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Tensor Product of Operators Satisfying Zariouh's Property (gaz), and Stability under Perturbations

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Abstract: For bounded linear operators defined on complex infinite-dimensional Banach space, H. Zariouh, in an article [1] introduced and studied the property (gaz). In this study, through techniques using the local spectral theory of operators, we discover the sufficient conditions that allow the transfer of the property (gaz) from two tensor factors $\mathcal{T}$ and $\mathcal{S}$ to their tensor product $\mathcal{T} \otimes \mathcal{S}$. The stability of the property (gaz) in the tensor product under perturbations is also investigated. The theory is exemplified by considering suitable classes of operators such as shift operators, convolution operators, and m-invertible contractions.

Keywords: property (gaz); commuting perturbations; tensor product

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1. Introduction

For a bounded linear operator $\mathcal{T}$ defined on a complex infinite-dimensional Banach space $\mathcal{X}$, i.e., $\mathcal{T} \in L(\mathcal{X})$, the spectrum (the set of scalars ξ such that $(\xi I - \mathcal{T})$ is not invertible) can be fractionated into parts (subsets) in several different ways. Some parts contained in the spectrum are studied using the classical Fredholm theory. In particular, the part of the Browder spectrum generated by the Browder operators (also classically known as Riesz–Schauder operators) has a appreciable role in the theory of operators. Among the various studies that exist on Browder's theorem (for a given $\mathcal{T}$, if its Weyl spectrum coincides with the Browder spectrum) and its variants, is the a-Browder theorem (for given $\mathcal{T}$, if its upper Weyl spectrum coincides with the upper Browder spectrum), or the property (gaz), which means that the set of all spectral points ξ of $\mathcal{T}$ for which $(\xi I - \mathcal{T})$ is upper semi B-Weyl coincides with the set of all eigenvalues ξ of $\mathcal{T}$, for which $(\xi I - \mathcal{T})$ is left Drazin invertible. This property (gaz) is relevant since it has recently been studied in [2], where the spectral structure of the operators that satisfy it, observed in [3], were transmitted from an invertible Drazin operator to its reverse Drazin. In addition, in [4], this property was studied using a topological approach; in particular, it was shown that the group of operators that belong to $L(\mathcal{X})$ and that satisfy the property (gaz) is closed in $L(\mathcal{X})$, because the property (gaz) is equivalent to the property (az) (see [2]), where the property (az) means that the set of all spectral points ξ of $\mathcal{T}$ for which $(\xi I - \mathcal{T})$ is upper semi-Weyl coincides with the set of all eigenvalues ξ of $\mathcal{T}$ for which $(\xi I - \mathcal{T})$ is upper semi-Fredholm with finite ascent.

The study of the tensor product of two vector spaces or of two linear operators has a role in the theory of operators, as seen in [5], where various properties for the spectra of tensor products of two linear operators are discussed. The study of the stability of Weyl-type properties or Browder-type properties in the tensor product was initiated by S.

Kubrusly and P. Duggal in [6]. Subsequently, these studies were continued by P. Duggal in [7], M. Rashid in [8], and M. Rashid and T. Prasad in [9].

Since the study of the tensor product of two linear operators has an important role in the theory of operators, in this article we consider the tensor product of two operators that satisfy the property (gaz). In Section 3, we present our main results and also present a new theoretical development linked to the property (gaz). Specifically, we focus our interest on obtaining the sufficient conditions that allow the transfer of the property (gaz), for a given two operators $\mathcal{T}$ and $\mathcal{S}$, to their tensor product $\mathcal{T} \otimes \mathcal{S}$. Furthermore, in Section 4, some of the conditions that guarantee the stability of the property (gaz) in the tensor product $\mathcal{T} \otimes \mathcal{S}$ under commutative perturbations are established.

2. Preliminaries

In the remainder of this paper, $L(\mathcal{X})$ denotes the algebra consistent with all bounded linear operators defined on the infinite-dimensional complex Banach space $\mathcal{X}$. For $\mathcal{T} \in L(\mathcal{X})$, let $\mathcal{T}^*$, $\sigma(\mathcal{T})$, $\sigma_a(\mathcal{T})$, $\sigma_s(\mathcal{T})$, $N(\mathcal{T})$, and $\mathcal{T}(\mathcal{X})$ denote the adjoint operator, the spectrum, the approximate point spectrum, the surjectivity spectrum, the null space, and the range of $\mathcal{T}$, respectively. We refer to [10] for other notations or terminology. However, for $\mathcal{T} \in L(\mathcal{X})$, we give the classical notations for spectral sets that will be useful in what follows. The spectrum may be:

