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General Pseudo Quasi-Overlap Functions on Lattices

Rui Eduardo Brasileiro Paiva [1,*] **and Benjamín René Callejas Bedregal** [2]

[1] Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Maracanaú 61939-140, Brazil
[2] Departamento de Informática e Matemática Aplicada, Universidade Federal do Rio Grande do Norte, Natal 59078-970, Brazil
* Correspondence: rui.brasileiro@ifce.edu.br

Abstract: The notion of general quasi-overlaps on bounded lattices was introduced as a special class of symmetric n-dimensional aggregation functions on bounded lattices satisfying some bound conditions and which do not need to be continuous. In this paper, we continue developing this topic, this time focusing on another generalization, called general pseudo-overlap functions on lattices, which in a given classification system measures the degree of overlapping of several classes and for any given object where symmetry is an unnecessarily restrictive condition. Moreover, we also provide some methods of constructing these functions, as well as a characterization theorem for them. Also, the notions of pseudo-t-norms and pseudo-t-conorms are used to generalize the concepts of additive and multiplicative generators for the context of general pseudo-quasi-overlap functions on lattices and we explore some related properties.

Keywords: aggregation functions; general pseudo quasi-overlap functions; bounded lattices; L-fuzzy sets; additive and multiplicative generators

MSC: 26A48; 03G10; 18B35; 47S40; 03E72

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1. Introduction

Widely studied in the fuzzy set theory, aggregation functions have importance not only in the measure and integration theory or theory of functional equations but also in several applied fields. For example, for medical diagnosis, a mathematical model is given by the equation $\mathcal{S} \otimes^t \mathcal{X} = \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are respectively the matricial forms of the fuzzy relations of symptoms and patients, with the product t-norm, then the diagnostic matrix is given by $\mathcal{D} = \mathcal{S}^{-1} \otimes_{g_n} \mathcal{T}$, where g_n denotes Goguen's implication (see [1]). In [2], a chapter is devoted to fuzzy decision-making in public health strategies based on fuzzy aggregation functions. In this perspective, for a given fuzzy measure, the authors used Sugeno integrals are used to determine the expected fuzzy value in the context of the analysis of traffic accidents in the city of São Paulo, Brazil.

In some situations, it is necessary to measure the degree of overlapping of an object in fuzzy rule-based classification systems with more than two classes. In this context, in order to develop a classifier that tackles the problem of determining the risk of a to be suffering from a cardiovascular disease within the next 10 years, in [3] the authors used rules of the type:

$$\text{Rule } R_j: \text{ If } x_{p1} \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_{pn} \text{ is } A_{jn} \text{ then Class } = C_j \text{ with } RW_j,$$

where the inference procedure computes $A_n(\mu_{A_{j1}}(x_{p1}), \ldots, \mu_{A_{jn}}(x_{pn}))$ for an aggregation function A_n.

Overlap functions, introduced in [4], are continuous and commutative bivariate aggregation functions on $[0,1]$ (not necessarily associative) satisfying appropriate boundary conditions and have been widely investigated in the literature, as for example in [5–10].

Since then, many generalizations and extensions of this notion arose, such as [11–17]. In particular, in [18,19], the authors studied a special type of n-ary aggregation function on $[0, 1]$, called general overlap functions, which measure the degree of overlapping (intersection for non-crisp sets) of n different classes, for computing the matching degree in a classification problem. In [20] the authors used a special class of binary general overlap functions to expand the notion of BL-algebras (which are the algebraic counterpart of a type of fuzzy logic modeled by Peter Hájek). This class of functions offers a promising field of research [7,21–24].

In [23], Paiva et al., introduced the concept of quasi-overlap and overlap functions on bounded lattices and investigated some important properties of them. Other important theoretical results on these subjects can be found in [25–29]. Moreover, in [30] Paiva and Bedregal introduced the notion of general overlap functions in the context of bounded lattices and proposed some construction methods and a characterization theorem for this class of functions. Recently, in his doctoral thesis, Batista [31] removed the commutativity requirement from the properties of overlap functions and introduced the notion of pseudo-overlap functions to define new generalizations of Choquet integrals, named Pseudo-Choquet Integrals and Absolute Choquet Integral. At the same time, but independently, Zhang and Liang gave a talk where some examples and construction methods of pseudo overlap functions and pseudo grouping functions, and the residual implication (co-implication) operators derived from them were investigated, as well as some applications of pseudo overlap (grouping) functions in multi-attribute group decision-making, fuzzy math morphology, and image processing were discussed [32]. A complete version of this preliminary work is derived in the paper [33]. Also recently, Liang and Zhang [34] formalized the notion of interval-valued pseudo overlap functions and a few of their properties, including migrativity and homogeneity, and give some construction theorems and specific examples.

In this paper, we continue to consider this research by proposing a generalization of these three types of variants of overlap functions on bounded lattices, called pseudo general quasi-overlap functions, which are special aggregation functions on bounded lattices, not necessarily bivariate, to be used in situations where symmetry and continuity are unnecessary or irrelevant. For example, in Multi-Criteria Decision Making (MCDM), criteria, in general, have different levels of importance or weights. Suppose that for each alternative x_i and criteria c_j the expert provides a score $s_{i,j}$ from a bounded lattice L. So, if we have m criteria and n alternatives, we can apply an m-dimensional aggregation function A on L to get an overall score for each alternative x_i, i.e., $score(x_i) = A(s_{i,1}, \ldots, s_{i,m})$, which can be used to rank alternatives. As the criteria have different weights, symmetry is not desirable, associativity is meaningless when $m > 2$, and continuity is irrelevant. Therefore, it is reasonable to use general pseudo-quasi-overlap functions on L as aggregation functions. It is worth mentioning that this simple method of MCDM meets the principles of the increasing nature, dominance, and insensibility to indexations pointed out as basic in [35] for each MCDM.

On the other hand, the study of such generalizations will help us define new Choquet and Sugeno integral classes on bounded lattices that can provide us with some potential applications in the above fields and can provide also more flexibility in applications, from uncertainty control to new forms of data fusion, since data fusion uses overlapping information to determine relationships among data (the data association function) [36].

The rest of this paper is organized as follows: In Section 2, we recall some basic concepts and terminologies over aggregation functions on bounded lattices and the algebra of quasigroups which are used throughout the paper. In Section 3, the notion of general pseudo-quasi-overlap functions is formalized, and characterization and construction methods of general pseudo-quasi-overlap functions are proposed. In Section 4, we use the notions of pseudo t-norms and pseudo-t-conorms to generalize the concepts of additive and multiplicative generators for the context of general pseudo-quasi-overlap functions on lattices and we explore some properties related. Finally, Section 5 contains some concluding remarks.

