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Global Attractivity of Symbiotic Model of Commensalism in Four Populations with Michaelis–Menten Type Harvesting in the First Commensal Populations

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Abstract: This article revisits the stability property of a symbiotic model of commensalism with Michaelis–Menten type harvesting in the first commensal populations. The model was proposed by Nurmaini Puspitasari et al. By constructing some suitable Lyapunov functions, we provide a thorough analysis of the dynamic behaviors of the subsystem composed of the second and third species. After that, by applying the stability results of this subsystem and combining with the differential inequality theory, sufficient conditions which ensure the global attractivity of the equilibria are obtained. The results obtained here essentially improve and generalize some known results.

Keywords: commensalism; Michaelis–Menten type harvesting; comparison theorem; global attractivity

MSC: 34C20; 92D25

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1. Introduction

In the past decade, many scholars [1–42] investigated the dynamic behaviors of the commensalism model. Some substantial progress has been made in the study of the commensalism system. Topics such as the stability of the system [1–4,19–21,39,40], the persistent and extinction properties of the system [3,19], the existence of a periodic solution or almost periodic solution [6–10,41], the influence of stage structure [10], the influence of Allee effect [11–19], the influence of feedback control [19–21], the influence of harvesting [5,27–38], the bifurcation phenomenon of the system [4,18–20,26,27,36,37], the influence of functional reactions [2,3,9,16,23–25,43], the model governed by the discrete equation [5–10,36,38,41,44], the influence of time delays [1,6,26,42–48], the influence of dispersal [46] etc., were extensively investigated.

Commensalism is a long-term biological interaction (symbiosis) in which members of one species gain benefits while those of the other species neither benefit nor are harmed. Though this kind of relationship is often observed in nature, the mathematical model was not developed until 2013. In ground breaking work, Sun and Sun [4] proposed the following commensalism system:

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1 - \frac{x}{K_1} + \alpha\frac{y}{K_1}\right),\\ \frac{dy}{dt} &= r_2 y\left(1 - \frac{y}{K_2}\right),\end{aligned} \tag{1}$$

where r_1, r_2, K_1, K_2, α are all positive constants. The system admits four equilibria. The authors showed that only positive equilibrium E_4 is a stable node and all the other three equilibria are unstable.

It is well known that the harvesting of species is an effective way for human beings to obtain resources. Already, there are many scholars investigateing the dynamic behaviors of the commensalism model with the influence of linear or nonlinear type harvesting [5,28–40,42,45].

Deng and Huang [30] proposed the following non-selective harvesting Lotka–Volterra commensalism model incorporating partial closure for the populations:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\Big(1 - \frac{x}{K_1} + \alpha\frac{y}{K_1}\Big) - q_1 Emx, \\ \frac{dy}{dt} &= r_2 y\Big(1 - \frac{y}{K_2}\Big) - q_2 Emy. \end{aligned} \tag{2}$$

Their study showed that depending on the fraction of the stock available for harvesting, the system may lead to extinction, partial survival or two species coexisting in a stable state. The dynamic behaviors of the system become complicated compared with the non-harvesting system. Liu, Xie and Lin [28] studied the nonautonomous case of system (2); they investigated the partial survival, extinction and global stability of the system.

Many scholars [8,27,32,34–41] argued that the nonlinear harvesting such as Michaelis–Menten type harvesting is more realistic from the biological and economical points of view. Chen [27] proposed and studied the following model for the first time, in which the first species is subject to Michaelis–Menten type harvesting:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\Big(1 - \frac{x}{K_1} + \alpha\frac{y}{K_1}\Big) - \frac{qEx}{m_1 E + m_2 x}, \\ \frac{dy}{dt} &= r_2 y\Big(1 - \frac{y}{K_2}\Big), \end{aligned} \tag{3}$$

where r_1, r_2, K_1, K_2, α, q, E, m_1, m_2 are all positive constants, and r_1, r_2, K_1, K_2, α have the same meaning as that of the system (1), E is the fishing effort used to harvest and q is the catchablity coefficient; m_1 and m_2 are suitable constants. The author showed that for the limited harvesting case (i.e., q is enough small), the system admits a unique globally stable positive equilibrium; for the over harvesting case, if the cooperation intensity of both species (α) and the capacity of the second species (K_2) are large enough, the two species could coexist in a stable state, otherwise, the first species will be driven to extinction.

Zhu et al. [38] proposed the following Lotka–Volterra commensal symbiosis model with non-selective Michaelis–Menten type harvesting:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\Big(1 - \frac{x}{K_1} + \alpha\frac{y}{K_1}\Big) - \frac{q_1 Ex}{m_1 E + m_2 x}, \\ \frac{dy}{dt} &= r_2 y\Big(1 - \frac{y}{K_2}\Big) - \frac{q_2 Ey}{m_3 E + m_4 y}, \end{aligned} \tag{4}$$

where $r_1, r_2, K_1, K_2, \alpha, q_1, q_2, E, m_1, m_2, m_3$ and m_4 are all positive constants. By establishing some new lemmas, the authors investigated the extinction, partial survival and global attractivity of the positive equilibrium of the system. Their results essentially improve and generalize the main results of Chen [27].

In [29], Liu et al. proposed the following nonautonomous Lotka–Volterra commensalism model with Michaelis–Menten type harvesting:

$$\begin{aligned} \frac{dN_1(t)}{dt} &= N_1(t)\Big(a(t) - b(t)N_1(t) + c(t)N_2(t)\Big), \\ \frac{dN_2(t)}{dt} &= N_2(t)\Big(d(t) - e(t)N_2(t)\Big) - \frac{q(t)E(t)N_2(t)}{m_1(t)E(t) + m_2(t)N_2(t)}. \end{aligned} \tag{5}$$

The authors investigated the existence and stability of the positive periodic solution of the system.

Zhou et al. [41] argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. They proposed the following discrete commensal symbiosis model with nonlinear harvesting:

$$\begin{aligned} N_1(k+1) &= N_1(k)\exp\left\{a_1(k) - b_1(k)N_1(k) + c_1(k)N_2(k)\right\}, \\ N_2(k+1) &= N_2(k)\exp\left\{a_2(k) - b_2(k)N_2(k) - \frac{q(k)E(k)}{m_1(k)E(k)+m_2(k)N_2(k)}\right\}, \end{aligned} \tag{6}$$

where $N_1(k)$ and $N_2(k)$ represent the densities of the first and second species of k-generation, respectively. Under the assumption that the coefficients of the system (6) are all periodic sequences with a common integer period, they obtained a set of sufficient conditions which ensure the existence of at least one positive periodic solution of the system.

Shireen Jawad [32] proposed the following commensalism model with Michaelis–Menten type harvesting and a Holling II functional response:

$$\begin{aligned} \frac{du}{dt} &= ru\left(1 - \frac{u}{k}\right) + \frac{\beta uv}{\alpha+u} - \frac{qEu}{cE+lu}, \\ \frac{dv}{dt} &= sv\left(1 - \frac{v}{m}\right) - dv, \end{aligned} \tag{7}$$

where $u(t)$ and $v(t)$ denote the densities of the first and second species at time t, respectively;

The author investigated the local stability, persistence and bifurcation of the system.