- Upper semi-Fredholm: $\sigma_{usf}(\mathcal{T})$.
- B-Fredholm: $\sigma_{Bf}(\mathcal{T})$.
- Upper semi B-Fredholm: $\sigma_{uBf}(\mathcal{T})$.
- Lower semi B-Fredholm: $\sigma_{lBf}(\mathcal{T})$.
- Browder: $\sigma_b(\mathcal{T})$.
- Upper semi-Browder: $\sigma_{ub}(\mathcal{T})$.
- Fredholm: $\sigma_e(\mathcal{T})$.
- Weyl: $\sigma_w(\mathcal{T})$.
- Upper semi-Weyl: $\sigma_{ea}(\mathcal{T})$.
- Lower semi-Weyl: $\sigma_{es}(\mathcal{T})$.
- Drazin invertible: $\sigma_d(\mathcal{T})$.
- Left Drazin invertible: $\sigma_{ld}(\mathcal{T})$.
- B-Weyl: $\sigma_{Bw}(\mathcal{T})$.
- Upper semi B-Weyl: $\sigma_{uBw}(\mathcal{T})$.

$\mathcal{H}(\sigma(\mathcal{T}))$ denotes the set of all analytic functions defined on an open neighborhood of $\sigma(\mathcal{T})$, and for each $f \in \mathcal{H}(\sigma(\mathcal{T}))$, we assume that $f(\mathcal{T})$ is defined as in the classical functional calculus. In addition, for $\mathcal{T} \in L(\mathcal{X})$, $acc(\sigma(\mathcal{T}))$ denotes the set of accumulation points of $\sigma(\mathcal{T})$.

The localized interpretation of the single-valued extension property was given by Finch in [11], and with respect to the Fredholm theory, this property has been related to in several ways (see Section 3 of [12]). Precisely, an operator $\mathcal{T} \in L(\mathcal{X})$ is said to have *the single valued extension property* at $\xi_0 \in \mathbb{C}$ (abbreviated, the SVEP at ξ_0) if, for every open disc $\mathbb{D}_{\xi_0} \subseteq \mathbb{C}$ centered at ξ_0, the only analytic function $f : \mathbb{D}_{\xi_0} \to \mathcal{X}$ which satisfies the equation

$$(\xi I - \mathcal{T})f(\xi) = 0 \quad \text{for all } \xi \in \mathbb{D}_{\xi_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\xi_0}$. The operator $\mathcal{T}$ is said to have the SVEP if $\mathcal{T}$ has the SVEP at every point $\xi \in \mathbb{C}$.

It was proved that $\mathcal{T} \in L(\mathcal{X})$ has the SVEP at every isolated point of the spectrum. Therefore, if $\sigma(\mathcal{T})$ has no accumulation points, then $\mathcal{T}$ has the SVEP.

3. Property (gaz) under Tensor Product

Let $\mathcal{X} \otimes \mathcal{Y}$ be the algebraic completion (in some reasonable uniform cross norm) of the tensor product of two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. The tensor product of $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ on $\mathcal{X} \otimes \mathcal{Y}$ is the operator defined as

$$(\mathcal{T} \otimes \mathcal{S})(\sum_i x_i \otimes y_i) = \sum_i \mathcal{T}(x_i) \otimes S(y_i), \text{ where } \sum_i x_i \otimes y_i \in \mathcal{X} \otimes \mathcal{Y}.$$

In this section, for $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$, which are two operators satisfying the property (gaz), we study and list some sufficient conditions to ensure that the property (gaz) is transmitted from $\mathcal{T}$ and $\mathcal{S}$ to the tensor product $\mathcal{T} \otimes \mathcal{S}$.

For $\mathcal{T} \in L(\mathcal{X})$, we define:

$$\Delta^+(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_{ea}(\mathcal{T}) \quad \text{and} \quad \Delta^g(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_{uBw}(\mathcal{T}).$$

$$\Pi_a^0(\mathcal{T}) := \sigma_a(\mathcal{T}) \setminus \sigma_{ub}(\mathcal{T}) \quad \text{and} \quad \Pi_a(\mathcal{T}) := \sigma_a(\mathcal{T}) \setminus \sigma_{ld}(\mathcal{T}).$$

Note that $\Pi_a(\mathcal{T}) \subseteq \Delta^g(\mathcal{T})$, since $\sigma_{uBw}(\mathcal{T}) \subseteq \sigma_{ld}(\mathcal{T})$. In addition, $\Pi_0^a(\mathcal{T}) \subseteq \Delta^+(\mathcal{T})$. The purpose of defining these sets is to define the properties (gaz) and (az) formally.