2. Terminology and Basic Notions

In this section, some basic concepts and terminologies used throughout the paper are remembered.

2.1. Aggregation Functions on Bounded Lattices

In this subsection, it is assumed that the notions of posets or partial orders are familiar to readers. For more details see [37–40].

We remember that a *lattice* is a poset $(X, \leq_X)$ where each pair of elements $x, y \in X$ has infimum and supremum, denoted respectively by $x \wedge y$ and $x \vee y$. Moreover, if there are $0_X, 1_X \in X$ such that for each $x \in X$, $x \wedge 1_X = x$ and $x \vee 0_X = x$, then $(X, \leq_X)$ is called *bounded lattice*. We will simply say that X is a lattice whenever the order $\leq_X$ is clear in the context. Also, if $(X_1, \leq_{X_1}), \ldots, (X_n, \leq_{X_n})$ are lattices and the Cartesian product of the underlying sets is $\prod_{i=1}^{n} X_i = X_1 \times \ldots \times X_n$, then the structure $\left(\prod_{i=1}^{n} X_i, \leq_{\text{comp}}\right)$ is also a lattice called *product lattice* of $(X_1, \leq_{X_1}), \ldots, (X_n, \leq_{X_n})$, where $\leq_{\text{comp}}$ is the componentwise partial order on the Cartesian product $\prod_{i=1}^{n} X_i$ given as follows: let $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ be two points of $\prod_{i=1}^{n} X_i$. Therefore, $\vec{x} \leq_{\text{comp}} \vec{y} \Leftrightarrow x_i \leq_{X_i} y_i$, for every $i = 1, \ldots, n$.

Let $n \in \mathbb{N}$ be fixed and X a bounded lattice. An n-ary map $\psi : X^n \longrightarrow X$ is increasing if $\psi(\vec{x}) \leq_X \psi(\vec{y})$ whenever $\vec{x} \leq_{\text{comp}} \vec{y}$. If the orders $\leq_X$ and $\leq_{\text{comp}}$ are respectively replaced by the strict orders $<_X$ and $<_{\text{comp}}$, then one obtains a stronger requirement. A map with this property is called *strictly increasing*. Moreover, if $\psi(\vec{y}) \leq_X \psi(\vec{x})$ whenever $\vec{x} \leq_{\text{comp}} \vec{y}$, then ψ is a decreasing map. Similarly, *strictly decreasing* maps are defined. Recent studies have focused on n-ary maps on bounded lattices [23,41,42].

Definition 1 ([41])**.** *Consider X a bounded lattice. A map $F : X^n \longrightarrow X$ is called an aggregation function on X whenever it is increasing and satisfies boundary conditions: $F(0_X, \ldots, 0_X) = 0_X$ and $F(1_X, \ldots, 1_X) = 1_X$.*

We also remember that an n-ary aggregation function F on a bounded lattice X is called *symmetric*, if its value does not depend on the permutation of the arguments, i.e., $F(x_1, x_2 \ldots, x_n) = F(x_{P(1)}, x_{P(2)}, \ldots, x_{P(n)})$, for every $\vec{x} = (x_1, x_2, \ldots, x_n)$ and every permutation $P = (P(1), P(2), \ldots, P(n))$ of $(1, 2, \ldots, n)$.

Important examples of aggregation functions for this paper are pseudo-t-norms and pseudo-t-conorms.

Definition 2 ([43])**.** *Let X be a bounded lattice. An operation $T : X^2 \longrightarrow X$ (resp. $S : X^2 \longrightarrow X$) is called a pseudo-t-norm (resp. pseudo-t-conorm) if it is associative, increasing with respect to the both variables and has a neutral element $e = 1_X$ (resp. $e = 0_X$), i.e., $T(1_X, x) = T(x, 1_X) = x$ (resp. $S(0_X, x) = S(x, 0_X) = x$), for all $x \in X$.*

Remark 1. *Although pseudo-t-norms and pseudo-t-conorms are introduced as binary operations, the associativity enables us to extend them to n-ary operations. For example, given $n \geq 2$ and a pseudo t-norm T, then T can be extended to $T : X^n \longrightarrow X$:*

$$T(x_1, \ldots, x_n) = \underbrace{T(\ldots T(T}_{n-1 \text{ times}}(x_1, x_2), x_3), \ldots, x_n).$$

Similarly, given a pseudo t-conorm S, then S can be extended to $S : X^n \longrightarrow X$:

$$S(x_1, \ldots, x_n) = \underbrace{S(\ldots S(S}_{n-1 \text{ times}}(x_1, x_2), x_3), \ldots, x_n).$$

Note that here we are using the overloading of operators (i.e., the same name for different functions).

Definition 3. *Let X be a bounded lattice. A n-dimensional pseudo t-norm $T : X^n \longrightarrow X$ is called positive if it satisfies the condition: $T(x_1, \ldots, x_n) = 0_X \Leftrightarrow x_i = 0_X$, for some $i \in \{1, \ldots, n\}$. Dually, a n-dimensional pseudo t-conorm $S : X^n \longrightarrow X$ is called positive if it satisfies the condition: $S(x_1, \ldots, x_n) = 1_X \Leftrightarrow x_i = 1_X$, for some $i \in \{1, \ldots, n\}$.*

Another important type of n-ary aggregation functions on lattice X are general quasi-overlap functions [30].

Definition 4 ([30])**.** *Consider X a bounded lattice. The map $\mathcal{GO} : X^n \longrightarrow X$ is a general quasi-overlap function on X, if:*

($\mathcal{GO}$1) *$\mathcal{GO}$ is symmetric;*
($\mathcal{GO}$2) *$\mathcal{GO}(x_1, \ldots, x_n) = 0_X$ if $x_i = 0_X$, for some $i \in \{1, \ldots, n\}$;*
($\mathcal{GO}$3) *$\mathcal{GO}(x_1, \ldots, x_n) = 1_X$ if $x_i = 1_X$, for all $i \in \{1, \ldots, n\}$;*
($\mathcal{GO}$4) *$\mathcal{GO}$ is increasing.*

In Section 3, the notion of general quasi-overlap functions will be extended by dropping the requirement of symmetry in its definition. To obtain characterization theorems for these functions, the next subsection will be dedicated to the algebraic structure used for this purpose.

