb{C})$ and X is a $k_{\mathbb{R}}$-space.*

Then, $\beta \circ f\colon U \to F$ is a $C^r_{\mathbb{K}}$-map.

Proof. The case $r = 0$ was treated in Lemma 8. We first assume that $r \in \mathbb{N}$.

(a) Assuming (a), let $x \in U$, $y \in X$, and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K} \setminus \{0\}$ such that $t_n \to 0$ as $n \to \infty$ and $x + t_n y \in U$ for all $n \in \mathbb{N}$. Using the components of $f = (f_1,\ldots,f_k)$, we can write the difference quotient $\frac{1}{t_n}(\beta(f(x+t_ny)) - \beta(f(x)))$ as the telescopic sum

$$\sum_{\nu=1}^{k} \beta\Big(f_1(x+t_ny),\ldots,f_{\nu-1}(x+t_ny),\frac{f_\nu(x+t_ny)-f_\nu(x)}{t_n},f_{\nu+1}(x),\ldots,f_k(x)\Big),$$

which converges to

$$\sum_{\nu=1}^{k}\beta(f_1(x),\ldots,f_{\nu-1}(x),df_\nu(x,y),f_{\nu+1}(x),\ldots,f_k(x))=d(\beta\circ f)(x,y) \tag{5}$$

as $n\to\infty$, using the sequential continuity of β. By Lemma 8, $d(\beta\circ f)$ is continuous, whence $\beta\circ f$ is $C^1_{\mathbb{K}}$. If $r\geq 2$, then

$$g_\nu\colon U\times X\to E,\quad (x,y)\mapsto(f_1(x),\ldots,f_{\nu-1}(x),df_\nu(x,y),f_{\nu+1}(x),\ldots,f_k(x))$$

is a $C^{r-1}_{\mathbb{K}}$-map and $d(\beta\circ f)=\sum_{\nu=1}^k\beta\circ g_\nu$ is $C^{r-1}_{\mathbb{K}}$ by induction; thus $\beta\circ f$ is $C^r_{\mathbb{K}}$. If $r=\infty$, the preceding shows that $\beta\circ f$ is $C^s_{\mathbb{K}}$ for each $s\in\mathbb{N}_0$, whence $\beta\circ f$ is $C^r_{\mathbb{K}}$.

(b) If $X\times X$ is a $k_{\mathbb{R}}$-space, then $U\times X$ and U are $k_{\mathbb{R}}$-spaces. By Lemma 5, β is sequentially continuous. The argument from (a) shows that $d(\beta\circ f)(x,y)$ exists for all $(x,y)\in U\times X$ and is given by (5). Thus $d(\beta\circ f)$ is continuous, by Lemma 8, and thus $\beta\circ f$ is $C^1_{\mathbb{K}}$. Let f be $C^{r+1}_{\mathbb{K}}$ now and assume $\beta\circ f$ is $C^r_{\mathbb{K}}$ with rth differential of the form

$$d^r(\beta\circ f)(x,y_1,\ldots,y_r)=\sum_{(I_1,\ldots,I_r)}\beta(d^{|I_1|}f_1(x,y_{I_1}),\ldots,d^{|I_k|}f_k(x,y_{I_k})) \tag{6}$$

for $x\in U$ and $y_1,\ldots,y_r\in X$, where $(I_1,\ldots,I_k)$ ranges through k-tuples of (possibly empty) disjoint sets $I_1,\ldots,I_k$ with $I_1\cup\cdots\cup I_k=\{1,\ldots,r\}$, and the following notation is used: For $\nu\in\{1,\ldots,k\}$, we let $|I_\nu|\in\mathbb{N}_0$ be the cardinality of I_ν and define $y_{I_\nu}:=(y_{i_1},\ldots,y_{i_m})\in X^m$ if $i_1<i_2<\cdots<i_m$ are the elements of I_ν, abbreviating $m:=|I_j|$ (if I_ν is empty, the symbol $y_\emptyset$ is to be ignored). Holding $y_1,\ldots,y_r$ fixed, we can apply the case $r=1$ to the function $d^rf(\cdot,y_1,\ldots,y_r)$ and find that, for each $x\in U$ and $y_{r+1}\in X$, the directional derivative at x in the direction y_{r+1} exists and is given by

$$\begin{aligned}d^{r+1}(\beta\circ f)(x,y_1,\ldots,y_{r+1}) &= \sum_{(I_1,\ldots,I_r)}\sum_{\nu=1}^{k}\beta(d^{|I_1|}f_1(x,y_{I_1}),\ldots,d^{|I_{\nu-1}|}f_{\nu-1}(x,y_{I_{\nu-1}}),\\ &\quad d^{|I_\nu|+1}f_\nu(x,y_{I_\nu},y_{r+1}),d^{|I_{\nu+1}|}f_{\nu+1}(x,y_{I_{\nu+1}}),\ldots,d^{|I_k|}f_k(x,y_{I_k})).\end{aligned}$$

Thus, also $d^{r+1}(\beta\circ f)$ is of the form (6), with $r+1$ in place of r. Using Lemma 8, we deduce from the preceding formula that the map

$$U\times E\to F,\quad (x,y)\mapsto d^{r+1}(\beta\circ f)(x,y,\ldots,y)$$

is continuous. Thus, $d^{r+1}(\beta\circ f)$ is continuous, by Lemma 2, and thus $\beta\circ f$ is $C^{r+1}_{\mathbb{K}}$.

If $(r,\mathbb{K})=(\infty,\mathbb{R})$, then $\beta\circ f$ is $C^s_{\mathbb{R}}$ for each $s\in\mathbb{N}_0$ and hence $C^\infty_{\mathbb{R}}$ (still assuming (b)).

If $(r,\mathbb{K})=(\infty,\mathbb{C})$ and X is only assumed $k_{\mathbb{R}}$, then $\beta\circ f$ is continuous by the case $r=0$. Moreover, the restriction $\beta\circ f|_{U\cap Y}$ is $C^\infty_{\mathbb{C}}$ for each finite-dimensional vector subspace $Y\subseteq X$, by case (a). Hence, f is $C^\omega_{\mathbb{C}}$ (and thus $C^\infty_{\mathbb{C}}$) as a mapping to a completion of F (see [18] (Theorem 6.2)). Then, f is also $C^\infty_{\mathbb{C}}$ as a map to F, as all of its iterated directional derivatives are in F.

Both in (a) and (b), it remains to consider the case $(r,\mathbb{K})=(\omega,\mathbb{R})$. Then, f admits a $\mathbb{C}$-analytic extension $\widetilde{f}\colon\widetilde{U}\to(E_1)_{\mathbb{C}}\times\cdots\times(E_k)_{\mathbb{C}}$, defined on an open neighbourhood $\widetilde{U}$ of U in $X_{\mathbb{C}}$. The complex k-linear extension $\beta_{\mathbb{C}}\colon(E_1)_{\mathbb{C}}\times\cdots\times(E_k)_{\mathbb{C}}\to F_{\mathbb{C}}$ of β is given by

$$z\mapsto\sum_{a_1,\ldots,a_k=0}^{1}i^{a_1+\cdots+a_k}\beta(x_{1,a_1},\ldots,x_{k,a_k})$$

for $z=(x_{1,0}+ix_{1,1},\ldots,x_{k,0}+ix_{k,1})$ with $x_{\nu,0}\in E_\nu$ and $x_{\nu,1}\in E_\nu$ for $\nu\in\{1,\ldots,k\}$. By the latter formula, $\beta_{\mathbb{C}}$ is sequentially continuous in the situation of (a), and c-hypocontinuous in its arguments $(j,\ldots,k)$ in the situation of (b). The case $(\infty,\mathbb{C})$ shows that $\beta_{\mathbb{C}}\circ\widetilde{f}$ is complex analytic. As this mapping extends $\beta \circ f$, the latter map is real analytic. In case (b), we used here that $X_{\mathbb{C}} \cong X \times X$ is a $k_{\mathbb{R}}$-space. $\square$

Moreover, the following variant will be useful.

Lemma 12. *Let X_1, X_2, E_1, E_2 and F be locally convex $\mathbb{K}$-vector spaces, and $U_1 \subseteq X_1$, $U_2 \subseteq X_2$ be open subsets. Let $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ and $\beta\colon E_1 \times E_2 \to F$ be a $\mathbb{K}$-bilinear map. Assume that X_1 is finite-dimensional and β is c-hypocontinuous in its first variable. Then, for all $C^r_{\mathbb{K}}$-maps $f_1\colon U_1 \to E_1$ and $f_2\colon U_1 \times U_2 \to E_2$, also the following map is $C^r_{\mathbb{K}}$:*

$$g\colon U_1 \times U_2 \to F\,, \quad (x_1, x_2) \mapsto \beta(f_1(x_1), f_2(x_1, x_2))\,.$$

Proof. We first prove the assertion for $r \in \mathbb{N}_0$ (from which the case $r = \infty$ follows). If $r = 0$, we have to show that g is continuous. If $(x_1, x_2) \in U_1 \times U_2$, then x_1 has a compact neighbourhood $W = W_{x_1}$ in U_1. Then, $f_1(W)$ is compact, and thus $\beta|_{f_1(W)\times E_2}$ is continuous, by c-hypocontinuity. Hence, $g|_{W\times U_2} = \beta|_{f_1(W)\times E_2} \circ (f_1 \circ \pi_W, f_2)$ is continuous, where $\pi_W\colon W \times U_2 \to W$ is the projection onto the first factor. Since $(W^0_{x_1} \times U_2)_{x_1 \in U_1}$ is an open cover of $U_1 \times U_2$, the map g is continuous.

Since β is sequentially continuous by Lemma 5, we see as in the preceding proof that the directional derivative $dg(x, y)$ exists for all $x = (x_1, x_2) \in U_1 \times U_2$ and $y = (y_1, y_2) \in X_1 \times X_2$, and is given by

$$dg(x, y) = \beta(df_1(x_1, y_1),\, f_2(x)) + \beta(f_1(x_1),\, df_2(x, y))\,. \tag{7}$$

Note that $(x_1, y_1) \mapsto f_1(x_1)$ and df_1 are $C^{r-1}_{\mathbb{K}}$-mappings $U_1 \times X_1 \to E_1$. Moreover, $((x_1, y_1), (x_2, y_2)) \mapsto f_2(x_1, x_2)$ and $((x_1, y_1), (x_2, y_2)) \mapsto df_2((x_1, x_2), (y_1, y_2))$ are $C^{r-1}_{\mathbb{K}}$-maps $(U_1 \times X_1) \times (U_2 \times X_2) \to E_2$ (cf. Remark 1). By induction, the right-hand side of (7) is a $C^{r-1}_{\mathbb{K}}$-map. Hence, g is $C^r_{\mathbb{K}}$.

The case $(r, \mathbb{K}) = (\omega, \mathbb{R})$ follows from the case $(\infty, \mathbb{C})$ as in the preceding proof. $\square$

Remark 5. *In a setting of differential calculus in which continuity on products is replaced with k-continuity (as championed by E. G. F. Thomas), every bilinear map β which is c-hypocontinuous in the second factor is smooth (see [28] (Theorem 4.1)); smoothness of $\beta \circ f$ for a smooth map f then follows from the Chain Rule (cf. also [29]). Likewise, β is smooth in the sense of convenient differential calculus.*

5. Differentiability Properties of $f^\wedge$

For $k = 1$, the following result is essential for our constructions of vector bundles.

Proposition 2. *Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $k \in \mathbb{N}$, $E_1, \ldots, E_k$ and F be locally convex $\mathbb{L}$-vector spaces, X be a locally convex $\mathbb{K}$-vector space, and $U \subseteq X$ be an open subset. Then, the following holds.*

(a) *If $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and $X \times E_1 \times \cdots \times E_k$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X \times E_1 \times \cdots \times E_k$ is a $k_{\mathbb{R}}$-space, or all of the vector spaces $E_1, \ldots, E_k$ are finite dimensional, then*

$$f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F\,, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)$$

is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$.

(b) *If $E := E_1 = E_2 = \cdots = E_k$ holds and, moreover, $(X \times E) \times (X \times E)$ is a $k_{\mathbb{R}}$-space or $r = 0$ and $X \times E$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $X \times E$ is a $k_{\mathbb{R}}$-space, then*

$$f^\wedge\colon U \times E^k \to F\,, \quad (x, y_1, \ldots, y_k) \mapsto f(x)(y_1, \ldots, y_k)$$

is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E, F)_c$ such that $f(x)$ is a symmetric k-linear map for each $x \in U$.

(c) *If X is finite-dimensional, $k = 1$, and $E := E_1$ is barrelled, then $f^\wedge\colon U \times E \to F$, $(x, y) \mapsto f(x)(y)$ is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L_{\mathbb{L}}(E, F)_c$.*

(d) *If all of the spaces $E_1, \ldots, E_k$ are normable, then $f^\wedge\colon U \times E_1 \times \cdots \times E_k \to F$ is $C^r_{\mathbb{K}}$ for each $C^r_{\mathbb{K}}$-map $f\colon U \to L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$.*

Proof. Let $\mathrm{ev}\colon L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c \times E_1 \times \cdots \times E_k \to F$ be the evaluation map, which is c-hypocontinuous in its arguments $(2, \ldots, k+1)$ by Lemma 7.

(a) Assuming the respective $k_{\mathbb{R}}$-property, the map $f^\wedge = \mathrm{ev} \circ (f \times \mathrm{id}_{E_1 \times \cdots \times E_k})$ is $C^r_{\mathbb{K}}$, by Lemma 11 (b). If $E_1, \ldots, E_k$ are finite-dimensional, then $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_c$ equals $L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b$, whence the conclusion of (a) is a special case of (d).

(b) By Lemma 11 (b), the map

$$g\colon U \times E \to F, \quad (x, y) \mapsto f^\wedge(x, y, \ldots, y)$$

is $C^r_{\mathbb{K}}$, as $g = \mathrm{ev} \circ (f \times \delta)$ with $\delta\colon E \to E^k$, $y \mapsto (y, \ldots, y)$, which is continuous $\mathbb{K}$-linear. Then, also $f^\wedge$ is $C^r_{\mathbb{K}}$, by Lemma 2.

(c) The bilinear map $\mathrm{ev}\colon L_{\mathbb{K}}(E, F)_c \times E \to F$ is c-hypocontinuous in its first argument, by Lemma 7. Hence, $f^\wedge = \mathrm{ev} \circ (f \times \mathrm{id}_E)$ is $C^r_{\mathbb{K}}$, by Lemma 12.

(d) If $E_1, \ldots, E_k$ are normable, then the evaluation map

$$\varepsilon\colon L^k_{\mathbb{L}}(E_1, \ldots, E_k, F)_b \times E_1 \times \cdots \times E_k \to F$$

is continuous $(k+1)$-linear and hence $C^r_{\mathbb{K}}$, whence also $f^\wedge = \varepsilon \circ (f \times \mathrm{id}_{E_1 \times \cdots \times E_k})$ is $C^r_{\mathbb{K}}$. $\square$

Remark 6. *If X and all of $E_1, \ldots, E_k$ are metrisable, then the topological space $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is metrisable and hence a k-space. If X and all of $E_1, \ldots, E_k$ are k_ω-spaces, then also $(X \times E_1 \times \cdots \times E_k) \times (X \times E_1 \times \cdots \times E_k)$ is a k_ω-space and hence a k-space. In either case, we are in the situation of (a).*

6. Infinite-Dimensional Vector Bundles

In this section, we provide foundational material concerning vector bundles modelled on locally convex spaces (cf. also [13] (Chapter 3)). Notably, we discuss the description of vector bundles via cocycles, and define equivariant vector bundles.

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$. The word "manifold" always refers to a manifold modelled on a locally convex space. Likewise, the Lie groups that we consider need not have finite dimension.

Definition 5. *Let M be a $C^r_{\mathbb{K}}$-manifold and F be a locally convex $\mathbb{L}$-vector space. An $\mathbb{L}$*-vector bundle of class $C^r_{\mathbb{K}}$ over M, *with typical fibre F, is a $C^r_{\mathbb{K}}$-manifold E, together with a surjective $C^r_{\mathbb{K}}$-map $\pi\colon E \to M$ and endowed with an $\mathbb{L}$-vector space structure on each fibre $E_x := \pi^{-1}(\{x\})$, such that, for each $x \in M$, there exists an open neighbourhood $U \subseteq M$ of x and a $C^r_{\mathbb{K}}$-diffeomorphism*

$$\psi\colon \pi^{-1}(U) \to U \times F$$

(called a "local trivialisation") such that $\psi(E_y) = \{y\} \times F$ for each $y \in U$ and the map $\mathrm{pr}_F \circ \psi|_{E_y}\colon E_y \to F$ is $\mathbb{L}$-linear (and hence an isomorphism of topological vector spaces, if we give E_y the topology induced by E), where $\mathrm{pr}_F\colon U \times F \to F$ is the projection.

In the situation of Definition 5, let $(\psi_i)_{i \in I}$ be an atlas of local trivialisations for E, *i.e.*, a family of local trivialisations

$$\psi_i\colon \pi^{-1}(U_i) \to U_i \times F$$

of E whose domains U_i cover M. Then, given $i, j \in I$, we have

$$\psi_i(\psi_j^{-1}(x, v)) = (x, g_{ij}(x)(v))$$

for $x \in U_i \cap U_j$, $v \in F$, for some function

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(F) \subseteq L(F).$$

Here,

$$G_{ij} : (U_i \cap U_j) \times F \to F, \quad (x, v) \mapsto g_{ij}(x)(v)$$

is $C^r_{\mathbb{K}}$, as $\psi_i(\psi_j^{-1}(x, v)) = (x, G_{ij}(x, v))$ is $C^r_{\mathbb{K}}$ in $(x, v) \in (U_i \cap U_j) \times F$. By Proposition 1, $g_{ij} : U_i \cap U_j \to L(F)_c$ is a $C^r_{\mathbb{K}}$-map, and as a map to $L(F)_b$, it is at least $C^{r-1}_{\mathbb{K}}$ (if $r \geq 1$). Note that the "transition maps" g_{ij} satisfy the "cocycle conditions"

$$\begin{cases} (\forall i \in I)\, (\forall x \in U_i) & g_{ii}(x) = \mathrm{id}_F \quad \text{and} \\ (\forall i, j, k \in I)\, (\forall x \in U_i \cap U_j \cap U_k) & g_{ij}(x) \circ g_{jk}(x) = g_{ik}(x)\,. \end{cases}$$

Proposition 3. *Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$. Assume that*

(a) *M is a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z;*
(b) *E is a set and $\pi : E \to M$ a surjective map;*
(c) *F is a locally convex $\mathbb{L}$-vector space;*
(d) *$(U_i)_{i \in I}$ is an open cover of M;*
(e) *$(\psi_i)_{i \in I}$ is a family of bijections $\pi^{-1}(U_i) \to U_i \times F$ such that $\psi_i(\pi^{-1}(\{x\})) = \{x\} \times F$ for all $x \in U_i$;*
(f) *$g_{ij}(x)(v) := \mathrm{pr}_F(\psi_i(\psi_j^{-1}(x, v)))$ depends $\mathbb{L}$-linearly on $v \in F$, for all $i, j \in I$, $x \in U_i \cap U_j$;*
(g) *$G_{ij} : (U_i \cap U_j) \times F \to F$, $G_{ij}(x, v) := g_{ij}(x)(v)$ is a $C^r_{\mathbb{K}}$-map.*

Then, there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on E making ψ_i a local trivialisation for each $i \in I$.