Definition 1 ([1])**.** *The operator $\mathcal{T} \in L(\mathcal{X})$ is said to satisfy:*

1. *Property (gaz), if $\Delta^g(\mathcal{T}) = \Pi_a(\mathcal{T})$.*
2. *Property (az), if $\Delta^+(\mathcal{T}) = \Pi_a^0(\mathcal{T})$.*

In the following result, we see that these two properties are equivalent.

Theorem 1 ([1], Corollary 3.5)**.** *$\mathcal{T} \in L(\mathcal{X})$ satisfies the property (az) if and only if $\mathcal{T}$ satisfies the property (gaz).*

We will use the following characterization.

Theorem 2 ([4], Corollary 3)**.** *$\mathcal{T} \in L(\mathcal{X})$ satisfies the property (gaz) if and only if $\sigma_b(\mathcal{T}) = \sigma_{ea}(\mathcal{T})$.*

Recall that $\mathcal{T}$ satisfies the a-Browder theorem if

$$\sigma_{ea}(\mathcal{T}) = \sigma_{ub}(\mathcal{T}).$$

This is an example where the tensor product of two operators that satisfy the property (gaz) does not itself satisfy the property (gaz).

Example 1. *Let $\mathcal{U} \in L(l^2)$ be the forward unilateral shift, defined as $\mathcal{U}(x_1, x_2, x_3, ...) := (0, x_1, x_2, x_3, ...)$ for each $(x_1, x_2, x_3, ...) \in l^2$. Let $\mathcal{T}$ and $\mathcal{S}$ be two operators in $L(l^2 \otimes l^2)$, such that:*

$$T := (I - \mathcal{U}\mathcal{U}^*) \oplus (0.5\mathcal{U} - I) \ \textit{and} \ S := -(I - \mathcal{U}\mathcal{U}^*) \oplus (0.5\mathcal{U}^* + I).$$

From [5], Remark 2, we know that $\mathcal{T}$ and $\mathcal{S}^*$ have the SVEP, whereby $\mathcal{T}$ satisfies the a-Browder theorem, and from [2], Corollary 3.7, we obtain that $\mathcal{S}$ satisfies the property (gaz). Furthermore, considering [13], Theorem 1, we obtain

$$1 \in \sigma(\mathcal{T} \otimes \mathcal{S}) \setminus \sigma_{ea}(\mathcal{T} \otimes \mathcal{S}).$$

On the other hand,

$$\sigma(\mathcal{T} \otimes \mathcal{S}) = \{\{0, 1\} \cup \{0.5\mathbb{D} - 1\}\}.\{\{0, -1\} \cup \{0.5\mathbb{D} + 1\}\}.$$

Note that $\sigma(\mathcal{T}) = \sigma_a(\mathcal{T})$, thus $\mathcal{T}$ satisfies the property (gaz) (see [1], Theorem 3.2). Since $1 \in acc(\sigma(\mathcal{S}))$, $1 \in acc(\mathcal{T} \otimes \mathcal{S})$, whereby $1 \in \sigma_b(\mathcal{T} \otimes \mathcal{S})$. Hence, $\sigma_b(\mathcal{T} \otimes \mathcal{S}) \neq \sigma_{ea}(\mathcal{T} \otimes \mathcal{S})$. This implies that $\mathcal{T} \otimes \mathcal{S}$ does not satisfy the property (gaz) (see Theorem 2).

Some properties of the tensor product are already known, for example those of the following theorem.

Theorem 3 ([5])**.** *Let $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ be two operators, then:*

1. $\sigma(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma(\mathcal{S})$, $\sigma_a(\mathcal{T} \otimes \mathcal{S}) = \sigma_a(\mathcal{T})\sigma_a(\mathcal{S})$.
2. $\sigma_{usf}(\mathcal{T} \otimes \mathcal{S}) = \sigma_a(\mathcal{S})\sigma_{usf}(\mathcal{T}) \cup \sigma_a(\mathcal{T})\sigma_{usf}(\mathcal{S})$.
3. $\sigma_{ub}(\mathcal{T} \otimes \mathcal{S}) = \sigma_a(\mathcal{T})\sigma_{ub}(\mathcal{S}) \cup \sigma_{ub}(\mathcal{T})\sigma_a(\mathcal{S})$.
4. $\sigma_b(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_b(\mathcal{S}) \cup \sigma_b(\mathcal{T})\sigma(\mathcal{S})$.

The upper semi-Weyl spectrum does not verify the identity of the spectrum for the tensor product, i.e., $\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) \neq \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}) \cup \sigma_{ea}(\mathcal{S})\sigma(\mathcal{T})$. In fact, we have the following lemma.