2.2. The Algebra of Quasigroups

In this subsection, we summarize some terminology and basic facts regarding quasigroups. For more details it is indicated [44–49]. The concept of quasigroup is a natural generalization of the concept of a group and is nothing more than a set X equipped with a binary operation $*$ on X (usually called multiplication) such that for any two elements a and b of X, there must be two other elements x and y of X that transform a into b through $*$. Quasigroups differ from groups mainly in that they need not be associative and need not have an identity element.

Definition 5 ([44])**.** *Let X be a non-empty set and $*$ be a binary operation on X, called multiplication. The algebraic structure $(X, *)$ is called a quasigroup if for any ordered pair $(a, b) \in X^2$ there exist a unique solution $x, y \in X$ to the equations $x * a = b$ and $a * y = b$.*

From Definition 5 it follows, that any two elements from the triple $(a, b, a * b)$ specify the third element in a unique way. Indeed, for any elements a and b there exists a unique element $a * b$. This follows from the definition of operation $*$. Elements a and $a * b$ determine the third element in a unique way since there exists a unique solution to the equation $a * y = b$. Elements $b, a * b$ determine the third element uniquely since there exists a unique solution to the equation $x * a = b$.

Example 1. *Consider the following:*

(1) *Let $\mathbb{Z}_4$ be the set of integers modulo 4, equipped with the operation of subtraction and consider the equation*

$$x - y = z \tag{1}$$

between elements x, y, z of $\mathbb{Z}_4$. If x and y are given, then (1) specifies z uniquely. If (1) holds, and y, z are given, then x is specified uniquely as $x = y + z$. Moreover, if (1) holds, and x, z are given, then y is specified uniquely as $y = x - z$. Therefore, $(\mathbb{Z}_4, -)$ is a quasigroup.

(2) *Consider $\mathbb{R}$ the set of real numbers equipped with the binary operator ∇, where for any two real numbers x and y, $x \nabla y = \frac{x+y}{2}$. Consider also the equation*

$$x \nabla y = z \tag{2}$$

between real numbers x, y, and z. Under these conditions, z is uniquely specified by x and y. Moreover, if y and z are given, then x is uniquely specified as $x = 2z - y$. Similarly, y is uniquely specified by (2) in terms of x and z. Therefore $(\mathbb{R}, \nabla)$ is a quasigroup.

As is well known, in some multiplicative structures such as rings and fields, division is not always possible. Indeed, we cannot divide by 0 in the field $\mathbb{R}$ of real numbers, nor by the element 2 within the ring $\mathbb{Z}$ of integers. However, quasigroups are defined so that division is always possible. In fact, there are two forms of division in a quasigroup: from the right and from the left.

Definition 6 ([46]–Quasigroup divisions)**.** *Let $(X, *)$ be a quasigroup and consider elements x and y of X. Under these conditions:*

(**i**) *The element $x \backslash y$ of X is defined as the unique solution z of the equation $x * z = y$. In other words,*

$$x * (x \backslash y) = y. \tag{3}$$

The element $x \backslash y$ may be read as "x dividing y" or "x backslash y". Moreover, the operation $\backslash$ on the set X is known as a left division in the quasigroup $(X, *)$.

(**ii**) *the element x/y of X is defined as the unique solution z of the equation $z * y = x$. In other words,*

$$(x/y) * y = x. \tag{4}$$

The element x/y may be read as "x divided by y" or "x slash y". Moreover, the operation $/$ on the set X is known as a right division in the quasigroup $(X, *)$.

Example 2. *Let $(\mathbb{R}, \nabla)$ be the arithmetic mean quasigroup structure on the real line, as given in item (2) of Example 1. In solving (4) for x in terms of y and z, it was shown there that $z/y = 2z - y$. This operation of right division in the arithmetic mean quasigroup has a geometrical interpretation, as the reflection of y in a mirror located at z. Similarly, in solving (3) for y in terms of x and z, it was shown there that $x \backslash z = 2z - x$. This operation of left division in the arithmetic mean quasigroup also has a geometrical interpretation, as the reflection of x in a mirror located at z.*

Lemma 1 ([46]–Characterization of quasigroups)**.** *A set X forms a quasigroup $(X, *)$ under a multiplication $*$ if and only if it is equipped with a left division $\backslash$ and a right division $/$ such that for all x, y in X one has:*

(C1) $x * (x \backslash y) = y$;
(C2) $x \backslash (x * y) = y$;
(C3) $(x/y) * y = x$;
(C4) $(y * x)/x = y$.

Theorem 1 ([46]–Divisions as quasigroup multiplications)**.** *Let $(X, *)$ be a quasigroup, with left division $\backslash$ and right division $/$. Then $(X, \backslash)$ and $(X, /)$ are quasigroups.*

Remark 2. *Theorem 1 gives an immediate proof for the content of item (1) in the Example 1, showing that the set $\mathbb{Z}_4$ of integers modulo 4 forms a quasigroup under subtraction. Notice that subtraction is the right division for the addition in any additive group like $(\mathbb{Z}_4, +)$.*

Theorem 2 ([45]). *If the structure $(X, *)$ is an associative quasigroup, then necessarily $(X, *)$ has a unique identity element e.*

In this way, it is concluded that every associative quasigroup is a group. In this perspective, a quasigroup is Abelian if it is commutative and associative, so is an Abelian group. In addition, given an Abelian group $(X, *)$, we remember that for an element $a \in X$, any other $b \in X$ is called inverse of a, denoted by $b = a^{-1}$, when $a * b = b * a = e$.

Remark 3 ([46]). *Let $(X, *)$ be a group, considered as a quasigroup. Then $x \backslash y = x^{-1} * y$ and $y/x = y * x^{-1}$. When a quasigroup is Abelian, since the operator $*$ is commutative, for any x, y in X, its left division $x \backslash y$ and right division y/x coincides.*

In the next section, the notions of quasigroups and groups will be useful to present a characterization theorem.

3. General Pseudo Quasi-Overlap Functions

In this section, the concept of general pseudo quasi-overlap functions is formalized and construction methods and characterization of general pseudo quasi-overlap functions are proposed.