Proof. For $i, j \in I$, let $\mathrm{pr}_{ij} : (U_i \cap U_j) \times F \to U_i \cap U_j$ be the projection onto the first component. As the maps

$$\psi_i \circ \psi_j^{-1}|_{(U_i \cap U_j) \times F} = (\mathrm{pr}_{ij}, G_{ij})$$

are $C^r_{\mathbb{K}}$, there is a uniquely determined $C^r_{\mathbb{K}}$-manifold structure on E making ψ_i a $C^r_{\mathbb{K}}$-diffeomorphism for each $i \in I$. Given $x \in M$, we pick $i \in I$ with $x \in U_i$; we give $E_x := \pi^{-1}(\{x\})$ the unique $\mathbb{L}$-vector space structure making the bijection $\mathrm{pr}_F \circ \psi_i|_{E_x} : E_x \to F$ an isomorphism of vector spaces. It is easy to see that the vector space structure on E_x is independent of the choice of ψ_i, and it is easily verified that we have turned E into an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ with the asserted properties. □

Remark 7. *Let M be a $C^r_{\mathbb{K}}$-manifold, F be a locally convex $\mathbb{L}$-vector space, $(U_i)_{i \in I}$ be an open cover of M, and $(g_{ij})_{i,j \in I}$ be a family of maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}(F)$ satisfying the cocycle conditions and such that*

$$G_{ij} : (U_i \cap U_j) \times F \to F\,, \quad (x, v) \mapsto g_{ij}(x)(v)$$

is $C^r_{\mathbb{K}}$, for all $i, j \in I$. Using Proposition 3, the usual construction familiar from the finite-dimensional case provides an $\mathbb{L}$-vector bundle $\pi : E \to M$ of class $C^r_{\mathbb{K}}$, with typical fibre F, and a family $(\psi_i)_{i \in I}$ of local trivialisations $\pi^{-1}(U_i) \to U_i \times F$, whose associated transition maps are the given g_{ij}'s. The bundle E is unique up to canonical isomorphism.

Combining Proposition 3 and Proposition 2, we obtain:

Corollary 2. *Retaining the hypotheses* (a)–(f) *from Proposition 3 but omitting* (g)*, consider the following conditions:*

(g)′ *$g_{ij}(x) \in L(F)$ for all $i, j \in I$, $x \in U_i \cap U_j$, and $g_{ij} : U_i \cap U_j \to L(F)_c$ is $C^r_{\mathbb{K}}$;*
(g)″ *$g_{ij}(x) \in L(F)$ for all $i, j \in I$, $x \in U_i \cap U_j$, and $g_{ij} : U_i \cap U_j \to L(F)_b$ is $C^r_{\mathbb{K}}$;*
(i) *$(Z \times F) \times (Z \times F)$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and $Z \times F$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $Z \times F$ is a $k_{\mathbb{R}}$-space;*

(ii) $\dim(M) < \infty$ *and F is barrelled;*
(iii) *F is normable.*

If (g)$'$ *holds as well as* (i) *or* (ii)*, then the conclusions of Proposition* 3 *remain valid. They also remain valid if* (g)$''$ *and* (iii) *hold.*

Example 2 below shows that Conditions (a)–(f) and (g)$'$ alone are not sufficient for the conclusion of Proposition 3, without extra conditions on Z and F. Note that (i) is satisfied if both Z and F are metrisable, or both Z and F are k_ω-spaces.

Equivariant vector bundles. Beyond vector bundles, we shall discuss *equivariant* vector bundles in the following, i.e., vector bundles together with an action of a (finite- or infinite-dimensional) Lie group G. Choosing $G = \{e\}$ as a trivial group, we obtain results about ordinary vector bundles (without a group action), as a special case.

For the remainder of this section, and also in Section 7, let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $r \leq s$. Let G be a $C^s_\mathbb{K}$-Lie group (modelled on a locally convex $\mathbb{K}$-vector space Y) and M be a $C^r_\mathbb{K}$-manifold. We assume that a $C^r_\mathbb{K}$-action

$$\alpha\colon G \times M \to M$$

is given. Then, (M, α) is called a *G-manifold of class* $C^r_\mathbb{K}$.

Definition 6. *An* equivariant $\mathbb{L}$-vector bundle of class $C^r_\mathbb{K}$ *over a G-manifold* (M, α) *of class* $C^r_\mathbb{K}$ *is an* $\mathbb{L}$*-vector bundle* $\pi\colon E \to M$ *of class* $C^r_\mathbb{K}$*, together with a* $C^r_\mathbb{K}$*-action*

$$\beta\colon G \times E \to E$$

such that $\beta(g, E_x) \subseteq E_{\alpha(g,x)}$ *for all* $(g, x) \in G \times M$*, and* $\beta(g, \cdot)|_{E_x}\colon E_x \to E_{\alpha(g,x)}$ *is* $\mathbb{L}$*-linear.*

In other words, $\beta(g, \cdot)$ takes fibres linearly to fibres and coincides with $\alpha(g, \cdot)$ on the zero section. The mapping π is then equivariant in the sense that $\alpha \circ (\mathrm{id}_G \times \pi) = \pi \circ \beta$.

Example 1. *If M is a G-manifold of class* $C^r_\mathbb{K}$*, with* $r \geq 1$*, then the tangent bundle TM is an equivariant* $\mathbb{L}$*-vector bundle of class* $C^{r-1}_\mathbb{K}$ *in a natural way, with* $\mathbb{L} := \mathbb{K}$*. In fact, the action* $\alpha\colon G \times M \to M$ *has a tangent map* $T\alpha\colon T(G \times M) \to TM$*, which is* $C^{r-1}_\mathbb{K}$*. Let* $0_G\colon G \to TG$ *be the* 0*-section. Identifying* $T(G \times M)$ *with* $TG \times TM$ *in the usual way, we obtain a* $C^{r-1}_\mathbb{K}$*-map* $\beta\colon G \times TM \to TM$ *via*

$$\beta := (T\alpha) \circ (0_G \times \mathrm{id}_{TM})\,.$$

It is easy to see that $\beta(g, v) = T_x(\alpha(g, \cdot))(v) \in T_{\alpha(g,x)}M$ *for* $g \in G$ *and* $v \in T_x M$*, whence* $\beta(g, T_x M) \subseteq T_{\alpha(g,x)}M$ *and* $\beta(g, \cdot)|_{T_x M} = T_x(\alpha(g, \cdot))$*. Clearly,* β *is an action making TM an equivariant* $\mathbb{K}$*-vector bundle of class* $C^{r-1}_\mathbb{K}$ *over the G-manifold M.*

Induced action on an invariant subbundle. Given an $\mathbb{L}$-vector bundle $\pi : E \to M$ of class $C^r_\mathbb{K}$, with typical fibre F, we call a subset $E_0 \subseteq E$ a *subbundle* if there exists a sequentially closed $\mathbb{L}$-vector subspace $F_0 \subseteq F$ such that for each $x \in M$ there exists a local trivialisation $\psi : \pi^{-1}(U) \to U \times F$ of E such that $\psi(E_0 \cap \pi^{-1}(U)) = U \times F_0$. It readily follows from [13] (Lemma 1.3.19 and Exercise 2.2.4) that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_\mathbb{K}$ on $\pi|_{E_0} : E_0 \to M$ making $\psi|_{\pi^{-1}(U) \cap E_0} : \pi^{-1}(U) \cap E_0 \to U \times F_0$ a local trivialisation of E_0, for each local trivialisation ψ as before. Then, the inclusion map $E_0 \to E$ is $C^r_\mathbb{K}$, and a mapping $N \to E$ from a $C^r_\mathbb{K}$-manifold N to E with image in E_0 is $C^r_\mathbb{K}$ as a mapping to E if and only if its co-restriction to E_0 is $C^r_\mathbb{K}$, by the facts just cited. In the preceding situation, suppose that a $C^s_\mathbb{K}$-Lie group G acts $C^s_\mathbb{K}$ on M and E is an equivariant vector bundle of class $C^r_\mathbb{K}$ with respect to the action $\beta\colon G \times E \to E$. If E_0 is invariant under the G-action, i.e., if $\beta(G \times E_0) \subseteq E_0$, as a special case of the preceding observations, we deduce from the $C^r_\mathbb{K}$-property of β that $\beta|_{G \times E_0}$ and thus also $\beta|_{G \times E_0}^{E_0}\colon G \times E_0 \to E_0$ is $C^r_\mathbb{K}$. We can summarise as follows.

Proposition 4. *If E is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over a G-manifold M, then the action induced on any G-invariant subbundle E_0 is $C^r_{\mathbb{K}}$ and thus makes the latter an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$.*

7. Completions of Vector Bundles

Let $\pi\colon E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, as in Definition 6, with typical fibre F and G-actions $\alpha\colon G \times M \to M$ and $\beta\colon G \times E \to E$. Assume that $r \geq 1$. Our goal is to complete the fibre of the bundle, i.e., to find a G-equivariant vector bundle $\widetilde{E}$ whose typical fibre is a completion of the locally convex space F, and which contains E as a dense subset.

Let $\widetilde{F}$ be a completion of F such that $F \subseteq \widetilde{F}$ and, for each $x \in M$, let $\widetilde{E}_x$ be a completion of E_x such that $E_x \subseteq \widetilde{E}_x$. We may assume that the sets $\widetilde{E}_x$ are pairwise disjoint for $x \in M$. Consider the (disjoint) union

$$\widetilde{E} := \bigcup_{x \in M} \widetilde{E}_x\,. \tag{8}$$

We shall turn $\widetilde{E}$ into an equivariant vector bundle. Consider the map $\widetilde{\beta}\colon G \times \widetilde{E} \to \widetilde{E}$, defined using the continuous extension $(\beta(g,\cdot)|_{E_x})\widetilde{\ }\colon \widetilde{E}_x \to \widetilde{E}_{\alpha(g,x)}$ of the linear map $\beta(g,\cdot)|_{E_x}\colon E_x \to E_{\alpha(g,x)}$ via

$$\widetilde{\beta}(g,v) := (\beta(g,\cdot)|_{E_x})\widetilde{\ }(v)$$

for $g \in G$, $x \in M$, and $v \in \widetilde{E}_x$. It is clear that $\widetilde{\beta}$ makes $\widetilde{E}$ a G-set. Let

$$\widetilde{\pi}\colon \widetilde{E} \to M \tag{9}$$

be the map taking elements from $\widetilde{E}_x$ to x. Then, $\widetilde{\pi}$ is G-equivariant. If $\psi\colon \pi^{-1}(U) \to U \times F$ is a local trivialisation of E and $\mathrm{pr}_F\colon U \times F \to F$, $(x,y) \mapsto y$, we define

$$\widetilde{\psi}\colon \widetilde{\pi}^{-1}(U) \to U \times \widetilde{F}\,, \quad \widetilde{E}_x \ni v \mapsto (x, (\mathrm{pr}_F \circ \psi|_{E_x})\widetilde{\ }(v))\,. \tag{10}$$

Then, the following holds:

Proposition 5. *$(\widetilde{E}, \widetilde{\beta})$ can be made an equivariant $\mathbb{L}$-vector bundle of class $C^{r-1}_{\mathbb{K}}$ over the G-manifold M, such that $\widetilde{\psi}$ is a local trivialisation of $\widetilde{E}$ for each local trivialisation ψ of E.*

Remark 8. *Omitting the hypothesis that $r \geq 1$, assume instead that E is an equivariant $\mathbb{L}$-vector bundle* of class *$LC^r_{\mathbb{K}}$. That is, both E and M are $LC^r_{\mathbb{K}}$-manifolds (each admitting an atlas with transition maps of class $LC^r_{\mathbb{K}}$), a family of local trivialisations can be chosen with $LC^r_{\mathbb{K}}$-transition maps, and the G-actions on E and M are $LC^r_{\mathbb{K}}$. Then, also $\widetilde{E}$ is an equivariant vector bundle of class $LC^r_{\mathbb{K}}$ (and hence of class $C^r_{\mathbb{K}}$).*

Extension of differentiable maps to subsets of the completions. To enable the proof of Proposition 5, we need to discuss conditions ensuring that a C^r-map $f\colon E \supseteq U \to F$ (with locally convex spaces E and F) can be extended to a C^r-map $\widetilde{U} \to \widetilde{F}$ on an open subset of the completion $\widetilde{E}$ of E, or at least to a C^{r-1}-map. Although this is not possible in general, it is possible if F is normed and r is finite. This will be sufficient for our purposes. The natural framework for the discussion of the problem is not C^r-maps, but Lipschitz differentiable maps, as in Definition 4.

Proposition 6. *Let E be a locally convex $\mathbb{K}$-vector space, $(F, \|\cdot\|)$ be a Banach space over $\mathbb{K}$, $U \subseteq E$ be open and $f\colon U \to F$ be an $LC^r_{\mathbb{K}}$-map, where $r \in \mathbb{N}_0$. Let $\widetilde{E}$ be a completion of E such that $E \subseteq \widetilde{E}$. Then, f extends to an $LC^r_{\mathbb{K}}$-map $\widetilde{f}\colon \widetilde{U} \to F$ on an open subset $\widetilde{U} \subseteq \widetilde{E}$ which contains U as a dense subset.*

The following lemma enables an inductive proof of Proposition 6.

Lemma 13. *Let $k \in \mathbb{N}$, X be a locally convex $\mathbb{K}$-vector space, and $E_1, \ldots, E_k, F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\widetilde{X}$, $\widetilde{E}_1, \ldots, \widetilde{E}_k$ and $\widetilde{F}$, respectively. Let $U \subseteq X$ be open and $f\colon U \times E_1 \times \cdots \times E_k \to F$ be a map such that $f^\vee(x) := f(x, \cdot)\colon E_1 \times \cdots \times E_k \to F$ is k-linear over $\mathbb{L}$ for each $x \in U$. Assume that there exists an $LC^r_{\mathbb{K}}$-map $h\colon W \to \widetilde{F}$ which extends f, defined on an open set $W \subseteq \widetilde{X} \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ in which $U \times E_1 \times \cdots \times E_k$ is dense. Then, there exists an $LC^r_{\mathbb{K}}$-map*

$$\widetilde{f}\colon \widetilde{U} \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F} \tag{11}$$

which extends f, for some open subset $\widetilde{U} \subseteq \widetilde{E}$ in which U is dense. The maps $(\widetilde{f})^\vee(x) := \widetilde{f}(x, \cdot)\colon \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}$ are k-linear over $\mathbb{L}$, for each $x \in \widetilde{U}$.

Proof. For each $x \in U$, there exists an open neighbourhood V_x of x in $\widetilde{X}$ and a balanced, open 0-neighbourhood $Q_x \subseteq \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ such that $V_x \times Q_x \subseteq W$. After shrinking V_x, we may assume that $X \cap V_x = U$, whence $U \cap V_x = X \cap V_x$ is dense in V_x. Given $z \in \mathbb{L}$ such that $|z| \leq 1$, consider the map

$$V_x \times Q_x \to \widetilde{F}, \quad (y, v) \mapsto h(y, zv) - z^k h(y, v)\,.$$

This map vanishes, because it is continuous and vanishes on the dense subset $(V_x \cap X) \times (Q_x \cap (E_1 \times \cdots \times E_k))$. As a consequence, we obtain a well-defined map

$$f_x\colon V_x \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}, \quad (y, v) \mapsto z^{-k} h(y, zv)$$

for $y \in V_x$, $v \in \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ and $z \in \mathbb{L} \setminus \{0\}$ with $zv \in Q_x$. As $f_x(y, v) = z^{-k} h(y, zv)$ is $LC^r_{\mathbb{K}}$ in $(y, v) \in V_x \times z^{-1} Q_x$ and these sets form an open cover of $V_x \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$, we see that f_x is $LC^r_{\mathbb{K}}$. Given $x, y \in U$, the set $U \cap V_x \cap V_y = X \cap V_x \cap V_y$ is dense in the open set $V_x \cap V_y \subseteq \widetilde{X}$. Since f_x, f_y, and f coincide on the set $(U \cap V_x \cap V_y) \times E_1 \times \cdots \times E_k$, it follows that the continuous maps f_x and f_y coincide on the set $(V_x \cap V_y) \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$ in which the former set is dense. Hence, setting $\widetilde{U} := \bigcup_{x \in U} V_x$, a well-defined map $\widetilde{f}$ as in (11) is obtained if we set

$$\widetilde{f}(y, v) := f_x(y, v) \quad \text{if } x \in U, y \in V_x \text{ and } v \in \widetilde{E}_1 \times \cdots \times \widetilde{E}_k.$$

The final assertion follows by continuity from the k-linearity of the mappings $f^\vee(x)$ for $x \in U$. □

Proof of Proposition 6. We proceed by induction on $r \in \mathbb{N}_0$.

The case $r = 0$. Given $x \in U$, there exists a continuous seminorm q on E such that $B^q_1(x) \subseteq U$ and

$$\|f(z) - f(y)\| \leq q(z - y) \quad \text{for all } y, z \in B^q_1(x). \tag{12}$$

Then, $N_q := \{y \in E\colon q(y) = 0\}$ is a closed vector subspace of E and $\|y + N_q\|_q := q(y)$ for $y \in E$ defines a norm on $E_q := E/N_q$ making the map $\alpha_q\colon E \to E_q$, $y \mapsto y + N_q$ continuous linear. By (12), we have $\|f(z) - f(y)\| = 0$ for all $y, z \in B^q_1(x)$ such that $y - z \in N_q$. Hence,

$$h\colon \alpha_q(B^q_1(x)) \to F, \quad y + N_q \mapsto f(y)$$

is a well-defined map. Note that $\alpha_q(B^q_1(x))$ is the open ball $B := \{y \in E_q\colon \|y - \alpha_q(x)\|_q < 1\}$ in E_q. Let $\widetilde{E}_q$ be the completion of the normed space E_q; the extended norm will again be denoted by $\|.\|_q$. Applying (12) to representatives, we see that

$$\|h(z) - h(y)\| \leq \|z - y\|_q \quad \text{for all } y, z \in B.$$

Hence, h satisfies a global Lipschitz condition (with Lipschitz constant 1), and hence h is uniformly continuous, entailing that h extends uniquely to a uniformly continuous map

$$\widetilde{h}\colon \widetilde{B} \to F$$

on the corresponding open ball $\widetilde{B}$ in $\widetilde{E}_q$. Then, $\|\widetilde{h}(z)-\widetilde{h}(y)\| \leq \|z-y\|_q$ for all $y,z\in\widetilde{B}$, by continuity. Let $\widetilde{\alpha}_q\colon \widetilde{E}\to\widetilde{E}_q$ be the continuous extension of the continuous linear map α_q. Then, $V_x := (\widetilde{\alpha}_q)^{-1}(\widetilde{B})$ is an open neighbourhood of x in $\widetilde{E}$ such that $V_x\cap E = B^q_1(x)\subseteq U$. Moreover, $f_x := \widetilde{h}\circ\widetilde{\alpha}_q|_{V_x}$ is a continuous map extending $f|_{V_x\cap E}$, which furthermore satisfies

$$\|f_x(z)-f_x(y)\| \;\leq\; \widetilde{q}(z-y) \quad \text{for all } y,z\in V_x, \tag{13}$$

where we use the continuous seminorm $\widetilde{q} := \|.\|_q\circ\widetilde{\alpha}_q\colon \widetilde{E}\to[0,\infty[$ extending q. Then

$$\widetilde{U} := \bigcup_{x\in U} V_x$$

is an open subset of $\widetilde{E}$ and $E\cap\widetilde{U} = U$ is dense in $\widetilde{U}$. Given $x,y\in U$, the set $U\cap V_x\cap V_y = E\cap V_x\cap V_y$ is dense in the open set $V_x\cap V_y\subseteq\widetilde{E}$. Since

$$f_x|_{U\cap V_x\cap V_y} \;=\; f|_{U\cap V_x\cap V_y} \;=\; f_y|_{U\cap V_x\cap V_y}\,,$$

it follows that $f_x|_{V_x\cap V_y} = f_y|_{V_x\cap V_y}$. Hence

$$\widetilde{f}\colon \widetilde{U}\to F\,,\quad z\mapsto f_x(z) \quad \text{for } x\in U \text{ such that } z\in V_x$$

is a well-defined map. Since $\widetilde{f}|_{V_x} = f_x$ is $LC^0_{\mathbb{K}}$ for each $x\in U$ (by (13)), the map $\widetilde{f}$ is $LC^0_{\mathbb{K}}$. Furthermore, $\widetilde{f}$ extends f by construction.