The above works are all focus on the two species commensalism model with nonlinear harvesting. Recently, Puspitasari, Kusumawinahyu and Trisilowati [31,39] began to study a three-species case and a four-species case. In [31], stimulated by the work of Chen [27] and also by observing the relationships among the mango tree (host), orchids (commensals) and parasites (parasites) attached to the trunk of the mango tree, Puspitasari, Kusumawinahyu and Trisilowati proposed the following three-species symbiotic model of commensalism and parasitism with harvesting in commensal populations:

$$\begin{aligned} \frac{dx}{dt} &= r_1x\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qEx}{m_1E+m_2x}, \\ \frac{dy}{dt} &= r_2y\left(1 - \frac{y}{k_2} - b\frac{z}{k_2}\right), \\ \frac{dz}{dt} &= r_3z\left(1 - \frac{z}{k_3} + c\frac{y}{k_3}\right), \end{aligned} \tag{8}$$

where $x(t), y(t)$ and $z(t)$ denote the commensal population, host population and parasite species, respectively. The model is based on the model of Chen [27], by adding a new population, namely the parasite population, which is denoted by z. The authors investigated the existence and local stability of the equilibria of system (8). Recently, by establishing three powerful Lemmas, Chen, Zhou and Lin [40] obtained sufficient conditions which ensure the global stability of the equilibria.

In [39], Puspitasari, Kusumawinahyu and Trisilowati further proposed the following symbiotic model of commensalism in four populations with Michaelis–Menten type harvesting in the first commensal populations:

$$\begin{aligned} \frac{dx}{dt} &= r_1x\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qEx}{m_1E+m_2x}, \\ \frac{dy}{dt} &= r_2y\left(1 - \frac{y}{k_2} - b\frac{z}{k_2}\right), \\ \frac{dz}{dt} &= r_3z\left(1 - \frac{z}{k_3} + c\frac{y}{k_3}\right), \\ \frac{dp}{dt} &= r_4p\left(1 - \frac{p}{k_4} + d\frac{y}{k_4}\right), \end{aligned} \tag{9}$$

where $x(t), y(t), z(t)$ and $p(t)$ denote the first commensal population, host population, parasite species and the second commensal population, respectively. All parameters used in this model are positive. $r_i, i = 1,2,3,4$ interpret the intrinsic growth of x, y, z and p. $k_i, i = 1,2,3,4$ interpret the carrying capacities of x, y, z and p, respectively. The parameter a is the relationship between x and y. The parameters b and c are the relationship between

y and z. d shows the relationship between y and p. The parameter E is a fishing business used for harvest, q is the catching power coefficient, m_1 and m_2 are the suitable constants. p is the second commensal population, it does not harm other populations. The authors showed that the system may have sixteen possible equilibria, and only four of them could be asymptotically stable if they meet the stability conditions that have been determined.

Those four equilibria could be expressed as below:

$$\begin{aligned} &T_4(0,0,k_3,k_4), \quad T_7\Big(0, \tfrac{k_2-bk_3}{1+bc}, \tfrac{k_3+ck_2}{1+bc}, k_4 + \tfrac{d(k_2-bk_3)}{1+bc}\Big), \\ &T_{12}(x_a^*,0,k_3,k_4), \quad T_{15}\Big(x_c^*, \tfrac{k_2-bk_3}{1+bc}, \tfrac{k_3+ck_2}{1+bc}, k_4 + \tfrac{d(k_2-bk_3)}{1+bc}\Big). \end{aligned} \tag{10}$$

Noting that the conclusions of Puspitasari, Kusumawinahyu and Trisilowati [39] are all local ones, whether we could obtain some sufficient conditions to ensure the global stability property of above four equilibria becomes an interesting problem.

When we talk about the global stability property, we naturally think of constructing an appropriate Lyapunov function, for example, to investigate the global stability property of the positive equilibrium:

$$T_{15}\Big(x_c^*, \frac{k_2-bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}, k_4 + \frac{d(k_2-bk_3)}{1+bc}\Big) \stackrel{def}{=} \Big(x_c^*, y^*, z^*, p^*\Big), \tag{11}$$

we may construct the following Lyapunov function:

$$\begin{aligned} V(t) &= l_1\Big(x - x_c^* - x_c^* \ln \tfrac{x}{x_c^*}\Big) + l_2\Big(y - y^* - y^* \ln \tfrac{y}{y^*}\Big) \\ &\quad + l_3\Big(z - z^* - z^* \ln \tfrac{z}{z^*}\Big) + l_4\Big(p - p^* - p^* \ln \tfrac{p}{p^*}\Big). \end{aligned} \tag{12}$$

Then we compute the derivative of V along the positive solution of system (9) to find out some suitable conditions to ensure the negative definite of $\frac{dV}{dt}$. However, the essentiality of this method is to make sure the intraspecific competition coefficient is large enough to control the coefficient of interspecific interaction, which cannot reflect the substantial characteristics of the system. Another troublesome thing is to estimate the term $\frac{qEx}{m_1E+m_2x}$ in system (9). It is well known that this kind of harvesting term is the main factor that leads to the complex dynamic behaviors of the system (see, for example, [32,34,37]); the ecosystem with the Michaelis–Menten type system, generally speaking, has very complex bifurcation behaviors. In the estimation of $\frac{dV}{dt}$, this term may need some extra assumptions, which are needed to ensure the negative definite of $\frac{dV}{dt}$; however, this is not the essential one for ensuring the global stability of the system.

This brings to our attention that in system (9), the second and third equations are independent of the variables x and p; also, the second and third equations are the Lokta–Voletera type systems; this motivated us to investigate the dynamic behaviors of this subsystem firstly, and then to investigate the dynamic behaviors of x and p. The aim of this paper is, by developing the analysis technique of Zhu et al. [38] and Chen, Zhou and Lin [40] , to obtain some very simple but essential conditions to ensure the global attractivity of the above four equilibria.

The rest of the paper is arranged as follows. We state the main conclusions in Section 2. To finish the proof of the main results, we provide several important lemmas in Section 3. A detailed proof of the main results is given in Section 4. Some numerical simulations are provided in Section 5 to show the feasibility of the main results. We end this paper with a brief discussion.

2. Main Results

Definition 1. *Let (x^*, y^*, z^*, p^*) be any equilibrium of system (9), if for any positive solution $(x(t), y(t), z(t), p(t))$ of system (9), the following equalities hold.*

$$\lim_{t\to+\infty} x(t) = x^*, \quad \lim_{t\to+\infty} y(t) = y^*, \quad \lim_{t\to+\infty} z(t) = z^*, \quad \lim_{t\to+\infty} p(t) = p^*.$$

Then we say (x^*, y^*, z^*, p^*) is globally attractive.

The main results of this paper are as follows.

Theorem 1. *Assume that:*

$$r_1 < \frac{qE}{m_1 E + m_2 k_1} \tag{13}$$

and

$$1 < \frac{bk_3}{k_2} \tag{14}$$

hold, then $T_4(0, 0, k_3, k_4)$ is globally attractive, i.e., any positive solution $(x(t), y(t), z(t), p(t))$ of system (9) satisfies

$$\lim_{t\to+\infty} x(t) = 0, \quad \lim_{t\to+\infty} y(t) = 0, \quad \lim_{t\to+\infty} z(t) = k_3, \quad \lim_{t\to+\infty} p(t) = k_4.$$

Theorem 2. *Assume that:*

$$q < r_1 m_1 \tag{15}$$

and

$$1 < \frac{bk_3}{k_2} \tag{16}$$

hold, then $T_{12}(x_a^*, 0, k_3, k_4)$ is globally attractive, i.e., any positive solution $(x(t), y(t), z(t), p(t))$ of system (9) satisfies

$$\lim_{t\to+\infty} x(t) = x_a^*, \quad \lim_{t\to+\infty} y(t) = 0, \quad \lim_{t\to+\infty} z(t) = k_3, \quad \lim_{t\to+\infty} p(t) = k_4.$$