Definition 7. *Consider X a bounded lattice. The map $\mathcal{GP} : X^n \longrightarrow X$ is a general pseudo quasi-overlap function on X, if:*

($\mathcal{GP}$1) $\mathcal{GP}(x_1, \ldots, x_n) = 0_X$ *if* $x_i = 0_X$*, for some* $i \in \{1, \ldots, n\}$;
($\mathcal{GP}$2) $\mathcal{GP}(x_1, \ldots, x_n) = 1_X$ *if* $x_i = 1_X$*, for all* $i \in \{1, \ldots, n\}$;
($\mathcal{GP}$3) $\mathcal{GP}$ *is increasing.*

Example 3. (1) *Let X be a bounded lattice and $a \in X$. The map $\mathcal{GP} : X^n \longrightarrow X$ given by*

$$\mathcal{GP}(x_1, \ldots, x_n) = \begin{cases} \bigwedge_{i=1}^{n} x_i & \text{if } \bigvee_{i=1}^{n} x_i \leq a \\ \bigvee_{i=1}^{n} x_i & \text{otherwise} \end{cases}$$

is a general pseudo quasi-overlap function on X. A variant of this map given by

$$\mathcal{GP}(x_1, \ldots, x_n) = \begin{cases} \bigwedge_{i=1}^{n} x_i & \text{if } x_i \leq a \text{ for each } i = 1, \ldots, n \\ \bigvee_{i=1}^{n} x_i & \text{otherwise} \end{cases}$$

is also a general quasi-overlap function on X.

(2) *Let X be a non-empty set and $(\wp(X), \subseteq)$ be the lattice of the powersets of X with the inclusion order. The map $\mathcal{GP} : \wp(X)^2 \longrightarrow \wp(X)$ given by*

$$\mathcal{GP}(X_1, \ldots, X_n) = \bigcup_{i=1}^{n} X_i - \overline{\bigcap_{i=1}^{n} X_i}$$

is a general pseudo quasi-overlap function on $\wp(X)$. Another general pseudo quasi-overlap function on $\wp(X)$ is given by

$$\mathcal{GP}_j(X_1, \ldots, X_n) = \begin{cases} \emptyset & \text{if } \bigcap_{i=1}^{n} X_i = \emptyset \\ X_j & \text{otherwise,} \end{cases}$$

where $j \in \{1, \ldots, n\}$ is fixed.

(3) *Consider X a non-empty set and let $\mathcal{F}(X)$ be the lattice of fuzzy sets on X, where the order considered is the inclusion of fuzzy sets. If $f : [0,1]^n \longrightarrow [0,1]$ is given by*

$$f(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \cdot \left(\max\left\{ \sum_{i=1}^{n} x_i - (n-1), 0 \right\} \right),$$

with integers $\alpha_i \geq 1$ for all $i \in \{1, \ldots, n\}$, then the map

$$\mathcal{GP}_f(A_1, \ldots, A_n) = \{(x, f(A_1(x), \ldots, A_n(x))) \mid x \in X\}$$

is a general pseudo quasi-overlap function on $\mathcal{F}(X)$.

(4) *The function f from the previous item is a general pseudo-quasi-overlap function on the n-dimensional cube $[0,1]^n$ which is not a general quasi-overlap function. Furthermore, the function $\mathcal{GP}_{GM} : [0,1]^n \longrightarrow [0,1]$ given by*

$$\mathcal{GP}_{GM}(x_1, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i^{\alpha_i}} \cdot \left(\max\left\{ \sum_{i=1}^{n} x_i - (n-1), 0 \right\} \right),$$

with integers $\alpha_i \geq 1$ for every $i \in \{1, \ldots, n\}$ is also a general pseudo quasi-overlap function which it is not a general quasi-overlap function.

Obviously, every general quasi-overlap function is a general pseudo-quasi-overlap function. A general pseudo-quasi-overlap is said to be *proper* if it is not commutative. In Example 3, the only proper general pseudo quasi-overlap functions are $\mathcal{GP}_j$, $\mathcal{GP}_f$ and $\mathcal{GP}_{GM}$. The following theorems show how to transform (proper) general pseudo-quasi-overlap functions into general quasi-overlap functions.

Theorem 3. *Let X be a bounded lattice and $\mathcal{GP} : X^n \longrightarrow X$ a general pseudo-quasi-overlap function. The map $\mathcal{GO} : X^n \longrightarrow X$ given by $\mathcal{GO}(x_1, x_2, \ldots, x_n) = \mathcal{GP}(x_{P(1)}, x_{P(2)}, \ldots, x_{P(n)})$, for every $\vec{x} = (x_1, x_2, \ldots, x_n)$ and arbitrary permutation $P = (P(1), P(2), \ldots, P(n))$ of $(1, 2, \ldots, n)$, is a general quasi-overlap function.*

Proof. Direct. □

Theorem 4. *Let X be a bounded lattice, the maps $A : X^{n!} \longrightarrow X$ an aggregation function and $\mathcal{GP} : X^n \longrightarrow X$ a general pseudo quasi-overlap function. The map $\mathcal{GO} : X^n \longrightarrow X$ given by $\mathcal{GO}(x_1, \ldots, x_n) = A\big(\mathcal{GP}(x_{P_1(1)}, \ldots, x_{P_1(n)}), \ldots, \mathcal{GP}(x_{P_{n!}(1)}, \ldots, x_{P_{n!}(n)})\big)$, for every $\vec{x} = (x_1, \ldots, x_n)$ and every permutation $P_j = \big(P_j(1), \ldots, P_j(n)\big)$ of $(1, 2, \ldots, n)$ and $j = 1, \ldots, n!$, is a general quasi-overlap function if satisfies:*

(i) *A is symmetric;*
(ii) *$A(x_1, \ldots, x_{n!}) = 0_X$ whenever $x_i = 0_X$, for some $i \in \{1, \ldots, n!\}$;*
(iii) *$A(x_1, \ldots, x_{n!}) = 1_X$ whenever $x_i = 1_X$, for all $i \in \{1, \ldots, n!\}$.*

Proof. Suppose A is an aggregation function that satisfies the properties **(i)**, **(ii)** and **(iii)**. Then, by the symmetry of A (**(i)**), for every $\vec{x} = (x_1, \ldots, x_n)$ and every permutation $P_j = \big(P_j(1), \ldots, P_j(n)\big)$ of $(1, 2, \ldots, n)$ and $j = 1, \ldots, n!$, one has that:

$$\begin{aligned} \mathcal{GO}(x_1, \ldots, x_n) &= A\Big(\mathcal{GP}(x_{P_1(1)}, \ldots, x_{P_1(n)}), \ldots, \mathcal{GP}(x_{P_{n!}(1)}, \ldots, x_{P_{n!}(n)})\Big) \\ &= \mathcal{GO}(x_{P_j(1)}, \ldots, x_{P_j(n)}). \end{aligned}$$