Induction step. If f is $LC^{r+1}_{\mathbb{K}}$, then f extends to an $LC^0_{\mathbb{K}}$-map $\widetilde{f}\colon\widetilde{U}\to F$ on an open subset $\widetilde{U}\subseteq\widetilde{E}$ such that $\widetilde{U}\cap E = U$, and $df\colon U\times E\to F$ extends to an $LC^r_{\mathbb{K}}$-map $h\colon W\to F$ on an open subset W of $\widetilde{E}\times\widetilde{E}$, by induction. Using Lemma 13, we find an open neighbourhood V of U in $\widetilde{E}$ and an $LC^r_{\mathbb{K}}$-map $g\colon V\times\widetilde{E}\to F$ which extends df. After replacing $\widetilde{U}$ and V with their intersection, we may assume that $\widetilde{U} = V$. If $x_0\in\widetilde{U}$ and $y_0\in\widetilde{E}$, there exist open neighbourhoods Q of x_0 and P of y_0 in $\widetilde{E}$, and $\varepsilon>0$ such that $Q+\mathbb{D}_\varepsilon P\subseteq\widetilde{U}$. Then, the map

$$\ell\colon Q\times P\times\mathbb{D}_\varepsilon\to F,\quad (x,y,t)\mapsto\int_0^1 g(x+sty,y)\,ds$$

is continuous, being given by a parameter-dependent weak integral with continuous integrand. For (x,y,t) in the dense subset $(Q\cap E)\times(P\cap E)\times(\mathbb{D}_\varepsilon\setminus\{0\})$ of the set $Q\times P\times(\mathbb{D}_\varepsilon\setminus\{0\})$, the Mean Value Theorem implies that

$$\ell(x,y,t) \;=\; \frac{f(x+ty)-f(x)}{t} \;=\; \frac{\widetilde{f}(x+ty)-\widetilde{f}(x)}{t}.$$

Then, $\ell(x,y,t) = \frac{\widetilde{f}(x+ty)-\widetilde{f}(x)}{t}$ for all $(x,y,t)\in Q\times P\times(\mathbb{D}_\varepsilon\setminus\{0\})$, by continuity. Thus,

$$\frac{f(x_0+ty_0)-f(x_0)}{t} \;=\; \ell(x_0,y_0,t) \;\to\; \ell(x_0,y_0,0) \;=\; g(x_0,y_0)$$

as $t\to0$. Hence, $d\widetilde{f}(x_0,y_0) = g(x_0,y_0)$. Since g is $LC^r_{\mathbb{K}}$, it follows that $\widetilde{f}$ is $LC^{r+1}_{\mathbb{K}}$. □

The conclusion of Proposition 6 becomes false in general if the Banach space F is replaced by a complete locally convex space. In fact, there exists a smooth map $E\to(\ell^1)^\Omega$ from a proper, dense vector subspace E of ℓ^1 to a suitable power of ℓ^1, which has no continuous extension to $E\cup\{x\}$ for any $x\in\ell^1\setminus E$ (see Appendix B). Nonetheless, we have the following result.

Proposition 7. *Let $k\in\mathbb{N}$, X be a locally convex $\mathbb{K}$-vector space, and $E_1,\ldots,E_k,F$ be locally convex $\mathbb{L}$-vector spaces, with completions $\widetilde{X}$, $\widetilde{E}_1,\ldots,\widetilde{E}_k$ and $\widetilde{F}$, respectively. Let $U\subseteq X$ be open and $f\colon U\times E_1\times\cdots\times E_k\to F$ be a mapping such that $f^\vee(x) := f(x,\cdot)\colon\ E_1\times\cdots\times E_k\to F$*

is k-linear over $\mathbb{L}$ for each $x \in U$. If f is $LC^r_{\mathbb{K}}$ for some $r \in \mathbb{N}_0 \cup \{\infty\}$ (resp., $C^r_{\mathbb{K}}$ for some $r \in \mathbb{N} \cup \{\infty, \omega\}$), then there exists a unique map

$$\widetilde{f}\colon U \times \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F} \tag{14}$$

which is $LC^r_{\mathbb{K}}$ (resp., $C^{r-1}_{\mathbb{K}}$) and extends f. The maps $\widetilde{f}^\vee(x) := \widetilde{f}(x, \cdot)\colon \widetilde{E}_1 \times \cdots \times \widetilde{E}_k \to \widetilde{F}$ are k-linear over $\mathbb{L}$, for each $x \in U$.

Proof. Abbreviate $E := E_1 \times \cdots \times E_k$ and $\widetilde{E} := \widetilde{E}_1 \times \cdots \times \widetilde{E}_k$. Assume first that $r \neq \omega$. Since $LC^r_{\mathbb{K}}$-maps are continuous and $U \times E$ is dense in $U \times \widetilde{E}$, there is at most one map $\widetilde{f}$ with the asserted properties. We may therefore assume that $r \in \mathbb{N}_0$. We may also assume that F is complete. Then, $F = \varprojlim F_j$ for some projective system $((F_j)_{j \in J}, (p_{ij})_{i \leq j})$ of Banach spaces F_j and continuous linear maps $p_{ij}\colon F_j \to F_i$, with limit maps $p_j\colon F \to F_j$. We claim that $p_j \circ f\colon U \times E \to F_j$ has an $LC^r_{\mathbb{K}}$-extension $g_j := (p_j \circ f)^{\widetilde{}}\colon U \times \widetilde{E} \to F_j$, for each $j \in J$. If this is true, then $p_{ij} \circ g_j = g_i$ for $i \leq j$, by uniqueness of continuous extensions. Hence, by the universal property of the projective limit, there exists a unique map $\widetilde{f}\colon U \times \widetilde{E} \to F$ such that $p_j \circ \widetilde{f} = g_j$. Then, $p_j \circ \widetilde{f}|_{U \times E} = g_j|_{U \times E} = p_j \circ f$ and hence $\widetilde{f}|_{U \times E} = f$. Furthermore, $\widetilde{f}$ is $LC^r_{\mathbb{K}}$, by Lemma 9 (d). To prove the claim, note that Proposition 6 yields an $LC^r_{\mathbb{K}}$-extension $h_j\colon W_j \to F_j$ of $p_j \circ f$ to an open subset $W_j \subseteq \widetilde{X} \times \widetilde{E}$, which contains $U \times E$ as a dense subset. Now, Lemma 13 yields an open subset $U_j \subseteq \widetilde{X}$ in which U is dense, and an $LC^r_{\mathbb{K}}$-extension $e_j\colon U_j \times \widetilde{E} \to F_j$ of $p_j \circ f$. Then, $g_j := e_j|_{U \times \widetilde{E}}$ is as desired.

We now consider the case $(r, \mathbb{K}) = (\omega, \mathbb{R})$. If $\mathbb{L} = \mathbb{C}$, by the density of $U \times E$ in $U \times \widetilde{E}$, for any real analytic extension $\widetilde{f}\colon U \times \widetilde{E} \to \widetilde{F}$ and $x \in U$, the map $\widetilde{f}(x, \cdot)$ will be k-linear over $\mathbb{L}$. We may therefore assume that $\mathbb{L} = \mathbb{R}$. Let $h\colon W \to F_{\mathbb{C}}$ be a $\mathbb{C}$-analytic extension of f, defined on an open subset $W \subseteq X_{\mathbb{C}} \times E_{\mathbb{C}}$ such that $U \times E \subseteq W$. For each $x \in U$, there exist an open x-neighbourhood $U_x \subseteq U$ and balanced open 0-neighbourhoods $V_x \subseteq X$ and $W_x \subseteq E_{\mathbb{C}}$ such that $(U_x + iV_x) \times W_x \subseteq W$. We claim that there exists a $\mathbb{C}$-analytic map $g_x\colon (U_x + iV_x) \times E_{\mathbb{C}} \to F_{\mathbb{C}}$ such that $g_x|_{U_x \times E} = f|_{U_x \times E}$. For $x, y \in U$, the intersection $((U_x + iV_x) \times E_{\mathbb{C}}) \cap ((U_y + iV_y) \times E_{\mathbb{C}}) = ((U_x \cap U_y) + i(V_x \cap V_y)) \times E_{\mathbb{C}}$ is connected and meets $U \times E$ whenever it is non-empty. Hence, by the Identity Theorem, g_x and g_y coincide on the intersection of their domains. We therefore obtain a well-defined $\mathbb{C}$-analytic map $g\colon Q \times E_{\mathbb{C}} \to F_{\mathbb{C}}$ such that $g|_{(U_x + iV_x) \times E_{\mathbb{C}}} = g_x$ for each $x \in U$, using the open subset $Q := \bigcup_{x \in U}(U_j + iV_j)$ of $X_{\mathbb{C}}$. For each $x \in U$, the map $g(x, \cdot)|_E = g_x(x, \cdot)|_E = f(x, \cdot)$ is k-linear over $\mathbb{R}$. Using the Identity Theorem, we see that $g(x, \cdot)$ is k-linear over $\mathbb{C}$ for each $x \in U$, and hence for each $x \in Q$ by the Identity Theorem. By the case $(\infty, \mathbb{C})$, g has a $\mathbb{C}$-analytic extension $\widetilde{g}\colon Q \times \widetilde{E}_{\mathbb{C}} \to \widetilde{F}_{\mathbb{C}}$. Since $g(U \times E) = f(U \times E) \subseteq F \subseteq \widetilde{F}$ and $U \times E$ is dense in $U \times \widetilde{E}$, we deduce that $\widetilde{g}(U \times \widetilde{E}) \subseteq \widetilde{F}$; we therefore obtain a map

$$\widetilde{f}\colon U \times \widetilde{E} \to \widetilde{F}, \quad (x, y) \mapsto \widetilde{g}(x, y)$$

for $x \in U$, $y \in \widetilde{E}$. Since $\widetilde{g}$ is a $\mathbb{C}$-analytic extension for $\widetilde{f}$, the function $\widetilde{f}$ is $\mathbb{R}$-analytic. To prove the claim, consider for $x \in U$ and $n \in \mathbb{N}$ the $\mathbb{C}$-analytic map

$$g_{x,n}\colon (U_x + iV_x) \times nW_x \to F_{\mathbb{C}}, \quad (z, y) \mapsto n^k h(z, (1/n)y).$$

If $n \leq m$ and $y \in nW_x \cap E$, we have for all $z \in U_x$

$$g_{x,m}(z, y) = m^k h(z, (1/m)y) = m^k f(z, (1/m)y) = f(z, y) = n^k f(t, (1/n)y) = g_{x,n}(z, y),$$

whence $g_{x,m}(z, y) = g_{x,n}(z, y)$ for all $z \in U_x + iV_x$ and $y \in nW_x$, by the Identity Theorem. Thus, $g_x\colon (U_x + iV_x) \times E_{\mathbb{C}} \to F_{\mathbb{C}}$, $(z, y) \mapsto g_{x,n}(z, y)$ if $y \in nW_x$ is a well-defined $\mathbb{C}$-analytic extension of $f|_{U_x \times E}$. $\square$

Proof of Proposition 5. It suffices to prove the strengthening described in Remark 8. Let $(\psi_i)_{i \in I}$ be a family of local trivialisations $\psi_i\colon \pi^{-1}(U_i) \to U_i \times F$ of an $LC^r_{\mathbb{K}}$-vector bundle E

such that each local trivialisation is some ψ_i. Let $(g_{ij})_{i,j\in I}$ be the corresponding cocycle and G_{ij} be the $LC^r_{\mathbb{K}}$-map $g_{ij}^{\wedge}\colon (U_i\cap U_j)\times F\to F$, which is $\mathbb{L}$-linear in the second argument. By Proposition 7, there is a unique $LC^r_{\mathbb{K}}$-map $\widetilde{G}_{ij}\colon U\times\widetilde{F}\to\widetilde{F}$ which extends G_{ij}, and $\widetilde{G}_{ij}$ is $\mathbb{L}$-linear in the second argument. Thus, we obtain a map

$$\widetilde{g}_{ij}\colon U_i\cap U_j\to L_{\mathbb{L}}(\widetilde{F}),\quad x\mapsto \widetilde{G}_{ij}(x,\cdot).$$

By continuity and density, for all $i\in I$, we have $\widetilde{G}_{ii}(x,y)=y$ for all $(x,y)\in U_i\times\widetilde{F}$. Thus, $\widetilde{g}_{ii}(x)=\mathrm{id}_{\widetilde{F}}$ for all $x\in U$. For all $i,j,k\in I$, we have

$$\widetilde{G}_{ij}(x,\widetilde{G}_{jk}(x,y))=\widetilde{G}_{ik}(x,y)\quad\text{for all } (x,y)\in(U_i\cap U_j\cap U_k)\times\widetilde{F},$$

as both sides are continuous in (x,y) and equality holds for y in the dense subset F of $\widetilde{F}$; thus, $\widetilde{g}_{ij}(x)\circ\widetilde{g}_{jk}(x)=\widetilde{g}_{ik}(x)$. Notably, $\widetilde{g}_{ij}(x)\circ\widetilde{g}_{ji}(x)=\widetilde{g}_{ii}(x)=\mathrm{id}_{\widetilde{F}}$ for all $x\in U$ and $i,j\in I$, entailing that $\widetilde{g}_{ij}(x)\in\mathrm{GL}(\widetilde{F})$. By the preceding, the $\widetilde{g}_{ij}$ satisfy the cocycle conditions. Let $\widetilde{E}$ and $\widetilde{\pi}$ be as in (8) and (9); define $\widetilde{\psi}_i\colon\widetilde{\pi}^{-1}(U_i)\to U_i\times\widetilde{F}$ as in (10), replacing ψ with ψ_i. For all $i,j\in I$ and $x\in U$, we then have that

$$\widetilde{\psi}_i(\widetilde{\psi}_j^{-1}(x,y))=(x,\widetilde{G}_{ij}(x,y))$$

holds for all $y\in\widetilde{F}$, as equality holds for all $y\in F$. As an analogue of Proposition 3 holds with $LC^r_{\mathbb{K}}$-maps in place of $C^r_{\mathbb{K}}$-maps, we get a unique $\mathbb{L}$-vector bundle structure of class $LC^r_{\mathbb{K}}$ on $\widetilde{E}$ making $\widetilde{\psi}_i$ a local trivialisation for each $i\in I$.

It is apparent that $\widetilde{\beta}\colon G\times\widetilde{E}\to\widetilde{E}$ is an action, and $\widetilde{E}_x$ is taken $\mathbb{L}$-linearly to $\widetilde{E}_{\alpha(g,x)}$ by $\widetilde{\beta}(g,\cdot)$, for each $g\in G$ and $x\in M$. It only remains to show that $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$. To this end, let $g_0\in G$ and $x_0\in M$; we show that $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$ on $U\times\widetilde{\pi}^{-1}(V)$ for some open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M. Indeed, there exists a local trivialisation $\psi\colon\pi^{-1}(W)\to W\times F$ of E over an open neighbourhood W of $\alpha(g_0,x_0)$ in M. The action α being continuous, we find an open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M over which E is trivial, such that $\alpha(U\times V)\subseteq W$. Let $\phi\colon\pi^{-1}(V)\to V\times F$ be a local trivialisation of E over V. Then,

$$\phi(\beta(g^{-1},\psi^{-1}(\alpha(g,x),v)))=(x,A(g,x,v))\quad\text{for all } g\in U, x\in V, \text{and } v\in F,$$

for an $LC^r_{\mathbb{K}}$-map $A\colon U\times V\times F\to F$, which is $\mathbb{L}$-linear in the third argument. By Proposition 7, there is a unique extension of A to an $LC^r_{\mathbb{K}}$-map

$$\widetilde{A}\colon U\times V\times\widetilde{F}\to\widetilde{F},$$

and the latter is $\mathbb{L}$-linear in its third argument. For all $g\in U$ and $x\in V$, we then have

$$\widetilde{\phi}(\widetilde{\beta}(g^{-1},\widetilde{\psi}^{-1}(\alpha(g,x),v)))=(x,\widetilde{A}(g,x,v))$$

for all $v\in\widetilde{F}$, as equality holds for all $v\in F$. Thus, $\widetilde{\beta}$ is $LC^r_{\mathbb{K}}$. □

8. Tensor Products of Vector Bundles

Throughout this section, let $\mathbb{L}\in\{\mathbb{R},\mathbb{C}\}$, $\mathbb{K}\in\{\mathbb{R},\mathbb{L}\}$, $s\in\{\infty,\omega\}$, and $r\in\mathbb{N}_0\cup\{\infty,\omega\}$ such that $r\leq s$. Let G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex $\mathbb{K}$-vector space Y, M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z, and $\alpha\colon G\times M\to M$ be a $C^r_{\mathbb{K}}$-action. For $k\in\{1,2\}$, let $\pi_k\colon E_k\to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M, whose typical fibre is a locally convex $\mathbb{L}$-vector space F_k. Let $\beta_k\colon G\times E_k\to E_k$ be the G-action of class $C^r_{\mathbb{K}}$. Consider the set $\mathcal{A}$ of all pairs of local trivialisations of E_1 and E_2 trivialising these over the same open subset of M. Using an index set I, we have

$\mathcal{A} = \{(\psi_i^1, \psi_i^2) : i \in I\}$, where $\psi_i^k : \pi_k^{-1}(U_i) \to U_i \times F_k$ is a local trivialisation of E_k for $k \in \{1,2\}$, for each $i \in I$. Apparently, $(U_i)_{i\in I}$ is an open cover of M.

For our first result concerning tensor products, Proposition 8, we assume that F_1 is finite-dimensional. Then, fixing a basis $e_1, \ldots, e_n$ for F_1, the map $\theta\colon (F_2)^n \to F_1 \otimes_{\mathbb{L}} F_2$, $(y_1, \ldots, y_n) \mapsto \sum_{\tau=1}^n e_\tau \otimes y_\tau$ is an isomorphism of $\mathbb{L}$-vector spaces. We give $F_1 \otimes_{\mathbb{L}} F_2$ the topology $\mathcal{T}$, making θ a homeomorphism. This topology makes $F_1 \otimes_{\mathbb{L}} F_2$ a locally convex $\mathbb{L}$-vector space and θ an isomorphism of topological $\mathbb{L}$-vector spaces. It is easy to check (and well known) that the topology $\mathcal{T}$ is independent of the chosen basis. Let $e_1^*, \ldots, e_n^* \in F_1'$ be the basis dual to $e_1, \ldots, e_n$. Our goal is to make the union

$$E_1 \otimes E_2 := \bigcup_{x\in M} (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$$

an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M, with typical fibre $F_1 \otimes_{\mathbb{L}} F_2$; the tensor products $(E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ are chosen pairwise disjoint here for $x \in M$. Let $\pi\colon E_1 \otimes E_2 \to M$ be the mapping which takes $v \in (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ to x.