Theorem 3. *Assume that:*

$$r_1\left(1 + \frac{ay^*}{k_1}\right) < \frac{qE}{m_1 E + m_2(k_1 + ay^*)} \tag{17}$$

and

$$1 > \frac{bk_3}{k_2} \tag{18}$$

hold, then $T_7\left(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$ is globally attractive, i.e., any positive solution $(x(t), y(t), z(t), p(t))$ of system (9) satisfies

$$\lim_{t\to+\infty} x(t) = 0, \quad \lim_{t\to+\infty} y(t) = \frac{k_2 - bk_3}{1+bc}, \quad \lim_{t\to+\infty} z(t) = \frac{k_3 + ck_2}{1+bc}, \quad \lim_{t\to+\infty} p(t) = k_4 + \frac{d(k_2 - bk_3)}{1+bc},$$

where

$$y^* = \frac{k_2 - bk_3}{1+bc}.$$

Theorem 4. *Assume that:*

$$r_1\left(1 + \frac{ay^*}{k_1}\right) > \frac{q}{m_1} \tag{19}$$

and

$$1 > \frac{bk_3}{k_2} \tag{20}$$

hold, then $T_{15}\left(x_c^*, \frac{k_2-bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}, k_4+\frac{d(k_2-bk_3)}{1+bc}\right)$ *is globally attractive, i.e., any positive solution* $(x(t), y(t), z(t), p(t))$ *of system (9) satisfies*

$$\lim_{t\to+\infty} x(t) = x_c^*, \ \lim_{t\to+\infty} y(t) = \frac{k_2-bk_3}{1+bc}, \ \lim_{t\to+\infty} z(t) = \frac{k_3+ck_2}{1+bc}, \ \lim_{t\to+\infty} p(t) = k_4 + \frac{d(k_2-bk_3)}{1+bc},$$

where

$$y^* = \frac{k_2-bk_3}{1+bc}.$$

3. Lemmas

To finish the proof of Theorems 1–4, we need several powerful Lemmas.

As a direct corollary of Lemma 2 of Chen [49], we have

Lemma 1. *If* $a > 0, b > 0$ *and* $\dot{x} \geq x(b-ax)$, *when* $t \geq 0$ *and* $x(0) > 0$, *we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a > 0, b > 0$ *and* $\dot{x} \leq x(b-ax)$, *when* $t \geq 0$ *and* $x(0) > 0$, *we have*

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}.$$

Now let us consider the following single species system.

$$\frac{dy}{dt} = ry\left(1-\frac{y}{K}\right) - \frac{qEy}{m_1E+m_2y}. \tag{21}$$

From Lemma 3 and Theorem 2 in Zhu et al. [38], we have:

Lemma 2. *Assume that*

$$r > \frac{q}{m_1} \tag{22}$$

holds, then system (21) admits a unique positive equilibrium y^*, *which is globally stable, where*

$$y^* = \frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}, \tag{23}$$

and

$$\begin{aligned} A_1 &= m_2r, \\ A_2 &= Em_1r - Km_2r, \\ A_3 &= EKq - EKm_1r. \end{aligned} \tag{24}$$

From Theorem 1 in Zhu et al. [38], we have:

Lemma 3. *Assume that*

$$r < \frac{qE}{m_1E+m_2K} \tag{25}$$

holds, then in system (21), species y will finally be driven to extinction, i.e.,

$$\lim_{t\to+\infty} y(t) = 0. \tag{26}$$

Now let us consider the system

$$\begin{aligned} \frac{dy}{dt} &= r_2 y\Big(1 - \frac{y}{k_2} - b\frac{z}{k_2}\Big), \\ \frac{dz}{dt} &= r_3 z\Big(1 - \frac{z}{k_3} + c\frac{y}{k_3}\Big). \end{aligned} \tag{27}$$

Lemma 4. *(i) Assume that:*

$$1 < \frac{bk_3}{k_2} \tag{28}$$

hold, then the boundary equilibrium $(0, k_3)$ in system (27) is globally stable.
(ii) Assume that:

$$1 > \frac{bk_3}{k_2} \tag{29}$$

hold, then system (27) admits a unique positive equilibrium (y^*, z^*), which is globally stable, where

$$y^* = \frac{k_2 - bk_3}{1 + bc}, z^* = \frac{k_3 + ck_2}{1 + bc}. \tag{30}$$

Proof. (i) Let us consider the Lyapunov function

$$V_1(x, y) = y + \frac{r_2 bk_3}{r_3 ck_2}\Big(z - k_3 - k_3 \ln\frac{z}{k_3}\Big). \tag{31}$$

By computation, we have:

$$\frac{dV_1}{dt} = r_2\Big(1 - \frac{bk_3}{k_2}\Big)y - \frac{r_2}{k_2}y^2 - \frac{r_2 b}{ck_2}(z - k_3)^2. \tag{32}$$

Hence, under the assumption (28) holding, $\frac{dV_1}{dt} < 0$ strictly for all $y, z > 0$ except the boundary equilibrium $(0, k_3)$, where $\frac{dV_1}{dt} = 0$. Thus, $V_1(x, y)$ satisfies Lyapunov's asymptotic stability theorem, and the boundary equilibrium $(0, k_3)$ of system (27) is globally stable.

(ii) We had proved this part in the draft, however, when we revising the paper, we found the paper recently published by Chen, Zhou and Lin [40] had proved the conclusion. So, one could refer to the proof of Lemma 4 in [40] for more detail.

The proof of Lemma 4 is finished. □

4. Proof of the Main Results

Proof of Theorem 1 . For small enough $\varepsilon > 0$, condition (14) implies that

$$r_1 + \frac{ar_1\varepsilon}{k_1} < \frac{qE}{m_1 E + m_2(k_1 + a\varepsilon)}. \tag{33}$$

Let $(x(t), y(t), z(t), p(t))^T$ be any positive solution of system (9), it follows from (14) and Lemma 4 (i) that:

$$\lim_{t\to+\infty} y(t) = 0, \ \lim_{t\to+\infty} z(t) = k_3. \tag{34}$$

From above equation, for $\varepsilon > 0$ enough small, which satisfies inequality (33), there exists a $T_1 > 0$, such that:

$$y(t) < \varepsilon \ \text{for all} \ t \geq T_1. \tag{35}$$

For $t \geq T_1$, from (35) and the first equation of system (9), we have:

$$\begin{aligned} \frac{dx}{dt} &\leq r_1 x\left(1 - \frac{x}{k_1} + a\frac{\varepsilon}{k_1}\right) - \frac{qEx}{m_1E + m_2x} \\ &= r_1\left(1 + a\frac{\varepsilon}{k_1}\right)x\left(1 - \frac{x}{k_1\left(1 + a\frac{\varepsilon}{k_1}\right)}\right) - \frac{qEx}{m_1E + m_2x}. \end{aligned} \tag{36}$$

Now let us consider the equation,

$$\frac{du}{dt} = r_1\left(1 + a\frac{\varepsilon}{k_1}\right)u\left(1 - \frac{u}{k_1\left(1 + a\frac{\varepsilon}{k_1}\right)}\right) - \frac{qEu}{m_1E + m_2u}. \tag{37}$$

From (33) and Lemma 3, it follows that:

$$\lim_{t \to +\infty} u(t) = 0. \tag{38}$$

From (36), (37) and the comparison theorem of differential equation, it immediately follows that:

$$\lim_{t \to +\infty} x(t) = 0. \tag{39}$$

From (35) and the forth equation of system (9), for $t \geq T_1$, one has:

$$\frac{dp}{dt} \leq r_4 z\left(1 - \frac{p}{k_4} + d\frac{\varepsilon}{k_4}\right). \tag{40}$$

Applying Lemma 1 to (40) leads to:

$$\limsup_{t \to +\infty} p(t) \leq k_4\left(1 + d\frac{\varepsilon}{k_4}\right). \tag{41}$$