Thus, $\mathcal{GO}$ satisfies ($\mathcal{GO}1$). Moreover, if

$$\mathcal{GO}(x_1, \ldots, x_n) = A\big(\mathcal{GP}(x_{P_1(1)}, \ldots, x_{P_1(n)}), \ldots, \mathcal{GP}(x_{P_{n!}(1)}, \ldots, x_{P_{n!}(n)})\big) = 0_X,$$

then by (**ii**) $\mathcal{GP}(x_{P_j(1)}, \ldots, x_{P_j(n)}) = 0_X$ for some $j \in \{1, \ldots, n!\}$, which implies by ($\mathcal{GP}1$) that $x_{P_j(i)} = 0_X$ for some $j \in \{1, \ldots, n!\}$ and any $i \in \{1, \ldots, n\}$ fixed. Similarly, using (**iii**) it is shown that $\mathcal{GO}$ satisfies ($\mathcal{GP}2$). Finally, if for each $j \in \{1, \ldots, n!\}$ and for any $i \in \{1, \ldots, n\}$ fixed, one has that $x_{P_j(i)} \leq z$ then, since A is increasing and ($\mathcal{GP}2$) is true, it follows that

$$\begin{aligned}
\mathcal{GO}(x_{P_j(1)}, \ldots, x_{P_j(i)}, \ldots, x_{P_j(n)}) &= A\Big(\mathcal{GP}(x_{P_1(1)}, \ldots, x_{P_1(i)}, \ldots, x_{P_1(n)}), \ldots, \mathcal{GP}(x_{P_{n!}(1)}, \ldots, x_{P_{n!}(i)}, \ldots, x_{P_{n!}(n)})\Big) \\
&\leq_X A\Big(\mathcal{GP}(x_{P_1(1)}, \ldots, z, \ldots, x_{P_1(n)}), \ldots, \mathcal{GP}(x_{P_{n!}(1)}, \ldots, z, \ldots, x_{P_{n!}(n)})\Big) \\
&= \mathcal{GO}(x_{P_j(1)}, \ldots, z, \ldots, x_{P_j(n)}).
\end{aligned}$$

Thus, $\mathcal{GO}$ satisfies ($\mathcal{GO}4$). Therefore, $\mathcal{GO}$ is a general quasi-overlap function on X. □

Theorem 5. *Let $\oplus$ and $\otimes$ be two increasing binary operations on a bounded lattice X, such that $\otimes$ distributes over $\oplus$ and consider that 0_X is the unique $u \in X$ such that $u \otimes v = 0_X$, for each $v \in X$. Suppose that $(X, \oplus, 0_X)$ is a quasigroup with identity element 0_X and right division $\ominus$ and that $(X, \otimes, 1_X)$ is an Abelian group with division $\big/$. The function $\mathcal{GP} : X^n \longrightarrow X$ is a general pseudo quasi-overlap function if and only if*

$$\mathcal{GP}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \Big/ [f(x_1, \ldots, x_n) \oplus g(x_1, \ldots, x_n)] \tag{5}$$

for some $f, g : X^n \longrightarrow X$ such that

(**i**) *$f(x_1, \ldots, x_n) = 0_X$ if $x_i = 0_X$, for some $i \in \{1, \ldots, n\}$;*
(**ii**) *$g(x_1, \ldots, x_n) = 0_X$ if $x_i = 1_X$, for all $i \in \{1, \ldots, n\}$;*
(**iii**) *f is increasing and g is decreasing;*
(**iv**) *$f(x_1, \ldots, x_n) \oplus g(x_1, \ldots, x_n) \neq 0_X$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{GP}$ is a general pseudo quasi-overlap function and for each $\vec{x} \in X^n$ take $f(\vec{x}) = \mathcal{GP}(\vec{x})$. Then, from the right division of $\oplus$ take $g(\vec{x}) = \mathcal{GP}(\vec{x}) \ominus 1_X$. By Lemma 1, item (C1), we have:

$$f(\vec{x}) \oplus g(\vec{x}) = \mathcal{GP}(\vec{x}) \oplus (\mathcal{GP}(\vec{x}) \ominus 1_X) = 1_X. \tag{6}$$

Thus, since 1_X is an identity element of $\otimes$, by Equation (6), we can write the following:

$$\begin{aligned}
\mathcal{GP}(\vec{x}) &= f(\vec{x}), \quad \text{because } f(\vec{x}) = \mathcal{GP}(\vec{x}) \text{ for each } \vec{x} \in X^n; \\
&= \Big(f(\vec{x}) \Big/ 1_X\Big) \otimes 1_X, \quad \text{by Lemma 1, item (C3);} \\
&= \Big(f(\vec{x}) \Big/ [f(\vec{x}) \oplus g(\vec{x})]\Big) \otimes 1_X \\
&= f(\vec{x}) \Big/ [f(\vec{x}) \oplus g(\vec{x})].
\end{aligned} \tag{7}$$

In particular, one easily verifies that conditions (**i**) and (**iv**) hold. For condition (**ii**), if $x_i = 1_X$, for all $i \in \{1, \ldots, n\}$, then by Equation (6) we have that $1_X \oplus g(1_X, \ldots, 1_X) = 1_X$ if and only if $g(1_X, \ldots, 1_X) = 1_X \ominus 1_X$. On the other hand, since 0_X is an identity element of $\oplus$, it follows that $w \oplus 0_X = w$ and therefore, because $\ominus$ is the right divisor, $0_X = w \ominus w$ for all $w \in X$. Hence, $1_X \ominus 1_X = 0_X$ and so $g(x_1, \ldots, x_n) = 0_X$ whenever $x_i = 1_X$, for all $i \in \{1, \ldots, n\}$. For condition (**iii**), because $f(\vec{x}) = \mathcal{GP}(\vec{x})$ for each $\vec{x} \in X^n$, it follows that f is increasing and so $\vec{x} \leq_{\text{comp}} \vec{y}$ implies $f(\vec{x}) \leq_X f(\vec{y})$. On the other hand, as $g(\vec{x}) = f(\vec{x}) \ominus 1_X$ for all $\vec{x} \in X^n$, it's easy to see that the closer $f(\vec{x})$ is to 1_X, the closer to 0_X is $g(\vec{x})$, thus $g(\vec{y}) \leq_X g(\vec{x})$ whenever $\vec{x} \leq_{\text{comp}} \vec{y}$ and so g is decreasing.