We define $\psi_i\colon \pi^{-1}(U_i) \to U_i \times (F_1 \otimes_{\mathbb{L}} F_2)$ via

$$\psi_i(v) := (x, ((\mathrm{pr}_{F_1} \circ \psi_i^1|_{(E_1)_x}) \otimes (\mathrm{pr}_{F_2} \circ \psi_i^2|_{(E_2)_x}))(v))$$

for $x \in U_i$ and $v \in (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$, where $\mathrm{pr}_{F_k}\colon M \times F_k \to F_k$ is the projection.

Given $i, j \in I$ and $x \in U_i \cap U_j$, we have $\psi_i^k((\psi_j^k)^{-1}(x,v)) = (x, G_{ij}^k(x,v))$ for all $k \in \{1,2\}$ and $v \in F_k$, where $G_{ij}^k\colon (U_i \cap U_j) \times F_k \to F_k$ is $C^r_{\mathbb{K}}$ and $g_{ij}^k(x) := G_{ij}^k(x,\cdot)$ an $\mathbb{L}$-linear mapping. Then, $c_{\sigma,\tau}\colon U_i \cap U_j \to \mathbb{K}$, $x \mapsto e_\sigma^*(G_{ij}^1(x, e_\tau))$ is $C^r_{\mathbb{K}}$, and $\psi_i((\psi_j)^{-1}(x,v)) = (x, G_{ij}(x,v))$ for $x \in U_i \cap U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_{\mathbb{L}} F_2$, where

$$\begin{aligned} G_{ij}(x,v) &= (g_{ij}^1(x) \otimes g_{ij}^2(x))(v) = \sum_{\tau=1}^n (g_{ij}^1(x)e_\tau) \otimes (g_{ij}^2(x)v_\tau) \\ &= \sum_{\sigma,\tau=1}^n e_\sigma \otimes (c_{\sigma,\tau}(x) g_{ij}^2(x) v_\tau) = \theta\left(\left(\sum_{\tau=1}^n c_{\sigma,\tau}(x) G_{ij}^2(x, v_\tau)\right)_{\sigma=1}^n\right). \end{aligned}$$

As $F_1 \otimes_{\mathbb{L}} F_2 \to F_2$, $v \mapsto v_\tau = \mathrm{pr}_\tau(\theta^{-1}(v))$ is a continuous linear map (where $\mathrm{pr}_\tau : (F_2)^n \to F_2$ is the projection onto the τ-component), in view of the preceding formula G_{ij} is $C^r_{\mathbb{K}}$. Thus, by Proposition 3, there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on $E_1 \otimes E_2$ making each ψ_i a local trivialisation.

Note that $\beta\colon G \times (E_1 \otimes E_2) \to E_1 \otimes E_2$, $(g,v) \mapsto (\beta_1(g,\cdot)|_{(E_1)_x}^{E_{\alpha(g,x)}} \otimes \beta_2(g,\cdot)|_{(E_2)_x}^{(E_2)_{\alpha(g,x)}})(v)$ for $g \in G$, $x \in M$, $v \in (E_1 \otimes E_2)_x$ defines an action of G on $E_1 \otimes E_2$ by $\mathbb{L}$-linear mappings, which makes $\pi\colon E_1 \otimes E_2 \to M$ an equivariant mapping and such that $\beta(g,\cdot)$ is $\mathbb{L}$-linear on $(E_1)_x \otimes_{\mathbb{L}} (E_2)_x$ for all $g \in G$ and $x \in M$.

To show that β is $C^r_{\mathbb{K}}$, let $g_0 \in G$ and $x_0 \in M$. We pick $i \in I$ such that $\alpha(g_0, x_0) \in U_i$. The mapping α being continuous, we find open neighbourhoods U of g_0 in G and V of x_0 in M such that $\alpha(U \times V) \subseteq U_i$. There is $j \in I$ such that $x_0 \in U_j \subseteq V$. For $k \in \{1,2\}$, $g \in U$, $x \in U_j$ and $v \in F_k$, we have

$$\psi_i^k(\beta_k(g, (\psi_j^k)^{-1}(x,v))) = (\alpha(g,x), a_k(g,x,v))$$

for some $C^r_{\mathbb{K}}$-map $a_k\colon U \times U_j \times F_k \to F_k$, which is $\mathbb{L}$-linear in the final argument. Define $b_{\sigma,\tau} : U \times U_j \to \mathbb{L}$, $(g,x) \mapsto e_\sigma^*(a_1(g,x,e_\tau))$; then, $b_{\sigma,\tau}$ is $C^r_{\mathbb{K}}$. If $g \in U$, $x \in U_j$ and $v = \sum_{\tau=1}^n e_\tau \otimes v_\tau \in F_1 \otimes_{\mathbb{L}} F_2$, then $\psi_i(\beta(g, \psi_j^{-1}(x,v)))$ equals

$$\left(\alpha(g,x), \sum_{\tau=1}^n a_1(g,x,e_\tau) \otimes a_2(g,x,v_\tau)\right) = \left(\alpha(g,x), \theta\left(\left(\sum_{\tau=1}^n b_{\sigma,\tau}(g,x) a_2(g,x,v_\tau)\right)_{\sigma=1}^n\right)\right),$$

which is a $C^r_{\mathbb{K}}$-function of (g,x,v). As a consequence, $\beta|_{U\times\pi^{-1}(U_j)}$ is $C^r_{\mathbb{K}}$ and thus β, being $C^r_{\mathbb{K}}$ locally, is $C^r_{\mathbb{K}}$. We summarise as follows.

Proposition 8. *Let G be a $C^s_{\mathbb{K}}$-Lie group and M be a G-manifold of class $C^r_{\mathbb{K}}$. Let E_1 and E_2 be equivariant $\mathbb{L}$-vector bundles of class $C^r_{\mathbb{K}}$ over M. If the typical fibre of E_1 is finite-dimensional, then $E_1\otimes E_2$, as defined above, is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

Instead of $\dim(F_1)<\infty$ (as before) assume that F_1 and F_2 are Fréchet spaces and the modelling spaces of G and M are metrisable. The completed projective tensor product

$$F := F_1\widehat{\otimes}_\pi F_2$$

over $\mathbb{L}$ then is a Fréchet space (cf. [30] (p. 438, lines after Definitions 43.4)). We define

$$E := E_1\widehat{\otimes}_\pi E_2 := \bigcup_{x\in M}(E_1)_x\widehat{\otimes}_\pi(E_2)_x,$$

where the $(E_1)_x\widehat{\otimes}_\pi(E_2)_x$ for $x\in M$ are chosen pairwise disjoint. Let $\pi\colon E\to M$ be the map taking $v\in E_x := (E_1)_x\widehat{\otimes}_\pi(E_2)_x$ to x. Define $\psi_i\colon \pi^{-1}(U_i)\to U_i\times(F_1\widehat{\otimes}_\pi F_2)$ via

$$\psi_i(v) := \big(x, ((\mathrm{pr}_{F_1}\circ\psi_i^1|_{(E_1)_x})\widehat{\otimes}_\pi(\mathrm{pr}_{F_2}\circ\psi_i^2|_{(E_2)_x}))(v)\big)$$

for $x\in U_i$ and $v\in(E_1)_x\widehat{\otimes}_\pi(E_2)_x$, where $\mathrm{pr}_{F_k}\colon M\times F_k\to F_k$ is the projection. Note that $\beta\colon G\times E\to E$, $(g,v)\mapsto(\beta_1(g,\cdot)|_{(E_1)_x}\widehat{\otimes}_\pi\beta_2(g,\cdot)|_{(E_2)_x})(v)$ for $g\in G$, $x\in M$, $v\in E_x$ defines an action of G on E which makes $\pi\colon E\to M$ an equivariant mapping. We show:

Proposition 9. *$\pi\colon E_1\widehat{\otimes}_\pi E_2\to M$ admits a unique structure of equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M such that ψ_i is a local trivialisation for each $i\in I$.*

Proof. The uniqueness for prescribed local trivialisations is clear. Let us show the existence of the structure. Given $i,j\in I$ and $x\in U_i\cap U_j$, we have $\psi_i^k((\psi_j^k)^{-1}(x,v)) = (x,G_{ij}^k(x,v))$ for all $k\in\{1,2\}$ and $v\in F_k$, where $G^k_{ij}\colon(U_i\cap U_j)\times F_k\to F_k$ is $C^r_{\mathbb{K}}$ and $g^k_{ij}(x) := G^k_{ij}(x,\cdot)$ an $\mathbb{L}$-linear mapping. By Proposition 1 (a), the map $g^k_{ij}\colon U_i\cap U_j\to L(F_k)_c$ is $C^r_{\mathbb{K}}$. Now,

$$L_{\mathbb{L}}(F_1)_c\times L_{\mathbb{L}}(F_2)\to L_{\mathbb{L}}(F_1\widehat{\otimes}_\pi F_2)_c,\quad (S,T)\mapsto S\widehat{\otimes}_\pi T$$

being continuous $\mathbb{L}$-bilinear (as recalled in Lemma 14), we deduce that

$$g_{ij}\colon U_i\cap U_j\to L_{\mathbb{L}}(F_1\widehat{\otimes}_\pi F_2)_c,\quad x\mapsto g^1_{ij}(x)\widehat{\otimes}_\pi g^2_{ij}(x)$$

is $C^r_{\mathbb{K}}$. Hence, $G_{ij} := g_{ij}^\wedge\colon(U_i\cap U_j)\times(F_1\widehat{\otimes}_\pi F_2)\to F_1\widehat{\otimes}_\pi F_2$, $(x,v)\mapsto g_{ij}(x)(v)$ is $C^r_{\mathbb{K}}$, by Proposition 2 (a). We easily check that $\psi_i((\psi_j)^{-1}(x,v)) = (x,G_{ij}(x,v))$ holds for G_{ij} as just defined, for all $x\in U_i\cap U_j$ and $v\in F_1\widehat{\otimes}_\pi F_2$. Hence, $E_1\widehat{\otimes}_\pi E_2$ can be made an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ in such a way that each ψ_i is a local trivialisation, by Proposition 3. Note that $\beta(g,\cdot)$ is $\mathbb{L}$-linear on E_x for all $g\in G$ and $x\in M$. To show that β is $C^r_{\mathbb{K}}$, let g_0, x_0, i, U, V, j and the $C^r_{\mathbb{K}}$-map a_k be as in the proof of Proposition 8. By Proposition 1 (a), $a_k^\vee\colon U\times U_j\to L(F_k)_c$, $(g,x)\mapsto a_k(g,x,\cdot)$ is $C^r_{\mathbb{K}}$. Hence,

$$a\colon U\times U_j\to L(F_1\widehat{\otimes}_\pi F_2)_c,\quad (g,x)\mapsto a_1^\vee(g,x)\widehat{\otimes}_\pi a_2^\vee(g,x)$$

is $C^r_{\mathbb{K}}$, by the Chain Rule and Lemma 14. Using Proposition 2 (a), we find that the map $a^\wedge\colon U\times U_j\times(F_1\widehat{\otimes}_\pi F)\to F_1\widehat{\otimes}_\pi F_2$, $(g,x,v)\mapsto a(g,x)(v)$ is $C^r_{\mathbb{K}}$. We easily verify that $\psi_i(\beta(g,(\psi_j)^{-1}(x,v))) = (\alpha(g,x),a^\wedge(g,x,v))$ for all $(g,x,v)\in U\times U_j\times(F_1\widehat{\otimes}_\pi F_2)$. Thus, $\psi_i(\beta(g,(\psi_j)^{-1}(x,v)))$ is $C^r_{\mathbb{K}}$ in (g,x,v), which completes the proof. □

We used the following fact:

Lemma 14. *Let E_1, E_2, F_1, and F_2 be Fréchet spaces over $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$. Then, the following bilinear map is continuous:*

$$\Xi\colon L_{\mathbb{L}}(E_1, F_1)_c \times L_{\mathbb{L}}(E_2, F_2)_c \to L_{\mathbb{L}}((E_1 \widehat{\otimes}_\pi E_2), (F_1 \widehat{\otimes}_\pi F_2))_c, \quad (S_1, S_2) \mapsto S_1 \widehat{\otimes}_\pi S_2.$$

Proof. Let $K \subseteq E_1 \widehat{\otimes} E_2$ be compact, q be a continuous seminorm on $F_1 \widehat{\otimes}_\pi F_2$, and $\varepsilon > 0$. After increasing q, we may assume that $q = q_1 \otimes q_2$ for continuous seminorms q_k on F_k for $k \in \{1, 2\}$. By [30] (p. 465, Corollary 2 to Theorem 45.2), K is contained in the closed, absolutely convex hull of $K_1 \otimes K_2$ for certain compact subsets $K_k \subseteq E_k$ for $k \in \{1, 2\}$. For all $S_k \in L(E_k, F_k)$ such that $\sup q_k(S_k(K_k)) \leq \sqrt{\varepsilon}$, we have

$$\sup q((S_1 \widehat{\otimes}_\pi S_2)(K)) \leq \sup q((S_1 \widehat{\otimes}_\pi S_2)(K_1 \otimes K_2)) = \sup q_1(S_1(K_1)) q_2(S_2(K_2)) \leq \sqrt{\varepsilon}^2 = \varepsilon,$$

using [30] (Proposition 43.1). The assertion follows. □

Remark 9. *If E_1 and E_2 are Hilbert spaces over $\mathbb{L}$ with Hilbert space tensor product $E_1 \widehat{\otimes} E_2$, and also F_1 and F_2 are Hilbert spaces over $\mathbb{L}$, then the bilinear map*

$$\Xi\colon L(E_1, F_1)_b \times L(E_2, F_2)_b \to L((E_1 \widehat{\otimes} E_2), (F_1 \widehat{\otimes} F_2))_b$$

is continuous, as $\|S_1 \widehat{\otimes} S_2\|_{\mathrm{op}} \leq \|S_1\|_{\mathrm{op}} \|S_2\|_{\mathrm{op}}$.

Replace the hypotheses in Proposition 9 with the requirements that G and M are modelled on metrisable locally convex spaces, $r \geq 1$ and F_1, F_2 are Hilbert spaces. We now use Remark 9 instead of Lemma 14, replace $F_1 \widehat{\otimes}_\pi F_2$ with the Hilbert space $F_1 \widehat{\otimes} F_2$, Proposition 1 (a) with Proposition 1 (b) (so that operator-valued maps are only $C^{r-1}_{\mathbb{K}}$) and use Proposition 2 (b) with $r - 1$ in place of r. Repeating the proof of Proposition 9, we get:

Proposition 10. *On $E_1 \widehat{\otimes} E_2 = \bigcup_{x \in M} (E_1)_x \widehat{\otimes} (E_2)_x$, there is a unique equivariant $\mathbb{L}$-vector bundle structure of class $C^{r-1}_{\mathbb{K}}$ over M whose typical fibre is the Hilbert space $F_1 \widehat{\otimes} F_2$, such that $\psi_i\colon \pi^{-1}(U_i) \to U_i \times (F_1 \widehat{\otimes} F_2)$ is a local trivialisation for each $i \in I$.*

Remark 10. *If $r \geq 1$, G and M are modelled on metrisable spaces and both F_1 and F_2 are pre-Hilbert spaces with Hilbert space completions $\widetilde{F}_1$ and $\widetilde{F}_2$, we can use the non-completed tensor product $F_1 \otimes_{\mathbb{L}} F_2 \subseteq \widetilde{F}_1 \widehat{\otimes} \widetilde{F}_2$ with the induced topology as the fibre and get an equivariant $\mathbb{L}$-vector bundle structure over M of class $C^{r-1}_{\mathbb{K}}$ over M on $E_1 \otimes E_2 = \bigcup_{x \in M} (E_1)_x \otimes_{\mathbb{L}} (E_2)_x$, exploiting that the $\mathbb{L}$-bilinear map $L_{\mathbb{L}}(F_1)_b \times L_{\mathbb{L}}(F_2)_b \to L_{\mathbb{L}}(F_1 \otimes_{\mathbb{L}} F_2)_b$, $(S_1, S_2) \mapsto S_1 \otimes S_2$ is continuous.*

9. Locally Convex Direct Sums of Vector Bundles

Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $r \leq s$, G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex space Y, and M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z, together with a $C^r_{\mathbb{K}}$-action $\alpha\colon G \times M \to M$.

Let $n \in \mathbb{N}$ and $\pi_k\colon E_k \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M for $k \in \{1, \ldots, n\}$, with typical fibre a locally convex $\mathbb{L}$-vector space F_k; let $\beta_k\colon G \times E_k \to E_k$ be the G-action and $\mathrm{pr}_{F_k}\colon M \times F_k \to F_k$ be the projection onto the second component. We easily check that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on the "Whitney sum"

$$E := E_1 \oplus \cdots \oplus E_n := \bigcup_{x \in M} (E_1)_x \times \cdots \times (E_n)_x,$$

with the apparent map $\pi : E \to M$, such that $\psi : \pi^{-1}(U) \to U \times F_1 \times \cdots \times F_n$, $v = (v_1, \ldots, v_n) \mapsto (\pi(v), \mathrm{pr}_{F_1}(\psi_1(v_1)), \ldots, \mathrm{pr}_{F_n}(\psi_n(v_n)))$ is a local trivialisation of E, for all families $(\psi_k)_{k=1}^n$ of local trivialisations $\psi_k\colon (\pi_k)^{-1}(U) \to U \times F_k$, which trivialise the E_ks over a joint open subset U of M. Then, $\beta(g, v) := (\beta_1(g, v_1), \ldots, \beta_n(g, v_n))$ for $g \in G$, $v = (v_1, \ldots, v_n) \in E$ yields an action of G on E. It is straightforward that β is $C^r_{\mathbb{K}}$. Thus,

Proposition 11. *If $E_1, \ldots, E_n$ are equivariant $\mathbb{L}$-vector bundles of class $C^r_{\mathbb{K}}$ over a G-manifold M of class $C^r_{\mathbb{K}}$, then also $E_1 \oplus \cdots \oplus E_n$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

The following lemma allows infinite direct sums to be tackled.