From the forth equation of system (9), we also have:

$$\frac{dp}{dt} \geq r_4 z\left(1 - \frac{p}{k_4}\right). \tag{42}$$

Applying Lemma 1 to (42) leads to:

$$\liminf_{t \to +\infty} p(t) \geq k_4. \tag{43}$$

(41) together with (43) leads to:

$$k_4 \leq \liminf_{t \to +\infty} p(t) \leq \limsup_{t \to +\infty} p(t) \leq k_4\left(1 + d\frac{\varepsilon}{k_4}\right). \tag{44}$$

Setting $\varepsilon \to 0$ in the above inequality, one has:

$$\lim_{t \to +\infty} p(t) = k_4. \tag{45}$$

(34), (39) and (45) show that $T_4(0, 0, k_3, k_4)$ is globally attractive.
This completes the proof of Theorem 1. □

Proof of Theorem 2. For $\varepsilon > 0$ enough small, condition (14) implies that

$$r_1 + \frac{a r_1 \varepsilon}{k_1} > \frac{q}{m_1}. \tag{46}$$

Let $(x(t), y(t), z(t), p(t))^T$ be any positive solution of system (9), it follows from (16) and Lemma 4 (i) that:

$$\lim_{t\to+\infty} y(t) = 0, \ \lim_{t\to+\infty} z(t) = k_3. \tag{47}$$

From above equation, for $\varepsilon > 0$ enough small, which satisfies inequality (47), there exists a $T_1 > 0$, such that:

$$y(t) < \varepsilon \ \ for \ \ all \ \ t \geq T_1. \tag{48}$$

For $t \geq T_1$, from (48) and the first equation of system (9), we have:

$$\begin{aligned} \frac{dx}{dt} &\leq r_1 x\left(1 - \frac{x}{k_1} + a\frac{\varepsilon}{k_1}\right) - \frac{qEx}{m_1E+m_2x} \\ &= r_1\left(1 + a\frac{\varepsilon}{k_1}\right)x\left(1 - \frac{x}{k_1\left(1+a\frac{\varepsilon}{k_1}\right)}\right) - \frac{qEx}{m_1E+m_2x}. \end{aligned} \tag{49}$$

Now let us consider the equation:

$$\frac{du}{dt} = r_1\left(1 + a\frac{\varepsilon}{k_1}\right)u\left(1 - \frac{u}{k_1\left(1+a\frac{\varepsilon}{k_1}\right)}\right) - \frac{qEu}{m_1E+m_2u}. \tag{50}$$

From (46) and Lemma 2, it follows that system (50) admits a unique positive equilibrium $u^*(\varepsilon)$, which is globally stable, i.e.,

$$\lim_{t\to+\infty} u(t) = u^*(\varepsilon), \tag{51}$$

where

$$u^*(\varepsilon) = \frac{-B_2 + \sqrt{B_2^2 - 4B_1B_3}}{2B_1}, \tag{52}$$

and

$$\begin{aligned} B_1 &= m_2 r_1\left(1 + a\frac{\varepsilon}{k_1}\right), \\ B_2 &= Em_1 r_1\left(1 + a\frac{\varepsilon}{k_1}\right) - k_1 m_2 r_1\left(1 + a\frac{\varepsilon}{k_1}\right)^2, \\ B_3 &= Ek_1 q\left(1 + a\frac{\varepsilon}{k_1}\right) - Ek_1 m_1 r_1\left(1 + a\frac{\varepsilon}{k_1}\right)^2. \end{aligned} \tag{53}$$

By comparison theorem of differential equation and (49), (51), we have:

$$\limsup_{t\to+\infty} x(t) \leq u^*(\varepsilon). \tag{54}$$

For $t \geq T_1$, from the first equation of system (9), we also have:

$$\frac{dx}{dt} \geq r_1 x\left(1 - \frac{x}{k_1}\right) - \frac{qEx}{m_1E + m_2x}. \tag{55}$$

Now let us consider the equation:

$$\frac{dv}{dt} = r_1 v\left(1 - \frac{v}{k_1}\right) - \frac{qEv}{m_1E+m_2x}. \tag{56}$$

From (56) and Lemma 2, it follows that system (56) admits a unique positive equilibrium $v^* = x_5^*$, which is globally stable, i.e.,

$$\lim_{t\to+\infty} v(t) = v^*, \tag{57}$$

where

$$v^* = \frac{-C_2 + \sqrt{C_2^2 - 4C_1C_3}}{2C_1}, \tag{58}$$

and

$$\begin{aligned} C_1 &= m_2r_1, \\ C_2 &= Em_1r_1 - k_1m_2r_1, \\ C_3 &= Ek_1q - Ek_1m_1r_1. \end{aligned} \tag{59}$$

By comparison theorem of differential equation and (55), (56), we have:

$$\liminf_{t\to+\infty} x(t) \geq v^*. \tag{60}$$

(54) and (60) lead to:

$$v^* \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq u^*(\varepsilon). \tag{61}$$

Since ε is enough small positive constant, setting $\varepsilon \to 0$ in (61) leads to:

$$\lim_{t\to+\infty} x(t) = x_5^*. \tag{62}$$

Similarly to the analysis of (40)–(45), by using inequality (48), we could obtain:

$$\lim_{t\to+\infty} p(t) = k_4. \tag{63}$$

(47), (62) and (63) show that $T_{12}(x_5^*, 0, k_3, k_4)$ is globally attractive. This completes the proof of Theorem 2. □

Proof of Theorem 3. For small enough $\varepsilon > 0$, condition (14) implies that

$$r_1 + \frac{ar_1(y^* + \varepsilon)}{k_1} < \frac{qE}{m_1E + m_2(k_1 + a(y * +\varepsilon))}. \tag{64}$$

Let $(x(t), y(t), z(t), p(t))^T$ be any positive solution of system (9), it follows from (18) and Lemma 4 (ii) that:

$$\lim_{t\to+\infty} y(t) = y^* = \frac{k_2 - bk_3}{1 + bc}, \quad \lim_{t\to+\infty} z(t) = z^* = \frac{k_3 + ck_2}{1 + bc}. \tag{65}$$

From above equation, for $\varepsilon > 0$ enough small, which satisfies inequality (64), without loss of generality, we may assume that $\varepsilon < \frac{1}{2}y^*$, there exists a $T_1 > 0$, such that:

$$y^* - \varepsilon < y(t) < y^* + \varepsilon \ \ for \ \ all \ \ t \geq T_1. \tag{66}$$

For $t \geq T_1$, from (66) and the first equation of system (9), we have:

$$\begin{aligned} \tfrac{dx}{dt} &\leq r_1x\left(1 - \tfrac{x}{k_1} + a\tfrac{(y^*+\varepsilon)}{k_1}\right) - \tfrac{qEx}{m_1E+m_2x} \\ &= r_1\left(1 + a\tfrac{(y^*+\varepsilon)}{k_1}\right)x\left(1 - \frac{x}{k_1\left(1+a\frac{y^*+\varepsilon}{k_1}\right)}\right) - \tfrac{qEx}{m_1E+m_2x}. \end{aligned} \tag{67}$$

Now let us consider the equation:

$$\tfrac{du}{dt} = r_1\left(1 + a\tfrac{(y^*+\varepsilon)}{k_1}\right)u\left(1 - \frac{u}{k_1\left(1+a\frac{y^*+\varepsilon}{k_1}\right)}\right) - \tfrac{qEu}{m_1E+m_2u}. \tag{68}$$

From (64) and Lemma 3, it follows that:

$$\lim_{t\to+\infty} u(t) = 0. \tag{69}$$

From (67) and (68) and the comparison theorem of differential equation, it immediately follows that:

$$\lim_{t\to+\infty} x(t) = 0. \tag{70}$$

From (66) and the fourth equation of system (9), for $t \geq T_1$, one has:

$$\frac{dp}{dt} \leq r_4 z\Big(1 - \frac{p}{k_4} + d\frac{y^* + \varepsilon}{k_4}\Big). \tag{71}$$