Lemma 1. *If $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$, then*

$$\begin{aligned} \sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) &\subseteq \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}) \cup \sigma_{ea}(\mathcal{S})\sigma(\mathcal{T}) \\ &\subseteq \sigma_b(\mathcal{T})\sigma(\mathcal{S}) \cup \sigma_b(\mathcal{S})\sigma(\mathcal{T}) = \sigma_b(\mathcal{T} \otimes \mathcal{S}). \end{aligned}$$

Proof. From [5], Lemma 5, we have $\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) \subseteq \sigma_{ea}(\mathcal{T})\sigma_a(\mathcal{S}) \cup \sigma_{ea}(\mathcal{S})\sigma_a(\mathcal{T})$. Thus, the first inclusion follows from the facts that $\sigma_a(\mathcal{T}) \subseteq \sigma(\mathcal{T})$ and $\sigma_a(\mathcal{S}) \subseteq \sigma(\mathcal{S})$. In addition, $\sigma_{ea}(\mathcal{T}) \subseteq \sigma_b(\mathcal{T})$ and $\sigma_{ea}(\mathcal{S}) \subseteq \sigma_b(\mathcal{S})$, and hence the second inclusion follows, and the equality also follows from part 4 of Theorem 3. $\square$

Under the effects of the property (gaz), the upper semi-Weyl spectrum verifies the identity of the spectrum for the tensor product, as seen in the next theorem.

Theorem 4. *Suppose that $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ satisfy the property (gaz). Then, $\mathcal{T} \otimes \mathcal{S}$ satisfies the property (gaz) if and only if*

$$\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}).$$

Proof. Assume that $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{T} \otimes \mathcal{S}$ satisfy the property (gaz). Equivalently, from Theorem 2, we have $\sigma_{ea}(\mathcal{T}) = \sigma_b(\mathcal{T})$, $\sigma_{ea}(\mathcal{S}) = \sigma_b(\mathcal{S})$, and $\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma_b(\mathcal{T} \otimes \mathcal{S})$.

Now, directly, from part 4 of Theorem 3, we obtain

$$\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma_b(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_b(\mathcal{S}) \cup \sigma_b(\mathcal{T})\sigma(\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}).$$

Hence,

$$\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}).$$

Conversely, from Theorem 2 and Theorem 3 part 4, we have

$$\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}) = \sigma(\mathcal{T})\sigma_b(\mathcal{S}) \cup \sigma_b(\mathcal{T})\sigma(\mathcal{S}) = \sigma_b(\mathcal{T} \otimes \mathcal{S}).$$

Therefore, again by Theorem 2, $\mathcal{T} \otimes \mathcal{S}$ satisfies the property (gaz). $\square$

Corollary 1. *Suppose that $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ satisfy the property (gaz). Then,*

$$\sigma_{uBw}(\mathcal{T} \otimes \mathcal{S}) = \sigma_d(\mathcal{T} \otimes \mathcal{S})$$

if and only if

$$\sigma_{ea}(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}).$$

Proof. From [2], Corollary 3, $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})$ if and only if $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz) and (from Theorem 4) if and only if $\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S})$. □

This is an example where the tensor product of two operators that satisfy the property (gaz) also satisfies the property (gaz).

Example 2. *Every multiplier operator $\mathcal{T}$ defined on a semi-simple commutative Banach algebra A, is $H(1)$ (see [14]), whereby $\mathcal{T}$ has the SVEP and so $\sigma_{usf}(\mathcal{T}) = \sigma_{ub}(\mathcal{T})$ and $\sigma_{ea}(\mathcal{T}) = \sigma_{ub}(\mathcal{T})$. In addition, if A is regular and Tauberian, then from [12], Corollary 5.88, $\sigma(\mathcal{T}) = \sigma_a(\mathcal{T})$. Particularly, this is true for two convolution operators $\mathcal{T}_\mu$ and $\mathcal{T}_\nu$ in $L^1(\mathcal{G})$, where $L^1(\mathcal{G})$, is the group algebra for a compact Abelian group $\mathcal{G}$. Thus, from Theorem 3 parts 3 and 2, we obtain:*

$$\sigma_{ub}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu) = \sigma_a(\mathcal{T}_\nu)\sigma_{ub}(\mathcal{T}_\mu) \cup \sigma_{ub}(\mathcal{T}_\nu)\sigma_a(\mathcal{T}_\mu) = \sigma_a(\mathcal{T}_\nu)\sigma_{usf}(\mathcal{T}_\mu) \cup \sigma_{usf}(\mathcal{T}_\nu)\sigma_a(\mathcal{T}_\mu) = \sigma_{usf}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu).$$