Lemma 15. *Let $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ be families of locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with locally convex direct sums $E := \bigoplus_{i \in I} E_i$ and $F := \bigoplus_{i \in I} F_i$, respectively. Let V be an open subset of a locally convex $\mathbb{K}$-vector space Z. Let $r \in \mathbb{N}_0 \cup \{\infty\}$, and assume that $f_i \colon V \times E_i \to F_i$ is a map which is linear in the second argument, for each $i \in I$. Moreover, assume that* (a) *or* (b) *holds:*

(a) *Z is finite-dimensional; or*
(b) *Z and each E_i is a k_ω-space and I is countable.*

If f_i is of class $C^r_{\mathbb{K}}$ for each $i \in I$, then also the following map is $C^r_{\mathbb{K}}$:

$$f \colon V \times E \to F, \quad (x, (v_i)_{i \in I}) \mapsto (f_i(x, v_i))_{i \in I}.$$

Proof. If (b) holds, we may assume that I is countably infinite, excluding a trivial case. Thus, assume that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, identify $E_1 \times \cdots \times E_n$ with a vector subspace of E, identifying $x \in E_1 \times \cdots \times E_n$ with $(x, 0)$. For each $n \in \mathbb{N}$, we then have

$$Z \times E = \bigcup_{n \in \mathbb{N}} (Z \times E_1 \times \cdots \times E_n) \quad \text{and} \quad V \times E = \bigcup_{n \in \mathbb{N}} (V \times E_1 \times \cdots \times E_n),$$

where $Z \times E_1 \times \cdots \times E_n$ is a k_ω-space in the product topology. The inclusion map

$$\lambda_n \colon F_1 \times \cdots \times F_n \to \bigoplus_{i \in \mathbb{N}} F_i, \quad v \mapsto (v, 0)$$

is continuous and $\mathbb{K}$-linear. Moreover,

$$g_n \colon V \times E_1 \times \cdots \times E_n \to F_1 \times \cdots \times F_n, \quad (x, v_1, \ldots, v_n) \mapsto (f_1(x, v_1), \ldots, f_n(x, v_n))$$

is a $C^r_{\mathbb{K}}$-map and so is $f|_{V \times E_1 \times \cdots \times E_n} = \lambda_n \circ g_n$, for each $n \in \mathbb{N}$. Hence, f is $C^r_{\mathbb{K}}$ on the open subset $V \times E$ of $Z \times E$, considered as the locally convex direct limit $\varinjlim (Z \times E_1 \times \cdots \times E_n)$, by [31] (Proposition 4.5 (a)). This locally convex space equals $Z \times \varinjlim (E_1 \times \cdots \times E_n) = Z \times E$ with the product topology (see [32] (Theorem 3.4)).

If (a) holds, it suffices to prove the assertion for $r \in \mathbb{N}_0$. We proceed by induction. *The case $r = 0$.* Let $(x, v) = (x, (v_i)_{i \in I}) \in V \times E$; we show that f is continuous at (x, v). To this end, let Q be an absolutely convex, open 0-neighbourhood in F. There is a finite subset $J \subseteq I$ such that $v_i = 0$ for all $i \in I \setminus J$. Let $N := |J| + 1$. For each $i \in I$, the intersection $Q_i := (\frac{1}{N} Q) \cap F_i$ is an absolutely convex, open 0-neighbourhood in F_i. For the absolutely convex hull, we get $\operatorname{absconv}(\bigcup_{i \in I} Q_i) \subseteq \frac{1}{N} Q$. Since f_i is continuous for each $i \in J$ and J is finite, we find a compact neighbourhood K of x in V such that $f_i(y, v_i) - f_i(x, v_i) \in Q_i$ for all $y \in K$ and $i \in J$. Since $f_i(K \times \{0\}) = \{0\}$, where K is compact and f_i is continuous, for each $i \in I$, there is an absolutely convex, open 0-neighbourhood P_i in E_i such that $f_i(K \times P_i) \subseteq Q_i$. Then, $W := v + \operatorname{absconv}(\bigcup_{i \in I} P_i)$ is an open neighbourhood of v in E. Let $y \in K$ and $w \in W$ be given, say $w = (w_i)_{i \in I} = v + (t_i p_i)_{i \in I}$, where $p_i \in P_i$ and $(t_i)_{i \in I} \in \bigoplus_{i \in I} \mathbb{R}$ such that $t_i \in [0, 1]$ and $\sum_{i \in I} t_i = 1$. Then, for each $i \in I \setminus J$, since $v_i = 0$, we obtain

$$f_i(y, w_i) - f(x, v_i) = f_i(y, t_i p_i) = t_i f_i(y, p_i) \in t_i Q_i\,.$$

For $i \in J$, on the other hand, we have

$$\begin{aligned} f_i(y, w_i) - f(x, v_i) &= f_i(y, w_i - v_i) + (f_i(y, v_i) - f_i(x, v_i)) \\ &= t_i f_i(y, p_i) + (f_i(y, v_i) - f_i(x, v_i)) \in t_i Q_i + Q_i. \end{aligned}$$

As a consequence, $f(y,w) - f(x,v) \in (\prod_{i\in I} t_i Q_i) + \sum_{i\in J} Q_i \subseteq \frac{1}{N}Q + \sum_{i\in J}\frac{1}{N}Q = Q$, using the convexity of Q. We have shown that f is continuous at (x,v).

Induction step. Let $r \geq 1$ and assume the assertion is true for all numbers $< r$. Given $u, v \in E$, $x \in V$, and $z \in Z$, we have $u, v \in \bigoplus_{i\in J} E_i = \prod_{i\in J} E_i$ for some finite subset $J \subseteq I$. The map $f_J \colon V \times \prod_{i\in J} E_i \to \prod_{i\in J} F_i$, $(x,(v_i)_{i\in J}) \mapsto (f_i(x, v_i))_{i\in J}$ is $C^1_{\mathbb{K}}$, whence

$$\begin{aligned} df_J((x,u),(z,v)) &= \lim_{t\to 0} t^{-1}(f_J((x,u) + t(z,v)) - f_J(x,u)) \\ &= \lim_{t\to 0} t^{-1}(f((x,u) + t(z,v)) - f(x,u)) = df((x,u),(z,v)) \end{aligned}$$

exists in $\prod_{i\in J} F_i$ and thus in F; its ith component is

$$df_i((x,u_i),(z,v_i)) = d_1 f_i(x, u_i, z) + d_2 f_i(x, u_i, v_i)$$

in terms of partial differentials. Note that the mappings $g_i : (V \times Z) \times (E_i \times E_i) \to F_i$, $(x, z, u_i, v_i) \mapsto d_1 f_i(x, u_i, z)$ and $h_i \colon (V \times Z) \times (E_i \times E_i) \to F_i$, $(x, z, u_i, v_i) \mapsto d_2 f_i(x, u_i, v_i) = f_i(x, v_i)$ are $C^{r-1}_{\mathbb{K}}$ and linear in (u_i, v_i). By induction, the mappings

$$g \colon (V \times Z) \times (E \times E) \to F, \quad (x, z, (u_i)_{i\in I}, (v_i)_{i\in I}) \mapsto (g_i(x, z, u_i, v_i))_{i\in I} \quad \text{and}$$

$$h \colon (V \times Z) \times (E \times E) \to F, \quad (x, z, (u_i)_{i\in I}, (v_i)_{i\in I}) \mapsto (h_i(x, z, u_i, v_i))_{i\in I}$$

are $C^{r-1}_{\mathbb{K}}$, using that $E \times E \cong \bigoplus_{i\in I}(E_i \times E_i)$. Hence, also $df : (V \times E) \times (Z \times E) \to F$ is $C^{r-1}_{\mathbb{K}}$, as $df((x,u),(z,v)) = g(x,z,u,v) + h(x,z,u,v)$. Since df exists and is $C^{r-1}_{\mathbb{K}}$, the continuous map f is $C^r_{\mathbb{K}}$. $\square$

Remark 11. *The conclusion of Lemma 15 does not hold for $(r,\mathbb{K}) = (\omega,\mathbb{R})$ in the example $I = \mathbb{N}$, $V = Z = \mathbb{R}$, $E_k = \mathbb{R}$, $f_k(r,t) := \frac{t}{1+kr^2}$, using that the Taylor series of $f_k(\cdot,t)$ around 0 has radius of convergence $\frac{1}{\sqrt{k}}$ for all $t \in \mathbb{R}\setminus\{0\}$.*

Assuming now $r \neq \omega$, consider a family $(E_i)_{i\in I}$ of equivariant $\mathbb{L}$-vector bundles $\pi_i \colon E_i \to M$ of class $C^r_{\mathbb{K}}$ with typical fibre F_i and G-action $\beta_i \colon G \times E_i \to E_i$. We assume that (a) or (b) is satisfied:

(a) G and M are finite-dimensional; or
(b) I is countable and each F_i as well as the modelling spaces of G and M are k_ω-spaces.

Moreover, we assume:

(c) For each $x \in M$, there exists an open neighbourhood U of x in M, such that, for each $i \in I$, the vector bundle E_i admits a local trivialisation $\psi_i \colon (\pi_i)^{-1}(U) \to U \times F_i$.

Thus, the $C^r_{\mathbb{K}}$-vector bundle $E_i|_U$ is trivialisable for each $i \in I$. Define $E := \bigcup_{x\in M}\bigoplus_{i\in I}(E_i)_x$ with pairwise disjoint direct sums and $\pi \colon E \to M$, $\bigoplus_{i\in I}(E_i)_x \ni v \mapsto x$. Then

$$\beta \colon G \times E \to E, \quad (g, (v_i)_{i\in I}) \mapsto (\beta_i(g, v_i))_{i\in I}$$

is a G-action such that $\beta(g,\cdot)|_{E_x} \colon E_x \to E_{\alpha(g,x)}$ is $\mathbb{L}$-linear for all $(g,x) \in G \times M$, where $E_x := \pi^{-1}(\{x\})$. We readily deduce from Proposition 3 and Proposition 15 that there is a unique $\mathbb{L}$-vector bundle structure of class $C^r_{\mathbb{K}}$ on E such that

$$\pi^{-1}(U) \to U \times \bigoplus_{i\in I} F_i, \quad E_x \ni (v_i)_{i\in I} \mapsto (x, (\mathrm{pr}_{F_i}(\psi_i(v_i)))_{i\in I})$$

is a local trivialisation for E, for each family $(\psi_i)_{i\in I}$ of local trivialisations as above. The latter makes E an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$. In fact, the $C^r_{\mathbb{K}}$-property of β can be checked using pairs of local trivialisations, as in the proofs of Propositions 5, 8, and 9. Then, apply Proposition 15, with F_i in place of E_i and $Y \times Z$ in place of Z. Thus,

Proposition 12. *In the preceding situation, $\bigoplus_{i\in I} E_i$ is an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$ over M.*

Remark 12. *If M is a $C^r_{\mathbb{R}}$-manifold, then every $x \in M$ has an open neighbourhood U which is $C^r_{\mathbb{R}}$-diffeomorphic to a convex open subset W in the modelling space Z of M. If W can be chosen $C^r_{\mathbb{R}}$-paracompact, then every $C^r_{\mathbb{R}}$-vector bundle over U is trivialisable (see [12] (Corollary 15.10)). The latter condition is satisfied, for example, if Z is finite-dimensional, a Hilbert space, or a countable direct limit of finite-dimensional vector spaces (and hence a nuclear Silva space), cf. [3] (Theorem 16.10 and Corollary 16.16). If $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and Z has finite dimension, then each finite-dimensional holomorphic vector bundle over a, say, polycylinder in Z is $C^\infty_{\mathbb{C}}$-trivialisable (cf. [33]). Under suitable hypotheses, holomorphic Banach vector bundles over contractible bases are $C^\infty_{\mathbb{C}}$-trivialisable as well [34].*

10. Dual Bundles and Cotangent Bundles

In this section, we discuss conditions ensuring that a vector bundle has a canonical dual bundle. Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$, and M be a $C^r_{\mathbb{K}}$-manifold modeled on a locally convex space Z.

Definition 7. *Let $\pi\colon E \to M$ be an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F. Consider the disjoint union*

$$E' := \bigcup_{x\in M} (E_x)'\,;$$

let $p\colon E' \to M$ be the map taking $\lambda \in (E_x)'$ to x, for each $x \in M$. Given $t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ such that $t \le r$, we say that E has a canonical dual bundle *of class $C^t_{\mathbb{K}}$ with respect to $\mathcal{S} \in \{b, c\}$ if E' can be made an $\mathbb{L}$-vector bundle of class $C^t_{\mathbb{K}}$ over M, with typical fibre $F'_{\mathcal{S}}$ and bundle projection p, such that*

$$\widetilde{\psi}\colon p^{-1}(U) \to U \times F'_{\mathcal{S}}\,,\quad (E')_x = (E_x)' \ni \lambda \mapsto (x, ((\mathrm{pr}_F \circ \psi|_{E_x})^{-1})'(\lambda)) \tag{15}$$

is a local trivialisation of E', for each local trivialisation $\psi\colon \pi^{-1}(U) \to U \times F$ of E.

To pinpoint situations where the dual bundle exists, we recall a fact concerning the formation of dual linear maps (see [8] (Proposition 16.30)):

Lemma 16. *Let E and F be locally convex spaces, and $\mathcal{S} \in \{b, c\}$. If the evaluation homomorphism $\eta_{F,\mathcal{S}}\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$, $\eta_{F,\mathcal{S}}(x)(\lambda) := \lambda(x)$ is continuous, then*

$$\Theta\colon L(E, F)_{\mathcal{S}} \to L(F'_{\mathcal{S}}, E'_{\mathcal{S}})_{\mathcal{S}}\,,\quad \alpha \mapsto \alpha'$$

is a continuous linear map.

Remark 13. *Let F be a locally convex $\mathbb{K}$-vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. It is known that $\eta_{F,b}$ is continuous if and only if F is quasi-barrelled, i.e., every bornivorous barrel in F is a 0-neighbourhood [35] (Proposition 2 in Section 11). In particular, $\eta_{F,b}$ is continuous if F is bornological or barrelled. It is also known that $\eta_{F,c}$ is continuous (and actually a topological embedding) if F is a $k_{\mathbb{R}}$-space. If $\mathbb{K} = \mathbb{R}$, this follows from [36] (Theorem 2.3) and [37] (Lemma 14.3) (cf. also [37] (Propositions 2.3 and 2.4)). If $\mathbb{K} = \mathbb{C}$ and F is a $k_{\mathbb{R}}$-space, then $\eta_{F_{\mathbb{R}},c}$ is a topological embedding for the real topological vector space $F_{\mathbb{R}}$ underlying F. Now, $(F'_c)_{\mathbb{R}} \cong (F_{\mathbb{R}})'$ as a real topological vector space, using that a continuous $\mathbb{C}$-linear functional $\lambda\colon F \to \mathbb{C}$ is determined by its real part. Transporting the complex vector space structure from F'_c to $(F_{\mathbb{R}})'$, the latter becomes a complex locally convex space. Thus,*

$((F'_c)'_c)_{\mathbb{R}}$ can be identified with $((F_{\mathbb{R}})'_c)'_c$, and it is easy to verify that $\eta_{F,c}$ corresponds to $\eta_{F_{\mathbb{R}},c}$ if we make the latter identification.

Proposition 13. *Let $\pi\colon E \to M$ be an $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F. Let $\mathcal{S} \in \{b,c\}$. If $\mathcal{S} = c$, let $r_- := r$; if $\mathcal{S} = b$, assume $r \geq 1$ and set $r_- := r-1$. Consider the following conditions:*

- (α) *The modelling space Z of M is finite-dimensional, $\eta_{F,\mathcal{S}}$ is continuous, and $F'_{\mathcal{S}}$ is barrelled.*
- (β) *$\eta_{F,\mathcal{S}}$ is continuous and, moreover, $(Z \times F'_{\mathcal{S}}) \times (Z \times F'_{\mathcal{S}})$ is a $k_{\mathbb{R}}$-space, or $r_- = 0$ and $Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space, or $(r,\mathbb{K}) = (\infty,\mathbb{C})$ and $Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space.*
- (γ) *F is normable.*

If (α) or (β) is satisfied with $\mathcal{S} = c$, then E has a canonical dual bundle of class $C^r_{\mathbb{K}}$ with respect to $\mathcal{S} = c$. If (α), (β), or (γ) is satisfied with $\mathcal{S} = b$, then E has a canonical dual bundle of class $C^{r-1}_{\mathbb{K}}$ with respect to $\mathcal{S} = b$.

For $\mathcal{S} = b$, condition (α) of Proposition 13 is satisfied, for example, if F is a reflexive locally convex space (then $\eta_{F,b}$ is continuous and F'_b is barrelled, being reflexive.)

Proof. Let E' be the disjoint union $\bigcup_{x\in M}(E_x)'$, and $p\colon E' \to M$ be as in Definition 7. Let $(\psi_i)_{i\in I}$ be a family such that the $\psi_i\colon \pi^{-1}(U_i) \to U_i \times F$ form the set of all local trivialisations of E. Let $(g_{ij})_{i,j\in I}$ be the associated cocycle (as explained before Proposition 3). Then, $G_{ij} := g_{ij}^{\wedge}$ is $C^r_{\mathbb{K}}$ and hence $g_{ij} = (G_{ij})^{\vee}$ is $C^{r_-}_{\mathbb{K}}$, by Proposition 1. Given $i \in I$, we define $\widetilde{\psi}_i\colon p^{-1}(U_i) \to U_i \times F'_{\mathcal{S}}$ as in (15), using ψ_i instead of ψ. Then,

$$\begin{aligned}\widetilde{\psi}_i(\widetilde{\psi}_j^{-1}(x,\lambda)) &= (x,((\mathrm{pr}_F \circ \psi_i|_{E_x})^{-1})' \circ (\mathrm{pr}_F \circ \psi_j|_{E_x})'(\lambda)) \\ &= (x,(\mathrm{pr}_F \circ \psi_j|_{E_x} \circ (\mathrm{pr}_F \circ \psi_i|_{E_x})^{-1})'(\lambda)) \;=\; (x, g_{ji}(x)'(\lambda))\end{aligned}$$

for all $x \in U_i \cap U_j$ and $\lambda \in F'$ shows that

$$(\widetilde{\psi}_i \circ \widetilde{\psi}_j^{-1})(x,\lambda) = (x, h_{ij}(x)(\lambda)),$$

where $h_{ij}(x) := g_{ji}(x)' \in \mathrm{GL}(F'_{\mathcal{S}})$. If ($\alpha$) or ($\beta$) holds, then $\eta_{F,\mathcal{S}}\colon F \to (F'_{\mathcal{S}})'_{\mathcal{S}}$ is continuous by hypothesis. If $\mathcal{S} = b$ and (γ) holds, then $\eta_{F,b}$ is an isometric embedding (as is well known) and hence continuous. Thus, $\Theta\colon L(F)_{\mathcal{S}} \to L(F'_{\mathcal{S}})_{\mathcal{S}}$, $\alpha \mapsto \alpha'$ is a continuous $\mathbb{L}$-linear map (Lemma 16). Since $g_{ji}\colon U_i \cap U_j \to L(F)_{\mathcal{S}}$ is $C^{r_-}_{\mathbb{K}}$, we deduce that $h_{ij} = \Theta \circ g_{ji}\colon U_i \cap U_j \to L(F'_{\mathcal{S}})_{\mathcal{S}}$ is $C^{r_-}_{\mathbb{K}}$. Thus Condition (g)$'$ of Corollary 2 is satisfied, with r_- in place of r. Conditions (a)–(f) being apparent, the cited corollary provides an $\mathbb{L}$-vector bundle structure of class $C^{r_-}_{\mathbb{K}}$ on E'. □

Without specific hypotheses, a canonical dual bundle need not exist.