Applying Lemma 1 to (71) leads to:

$$\limsup_{t\to+\infty} p(t) \leq k_4\Big(1 + d\frac{y^* + \varepsilon}{k_4}\Big). \tag{72}$$

From (66) and the forth equation of system (9), for $t \geq T_1$, we also have:

$$\frac{dp}{dt} \geq k_4 z\Big(1 - \frac{p}{k_4} + d\frac{y^* - \varepsilon}{k_4}\Big). \tag{73}$$

Applying Lemma 1 to (73) leads to:

$$\liminf_{t\to+\infty} p(t) \geq k_4\Big(1 + d\frac{y^* - \varepsilon}{k_4}\Big). \tag{74}$$

(72) together with (74) leads to:

$$k_4\Big(1 + d\frac{y^* - \varepsilon}{k_4}\Big) \leq \liminf_{t\to+\infty} p(t) \leq \limsup_{t\to+\infty} p(t) \leq k_4\Big(1 + d\frac{y^* + \varepsilon}{k_4}\Big). \tag{75}$$

Setting $\varepsilon \to 0$ in the above inequality, it follows that:

$$\lim_{t\to+\infty} p(t) = k_4\Big(1 + d\frac{y^*}{k_4}\Big) = k_4 + dy^* = k_4 + \frac{d(k_2 - bk_3)}{1 + bc}. \tag{76}$$

(65), (70) and (76) show that $T_7\Big(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\Big)$ is globally attractive. This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. For $\varepsilon > 0$ enough small, condition (19) implies that

$$r_1 + \frac{ar_1(y^* - \varepsilon)}{k_1} > \frac{q}{m_1}. \tag{77}$$

Let $(x(t), y(t), z(t), p(t))^T$ be any positive solution of system (9), it follows from (20) and Lemma 4 (ii) that:

$$\lim_{t\to+\infty} y(t) = y^* = \frac{k_2 - bk_3}{1 + bc}, \ \lim_{t\to+\infty} z(t) = z^* = \frac{k_3 + ck_2}{1 + bc}. \tag{78}$$

From above equation, for $\varepsilon > 0$ enough small, which satisfies inequality (77), without loss of generality, we may assume that $\varepsilon < \frac{1}{2}y^*$, there exists a $T_1 > 0$, such that:

$$y^* - \varepsilon < y(t) < y^* + \varepsilon \ \ for \ \ all \ \ t \geq T_1. \tag{79}$$

For $t \geq T_1$, from (79) and the first equation of system (9), we have:

$$\begin{aligned} \frac{dx}{dt} &\leq r_1 x\left(1 - \frac{x}{k_1} + a\frac{(y^*+\varepsilon)}{k_1}\right) - \frac{qEx}{m_1E+m_2x} \\ &= r_1\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right)x\left(1 - \frac{x}{k_1\left(1+a\frac{y^*+\varepsilon}{k_1}\right)}\right) - \frac{qEx}{m_1E+m_2x}. \end{aligned} \tag{80}$$

Now let us consider the equation:

$$\frac{du_1}{dt} = r_1\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right)u_1\left(1 - \frac{u}{k_1\left(1+a\frac{y^*+\varepsilon}{k_1}\right)}\right) - \frac{qEu_1}{m_1E+m_2u_1}. \tag{81}$$

From (77) and Lemma 2, it follows that:

$$\lim_{t\to+\infty} u(t) = u_1^*(\varepsilon), \tag{82}$$

where

$$u_1^*(\varepsilon) = \frac{-D_2 + \sqrt{D_2^2 - 4D_1D_3}}{2D_1}, \tag{83}$$

and

$$\begin{aligned} D_1 &= m_2r_1\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right), \\ D_2 &= Em_1r_1\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right) - k_1m_2r_1\left(1 + a\frac{\varepsilon}{k_1}\right)^2, \\ D_3 &= Ek_1q\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right) - Ek_1m_1r_1\left(1 + a\frac{(y^*+\varepsilon)}{k_1}\right)^2. \end{aligned} \tag{84}$$

By the comparison theorem of the differential equation, (80) and (81), we have:

$$\limsup_{t\to+\infty} x(t) \leq u_1^*(\varepsilon). \tag{85}$$

For $t \geq T_1$, from the first equation of system (9), we also have:

$$\frac{dx}{dt} \geq r_1x\left(1 - \frac{x}{k_1} + a\frac{(y^* - \varepsilon)}{k_1}\right) - \frac{qEx}{m_1E + m_2x}. \tag{86}$$

Now let us consider the equation

$$\frac{dv_1}{dt} = r_1v_1\left(1 - \frac{v_1}{k_1} + a\frac{(y^*-\varepsilon)}{k_1}\right) - \frac{qEv_1}{m_1E+m_2v_1}. \tag{87}$$

From (77) and Lemma 2, it follows that system (87) admits a unique positive equilibrium $v_1^*(\varepsilon)$, which is globally stable, i.e.,

$$\lim_{t\to+\infty} v(t) = v_1^*(\varepsilon), \tag{88}$$

where

$$v_1^*(\varepsilon) = \frac{-E_2 + \sqrt{E_2^2 - 4E_1E_3}}{2E_1}, \tag{89}$$

and

$$\begin{aligned} E_1 &= m_2 r_1 \left(1 + a\frac{(y^* - \varepsilon)}{k_1}\right), \\ E_2 &= E m_1 r_1 \left(1 + a\frac{(y^* - \varepsilon)}{k_1}\right) - k_1 m_2 r_1 \left(1 + a\frac{y^* - \varepsilon}{k_1}\right)^2, \\ E_3 &= E k_1 q - E k_1 m_1 r_1 \left(1 + a\frac{(y^* - \varepsilon)}{k_1}\right)^2. \end{aligned} \tag{90}$$

By comparison theorem of differential equation, (86) and (87), we have:

$$\liminf_{t \to +\infty} x(t) \geq v_1^*(\varepsilon). \tag{91}$$

(85) and (91) lead to:

$$v_1^*(\varepsilon) \leq \liminf_{t \to +\infty} x(t) \leq \limsup_{t \to +\infty} x(t) \leq u_1^*(\varepsilon). \tag{92}$$

Since ε is enough small positive constant, setting $\varepsilon \to 0$ in (92) leads to:

$$\lim_{t \to +\infty} x(t) = x_c^*, \tag{93}$$

where

$$x_c^*(\varepsilon) = \frac{-F_2 + \sqrt{F_2^2 - 4F_1 F_3}}{2F_1}, \tag{94}$$

and

$$\begin{aligned} F_1 &= m_2 r_1 \left(1 + a\frac{y^*}{k_1}\right), \\ F_2 &= E m_1 r_1 \left(1 + a\frac{y^*}{k_1}\right) - k_1 m_2 r_1 \left(1 + a\frac{y^*}{k_1}\right)^2, \\ F_3 &= E k_1 q \left(1 + a\frac{y^*}{k_1}\right) - E k_1 m_1 r_1 \left(1 + a\frac{y^*}{k_1}\right)^2. \end{aligned} \tag{95}$$

Similarly to the analysis of (71)–(76), by using inequality (79), we could obtain:

$$\lim_{t \to +\infty} p(t) = k_4 \left(1 + d\frac{y^*}{k_4}\right) = k_4 + dy^* = k_4 + \frac{d(k_2 - bk_3)}{1 + bc}. \tag{96}$$

(78), (93) and (96) show that $T_{15}\left(x_c^*, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}, k_4 + \frac{d(k_2 - bk_3)}{1+bc}\right)$ is globally attractive. This completes the proof of Theorem 4. □

5. Numeric Simulations

Now let us consider the following examples.