Hence, $\sigma_{ub}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu) = \sigma_{ea}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu)$. Thus:

$$\sigma_a(\mathcal{T}_\nu)\sigma_{ea}(\mathcal{T}_\mu) \cup \sigma_{ea}(\mathcal{T}_\nu)\sigma_a(\mathcal{T}_\mu) = \sigma_a(\mathcal{T}_\nu)\sigma_{ub}(\mathcal{T}_\mu) \cup \sigma_{ub}(\mathcal{T}_\nu)\sigma_a(\mathcal{T}_\mu) = \sigma_{ub}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu) = \sigma_{ea}(\mathcal{T}_\nu\otimes\mathcal{T}_\mu).$$

Therefore, by Theorem 4, $\mathcal{T}_\nu\otimes\mathcal{T}_\mu$ satisfies the property (gaz).

It is well known that the property (gaz) implies the a-Browder theorem but not vice versa (see [2], Theorem 3.2). The following theorem gives an equivalence between these two.

Theorem 5. *Suppose that $\mathcal{T}\in L(\mathcal{X})$ satisfies the property (gaz) and also $\mathcal{S}\in L(\mathcal{Y})$. $\mathcal{T}\otimes\mathcal{S}$ satisfies the a-Browder theorem if and only if $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz).*

Proof. Assume that $\mathcal{T}\otimes\mathcal{S}$ satisfies the a-Browder theorem. Then, from Theorem 3 part 3,

$$\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma_{ub}(\mathcal{T}\otimes\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ub}(\mathcal{S}) \cup \sigma(\mathcal{S})\sigma_{ub}(\mathcal{T}). \tag{1}$$

Since $\mathcal{T}$ and $\mathcal{S}$ satisfy the property (gaz), they also satisfy the a-Browder theorem, whereby $\sigma_{ea}(\mathcal{T}) = \sigma_{ub}(\mathcal{T})$, and $\sigma_{ea}(\mathcal{S}) = \sigma_{ub}(\mathcal{S})$, and from Equation (1) we obtain

$$\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S}).$$

Therefore, by Theorem 4, $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz). □

Corollary 2. *Suppose that $\mathcal{T}\in L(\mathcal{X})$ satisfies the property (gaz) and also $\mathcal{S}\in L(\mathcal{Y})$. Then the operator $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz) if $\mathcal{T}\otimes\mathcal{S}$ has the SVEP.*

Proof. Since the SVEP implies the a-Browder theorem, from Theorem 5, $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz). □

Corollary 3. *Suppose that $\mathcal{T}\in L(\mathcal{X})$ satisfies the property (gaz) and also $\mathcal{S}\in L(\mathcal{Y})$. If the operator $\mathcal{T}\otimes\mathcal{S}$ has the SVEP, then:*

1. $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})$.
2. $(\mathcal{T}\otimes\mathcal{S})^*$ *satisfies the property* (gaz).
3. $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S}) = \sigma_{lBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_{Bw}(\mathcal{T}\otimes\mathcal{S})$.
4. $\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma_b(\mathcal{T}\otimes\mathcal{S}) = \sigma_{es}(\mathcal{T}\otimes\mathcal{S}) = \sigma_w(\mathcal{T}\otimes\mathcal{S})$.
5. $f(\mathcal{T}\otimes\mathcal{S})$ *satisfies the property* (gaz), *for each* $f\in\mathcal{H}(\sigma(\mathcal{T}\otimes\mathcal{S}))$.
6. $\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma_{ea}(\mathcal{T})\sigma(\mathcal{S})$.
7. $f(\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S})) = \sigma_{uBw}(f(\mathcal{T}\otimes\mathcal{S}))$, *for each* $f\in\mathcal{H}(\sigma(\mathcal{T}\otimes\mathcal{S}))$.

Proof. 1. From Corollary 2, $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz), so equivalently from [2], Corollary 3, $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})$.