Example 2. *Let A be a unital, associative, locally convex topological $\mathbb{K}$-algebra whose group of units $A^\times$ is open in A, and such that the inversion map $\iota\colon A^\times \to A^\times$ is continuous. Then, ι is smooth (and indeed $\mathbb{K}$-analytic); see, e.g., [13] (Propositions 10.1.12 and 10.1.13). We assume that the locally convex space underlying A is a non-normable Fréchet–Schwartz space and hence Montel, ensuring that $L(A)_b = L(A)_c$. For example, we might take $A := C^\infty(K,\mathbb{K})$, where K is a connected, compact, smooth manifold of positive dimension (cf. [13] (Lemma 10.2.2 (c))). Let $r,t \in \mathbb{N}_0 \cup \{\infty,\omega\}$ with $t \leq r$ and $\mathcal{S} \in \{b,c\}$. We consider the trivial vector bundle*

$$\mathrm{pr}_1\colon E := A^\times \times A \to A^\times.$$

(Thus, $E \cong TA^\times$, the tangent bundle). Then, E is a $\mathbb{K}$-vector bundle of class $C^r_{\mathbb{K}}$ over the base $A^\times$, with typical fibre A. Both $\psi_1 := \mathrm{id}\colon A^\times \times A \to A^\times \times A$ and $\psi_2\colon A^\times \times A \to A^\times \times A$, $(a,v) \mapsto (a,av)$ are global trivialisations of E. Identifying $E' := \bigcup_{a\in A^\times}(E_a)'$ with the set $A^\times \times A'$, we consider the associated bijections $\widetilde{\psi}_i\colon E' = A^\times \times A' \to A^\times \times A'$ for $i \in \{1,2\}$

(cf. (15)). Thus, $\widetilde{\psi}_1 = \mathrm{id}$, and $\widetilde{\psi}_2(a,\lambda) = (a, \lambda(a^{-1}\cdot))$ for $a \in A^\times$, $\lambda \in A'$. The map $G_{ij}\colon A^\times \times A \to A$, $(a,v) \mapsto \mathrm{pr}_2(\psi_i(\psi_j^{-1}(a,v)))$ is $C^r_{\mathbb{K}}$ for $i,j \in \{1,2\}$, where $\mathrm{pr}_2\colon A^\times \times A \to A$ is the projection onto the second factor. Then, also $g_{ij}\colon A^\times \to L(A)_c = L(A)_b$, $a \mapsto G_{ij}(a,\cdot)$ is $C^r_{\mathbb{K}}$, by Proposition 1 (a). Now, A being Fréchet and thus barrelled, the evaluation homomorphism $\eta_{A,b}$ is continuous; since A is metrisable and hence a k-space, also $\eta_{A,c}$ is continuous (see Remark 13). Since g_{ij} is $C^r_{\mathbb{K}}$, we deduce with Lemma 16 that also $h_{ij}\colon A^\times \to L(A'_{\mathcal{S}})_{\mathcal{S}}$, $a \mapsto (g_{ji}(a))'$ is $C^r_{\mathbb{K}}$. Define

$$H_{ij}\colon A^\times \times A'_{\mathcal{S}} \to A'_{\mathcal{S}} \quad (a,\lambda) \mapsto h_{ij}(a)(\lambda)$$

for $i,j \in \{1,2\}$. Then, H_{12} is discontinuous. To see this, we compose H_{12} with the map $\mathrm{ev}_1\colon A'_b \to \mathbb{K}$, $\lambda \mapsto \lambda(1)$, which evaluates functionals at the identity element $1 \in A$, and recall that ev_1 is continuous. Then, $\mathrm{ev}_1(H_{12}(a,\lambda)) = \lambda(g_{21}(a)(1)) = \lambda(a)$ for $a \in A^\times$ and $\lambda \in A'$. However, A being a non-normable locally convex space, the bilinear, separately continuous evaluation map $\varepsilon\colon A \times A'_b \to \mathbb{K}$, $(a,\lambda) \mapsto \lambda(a)$ is discontinuous, and hence so is its restriction $\varepsilon|_{A^\times \times A'_b} = \mathrm{ev}_1 \circ H_{12}$ to the non-empty open subset $A^\times \times A'_b$, as is readily verified. Now, $\mathrm{ev}_1 \circ H_{12}$ being discontinuous, also H_{12} is discontinuous (and therefore not $C^t_{\mathbb{K}}$). As a consequence, also $\widetilde{\psi}_1 \circ \widetilde{\psi}_2^{-1} = (\mathrm{pr}_1, H_{12})$ is discontinuous. Summing up:

There is no canonical vector bundle structure of class $C^t_{\mathbb{K}}$ on E' because the two vector bundle structures on E' making $\widetilde{\psi}_1$ (resp., $\widetilde{\psi}_2$) a global trivialisation do not coincide.

Remark 14. *In the preceding situation, set $M := A^\times$, $F := A'_b$, $I := \{1,2\}$, $U_i := M$ for $i \in I$, and $\pi := \mathrm{pr}_1\colon M \times F \to M$. If we let $M \times A'_b$ play the role of E in Proposition 3 and $\widetilde{\psi}_i\colon \pi^{-1}(U_i) \to U_i \times F$ the role of ψ_i in Proposition 3 (e), then all of Conditions (a)–(f) of Proposition 3 and Condition (g)′ of Corollary 2 are satisfied for $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ (with $\mathbb{L} := \mathbb{K}$). However, there is no $C^r_{\mathbb{K}}$-vector bundle structure on $M \times F$ making each $\widetilde{\psi}_i$ a trivialisation, as just observed, i.e., the conclusion of Corollary 2 becomes false.*

Remark 15. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N} \cup \{\infty, \omega\}$, $t \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $t \le r$ and M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex space Z. Then, the tangent bundle TM is a $\mathbb{K}$-vector bundle of class $C^{r-1}_{\mathbb{K}}$ over M, with typical fibre Z. Pick a locally convex vector topology $\mathcal{T}$ on Z'. Let $\mathcal{A}$ be the set of all maps $\widetilde{\psi}$ as in (15), with $(Z', \mathcal{T})$ in place of $F'_{\mathcal{S}}$, for ψ ranging through the set of all local trivialisations of TM (alternatively, only those of the form $(\pi_{TU}, d\phi)$ for charts $\phi\colon U \to V \subseteq Z$ of M, using the bundle projection $\pi_{TU}\colon TU \to U$). Let us say that M has a* canonical cotangent bundle *of class $C^t_{\mathbb{K}}$ with respect to $\mathcal{T}$ if $T'M := \bigcup_{x \in M} (T_x M)'$ admits a $\mathbb{K}$-vector bundle structure of class $C^t_{\mathbb{K}}$ over M with typical fibre $(Z', \mathcal{T})$, which makes each $\widetilde{\psi}\colon p^{-1}(U) \to U \times (Z', \mathcal{T})$ a local trivialisation (with $p\colon T'M \to M$, $(T_x M)' \ni \lambda \mapsto x$). Then, the evaluation map*

$$\varepsilon\colon (Z', \mathcal{T}) \times Z \to \mathbb{K}, \quad (\lambda, x) \mapsto \lambda(x)$$

must be continuous and hence Z normable. For $\mathbb{K} = \mathbb{R}$, this is explained in [17] (Remark 1.3.9) (written after Example 2 was found) if $r = \infty$. This implies the case $r \in \mathbb{N}$. As the diffeomorphism f employed as a change of charts is real analytic, the case $(\omega, \mathbb{R})$ follows and also the complex case, using a $\mathbb{C}$-analytic extension of f. When $\mathcal{T}$ is the compact-open topology, existence of a canonical cotangent bundle for M even implies that Z is finite-dimensional. (If ε is continuous on $Z'_c \times Z$, then there exists a compact subset $K \subseteq Z$ and a 0-neighbourhood $W \subseteq Z$ such that $\varepsilon((K^\circ) \times W) \subseteq \mathbb{D}$. Hence, $K^\circ \subseteq W^\circ$. Since K° is a 0-neighbourhood in Z'_c and W° compact (by Ascoli's Theorem), Z'_c is locally compact and hence finite-dimensional. As Z'_c separates points on Z, also Z must be finite-dimensional.)

Cotangent bundles are not needed to define 1-forms on an infinite-dimensional manifold M. Following [38], these can be considered as smooth maps on TM which are linear on the fibres (and a similar remark applies to differential forms of higher order).

Differentiability properties of the G-action on the dual bundle. Let $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{L}\}$, $s \in \{\infty, \omega\}$, $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ with $r \le s$, and G be a $C^s_{\mathbb{K}}$-Lie group modelled on a locally convex $\mathbb{K}$-vector space Y. Let M be a $C^r_{\mathbb{K}}$-manifold modelled on a locally convex $\mathbb{K}$-vector space Z and $\alpha\colon G \times M \to M$ be a G-action of class $C^r_{\mathbb{K}}$.

Proposition 14. *Let $\pi\colon E \to M$ be an equivariant $\mathbb{L}$-vector bundle of class $C^r_{\mathbb{K}}$, with typical fibre F and G-action $\beta\colon G \times E \to E$ of class $C^r_{\mathbb{K}}$. Let $\mathcal{S} \in \{b, c\}$. If $\mathcal{S} = c$, set $r_- := r$; if $\mathcal{S} = b$, assume $r \geq 1$ and set $r_- := r - 1$. Consider the following conditions:*

(a) *$\eta_{F,\mathcal{S}}$ is continuous, and, moreover, $(Y \times Z \times F'_{\mathcal{S}}) \times (Y \times Z \times F'_{\mathcal{S}})$ is a $k_{\mathbb{R}}$-space, or $r_- = 0$ and $Y \times Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space, or $(r, \mathbb{K}) = (\infty, \mathbb{C})$ and $Y \times Z \times F'_{\mathcal{S}}$ is a $k_{\mathbb{R}}$-space;*
(b) *M and G are finite-dimensional, $\eta_{F,\mathcal{S}}$ is continuous, and $F'_{\mathcal{S}}$ is barrelled; or*
(c) *F is normable.*

If $\mathcal{S} = c$ and (a) *or* (b) *holds, then E has a canonical dual bundle E' of class $C^{r_-}_{\mathbb{K}}$ with respect to $\mathcal{S}$, and the map $\beta^*\colon G \times E' \to E'$, defined using adjoint linear maps via*

$$\beta^*(g, \lambda) := (\beta(g^{-1}, \cdot)|^{E_x}_{E_{\alpha(g,x)}})'(\lambda)$$

for $g \in G$, $\lambda \in (E_x)'$, turns E' into an equivariant $\mathbb{L}$-vector bundle of class $C^{r_-}_{\mathbb{K}}$ over the G-manifold M. If $\mathcal{S} = b$ and (a), (b), *or* (c) *is satisfied, then the same conclusion holds.*

Proof. In view of Proposition 13, the hypotheses imply that E has a canonical dual bundle $p\colon E' \to M$ of class $C^{r_-}_{\mathbb{K}}$. It is apparent that $\beta^*\colon G \times E' \to E'$ is an action, and E'_x is taken $\mathbb{L}$-linearly to $E'_{\alpha(g,x)}$ by $\beta^*(g, \cdot)$, for each $g \in G$ and $x \in M$. It therefore only remains to show that β^* is $C^{r_-}_{\mathbb{K}}$. To this end, let $g_0 \in G$ and $x_0 \in M$; we show that β^* is $C^{r_-}_{\mathbb{K}}$ on $U \times p^{-1}(V)$, for some open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M. Indeed, there exists a local trivialisation $\psi\colon \pi^{-1}(W) \to W \times F$ of E over an open neighbourhood W of $\alpha(g_0, x_0)$ in M. The action α being continuous, we find an open neighbourhood U of g_0 in G and an open neighbourhood V of x_0 in M over which E is trivial, such that $\alpha(U \times V) \subseteq W$. Let $\phi\colon \pi^{-1}(V) \to V \times F$ be a local trivialisation of E over V. Then

$$\phi(\beta(g^{-1}, \psi^{-1}(\alpha(g, x), v))) = (x, A(g, x, v)) \quad \text{for all } g \in U, x \in V, \text{and } v \in F,$$

for a $C^r_{\mathbb{K}}$-map $A\colon U \times V \times F \to F$, which is $\mathbb{L}$-linear in the third argument. By Corollary 1, the map $a\colon U \times V \to L(F)_{\mathcal{S}}$, $(g, x) \mapsto A(g, x, \cdot)$ is $C^{r_-}_{\mathbb{K}}$. In view of the hypotheses, Lemmas 16 and 13 entail that also $a^*\colon U \times V \to L(F'_{\mathcal{S}})_{\mathcal{S}}$, $(g, x) \mapsto (a(g, x))'$ is $C^{r_-}_{\mathbb{K}}$-map. Now, again using the specific hypotheses, Proposition 2 shows that also the mapping $A^*\colon U \times V \times F'_{\mathcal{S}} \to F'_{\mathcal{S}}$, $(g, x, \lambda) \mapsto a^*(g, x)(\lambda)$ is $C^{r_-}_{\mathbb{K}}$. However, for $g \in U$, $x \in V$, and $\lambda \in F'$, we calculate

$$\begin{aligned}
\widetilde{\psi}(\beta^*(g, \widetilde{\phi}^{-1}(x, \lambda))) &= \left(\alpha(g, x), \left(\mathrm{pr}_F \circ \phi|_{E_x} \circ \beta(g^{-1}, \cdot)|^{E_x}_{E_{\alpha(g,x)}} \circ (\mathrm{pr}_F \circ \psi|_{E_{\alpha(g,x)}})^{-1}\right)'(\lambda)\right) \\
&= (\alpha(g, x), A^*(g, x, \lambda)),
\end{aligned}$$

using the notation as in (15). We conclude that $\beta^*|_{U \times p^{-1}(V)}$ is $C^{r_-}_{\mathbb{K}}$. □

Example 3. *For elementary examples, recall that the group* $\mathrm{Diff}(M)$ *of all smooth diffeomorphisms of a σ-compact, finite-dimensional smooth manifold M can be made a smooth Lie group, modelled on the (LF)-space $\Gamma_c(TM)$ of compactly supported smooth vector fields on M (see [13,15]). The natural action* $\mathrm{Diff}(M) \times M \to M$ *is smooth [13]. In view of Example 1, Proposition 14* (b)*, Proposition 8 and Proposition 4, we readily deduce that also the natural action of* $\mathrm{Diff}(M)$ *on TM is smooth, as well as the natural actions on $T^*M := (TM)'$, $TM^{\otimes n} \otimes (T^*M)^{\otimes m}$ for all $n, m \in \mathbb{N}_0$, and the natural action on the subbundles $S^n(T^*M)$ and $\bigwedge^n T^*M$ of $(T^*M)^{\otimes n}$ given by symmetric and exterior powers, respectively.*

11. Locally Convex Poisson Vector Spaces

We discuss a slight generalisation of the concept of a locally convex Poisson vector space introduced in [8]. Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

A bounded set-functor $\mathcal{S}$ associates with each locally convex $\mathbb{K}$-vector space E a set $\mathcal{S}(E)$ of bounded subsets of E, such that $\{\lambda(M) : M \in \mathcal{S}(E)\} \subseteq \mathcal{S}(F)$ for each continuous $\mathbb{K}$-linear map $\lambda : E \to F$ between locally convex $\mathbb{K}$-vector spaces (cf. [8] (Definition 16.15)). Given locally convex $\mathbb{K}$-vector spaces E and F, we shall write $L(E,F)_{\mathcal{S}}$ as a shorthand for $L_{\mathbb{K}}(E,F)_{\mathcal{S}(E)}$. We write $E'_{\mathcal{S}} := L_{\mathbb{K}}(E,\mathbb{K})_{\mathcal{S}}$.

Throughout this section, we let $\mathcal{S}$ be a bounded set-functor such that, for each locally convex space E, we have

$$\{K \subseteq E : K \text{ is compact}\} \subseteq \mathcal{S}(E). \tag{16}$$

Then, $\{x\} \in \mathcal{S}(E)$ for each $x \in E$, and we get a continuous linear point evaluation

$$\eta_{E,\mathcal{S}}(x) : E'_{\mathcal{S}} \to \mathbb{K}, \quad \lambda \mapsto \lambda(x).$$

Definition 8. *A* locally convex Poisson vector space *with respect to $\mathcal{S}$ is a locally convex $\mathbb{K}$-vector space E such that $E \times E$ is a $k_{\mathbb{R}}$-space and*

$$\eta_{E,\mathcal{S}} : E \to (E'_{\mathcal{S}})'_{\mathcal{S}}, \quad x \mapsto \eta_{E,\mathcal{S}}(x)$$

a topological embedding, together with a bilinear map $[.,.] : E'_{\mathcal{S}} \times E'_{\mathcal{S}} \to E'_{\mathcal{S}}$, $(\lambda,\eta) \mapsto [\lambda,\eta]$, which makes $E'_{\mathcal{S}}$ a Lie algebra, is $\mathcal{S}(E'_{\mathcal{S}})$-hypocontinuous in its second argument, and satisfies

$$\eta_{E,\mathcal{S}}(x) \circ \mathrm{ad}_{\lambda} \in \eta_{E,\mathcal{S}}(E) \quad \textit{for all } x \in E \textit{ and } \lambda \in E', \tag{17}$$

writing $\mathrm{ad}_{\lambda} := \mathrm{ad}(\lambda) := [\lambda, .] : E' \to E'$.

Remark 16. *(a) Definition 16.35 in [8] was more restrictive; E was assumed to be a k^{∞}-space there.*

(b) In [8] (16.31 (b)), the following additional condition was imposed: For each $M \in \mathcal{S}(E'_{\mathcal{S}})$ and $N \in \mathcal{S}(E)$, the set $\varepsilon(M \times N)$ is bounded in $\mathbb{K}$, where $\varepsilon : E' \times E \to \mathbb{K}$ is the evaluation map. As we assume (16), the latter condition is automatically satisfied, by [8] (Proposition 16.11 (a) and Proposition 16.14).

(c) Let us say that a locally convex space E is $\mathcal{S}$-reflexive *if $\eta_{E,\mathcal{S}} : E \to (E'_{\mathcal{S}})'_{\mathcal{S}}$ is an isomorphism of topological vector spaces.*

(d) Of course, we are mostly interested in the case where $[.,.]$ is continuous, but only hypocontinuity is required for the basic theory.

Definition 9. *Let $(E,[.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S}$, and $U \subseteq E$ be open. Given $f, g \in C^{\infty}_{\mathbb{K}}(U,\mathbb{K})$, we define a function $\{f,g\} : U \to \mathbb{K}$ via*

$$\{f,g\}(x) := \langle [f'(x), g'(x)], x \rangle \quad \textit{for } x \in U, \tag{18}$$

where $\langle .,. \rangle : E' \times E \to \mathbb{K}$, $\langle \lambda, x \rangle := \lambda(x)$ is the evaluation map and $f'(x) = df(x,.)$.

Condition (17) in Definition 8 enables us to define a map $X_f : U \to E$ via

$$X_f(x) := \eta^{-1}_{E,\mathcal{S}}(\eta_{E,\mathcal{S}}(x) \circ \mathrm{ad}(f'(x))) \quad \textit{for } x \in U. \tag{19}$$

Using Lemma 11 instead of [8] (Theorem 16.26), we see as in the proof of [8] (Theorem 16.40 (a)) that the function $\{f,g\} : U \to \mathbb{K}$ is $C^{\infty}_{\mathbb{K}}$. The $C^{\infty}_{\mathbb{K}}$-function $\{f,g\}$ is called the *Poisson bracket* of f and g. Using Lemma 11 instead of [8] (Theorem 16.26), we see as in the proof of [8] (Theorem 16.40 (b)) that $X_f : U \to E$ is a $C^{\infty}_{\mathbb{K}}$-map; it is called the *Hamiltonian vector field* associated with f. As in [8] (Remark 16.43), we see that the Poisson bracket just defined makes $C^{\infty}_{\mathbb{K}}(U,\mathbb{K})$ a Poisson algebra.

We shall write "b" and "c" in place of $\mathcal{S}$ if $\mathcal{S}$ is the bounded set functor, taking a locally convex space E to the set $\mathcal{S}(E)$ of all bounded subsets and compact subsets of E, respectively. Both of these satisfy the hypothesis (16).