Example 1. *Consider the following system*

$$\begin{aligned} \frac{dx}{dt} &= r_1 x(1 - x + y) - \frac{x}{2+x}, \\ \frac{dy}{dt} &= y(1 - y - 2z), \\ \frac{dz}{dt} &= z(1 - z + y), \\ \frac{dp}{dt} &= p(1 - p + y). \end{aligned} \tag{97}$$

Here, corresponding to system (9), we choose $r_2 = r_3 = r_4 = k_1 = k_2 = k_3 = k_4 = c = a = d = E = m_2 = 1, q = 1, b = m_1 = 2$, then by simple computation, we have:
(1) For $r_1 = 1$,

$$q = 1 < 2 = r_1 m_1 \tag{98}$$

and

$$1 < 2 = \frac{bk_3}{k_2} \tag{99}$$

hold, then it follows from Theorem 2 that $T_{12}\left(\frac{\sqrt{5}}{2} - \frac{1}{2}, 0, 1, 1\right)$ *is globally attractive. Figures 1–3 support this assertion;*
(2) For $r_1 = \frac{1}{4}$,

$$r_1 = \frac{1}{4} < \frac{1}{3} = \frac{qE}{m_1 E + m_2 k_1} \tag{100}$$

and

$$1 < 2 = \frac{bk_3}{k_2} \tag{101}$$

hold, then it follows from Theorem 2 that $T_4(0,0,1,1)$ *is globally attractive. In this case, we will only be concerned with the dynamic behaviors of species* x, *Figure 4 supports this assertion;*
(3) For $r_1 = \frac{9}{24}$, *one has:*

$$\frac{qE}{m_1 E + m_2 k_1} = \frac{1}{3} < r_1 = \frac{9}{24} < \frac{1}{2} = \frac{q}{m_1} \tag{102}$$

and

$$1 < 2 = \frac{bk_3}{k_2}; \tag{103}$$

in this case, conditions of Theorems 1 and 2 are not satisfied, and we have no idea about the global dynamic behaviors of the system (97); however, the numeric simulation (Figure 5) shows that, in this case, the first species is still extinct.

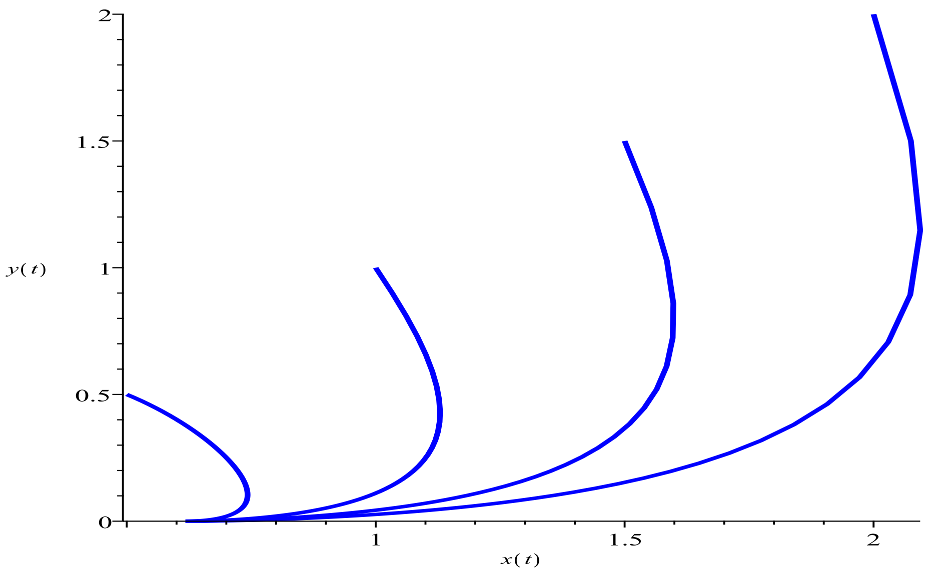

Figure 1. Dynamic behaviors of the first and second components x and y) in system (97) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1,1,1,1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2,2,2,2)$, respectively. One could see that $x(t) \to \frac{\sqrt{5}}{2} - \frac{1}{2}$, $y(t) \to 0$.

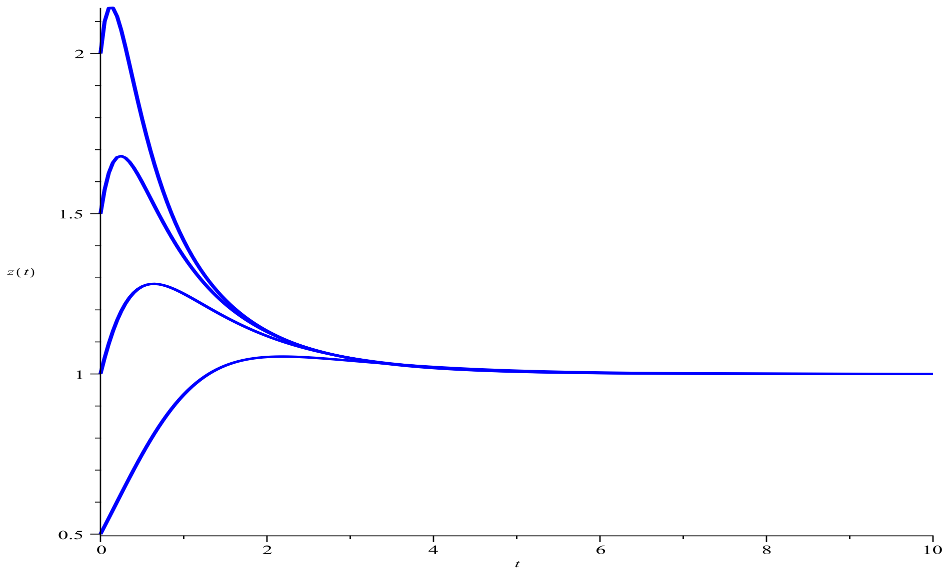

Figure 2. Dynamic behaviors of the third component z in system (97) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $z(t) \to 1$.

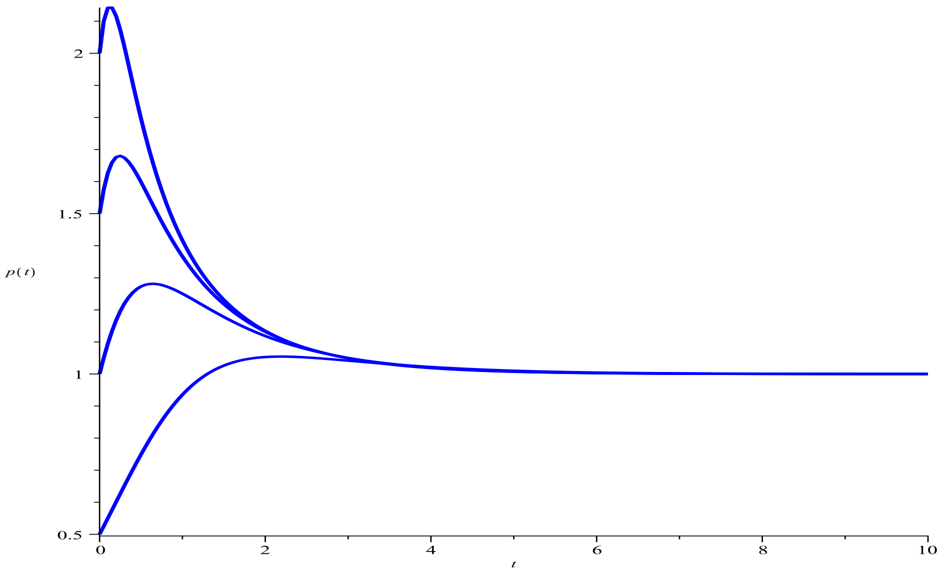

Figure 3. Dynamic behaviors of the forth component p in system (97) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $p(t) \to 1$.