2. With the hypothesis that $\mathcal{T}\otimes\mathcal{S}$ has the SVEP, we obtain from [2], Corollary 3.7, that $(\mathcal{T}\otimes\mathcal{S})^*$ satisfies the property (gaz).
3. The Drazin spectrum of an operator matches the Drazin spectrum of its dual, so from part 1, we obtain $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})^*$, and from part 2, $(\mathcal{T}\otimes\mathcal{S})^*$ satisfies the property (gaz), so $\sigma_d(\mathcal{T}\otimes\mathcal{S})^* = \sigma_{lBw}(\mathcal{T}\otimes\mathcal{S})$.
4. From part 2 and Corollary 2, we obtain that $\mathcal{T}\otimes\mathcal{S}$ and $(\mathcal{T}\otimes\mathcal{S})^*$ satisfy the property (gaz), or equivalently, satisfy the property (az). Hence, from [4], Theorem 2, we obtain that
$$\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma_b(\mathcal{T}\otimes\mathcal{S}) = \sigma_{es}(\mathcal{T}\otimes\mathcal{S}) = \sigma_w(\mathcal{T}\otimes\mathcal{S}).$$
5. From [12], Theorem 2.40, for each $f \in \mathcal{H}(\sigma(\mathcal{T}\otimes\mathcal{S}))$, $f(\mathcal{T}\otimes\mathcal{S})$ has the SVEP and thus satisfies the a-Browder theorem. Note that $\sigma(f(\mathcal{T}\otimes\mathcal{S})) = \sigma_a(f(\mathcal{T}\otimes\mathcal{S}))$. Hence, from [1], Theorem 3.2, $f(\mathcal{T}\otimes\mathcal{S})$ satisfies the property (gaz).
6. From part 1, $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})$, so by Corollary 1,
$$\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma(\mathcal{T})\sigma_{ea}(\mathcal{S}) \cup \sigma(\mathcal{S})\sigma_{ea}(\mathcal{T}).$$
7. Let $f \in \mathcal{H}(\sigma(\mathcal{T}\otimes\mathcal{S}))$. From part 5, $f(\mathcal{T}\otimes\mathcal{S})$ satisfies the property (gaz), or equivalently, by [2], Corollary 3, $\sigma_{uBw}(f(\mathcal{T}\otimes\mathcal{S})) = \sigma_d(f(\mathcal{T}\otimes\mathcal{S}))$. Now, from part 1, $\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S}) = \sigma_d(\mathcal{T}\otimes\mathcal{S})$. Hence, $f(\sigma_{uBw}(\mathcal{T}\otimes\mathcal{S})) = f(\sigma_d(\mathcal{T}\otimes\mathcal{S})) = \sigma_d(f(\mathcal{T}\otimes\mathcal{S})) = \sigma_{uBw}(f(\mathcal{T}\otimes\mathcal{S}))$, since Drazin's resolvent is a regularity. Hence, 7 follows.
□

Example 3. *Let $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ be such that $\sigma(T)$ and $\sigma(S)$ do not have accumulation points (for example algebraic operators). Hence, by [4], Corollary 4, $\mathcal{T}$ and $\mathcal{S}$ satisfy the property (az), or equivalently, satisfy the property (gaz). Note that $\sigma(\mathcal{T}\otimes\mathcal{S})$ does not have accumulation points, so $\mathcal{T}\otimes\mathcal{S}$ has the SVEP. Therefore, Corollaries 2 and 3 apply to $\mathcal{T}\otimes\mathcal{S}$.*

4. Stability of the Property (gaz) in the Tensor Product

In this section, we discover some sufficient conditions to show the stability of the property (gaz) in the tensor product under commuting perturbation, which is quasi-nilpotent and a Riesz operator.

Recall that an operator $\mathcal{R} \in L(\mathcal{X})$ is a Riesz operator if for all $\xi \neq 0$, $\xi I - \mathcal{R}$ is a Fredholm operator. In addition, an operator $\mathcal{Q} \in L(\mathcal{X})$ is quasi-nilpotent if for all $\xi \neq 0$, $\xi I - \mathcal{Q}$ is invertible, that is, $\sigma(\mathcal{Q}) = \{0\}$. On the other hand, if $\mathcal{Q}_1 \in L(\mathcal{X})$ and $\mathcal{Q}_2 \in L(\mathcal{Y})$ are quasi-nilpotent operators commuting with $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$, respectively, then
$$(\mathcal{T}+\mathcal{Q}_1)\otimes(\mathcal{S}+\mathcal{Q}_2) = (\mathcal{T}\otimes\mathcal{S}) + \mathcal{Q},$$
where $\mathcal{Q} := \mathcal{Q}_1\otimes\mathcal{S} + \mathcal{T}\otimes\mathcal{Q}_2 + \mathcal{Q}_1\otimes\mathcal{Q}_2 \in L(\mathcal{X}\otimes\mathcal{Y})$ is a quasi-nilpotent operator (see [9]).

Note that $\mathcal{Q}$ commutes with $(\mathcal{T}\otimes\mathcal{S})$. Thus, we obtain the next theorem.