($\Leftarrow$) Consider $f, g : X^n \longrightarrow X$ satisfying the conditions (**i**)–(**iv**). We show that the map of Equation (5) is a general pseudo-quasi-overlap function on X. Let us prove that the conditions $(\mathcal{GP}1)$, $(\mathcal{GP}2)$ and $(\mathcal{GP}3)$ hold. Let $\vec{x} \in X^n$ be such that $x_i = 0_X$ for some $i \in \{1, \dots n\}$. Due to conditions **(i)** and **(iv)**, it holds that $f(\vec{x}) = 0_X$ and $f(\vec{x}) \oplus g(\vec{x}) \neq 0_X$. Then, $\mathcal{GP}(\vec{x}) \otimes [f(\vec{x}) \oplus g(\vec{x})] = 0_X$ and consequently $\mathcal{GP}(\vec{x}) = 0_X$. Similarly, let $\vec{x} \in X^n$ be such that $x_i = 1_X$ for all $i \in \{1, \dots n\}$. Due to conditions **(ii)** and **(iv)**, it holds that $g(\vec{x}) = 0_X$ and $f(\vec{x}) \oplus g(\vec{x}) \neq 0_X$. So, $f(x) \neq 0_X$ and consequently, $\mathcal{GP}(\vec{x}) = f(\vec{x}) \Big/ f(\vec{x})$. However, since 1_X is an identity element of $\otimes$, it follows that $r \otimes 1_X = r$ and therefore $1_X = r \Big/ r$ for all $r \in X$. Thus, $f(\vec{x}) \Big/ f(\vec{x}) = 1_X$ and so $\mathcal{GP}(\vec{x}) = 1_X$ whenever $x_i = 1_X$, for all $i \in \{1, \dots, n\}$. Finally, let us see that $(\mathcal{GP}3)$ also holds. Consider $\vec{x}, \vec{y} \in X^n$. Without loss of generality, suppose that $\vec{x} \leq_{\text{comp}} \vec{y}$. Due to condition **(iii)**, it holds that $f(\vec{x}) \leq_X f(\vec{y})$ and $g(\vec{y}) \leq_X g(\vec{x})$. Similarly, we find that $f(\vec{x}) \otimes g(\vec{y}) \leq_X f(\vec{y}) \otimes g(\vec{x})$. Since $\oplus$ also is increasing, we have $[f(\vec{x}) \otimes f(\vec{y})] \oplus [f(\vec{x}) \otimes g(\vec{y})] \leq_X [f(\vec{x}) \otimes f(\vec{y})] \oplus [f(\vec{y}) \otimes g(\vec{x})]$. Moreover, since $\otimes$ distributes over $\oplus$, $f(\vec{x}) \otimes [f(\vec{y}) \oplus g(\vec{y})] \leq_X f(\vec{y}) \otimes [f(\vec{x}) \oplus g(\vec{x})]$. Thus, since $(X, \otimes)$ is an Abelian group, we have $\mathcal{GP}(\vec{x}) = f(\vec{x}) \Big/ [f(\vec{x}) \oplus g(\vec{x})] \leq_X f(\vec{y}) \Big/ [f(\vec{y}) \oplus g(\vec{y})] = \mathcal{GP}(\vec{y})$. $\square$

Example 4. *Let f the function of Example 3. For each $A, B \in \mathcal{F}(X)$ and $x \in X$ define $(A \otimes B)(x) = A(x)B(x)$, $(A \oplus B)(x) = \max(A(x), B(x))$. Then, $\left(A \Big/ B\right)(x) = \frac{A(x)}{B(x)}$. Moreover, if $F, G : \mathcal{F}(X)^n \longrightarrow \mathcal{F}(X)$ are defined for each $A_1, \dots, A_n \in \mathcal{F}(X)$ and $x \in X$ by $F(A_1, \dots, A_n)(x) = f(A_1(x), \dots, A_n(x))$ and $G(A_1, \dots, A_n)(x) = (A_1(x), \dots, A_n(x))$ then*

$$\begin{aligned}
\mathcal{GP}_f(A_1, \dots, A_n)(x) &= \left(F(A_1, \dots, A_n) \Big/ (F(A_1, \dots, A_n) \oplus G(A_1, \dots, A_n))\right)(x) \\
&= \frac{F(A_1, \dots, A_n)(x)}{\max(F(A_1, \dots, A_n)(x), G(A_1, \dots, A_n)(x))} \\
&= \frac{f(A_1(x), \dots, A_n(x))}{\max(f(A_1(x), \dots, A_n(x)), \max(1 - A_1(x), \dots, 1 - A_n(x)))} \\
&= \frac{\prod_{i=1}^n A_i(x)}{\max(\prod_{i=1}^n A_i(x), \max(1 - A_1(x), \dots, 1 - A_n(x)))} \\
&= \min\left(1, \frac{\prod_{i=1}^n A_i(x)}{1 - A_1(x)}, \dots, \frac{\prod_{i=1}^n A_i(x)}{1 - A_n(x)}\right)
\end{aligned}$$

Theorem 6. *Consider X a totally ordered bounded lattice and let $\mathcal{GP} : X^n \longrightarrow X$ be an aggregation function. Under these conditions, if $\mathcal{GP} \leq \min$, then $\mathcal{GP}$ is a general pseudo quasi-overlap function.*

Proof. First, since $\mathcal{GP}$ is an aggregation function, by boundary condition $(\mathcal{GP}2)$ is satisfied. In addition, it follows that $\mathcal{GP}$ is increasing and so $(\mathcal{GP}3)$ is satisfied. Moreover, if $x_i = 0_X$, for some $i \in \{1, \dots, n\}$, then $\mathcal{GP}(x_1, \dots, x_n) \leq \min(x_1, \dots, x_n) = 0_X$. Thus $\mathcal{GP}(x_1, \dots, x_n) = 0_X$, hence $(\mathcal{GP}1)$ is satisfied. Therefore, $\mathcal{GP}$ is a general pseudo quasi-overlap function on X. $\square$

Corollary 1. *Consider X a totally ordered bounded lattice and let $\psi : X \longrightarrow X$ be an increasing function satisfying $\psi(0_X) = 0_X$ and $\psi(1_X) = 1_X$. Hence, if $\psi(F(1_X, \dots, 1_X, t, 1_X, \dots, 1_X)) \leq t$, for each $t \in X$, and at any position, then $\mathcal{GP} = \psi \circ F$ is a general pseudo quasi-overlap function.*