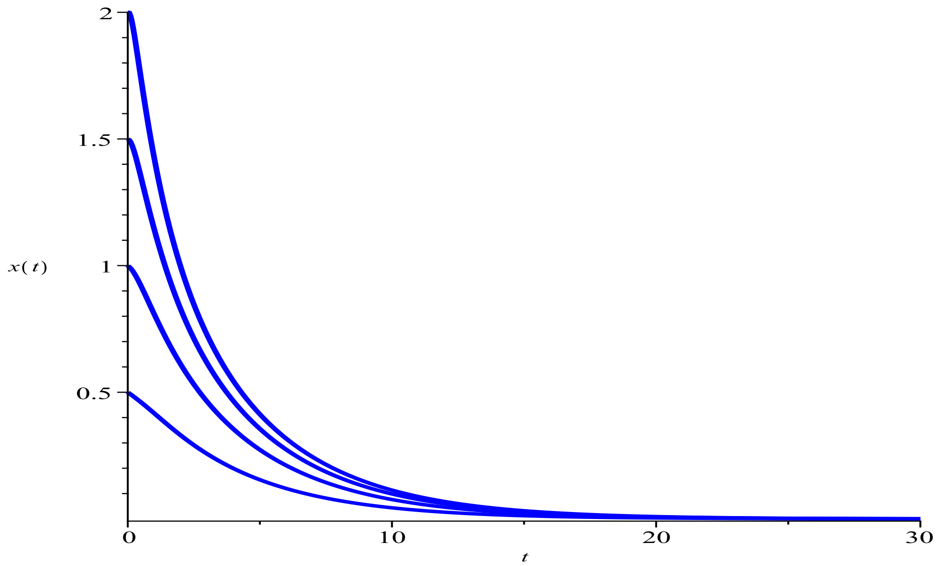

Figure 4. Dynamic behaviors of the first species x in system (97) with $r_1 = \frac{1}{4}$ and the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $x(t) \to 0$.

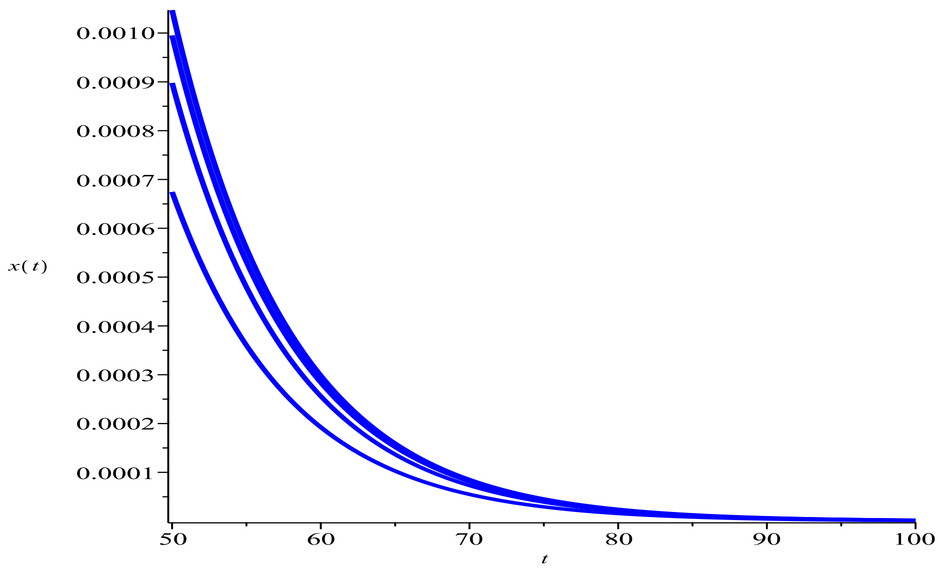

Figure 5. Dynamic behaviors of the first species x in system (97) with $r_1 = \frac{9}{24}$ and the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $x(t) \to 0$.

Example 2. *Consider the following system:*

$$\begin{aligned} \frac{dx}{dt} &= r_1 x(1 - x + y) - \frac{x}{2+x}, \\ \frac{dy}{dt} &= y(1 - \frac{y}{2} - \frac{z}{2}), \\ \frac{dz}{dt} &= z(1 - z + y), \\ \frac{dp}{dt} &= z(1 - p + y). \end{aligned} \tag{104}$$

Here, corresponding to system (9), we choose $r_2 = r_3 = r_4 = k_1 = k_3 = k_4 = b = c = a = d = E = m_2 = 1, q = 1, k_2 = m_1 = 2$, by simple computation, we have:
(1) For $r_1 = 1$,

$$r_1\left(1 + \frac{ay^*}{k_1}\right) = \frac{3}{2} > \frac{1}{2} = \frac{q}{m_1} \tag{105}$$

and

$$1 > \frac{bk_3}{k_2} \tag{106}$$

hold, then $T_{15}\Big(1.186, 0.5, 1.5, 1.5\Big)$ *is globally attractive. Figures 6–8 support this assertion;*

(2) For $r_1 = \frac{3}{21}$,

$$r_1\Big(1 + \frac{ay^*}{k_1}\Big) = \frac{9}{42} < \frac{2}{7} = \frac{qE}{m_1E + m_2(k_1 + ay^*)} \tag{107}$$

and

$$1 > \frac{bk_3}{k_2} \tag{108}$$

hold, then it follows from Theorem 3 that $T_7(0, 0.5, 1, 1)$ *is globally attractive. In this case, we will only be concerned with the dynamic behaviors of species* x*, Figure 9 supports this assertion;*

(3) For $r_1 = \frac{6}{21}$*, one has*

$$\frac{qE}{m_1E + m_2(k_1 + ay^*)} = \frac{4}{21} < r_1\Big(1 + \frac{ay^*}{k_1}\Big) = \frac{18}{42} < \frac{1}{2} = \frac{q}{m_1} \tag{109}$$

and

$$1 > \frac{bk_3}{k_2}; \tag{110}$$

in this case, the conditions of Theorems 3 and 4 are not satisfied, and we have no idea about the global dynamic behaviors of the system (104); however, the numeric simulation (Figure 10) shows that, in this case, the first species is still extinct.

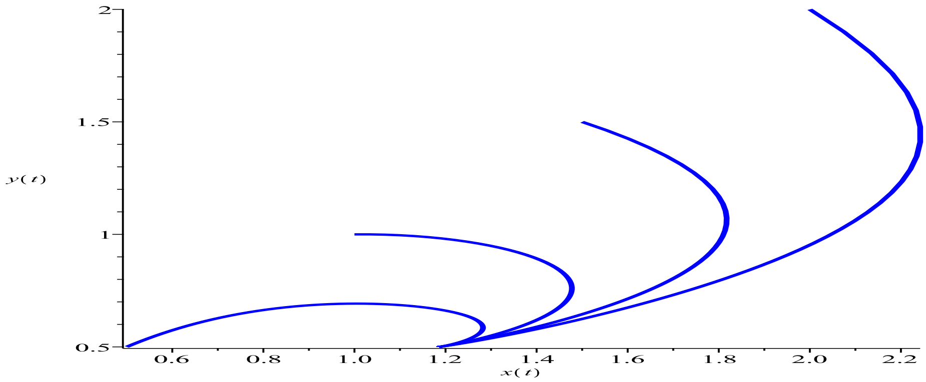

Figure 6. Dynamic behaviors of the first and second components (x and y) in system (104) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $x(t) \to 1.186, y(t) \to 0.5$.