Theorem 6. *Let $\mathcal{Q}_1 \in L(\mathcal{X})$ and $\mathcal{Q}_2 \in L(\mathcal{Y})$ be two quasi-nilpotent operators commuting with $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$, respectively. If $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz), then $(\mathcal{T}+\mathcal{Q}_1)\otimes(S+\mathcal{Q}_2)$ satisfies the property (gaz).*

Proof. Let us define $\mathcal{T}_1 := (\mathcal{T}+\mathcal{Q}_1)\otimes(S+\mathcal{Q}_2)$. Note that every quasi-nilpotent operator is a Riesz operator. Hence, by corollaries 3.24, 3.18, and 3.9 of [10], $\sigma(\mathcal{T}_1) = \sigma(\mathcal{T}\otimes\mathcal{S})$ and $\sigma_a(\mathcal{T}_1) = \sigma_a(\mathcal{T}\otimes\mathcal{S})$, $\sigma_{ea}(\mathcal{T}_1) = \sigma_{ea}(\mathcal{T}\otimes\mathcal{S})$, and $\sigma_{ub}(\mathcal{T}_1) = \sigma_{ub}(\mathcal{T}\otimes\mathcal{S})$, respectively. Since $\mathcal{T}\otimes\mathcal{S}$ satisfies the property (gaz) and hence satisfies the property (az):
$$\sigma(\mathcal{T}_1)\setminus\sigma_{ea}(\mathcal{T}_1) = \sigma(\mathcal{T}\otimes\mathcal{S})\setminus\sigma_{ea}(\mathcal{T}\otimes\mathcal{S}) = \sigma_a(\mathcal{T}\otimes\mathcal{S})\setminus\sigma_{ub}(\mathcal{T}\otimes\mathcal{S}) = \sigma_a(\mathcal{T}_1)\setminus\sigma_{ub}(\mathcal{T}_1).$$

Therefore, $\mathcal{T}_1 = (\mathcal{T}+\mathcal{Q}_1)\otimes(S+\mathcal{Q}_2)$ satisfies the property (az), or equivalently, satisfies the property (gaz). □

From Corollary 2 and Theorem 6, we obtain the following result.

Corollary 4. *Let $\mathcal{Q}_1 \in L(\mathcal{X})$ and $\mathcal{Q}_2 \in L(\mathcal{Y})$ be quasi-nilpotent operators commuting with $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$, respectively. Suppose that $\mathcal{T}$ and $\mathcal{S}$ satisfy the property (gaz). If $\mathcal{T} \otimes \mathcal{S}$ has the SVEP, then $(\mathcal{T} + \mathcal{Q}_1) \otimes (\mathcal{S} + \mathcal{Q}_2)$ satisfies the property (gaz).*

We put the set $\Delta_+(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_{usf}(\mathcal{T})$, so that $\Delta^+(\mathcal{T}) \subseteq \Delta_+(\mathcal{T})$, and hence

$$\mathrm{int}\,(\Delta_+(\mathcal{T})) = \emptyset \Rightarrow \mathrm{int}\,(\Delta^+(\mathcal{T})) = \emptyset.$$

In this case, from [4], Theorem 7, we obtain the result that $\mathcal{T}$ has the SVEP at each $\xi \notin \sigma_{ea}(\mathcal{T})$. Thus, for Riesz perturbations, we have the following result.

Theorem 7. *Let $\mathcal{T} \in L(\mathcal{X})$ and $\mathcal{S} \in L(\mathcal{Y})$ such that $\mathrm{int}(\Delta_+(\mathcal{T} \otimes \mathcal{S})) = \emptyset$. Let $\mathcal{R}_1 \in L(\mathcal{X})$ and $\mathcal{R}_2 \in L(\mathcal{Y})$ be two Riesz operators commuting with $\mathcal{T}$ and $\mathcal{S}$, respectively. Suppose that $\sigma(\mathcal{T} + \mathcal{R}_1) = \sigma(\mathcal{T})$, $\sigma(\mathcal{S} + \mathcal{R}_2) = \sigma(\mathcal{S})$, equally, for the approximate point spectrum. If $\mathcal{S}_0 := (\mathcal{T} + \mathcal{R}_1) \otimes (\mathcal{S} + \mathcal{R}_2)$, then:*

1. *$\mathcal{S}_0$ satisfies the property (gaz).*
2. *$\sigma_{usf}(\mathcal{S}_0) = \sigma_e(\mathcal{S}_0) = \sigma_{ea}(\mathcal{S}_0) = \sigma_w(\mathcal{S}_0) = \sigma_{ub}(\mathcal{S}_0) = \sigma_b(\mathcal{S}_0)$.*
3. *$\sigma_{uBf}(\mathcal{S}_0) = \sigma_{Bf}(\mathcal{S}_0) = \sigma_{uBw}(\mathcal{S}_0) = \sigma_{Bw}(\mathcal{S}_0) = \sigma_{ld}(\mathcal{S}_0) = \sigma_d(\mathcal{S}_0)$.*