Proof. In fact, for any fixed position i, and any $\vec{x} = (x_1, \dots, x_n)$ one has that $\mathcal{GP}(\vec{x}) \leq \max\limits_{x_j \in X, j \neq i} \mathcal{GP}(\vec{x}) = \mathcal{GP}(1_X, \dots, 1_X, x_i, 1_X, \dots, 1_X) \leq x_i$. This holds for every i, thus $\mathcal{GP}(\vec{x}) \leq \min(\vec{x})$. By applying Theorem 6 we complete the proof. $\square$

Corollary 2. *Consider X a totally ordered bounded lattice and let $\psi : X \longrightarrow X$ be an increasing function satisfying $\psi(0_X) = 0_X$ and $\psi(1_X) = 1_X$. If $\psi(F(1_X, \dots, 1_X, t, 1_X, \dots, 1_X)) \leq t$, for each $t \in X$, and at any position, then 0_X is an annihilator of $\psi \circ F$.*

Theorem 7. *Consider X a bounded lattice and let $\rho_1, \ldots, \rho_n, \psi : X \longrightarrow X$ be increasing bijections. For any general pseudo quasi-overlap function $\mathcal{GP}$, the map*

$$\widetilde{\mathcal{GP}}(x_1, \ldots, x_n) = \psi\Big(\mathcal{GP}(\rho_1(x_1), \ldots, \rho_n(x_n))\Big)$$

is a general pseudo quasi-overlap function.

Proof. In fact, the property $(\mathcal{GP}3)$, it follows from the fact that the maps $\rho_1, \ldots, \rho_n, \psi$ are strictly increasing and $\mathcal{GP}$ is increasing. Moreover, as for the properties $(\mathcal{GP}1)$ and $(\mathcal{GP}2)$, they follows from the fact that for each increasing bijection $\varphi : X \longrightarrow X$ one has that $\varphi(x) = 0_X$ iff $x = 0_X$ and $\varphi(x) = 1_X$ iff $x = 1_X$. Thus, since $\mathcal{GP}$ is a general pseudo quasi-overlap function, if $x_i = 0_X$ for some $i \in \{1, \ldots, n\}$, it follows that $\rho_i(x_i) = 0_X$, which by $(\mathcal{GP}1)$ implies that $\mathcal{GP}(\rho_1(x_1), \ldots, \rho_n(x_n)) = 0_X$ and so $\psi\Big(\mathcal{GP}(\rho_1(x_1), \ldots, \rho_n(x_n))\Big) = 0_X$. Therefore, $\widetilde{\mathcal{GP}}(x_1, \ldots, x_n) = 0_X$. On the other hand, if $x_i = 1_X$ for every $i \in \{1, \ldots, n\}$, it follows that $\rho_i(x_i) = 1_X$, which by $(\mathcal{GP}2)$ implies that $\mathcal{GP}(\rho_1(x_1), \ldots, \rho_n(x_n)) = 1_X$. Then $\psi\Big(\mathcal{GP}(\rho_1(x_1), \ldots, \rho_n(x_n))\Big) = 1_X$, which implies that $\widetilde{\mathcal{GP}}(x_1, \ldots, x_n) = 1_X$. □

4. General Pseudo Quasi-Overlap Generated by Pseudo t-Norms and Pseudo t-Conorms

The importance to define multivalued functions by means of its one-place additive/multiplicative generators is to provide less computational cost in applications. In this section, we use the notions of pseudo t-norms and pseudo-t-conorms to generalize the concepts of additive and multiplicative generators for the context of general pseudo quasi-overlap functions on lattices and we explore some properties related.

Definition 8. *Let X be a bounded lattice, $T : X^n \longrightarrow X$ an n-dimensional pseudo t-norm and $\eta, \zeta : X \longrightarrow X$ two increasing functions. If a n-dimensional function $\mathcal{GP}_{\zeta,\eta} : X^n \longrightarrow X$ is given for each $(x_1, \ldots, x_n) \in X^n$ by*

$$\mathcal{GP}_{\zeta,\eta}(x_1, \ldots, x_n) = \zeta(T(\eta(x_1), \ldots, \eta(x_n))) \tag{8}$$

then, the pair (ζ, η) is called a pseudo-multiplicative generator pair of $\mathcal{GP}_{\zeta,\eta}$ while $\mathcal{GP}_{\zeta,\eta}$ is said to be pseudo-multiplicatively generated function by the pair (ζ, η).

In the following theorem, we show in which conditions the two increasing functions $\eta, \zeta : X \longrightarrow X$ can pseudo-multiplicatively generate a general pseudo-quasi-overlap function.

Theorem 8. *Let X be a bounded lattice, $T : X^n \longrightarrow X$ a positive pseudo-t-norm and two increasing mappings $\eta, \zeta : X \longrightarrow X$ such that*

(i) $\eta(x) = 0_X$ *if* $x = 0_X$;
(ii) $\eta(x) = 1_X$ *if* $x = 1_X$;
(iii) $\zeta(x) = 0_X$ *if* $x = 0_X$;
(iv) $\zeta(x) = 1_X$ *if* $x = 1_X$.

Then, the n-dimensional function $\mathcal{GP}_{\zeta,\eta} : X^n \longrightarrow X$ given in Equation (8) is a general pseudo quasi-overlap function.

Proof. $(\mathcal{GP}1)$:

$$\begin{aligned} \mathcal{GP}_{\zeta,\eta}(x_1,\dots,x_n) = 0_X &\Leftrightarrow \zeta(T(\eta(x_1),\dots,\eta(x_n))) = 0_X \\ &\Leftrightarrow T(\eta(x_1),\dots,\eta(x_n)) = 0_X, \text{ by item (iii)} \\ &\Leftrightarrow \eta(x_i) = 0_X, \text{ for some } i \in \{1,\dots,n\}, \text{ by Definition 3} \\ &\Leftrightarrow x_i = 0_X, \text{ for some } i \in \{1,\dots,n\}, \text{ by item (i)} \end{aligned}$$

$(\mathcal{GP}2)$:

$$\begin{aligned} \mathcal{GP}_{\zeta,\eta}(x_1,\dots,x_n) = 1_X &\Leftrightarrow \zeta(T(\eta(x_1),\dots,\eta(x_n))) = 1_X \\ &\Leftrightarrow T(\eta(x_1),\dots,\eta(x_n)) = 1_X, \text{ by item (iv)} \\ &\Leftrightarrow \eta(x_i) = 1_X, \text{ for all } i \in \{1,\dots,n\}, \text{ since } 1_X \text{ is neutral element of } T \\ &\Leftrightarrow x_i = 1_X, \text{ for all } i \in \{1,\dots,n\}, \text{ by item (ii)} \end{aligned}$$