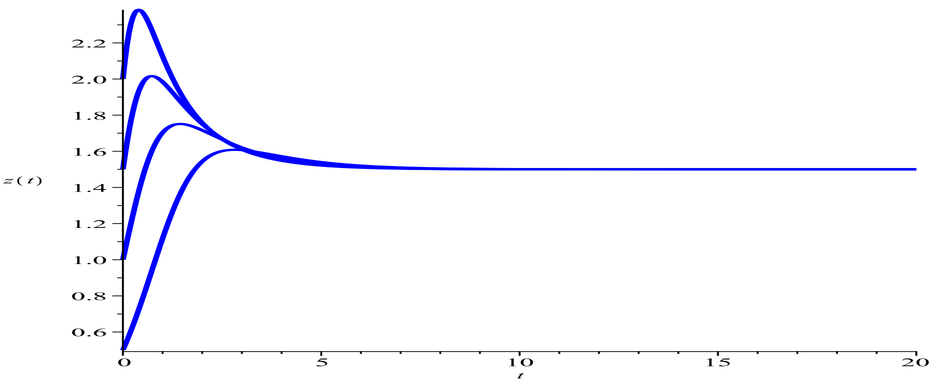

Figure 7. Dynamic behaviors of the third component z in system (104) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $z(t) \to 1.5$.

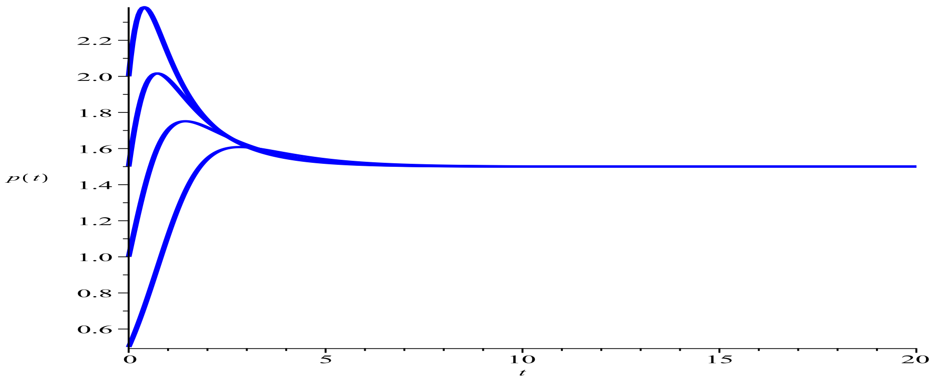

Figure 8. Dynamic behaviors of the forth component p in system (104) with the initial condition $(x(0), y(0), z(0), p(0)) = (0.5, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(1.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $p(t) \to 1.5$.

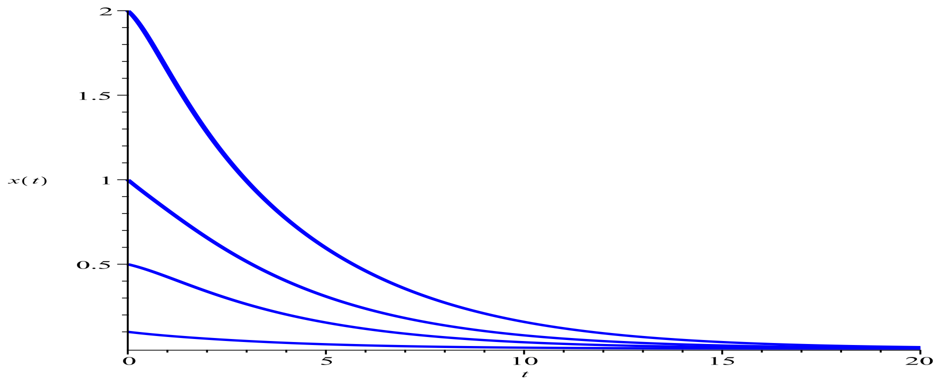

Figure 9. Dynamic behaviors of the first component x in system (104) with $r_1 = \frac{3}{21}$ and the initial condition $(x(0), y(0), z(0), p(0)) = (0.1, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(0.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $x(t) \to 0$.

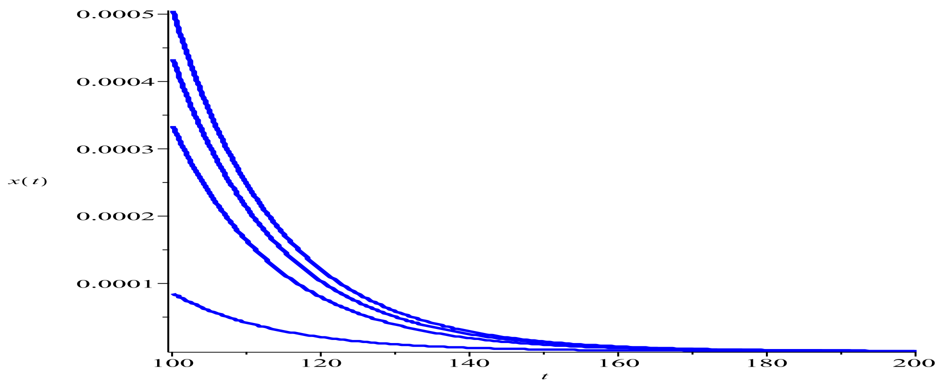

Figure 10. Dynamic behaviors of the first component x in system (104) with $r_1 = \frac{6}{21}$ and the initial condition $(x(0), y(0), z(0), p(0)) = (0.1, 0.5, 0.5, 0.5)$, $(1, 1, 1, 1)$, $(0.5, 1.5, 1.5, 1.5)$ and $(2, 2, 2, 2)$, respectively. One could see that $x(t) \to 0$.

6. Conclusions

Puspitasari, Kusumawinahyu and Trisilowati [39] proposed system (9). The system has sixteen equilibria. By computation, they showed that only four of them—T_4, T_7, T_{12}, T_{15}—could be locally asymptotically stable under some suitable assumptions. This brings to our attention that the second and third equations in system (9) are independent of the variables x and p; therefore, we could study this subsystem previously. With the help of two Lemmas recently proved by Zhu et al. [38], and the stability results for the subsystem (see Lemma 4), we finally obtained the sufficient conditions which ensure the global attractivity of those four equilibria. Noting that the results of [39] are locally one, while ours are globally one, it is in this sense that the results obtained here essentially improve and generalize the main results.

It should be pointed out that all of our results are sufficient, that is, under the assumption of the Theorem, globally attractive results followed. However, one could see from Theorems 1 and 2 that for the parameter r_1, which satisfies the inequality $\frac{qE}{m_1E+m_2k_1} < r_1 < \frac{q}{m_1}$, we have no idea about the dynamic behaviors of the system. One of the reviewers pointed out that we should add some numeric simulations to show the dynamic behaviors of the system if the conditions of Theorem are not satisfied. Our numeric simulation (Figure 5) shows that, in this case, the first species could still be driven to extinction, which means that Theorem 1 still has room to improve. Similarly, Theorem 3 also has room to improve. However, at present, due to the restriction of our method, we could only give numeric simulations. It is necessary for us to provide a thorough analysis of the dynamic behaviors of the system (9), and we will try this later.

One could see that, for the two species commensalism model with Michaelis–Menten type harvesting, the equilibria is increasing and the dynamic behaviors become complicated. For example, Jawad [32] showed that the system may have saddle-node bifurcation. However, for the higher dimensional system, whether the system could have saddle-node bifurcation is unknown. From our numeric simulation, it seems that the dynamic behaviors are not complex—is this the real case? At present we have no answer on this matter.

It is well known that a more plausible system should be a non-autonomous one, since the coefficients of the system are variable with time, therefore, it is also necessary to investigate the dynamic behaviors of the non-autonomous system; we leave this for future study.

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