Proof. 1. To prove that $\mathcal{S}_0 := (\mathcal{T} + \mathcal{R}_1) \otimes (\mathcal{S} + \mathcal{R}_2)$ satisfies the property (gaz), or equivalently, the property (az), it is sufficient to prove that $\mathrm{int}(\Delta^+(\mathcal{S}_0)) = \emptyset$ (see [4], Theorem 7). It is clear that $\Delta^+(\mathcal{S}_0) \subseteq \Delta_+(\mathcal{S}_0)$. By hypothesis, $\sigma(\mathcal{T} + \mathcal{R}_1) = \sigma(\mathcal{T})$, and $\sigma(\mathcal{S} + \mathcal{R}_2) = \sigma(\mathcal{S})$. From [10], Corollary 3.18, the upper semi-Fredholm spectrum is stable under Riesz commuting perturbation, and from Theorem 3 parts 1 and 2, we obtain

$$\Delta^+(\mathcal{S}_0) \subseteq \Delta_+(\mathcal{S}_0) = \Delta_+(\mathcal{T} \otimes \mathcal{S}).$$

Hence, it follows from the hypothesis, that $\mathrm{int}(\Delta^+(\mathcal{S}_0) = \emptyset$. Therefore, $\mathcal{S}_0 = (\mathcal{T} + \mathcal{R}_1) \otimes (\mathcal{S} + \mathcal{R}_2)$ satisfies the property (gaz).

2–3. As for the proof in part 1, for $\mathcal{S}_0 = (\mathcal{T} + \mathcal{R}_1) \otimes (\mathcal{S} + \mathcal{R}_2)$, we have $\mathrm{int}(\Delta_+(\mathcal{S}_0)) = \emptyset = \mathrm{int}(\Delta^+(\mathcal{S}_0))$. Thus, 2 and 3 follow from [4], Section 5. □

Example 4. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two left m-invertible contractions such that $\sigma(\mathcal{T}_i) \subseteq \Gamma$ for $i \in \{1, 2\}$, then $\xi \in \sigma(\mathcal{T}_i)$ is a pole of $\mathcal{T}_i$ if and only if $(\xi I - \mathcal{T}_i)(\mathcal{X})$ is closed (see [15] for definition and details). However, $\forall \xi \in \Delta_+(\mathcal{T}_i)$, $(\xi I - \mathcal{T}_i)(\mathcal{X})$ is closed, so $\Delta^+(\mathcal{T}_i) \subseteq \Delta_+(\mathcal{T}_i) \subseteq \Pi(\mathcal{T}_i) \subseteq \mathrm{iso}\,\sigma_a(\mathcal{T}_i)$, and hence $\mathrm{int}(\Delta_+(\mathcal{T}_i)) = \emptyset$, so $\mathcal{T}_i$ satisfies the property (gaz), whereby $\sigma(\mathcal{T}_i) = \sigma_a(\mathcal{T}_i)$. In addition, $\mathrm{int}(\Delta_+(\mathcal{T}_1 \otimes \mathcal{T}_2)) = \emptyset$. Therefore, applying Theorem 7, we obtain*

$$\mathcal{S}_{00} := (\mathcal{T}_1 + 0) \otimes (\mathcal{T}_2 + 0).$$

In particular, $\mathcal{T}_1 \otimes \mathcal{T}_2$ satisfies the property (gaz), so from Theorem 6,

$$\mathcal{S}_1 := (\mathcal{T}_1 + \mathcal{Q}_1) \otimes (\mathcal{T}_2 + \mathcal{Q}_2),$$

satisfies the property (gaz), where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two quasi-nilpotent operators commuting with $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively.

5. Conclusions

(1) The property (gaz), in general, does not transfer from two tensor factors to the tensor product of the two factors (see Example 1). However, we can conclude that it does so if the upper Weyl spectrum satisfies the identity of the spectrum of the tensor product (see Theorem 4).

(2) The a-Browder theorem is equivalent to the property (gaz) for the tensor product of two operators that satisfy the property (gaz) (see Theorem 5).
(3) Under certain conditions, the property (gaz) is stable in the tensor product of two operators that satisfy it with commutative perturbations in the factors, which can be quasi-nilpotent or Riesz (see Section 4).

Further Work: The study of the stability of tensor products under algebraic perturbations is in progress.

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