In the following, we describe new results for locally convex Poisson vector spaces over $\mathcal{S} = c$. We mention that the embedding property of $\eta_{E,c}$ is automatic in this case, as $E \times E$ is a $k_{\mathbb{R}}$-space in Definition 9; thus, E is a $k_{\mathbb{R}}$-space and Remark 13 applies.

Example 4. *Let $(\mathfrak{g}_j)_{j\in J}$ be a family of finite-dimensional real Lie algebras $\mathfrak{g}_j$. Endow $\mathfrak{g} := \bigoplus_{j\in J} \mathfrak{g}_j$ with the locally convex direct sum topology, which coincides with the finest locally convex vector topology. Then, $\mathfrak{g}$ is c-reflexive, as with every vector space with its finest locally convex vector topology (see [39] (Theorem 7.30 (a))). As a consequence, also $\mathfrak{g}'_c$ is c-reflexive (cf. [39] (Proposition 7.9 (iii))). Using [40] (Proposition 7.1), we see that the component-wise Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous on the locally convex space $\mathfrak{g} \times \mathfrak{g}$, which is naturally isomorphic to the locally convex direct sum $\bigoplus_{j\in J}(\mathfrak{g}_j \times \mathfrak{g}_j)$. We set $E := \mathfrak{g}'_c$ and give E'_c the continuous Lie bracket $[.,.]$ making $\eta_{\mathfrak{g},c}\colon \mathfrak{g} \to (\mathfrak{g}'_c)'_c = E'_c$ an isomorphism of topological Lie algebras. Then*

$$E = \mathfrak{g}'_c \cong \prod_{j\in J} (\mathfrak{g}_j)'_c$$

and $E \times E$ are $k_{\mathbb{R}}$-spaces, being Cartesian products of locally compact spaces (see [22]). Thus, $(E, [.,.])$ is a locally convex Poisson vector space over $\mathcal{S} = c$, in the sense of Definition 8. If J has cardinality $\geq 2^{\aleph_0}$ and $\mathfrak{g}_j \neq \{0\}$ for all $j \in J$ (e.g., if we take an abelian 1-dimensional Lie algebra $\mathfrak{g}_j$ for each $j \in J$), then $E \cong \mathbb{R}^J$ is not a k-space. Hence, E is not a k^∞-space, and hence it is not a Poisson vector space in the more restrictive sense of [8].

12. Continuity Properties of the Poisson Bracket

If E and F are locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ an open subset, we endow $C^\infty(U, F)$ with the compact-open C^∞-topology. Our goal is the following result:

Theorem 1. *Let $(E, [.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be open. Then, the Poisson bracket*

$$\{.,.\}\colon C^\infty_{\mathbb{K}}(U, \mathbb{K}) \times C^\infty_{\mathbb{K}}(U, \mathbb{K}) \to C^\infty_{\mathbb{K}}(U, \mathbb{K})$$

is c-hypocontinuous in its second variable. If $[.,.]\colon E'_c \times E'_c \to E'_c$ is continuous, then also the Poisson bracket is continuous.

Various auxiliary results are needed to prove Theorem 1. With little risk of confusion with subsets of spaces of operators, given a 0-neighbourhood $W \subseteq F$ and a compact set $K \subseteq U$, we shall write $\lfloor K, W \rfloor := \{f \in C(U, F)\colon f(K) \subseteq W\}$.

Lemma 17. *Let E, F be locally convex spaces and $U \subseteq E$ be open. Then, the linear map*

$$D\colon C^\infty_{\mathbb{K}}(U, F) \to C^\infty_{\mathbb{K}}(U, L(E, F)_c)\,, \quad f \mapsto f'$$

is continuous.

Proof. By Corollary 1, $f' \in C^\infty_{\mathbb{K}}(U, L(E, F)_c)$ for each $f \in C^\infty_{\mathbb{K}}(U, F)$. As D is linear and also $C^\infty(U, L(E, F)_c) \to C(U \times E^k, L(E, F)_c)$, $f \mapsto d^k f$ is linear for each $k \in \mathbb{N}_0$,

$$d^k \circ D\colon C^\infty(U, F) \to C(U \times E^k, L(E, F)_c)_{c.o.} \tag{20}$$

is linear, whence it will be continuous if it is continuous at 0. We pick a typical 0-neighbourhood in $C(U \times E^k, L(E, F)_c)_{c.o.}$, say $\lfloor K, V \rfloor$ with a compact subset $K \subseteq U \times E^k$ and a 0-neighbourhood $V \subseteq L(E, F)_c$. After shrinking V, we may assume that $V = \lfloor A, W \rfloor$ for some compact set $A \subseteq E$ and 0-neighbourhood $W \subseteq F$.

We now recall that for $f \in C^\infty_{\mathbb{K}}(U,F)$, we have

$$d^k(f')(x,y_1,\ldots,y_k) \;=\; d^{k+1}f(x,y_1,\ldots,y_k,\cdot)\colon E \to F \tag{21}$$

for all $k \in \mathbb{N}_0$, $x \in U$ and $y_1,\ldots,y_k \in E$ (cf. Corollary 1). Since $\lfloor K \times A, W \rfloor$ is an open 0-neighbourhood in $C(U \times E^{k+1}, F)$ and the map $C^\infty(U,F) \to C(U \times E^{k+1}, F)_{c.o.}$, $f \mapsto d^{k+1}f$ is continuous, we see that the set Ω of all $f \in C^\infty(U,F)$ such that $d^{k+1}f \in \lfloor K \times A, W\rfloor$ is a 0-neighbourhood in $C^\infty(U,F)$. In view of (21), we have $d^k(f') \in \lfloor K, \lfloor A, W \rfloor \rfloor$ for each $f \in \Omega$. Hence, $d^k \circ D$ from (20) is continuous at 0, as required. □

Lemma 18. *Let X be a Hausdorff topological space, F be a locally convex space, $K \subseteq X$ be compact and $M \subseteq C(X,F)_{c.o.}$ be compact. Let $\mathrm{ev}\colon C(X,F) \times X \to F$, $(f,x) \mapsto f(x)$ be the evaluation map. Then, $\mathrm{ev}(M \times K)$ is compact.*

Proof. The map $\rho\colon C(X,F)_{c.o.} \to C(K,F)_{c.o.}$, $f \mapsto f|_K$ is continuous by [20] (§3.2 (2)). Thus, $\rho(M)$ is compact in $C(K,F)_{c.o.}$. The map $\varepsilon\colon C(K,F) \times K \to F$, $(f,x) \mapsto f(x)$ is continuous by [20] (Theorem 3.4.2). Hence, $\mathrm{ev}(M \times K) = \varepsilon(\rho(M) \times K)$ is compact. □

Lemma 19. *Let E, F_1, F_2, and G be locally convex $\mathbb{K}$-vector spaces and $\beta\colon F_1 \times F_2 \to G$ be a bilinear map which is c-hypocontinuous in its second argument. Let $U \subseteq E$ be an open subset and $r \in \mathbb{N}_0 \cup \{\infty\}$. Assume that $E \times E$ is a $k_{\mathbb{R}}$-space, or $r = 0$ and E is a $k_{\mathbb{R}}$-space, or $(r,\mathbb{K}) = (\infty,\mathbb{C})$ and E is a $k_{\mathbb{R}}$-space. Then, the following holds:*

(a) *We have $\beta \circ (f,g) \in C^r_{\mathbb{K}}(U,G)$ for all $(f,g) \in C^r_{\mathbb{K}}(U,F_1) \times C^r_{\mathbb{K}}(U,F_2)$. The map*

$$C^r_{\mathbb{K}}(U,\beta)\colon C^r_{\mathbb{K}}(U,F_1) \times C^r_{\mathbb{K}}(U,F_2) \to C^r_{\mathbb{K}}(U,G),\;\; (f,g) \mapsto \beta \circ (f,g)$$

is bilinear. For each compact subset $M \subseteq C^r_{\mathbb{K}}(U,F_2)$ and 0-neighbourhood $W \subseteq C^r_{\mathbb{K}}(U,G)$, there is a 0-neighbourhood $V \subseteq C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(V \times M) \subseteq W$.

(b) *For each $g \in C^r_{\mathbb{K}}(U,F_2)$, the map $C^r_{\mathbb{K}}(U,F_1) \to C^r_{\mathbb{K}}(U,G)$, $f \mapsto \beta \circ (f,g)$ is continuous and linear.*

(c) *If β is also c-hypocontinuous in its first argument, then $C^r_{\mathbb{K}}(U,\beta)$ is c-hypocontinuous in its second argument and c-hypocontinuous in its first argument.*

(d) *If β is continuous, then $C^r_{\mathbb{K}}(U,\beta)$ is continuous.*

Proof. (a) By Lemma 11, $\beta \circ (f,g) \in C^r_{\mathbb{K}}(U,G)$. The bilinearity of $C^r(U,\beta)$ is clear. It suffices to prove the remaining assertion for each $r \in \mathbb{N}_0$. To see this, let $M \subseteq C^\infty_{\mathbb{K}}(U,F_2)$ be a compact subset and $W \subseteq C^\infty_{\mathbb{K}}(U,G)$ be a 0-neighbourhood. Since the topology on $C^\infty_{\mathbb{K}}(U,G)$ is initial with respect to the family of inclusion maps $C^\infty_{\mathbb{K}}(U,G) \to C^r_{\mathbb{K}}(U,G)$ for $r \in \mathbb{N}_0$, there exists $r \in \mathbb{N}_0$ and a 0-neighbourhood Q in $C^r_{\mathbb{K}}(U,G)$ such that $C^\infty_{\mathbb{K}}(U,G) \cap Q \subseteq W$. If the assertion holds for r, we find a 0-neighbourhood $P \subseteq C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(P \times M) \subseteq Q$. Then, $V := C^\infty_{\mathbb{K}}(U,F_1) \cap P$ is a 0-neighbourhood in $C^\infty_{\mathbb{K}}(U,F_1)$ and $C^\infty_{\mathbb{K}}(U,\beta)(V \times M) \subseteq C^\infty_{\mathbb{K}}(U,G) \cap C^r_{\mathbb{K}}(U,\beta)(P \times M) \subseteq C^\infty_{\mathbb{K}}(U,G) \cap Q \subseteq W$.

The case $r = 0$. Let $M \subseteq C(U,F_2)$ be compact and $W \subseteq C(U,G)$ be a 0-neighbourhood. Then, $\lfloor K, Q \rfloor \subseteq W$ for some compact subset $K \subseteq U$ and some 0-neighbourhood $Q \subseteq G$. By Lemma 18, the set $N := \mathrm{ev}(M \times K) \subseteq F_2$ is compact, where $\mathrm{ev}\colon C(U,F_2) \times U \to F_2$ is the evaluation map. Since β is c-hypocontinuous in its second argument, there exists a 0-neighbourhood $P \subseteq F_1$ with $\beta(P \times N) \subseteq Q$. Then, $\beta \circ (\lfloor K, P \rfloor \times M) \subseteq \lfloor K, Q \rfloor \subseteq W$.

Induction step. Let $M \subseteq C^r_{\mathbb{K}}(U,F_2)$ be a compact subset and $W \subseteq C^r_{\mathbb{K}}(U,G)$ be a 0-neighbourhood. The topology on $C^r(U,G)$ is initial with respect to the linear maps $\lambda_1\colon C^r_{\mathbb{K}}(U,G) \to C(U,G)_{c.o.}$, $f \mapsto f$ and $\lambda_2\colon C^r_{\mathbb{K}}(U,G) \to C^{r-1}_{\mathbb{K}}(U \times E, G)$, $f \mapsto df$ (by [26] (Lemma A.1 (d))). Note that the ordinary C^r-topology is used there, by [26] (Proposition 4.19 (d) and Lemma A2). After shrinking W, we may therefore assume that

$$W = (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2)$$

with absolutely convex 0-neighbourhoods $W_1 \subseteq C(U,G)$ and $W_2 \subseteq C^{r-1}_{\mathbb{K}}(U \times E, G)$. Applying the case $r = 0$ to $C(U,\beta)$, we find a 0-neighbourhood $V_1 \subseteq C(U,F_1)$ such that $C(U,\beta)(V_1 \times M) \subseteq W_1$. The map $\delta_j\colon C^r_{\mathbb{K}}(U,F_j) \to C^{r-1}_{\mathbb{K}}(U \times E, F_j)$, $f \mapsto df$ is continuous linear and $\pi\colon U \times E \to U$, $(x,y) \mapsto x$ is smooth, whence $\rho_j\colon C^r_{\mathbb{K}}(U,F_j) \to C^{r-1}_{\mathbb{K}}(U \times E, F_j)$, $f \mapsto f \circ \pi$ is continuous linear (cf. [26] (Lemma 4.4) or [13] (Proposition 1.7.11)). By (5),

$$\lambda_2 \circ C^r_{\mathbb{K}}(U,\beta) \;=\; C^{r-1}_{\mathbb{K}}(U \times E,\beta) \circ (\delta_1 \times \rho_2) \;+\; C^{r-1}_{\mathbb{K}}(U \times E,\beta) \circ (\rho_1 \times \delta_2)\,. \tag{22}$$

The subsets $\rho_2(M) \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_2)$ and $\delta_2(M) \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_2)$ are compact. Using the case $r-1$ (with $U \times E$ in place of U), which holds as the inductive hypothesis, we find 0-neighbourhoods $V_2, V_3 \subseteq C^{r-1}_{\mathbb{K}}(U \times E, F_1)$ such that $C^{r-1}_{\mathbb{K}}(U,\beta)(V_2 \times \rho_2(M)) \subseteq (1/2)W_2$ and $C^{r-1}_{\mathbb{K}}(U,\beta)(V_3 \times \delta_2(M)) \subseteq (1/2)W_2$. Then, $Q := (\delta_1)^{-1}(V_2) \cap (\rho_1)^{-1}(V_3)$ is an open 0-neighbourhood in $C^r_{\mathbb{K}}(U,F_1)$. Since $(1/2)W_2 + (1/2)W_2 = W_2$, we deduce from (22) that

$$\lambda_2(C^r_{\mathbb{K}}(U,\beta)(Q \times M)) \subseteq C^{r-1}_{\mathbb{K}}(U \times E,\beta)(V_2 \times \rho_2(M)) + C^{r-1}_{\mathbb{K}}(U \times E,\beta)(V_3 \times \delta_2(M)) \subseteq W_2.$$

Thus, $C^r_{\mathbb{K}}(U,\beta)(Q \times M) \subseteq (\lambda_2)^{-1}(W_2)$. Now, $V := V_1 \cap Q$ is a 0-neighbourhood in $C^r_{\mathbb{K}}(U,F_1)$ such that $C^r_{\mathbb{K}}(U,\beta)(V \times M) \subseteq (\lambda_1)^{-1}(W_1) \cap (\lambda_2)^{-1}(W_2) = W$.

(b) Since $C^r_{\mathbb{K}}(U,\beta)$ is bilinear, the map $f \mapsto \beta \circ (f,g)$ is linear. Its continuity follows from (a), applied with the singleton $M := \{g\}$.

(c) By (a) just established, the condition in Lemma 4 (a) is satisfied. By (b), the map $C^r_{\mathbb{K}}(U,\beta)$ is continuous in its first argument. Interchanging the roles of F_1 and F_2, we see that $C^r_{\mathbb{K}}(M,\beta)$ is also continuous in its second argument and hence c-hypocontinuous in its second argument. Likewise, $C^r_{\mathbb{K}}(U,\beta)$ is c-hypoocontinuous in its first argument.

(d) If β is continuous and hence smooth, then $C^r(U,\beta)$ is smooth and hence continuous, as a very special case of [26] (Proposition 4.16) or [13] (Corollary 1.7.13). □

Proof of Theorem 1. By Lemma 17, the mapping $D\colon C^\infty(U,\mathbb{K}) \to C^\infty(U,E'_c)$, $f \mapsto f'$ is continuous and linear. By Lemma 19 (c), the bilinear map

$$C^\infty(U,[.,.])\colon C^\infty(U,E') \times C^\infty(U,E') \to C^\infty(U,E')\,, \quad (f,g) \mapsto (x \mapsto [f(x),g(x)])$$

is c-hypocontinuous in its second argument; if $[.,.]$ is continuous, then also $C^\infty(U,[.,.])$, by Lemma 19 (d). The evaluation map $\beta\colon E \times E'_c \to \mathbb{K}$, $(x,\lambda) \mapsto \lambda(x)$ is c-hypocontinuous in its first argument, by Proposition 7. As a consequence, $\beta_*\colon C^\infty(U,E'_c) \to C^\infty(U,\mathbb{K})$, $f \mapsto \beta \circ (\mathrm{id}_U, f)$ is continuous linear by Lemma 19 (b). Since

$$\{.,.\} \;=\; \beta_* \circ C^\infty(U,[.,.]) \circ (D \times D)$$

by definition, we see that $\{.,.\}$ is a composition of continuous maps if $[.,.]$ is continuous, and hence continuous. In the general case, $\{.,.\}$ is a composition of a bilinear map which is c-hypocontinuous in its second argument and continuous linear maps, whence $\{.,.\}$ is c-hypocontinuous in its second arguemnt. □

13. Continuity of the Map Taking f to the Hamiltonian Vector Field X_f

In this section, we show the continuity of the mapping which takes a smooth function to the corresponding Hamiltonian vector field, in the case $\mathcal{S} = c$.

Theorem 2. *Let $(E,[.,.])$ be a locally convex Poisson vector space with respect to $\mathcal{S} = c$. Let $U \subseteq E$ be an open subset. Then, the map*

$$\Psi\colon C^\infty_{\mathbb{K}}(U,\mathbb{K}) \to C^\infty_{\mathbb{K}}(U,E)\,, \quad f \mapsto X_f \tag{23}$$

is continuous and linear.

Proof. Let $\eta_E\colon E \to (E'_c)'_c$ be the evaluation homomorphism and $V := \{A \in L(E'_c, E'_c)\colon (\forall x \in E)\ \eta_E(x) \circ A \in \eta_E(E)\}$. Then, V is a vector subspace of $L(E'_c, E'_c)$ and $\operatorname{ad}(E') \subseteq V$. The composition map $\Gamma\colon (E'_c)'_c \times L(E'_c, E'_c)_c \to (E'_c)'_c$, $(\alpha, A) \mapsto \alpha \circ A$ is hypocontinuous with respect to equicontinuous subsets of $(E'_c)'_c$, by Proposition 9 in [11] (Chapter III, §5, no. 5). If $K \subseteq E$ is compact, then the polar K° is a 0-neighbourhood in E'_c, entailing that $(K^\circ)^\circ \subseteq (E'_c)'$ is equicontinuous. Hence, η_E takes compact subsets of E to equicontinuous subsets of $(E'_c)'$, and hence

$$\beta\colon E \times V \to E\,, \quad (x, A) \mapsto \eta_E^{-1}(\Gamma(\eta_E(x), A))$$

is c-hypocontinuous in its first argument. By Lemma 19 (c), $\beta_*\colon C^\infty(U, V) \to C^\infty(U, E)$, $f \mapsto \beta \circ (\mathrm{id}_U, f)$ is continuous linear. Moreover, the map $D\colon C^\infty(U, \mathbb{K}) \to C^\infty(U, E'_c)$, $f \mapsto f'$ is continuous linear by Lemma 17. Furthermore, $\operatorname{ad} = [.,.]^\vee\colon E'_c \to L(E'_c, E'_c)_c$ is continuous linear since $[.,.]$ is c-hypocontinuous in its second argument (see Lemma 4 (b)), whence

$$C^\infty(U, \operatorname{ad})\colon C^\infty(U, E'_c) \to C^\infty(U, L(E'_c, E'_c)_c)\,, \quad f \mapsto \operatorname{ad} \circ f$$

is continuous linear (see, e.g., [26] (Lemma 4.13), or [13] (Corollary 1.7.13)). Hence, $\Psi = \beta_* \circ C^\infty(U, \operatorname{ad}) \circ D$ is continuous and linear. □

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Appendix A. Proofs for Some Basic Facts

We give proofs for various facts stated in Section 2.

Proof of Lemma 1. Let $E := E_1 \times \cdots \times E_k$. Since $df\colon U \times E \times X \times E \to F$ is continuous and $df(x, 0, 0, 0) = 0$, given q, there exists a continuous seminorm p on X such that $B_1^p(x) \subseteq U$, and continuous seminorms p_j on E_j for $j \in \{1, \ldots, k\}$ such that

$$\|df(y, v_1, \ldots, v_k, z, w_1, \ldots, w_k)\|_q \leq 1 \tag{A1}$$

for all $v_j, w_j \in B_1^{p_j}(0)$, $y \in B_1^p(x)$, and $z \in B_1^p(0)$. For $y \in B_1^p(x)$ and $(v_1, \ldots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$, the Mean Value Theorem (see [13] (Proposition 1.2.6)) shows that

$$f(y, v_1, \ldots, v_k) = \int_0^1 df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k)\, dt\,.$$

Since $\|df(y, tv_1, \ldots, tv_k, 0, v_1, \ldots, v_k)\|_q \leq 1$ for each t, it follows that $\|f(y, v_1, \ldots, v_k)\|_q \leq 1$ in the preceding situation. Because $f(y, \cdot)$ is k-linear, we deduce that (1) holds. To prove (2), we first note that (A1) implies that

$$\|df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)\|_q \leq \|z\|_p \tag{A2}$$

for all $y \in B_1^p(x)$, $(v_1, \ldots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$ and $z \in X$, exploiting the linearity of $df(y, v_1, \ldots, v_k, z, 0, \ldots, 0)$ in z. We now use the Mean Value Theorem to write

$$f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k) = \int_0^1 df(x + t(y - x), v_1, \ldots, v_k, y - x, 0, \ldots, 0)\, dt$$

for $y \in B_1^p(x)$ and $(v_1, \ldots, v_k) \in B_1^{p_1}(0) \times \cdots \times B_1^{p_k}(0)$. By (A2), we have

$$\|df(x + t(y - x), v_1, \ldots, v_k, y - x, 0, \ldots, 0)\|_q \leq \|y - x\|_p$$

and hence $\|f(y, v_1, \ldots, v_k) - f(x, v_1, \ldots, v_k)\|_q \leq \|y - x\|_p$. Now, (2) follows, using the k-linearity of the map $f(y, \cdot) - f(x, \cdot) \colon E_1 \times \cdots \times E_k \to F$. □

Proof of Lemma 2. By the Polarisation Formula for symmetric k-linear maps (see, e.g., ([13], Proposition 1.6.19)), we have

$$f(x, y_1, \ldots, y_k) = \frac{1}{k!2^k} \sum_{\varepsilon_1, \ldots, \varepsilon_k \in \{-1,1\}} \varepsilon_1 \cdots \varepsilon_k \, h(x, \varepsilon_1 y_1 + \cdots + \varepsilon_k y_k)$$

for all $x \in U$ and $y_1, \ldots, y_k \in E$. Thus, f is $C^r_{\mathbb{K}}$ if h is so. □

Proof of Lemma 3. (a) Let $\mathrm{pr}_2 \colon X_1 \times X_2 \to X_2$, $(x, y) \mapsto y$ be the projection onto the second component and pick $x_0 \in X_1$. Since pr_2 is continuous, every k-continuous function $f \colon X_2 \to \mathbb{R}$ yields a k-continuous function $f \circ \mathrm{pr}_2$ on X. Then, $f \circ \mathrm{pr}_2$ is continuous and hence also $f = (f \circ \mathrm{pr}_2)(x_0, \cdot)$.

(b) Let $f \colon U \to \mathbb{R}$ be k-continuous and $x \in U$. As X is completely regular, we find a continuous function $g \colon X \to \mathbb{R}$ with $g(x) \neq 0$ and support $\mathrm{supp}(g) \subseteq U$. Define $h \colon X \to \mathbb{R}$ via $h(y) := f(y)g(y)$ if $y \in U$, $h(y) := 0$ if $y \in X \setminus \mathrm{supp}(g)$. If $K \subseteq X$ is a compact subset, then each $x \in K$ has a compact neighbourhood K_x in K which is contained in U or in $X \setminus \mathrm{supp}(g)$. In the first case, $h|_{K_x} = f|_{K_x} g|_{K_x}$ is continuous by k-continuity of f. In the second case, $h|_{K_x} = 0$ is continuous as well. Thus, $h|_K$ is continuous. Since X is a $k_{\mathbb{R}}$-space, continuity of h follows. Thus, f is continuous on the open x-neighbourhood $g^{-1}(\mathbb{R} \setminus \{0\})$. □

A simple fact will be useful (see, e.g., [8] (Lemma 1.13)).

Lemma A1. *Let X be a topological space, F be a locally convex space, and $BC(X, F)$ be the space of bounded F-valued continuous functions on X, endowed with the topology of uniform convergence. Then, $\mu \colon BC(X, F) \times X \to F$, $(f, x) \mapsto f(x)$ is continuous.*

Proof of Lemma 4. (If $k = 2$, see Proposition 3 and 4 in [11] (Chapter III, §5, no. 3) for the equivalence (a)⇔(b) and the implication (b)⇒(c); (c)⇒(a) can be found in [8] (Proposition 1.8).) (a)⇔(b): $\beta(V \times M) \subseteq W$ is equivalent to $\beta^\vee(V) \in \lfloor M, W \rfloor$. Hence, (a) is equivalent to continuity of $\beta^\vee$ in 0 and hence to its continuity (see Proposition 5 in [11] (Chapter I, §1, no. 6)).

(b)⇒(c): If $M \in \mathcal{S}$, then $\varepsilon \colon L^{k-j+1}(E_j, \ldots, E_k, F)_{\mathcal{S}} \times M \to F$, $\varepsilon(\alpha, x) := \alpha(x)$ is continuous as a consequence of Lemma A1. Hence, $\beta|_{E_1 \times \cdots \times E_{j-1} \times M} = \varepsilon \circ (\beta^\vee \times \mathrm{id}_M)$ is continuous.

(c)⇒(a) if (3) holds: Given $M \in \mathcal{S}$ and a 0-neighbourhood $W \subseteq F$, by hypothesis, we can find $N \in \mathcal{S}$ such that $\mathbb{D}M \subseteq N$. By continuity of $\beta|_{E_1 \times \cdots \times E_{j-1} \times N}$, there exist 0-neighbourhoods $V_i \subseteq E_i$ for $i \in \{1, \ldots, k\}$ such that $\beta(V \times (N \cap U)) \subseteq W$, where $V := V_1 \times \cdots \times V_{j-1}$ and $U := V_j \times \cdots \times V_k$. Set $a := \frac{j-1}{k-j+1}$. Since M is bounded, $M \subseteq n^a U$ for some $n \in \mathbb{N}$. Then, $\frac{1}{n^a} M \subseteq N \cap U$. Using that β is k-linear, we obtain $\beta((\frac{1}{n} V) \times M) = \beta(V \times (\frac{1}{n^a} M)) \subseteq \beta(V \times (N \cap U)) \subseteq W$. □

Proof of Lemma 7. Given $\alpha \in L^k(E_1, \ldots, E_k, F)$, we have $\varepsilon^\vee(\alpha) = \varepsilon(\alpha, \cdot) = \alpha$, which is a continuous k-linear map. The map ε is also continuous in its first argument, as the topology on $L^k(E_1, \ldots, E_k, F)_{\mathcal{S}}$ is finer than the topology of pointwise convergence, by the hypothesis on $\mathcal{S}$. The linear map $\varepsilon^\vee \colon L^k(E_1, \ldots, E_k)_{\mathcal{S}} \to L^k(E_1, \ldots, E_k)_{\mathcal{S}}$, $\alpha \mapsto \alpha$ being continuous, condition (b) of Lemma 4 is satisfied by ε in place of β and hence also the equivalent condition (a), whence ε is $\mathcal{S}$-hypocontinuous in its arguments $(2, \ldots, k+1)$.

Now, assume that $k = 1$. Since $\mathcal{O}$ is finer than the topology of pointwise convergence, the map ε remains separately continuous in the situation described at the end of the lemma. Hence, if E is barrelled, Lemma 6 ensures hypocontinuity with respect to $\mathcal{T}$. $\square$

Proof of Lemma 8. (a) The composition $\beta \circ f$ is sequentially continuous and hence continuous, its domain X being first countable.

(b) Write $f = (f_1, \ldots, f_k)$ with components $f_j \colon X \to E_\nu$ for $\nu \in \{1, \ldots, k\}$. If K is a compact subset of X, then $M := (f_j, \ldots, f_k)(K)$ is a compact subset of $E_j \times \cdots \times E_k$. Since $\beta|_{E_1 \times \cdots \times E_{j-1} \times M}$ is continuous by Lemma 4 (c), the composition

$$\beta \circ f|_K = \beta|_{E_1 \times \cdots \times E_{j-1} \times M} \circ f|_K$$

is continuous. Thus, $\beta \circ f$ is k-continuous and hence continuous, as X is a $k_{\mathbb{R}}$-space and F is completely regular. $\square$

Proof of Lemma 9. (a) The case $r = 0$: Let q be a continuous seminorm on $F := \prod_{j \in J} F_j$, and $x \in U$. After increasing q, we may assume that

$$q(y) = \max\{q_j(y_j) \colon j \in \Phi\} \quad \text{for all } y = (y_j)_{j \in J} \in F, \tag{A3}$$

for some non-empty, finite subset $\Phi \subseteq J$ and continuous seminorms q_j on F_j for $j \in \Phi$. If each f_j is $LC^0_{\mathbb{K}}$, then we find a continuous seminorm p_j on E for each $j \in \Phi$ such that $B^{p_j}_1(x) \subseteq U$ and $q_j(f_j(z) - f_j(y)) \leq p_j(z - y)$ for all $z, y \in B^{p_j}_1(x)$. Then

$$p \colon E \to [0, \infty[, \quad y \mapsto \max\{p_j(y) \colon j \in \Phi\}$$

is a continuous seminorm on E such that $B^p_1(x) \subseteq U$ and $q(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_1(x)$. If f is $LC^0_{\mathbb{K}}$, let us show that f_j is $LC^0_{\mathbb{K}}$ for each $j \in J$. Let q be a continuous seminorm on F_j and $x \in U$. Let $\mathrm{pr}_j \colon F \to F_j$, $(y_i)_{i \in J} \mapsto y_j$ be the continuous linear projection onto the jth component. Then, $q \circ \mathrm{pr}_j$ is a continuous seminorm on F, whence we find a continuous seminorm p on E such that $B^p_1(x) \subseteq U$ and $(q \circ \mathrm{pr}_j)(f(z) - f(y)) \leq p(z - y)$ for all $z, y \in B^p_1(x)$. Since $(q \circ \mathrm{pr}_j)(f(z) - f(y)) = q(f_j(z) - f_j(y))$, we see that f_j is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$, then f is $C^r_{\mathbb{K}}$ if and only if each f_j is $C^r_{\mathbb{K}}$, and $d^k f = (d^k f_j)_{j \in J}$ in this case for all $k \in \mathbb{N}_0$ such that $k \leq r$ (see [13] (Lemma 1.3.3)). By the case $r = 0$, the map $d^k f$ is $LC^0_{\mathbb{K}}$ if and only if $d^k(f_j)$ is $LC^0_{\mathbb{K}}$ for all $j \in J$. The assertion follows.

(b) Let E, F, and Y be locally convex $\mathbb{K}$-vector spaces and $f \colon U \to F$ as well as $g \colon V \to Y$ be $LC^r_{\mathbb{K}}$-maps on open subsets $U \subseteq E$ and $V \subseteq F$, such that $f(U) \subseteq V$.

If $r = 0$, let $x \in U$ and q be a continuous seminorm on Y. There exists a continuous seminorm p on F such that $B^p_1(f(x)) \subseteq V$ and $q(g(b) - g(a)) \leq p(b - a)$ for all $a, b \in B^p_1(f(x))$. There exists a continuous seminorm P on E with $B^P_1(x) \subseteq U$ and $p(f(z) - f(y)) \leq P(z - y)$ for all $z, y \in B^P_1(x)$. Then, $f(B^P_1(x)) \subseteq B^p_1(f(x))$ and hence

$$q(g(f(z)) - g(f(y))) \leq p(f(z) - f(y)) \leq P(z - y)$$

for all $y, z \in B^P_1(x)$. Thus, $g \circ f \colon U \to Y$ is $LC^0_{\mathbb{K}}$.

If $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$ such that $k \leq r$, we can use Faà di Bruno's Formula

$$d^k(g \circ f)(x, y) = \sum_{j=1}^{k} \sum_{P \in P_{k,j}} d^j g(f(x), d^{|I_1|}(x, y_{I_1}), \ldots, d^{|I_j|}(x, y_{I_j})) \tag{A4}$$

for $x \in U$ and $y = (y_1, \ldots, y_k) \in E^k$, as in [13] (Theorem 1.3.18). Here, $P_{k,j}$ is the set of all partitions $P = \{I_1, \ldots, I_j\}$ of $\{1, \ldots, k\}$ into j disjoint, non-empty subsets $I_1, \ldots, I_j \subseteq \{1, \ldots, k\}$. For a non-empty subset $J \subseteq \{1, \ldots, k\}$ with elements $j_1 < \cdots < j_m$, let $y_J := (y_{j_1}, \ldots, y_{j_m})$. Using (a) and the case $r = 0$, we deduce from (A4) that $d^k(g \circ f)$ is $LC^0_{\mathbb{K}}$.

(c) For each continuous seminorm q on F, the restriction $q|_{F_0}$ is a continuous seminorm on F_0, and each continuous seminorm Q on F_0 arises in this way. In fact, we find an open, absolutely convex 0-neighbourhood $V \subseteq F$ such that $V \cap F_0 \subseteq B_1^Q(0)$. Then, the absolutely convex hull W of $V \cup B_1^Q(0)$ is a 0-neighbourhood in F with $W \cap F_0 = B_1^Q(0)$, whence $q|_{F_0} = Q$ holds for the Minkowski funtional q of W. The case $r = 0$ follows.

If $r \in \mathbb{N} \cup \{\infty\}$, let $\iota \colon F_0 \to F$ be the inclusion map and $f \colon U \to F_0$ be a map on an open subset $U \subseteq E$. Then, f is $C^r_{\mathbb{K}}$ if and only if $\iota \circ f$ is $C^r_{\mathbb{K}}$, and $d^k(\iota \circ f) = \iota \circ (d^k f)$ for all $k \in \mathbb{N}_0$ such that $k \leq r$ (see [13] (Lemma 1.3.19)). By the case $r = 0$, each of the maps $d^k f$ is $LC^0_{\mathbb{K}}$ if and only if $\iota \circ (d^k f)$ is so, from which the assertion follows.

(d) is immediate from (a) and (c). □

Appendix B. Smooth Maps Need Not Extend to the Completion

Let $E := \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \colon (\exists N \in \mathbb{N})(\forall n \geq N)\; x_n = 0\}$ be the space of finite sequences, endowed with the topology induced by the real Banach space ℓ^1 of absolutely summable real sequences. Then, E is a dense proper vector subspace of ℓ^1, and ℓ^1 is a completion of E. In this appendix, we provide a smooth map with the following pathological properties.

Proposition A1. *There exists a smooth map $f \colon E \to F$ to a complete locally convex space F which does not admit a continuous extension to $E \cup \{z\}$ for any $z \in \ell^1 \setminus E$.*

Proof. Given $z = (z_n)_{n\in\mathbb{N}} \in \ell^1 \setminus E$, the set $S := \{n \in \mathbb{N} \colon z_n \neq 0\}$ is infinite. For each $n \in \mathbb{N}$, we pick a smooth map $h_n \colon \mathbb{R} \to \mathbb{R}$ such that $h_n(z_n) = 1$; if $n \in S$, we also require that h_n vanishes on some 0-neighbourhood. Endow $\mathbb{R}^{\mathbb{N}}$ with the product topology. Then

$$g \colon \ell^1 \to \mathbb{R}^{\mathbb{N}}, \quad x = (x_n)_{n\in\mathbb{N}} \mapsto (h_1(x_1) \cdots h_n(x_n))_{n\in\mathbb{N}}$$

is a smooth map, as its components $g_n \colon \ell^1 \to \mathbb{R}$, $x \mapsto h_1(x_1) \cdots h_n(x_n)$ are smooth. If $x = (x_n)_{n\in\mathbb{N}} \in E$, then there is $N \in S$ such that $x_n = 0$ for all $n \geq N$. Thus, $g_n(x) = 0$ for all $n \geq N$ and hence $g(x) \in E$. Notably, $g(x) \in \ell^1$. It therefore makes sense to define

$$f_z \colon E \to \ell^1, \quad x \mapsto g(x).$$

We now show: $f_z \colon E \to \ell^1$ *is a smooth map to ℓ^1 which does not admit a continuous extension to $E \cup \{z\}$.*

In fact, for x and N as above, there exists $\varepsilon > 0$ such that $h_N(t) = 0$ for each $t \in\,]-\varepsilon, \varepsilon[$. Identify $\mathbb{R}^N$ with the closed vector subspace $\mathbb{R}^N \times \{0\}$ of E and $\mathbb{R}^{\mathbb{N}}$. Then,

$$U := \{y = (y_n)_{n\in\mathbb{N}} \in E \colon |y_N| < \varepsilon\}$$

is an open neighbourhood of x in E such that $f_z(U) \subseteq \mathbb{R}^N$. Thus, $f_z|_U$ is smooth as a map to $\mathbb{R}^N$ and hence also as a map to ℓ^1. As a consequence, $f_z \colon E \to \ell^1$ is smooth.

Now, suppose that $p = (p_n)_{n\in\mathbb{N}} \colon E \cup \{z\} \to \ell^1$ was a continuous extension of f_z; we shall derive a contradiction. To this end, set $y_k := (z_1, \ldots, z_k, 0, 0, \ldots) \in E$ for $k \in \mathbb{N}$. Then, $y_k \to z$ in E as $k \to \infty$. The inclusion map $\ell^1 \to \mathbb{R}^{\mathbb{N}}$ being continuous, we deduce that

$$p_n(y_k) \to p_n(z) \quad \text{as } k \to \infty,$$

for each $n \in \mathbb{N}$. Since $p_n(y_k) = g_n(y_k) = h_1(z_1) \cdots h_n(z_n) = 1$ for all $k \geq n$, it follows that $p_n(z) = 1$ for all $n \in \mathbb{N}$ and thus $(1, 1, \ldots) = p(z) \in \ell^1$, which is absurd. Therefore, f_z has all of the asserted properties.

We now define $\Omega := \ell^1 \setminus E$ and endow $F := (\ell^1)^\Omega$ with the product topology. We let $f := (f_z)_{z\in\Omega} \colon E \to F$ be the map with components f_z as defined before. By construction, f has the properties described in Proposition A1. □

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