$(\mathcal{GP}2)$: It follows immediately from the fact that T, η, and ζ are increasing mappings. □

Example 5. *Consider $(X, \leq, 0_X, 1_X)$ a bounded lattice and let $a \in X - \{0_X, 1_X\}$ such that $I_a = \{x \in X \mid a \parallel x\} = \emptyset$. Then $T_a : X^2 \longrightarrow X$ defined for all $x, y \in X$ by*

$$T_a(x,y) = \begin{cases} x \wedge y & \text{if } x \leq a \text{ or } 1_X \in \{x,y\} \\ a \wedge y & \text{otherwise} \end{cases}$$

then T_a is a positive pseudo t-norm on X. Let $\eta, \zeta : X \longrightarrow X$ be increasing functions such that

1. $\eta(0_X) = \zeta(0_X) = 0_X$,
2. $\eta(1_X) = \zeta(1_X) = 1_X$,
3. $\eta(a) = a$.

So, $\mathcal{GP}_{\zeta,\eta} : X^2 \longrightarrow X$ given for each $x, y \in X$ by

$$\mathcal{GP}_{\zeta,\eta}(x,y) = \zeta(T_a(\eta(x),\eta(y))) = \begin{cases} \zeta(\eta(x) \wedge \eta(y)) & \text{if } x \leq a \text{ or } 1_X \in \{x,y\} \\ \zeta(a \wedge \eta(y)) & \text{otherwise} \end{cases}$$

is a general pseudo-quasi-overlap function pseudo-multiplicatively generated by the pair (ζ, η).

Theorem 9. *Let X be a bounded lattice, $T : X^n \longrightarrow X$ a positive pseudo-t-norm and two increasing mappings $\eta, \zeta : X \longrightarrow X$ such that*

(i) $x = 0_X$ *whenever* $\zeta(x) = 0_X$;
(ii) $x = 1_X$ *whenever* $\zeta(x) = 1_X$;
(iii) $\mathcal{GP}_{\zeta,\eta} : X^n \longrightarrow X$ *defined in Equation (8) is a general pseudo quasi-overlap function.*

Then the following statements hold:

(1) $\eta(x) = 0_X$ *whenever* $x = 0_X$;
(2) $\eta(x) = 1_X$ *whenever* $x = 1_X$.

Proof. (**1**): If $x = 0_X$ then, by item (**iii**), $\mathcal{GP}_{\zeta,\eta}(x,\dots,x) = \zeta(T(\eta(x),\dots,\eta(x))) = 0_X$. Moreover, by item (**i**), $T(\eta(x),\dots,\eta(x)) = 0_X$. Thus, as T is a positive pseudo t-norm, it follows that $\eta(x) = 0_X$.
(**2**): If $x = 1_X$ then, by item (**iii**), $\mathcal{GP}_{\zeta,\eta}(x,\dots,x) = \zeta(T(\eta(x),\dots,\eta(x))) = 1_X$. Moreover, by item (**ii**), $T(\eta(x),\dots,\eta(x)) = 1_X$. Thus, as 1_X is the neutral element of T, it follows that $\eta(x) = 1_X$. □

Definition 9. *Let X and Y be two bounded lattices, $S : Y^n \longrightarrow Y$ a pseudo-t-conorm and consider the two decreasing mappings $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$. If a n-dimensional function $\mathcal{GP}_{\vartheta,\xi} : X^n \longrightarrow X$ is given for each $(x_1, \ldots, x_n) \in X^n$ by*

$$\mathcal{GP}_{\vartheta,\xi}(x_1, \ldots, x_n) = \vartheta(S(\xi(x_1), \ldots, \xi(x_n))) \tag{9}$$

then, the pair (ϑ, ξ) is called a pseudo-additive generator pair of $\mathcal{GP}_{\vartheta,\xi}$ while $\mathcal{GP}_{\vartheta,\xi}$ is said to be pseudo-additively generated by the pair (ϑ, ξ).

In the following results, we show in which conditions the two decreasing functions $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$ can pseudo-additively generate a general pseudo-quasi-overlap function.

Lemma 2. *Let X and Y be two bounded lattices, $S : Y^n \longrightarrow Y$ a positive pseudo t-conorm and $\xi : X \longrightarrow Y$ a decreasing mapping such that:*

(i) $S(\xi(x_1), \ldots, \xi(x_n)) \in Ran(\xi)$, *for $x_i \in X$ and $i = 1, \ldots, n$;*
(ii) *if $\xi(x) = \xi(0_X)$ then $x = 0_X$.*

Under these conditions, $\xi(0_X) \leq_Y S(\xi(x_1), \ldots, \xi(x_n))$ whenever $x_i = 0_X$, for some $i \in \{1, \ldots, n\}$.

Proof. Since ξ is decreasing and $S(\xi(x_1), \ldots, \xi(x_n)) \in Ran(\xi)$, for $x_i \in X$ with $i = 1, \ldots, n$ then one has that $S(\xi(x_1), \ldots, \xi(x_n)) \leq_Y \xi(0_X)$. Therefore, if $S(\xi(x_1), \ldots, \xi(x_n)) \geq_Y \xi(0_X)$, then it holds that $S(\xi(x_1), \ldots, \xi(x_n)) = \xi(0_X)$. Suppose that $\xi(0_X) = 0_Y$. Then, since ξ is decreasing, one has that $\xi(x_i) = 0_Y$ for each $x_i \in X$ with $i = 1, \ldots, n$, which is contradiction with condition (ii), and, therefore, it holds that $0_Y <_Y \xi(0_X)$. Now, suppose that $\xi(0_X) \neq 0_Y$ and $\xi(0_X) \neq 1_Y$. Then, since $\xi(0_X) \neq 0_Y$ one has that $S(\xi(0_X), \ldots, \xi(0_X)) >_Y \xi(0_X)$, which is also a contradiction. So, it follows that $\xi(0_X) = 1_Y$ and, therefore, since S is positive and $S(\xi(x_1), \ldots, \xi(x_n)) = \xi(0_X)$, we have that $\xi(x_i) = 1_Y$ for some $x_i \in X$, with $i \in \{1, \ldots, n\}$. Hence, by condition (ii), one has that $x_i = 0_X$ for some $i \in \{1, \ldots, n\}$. □

Lemma 3. *Let X and Y be two bounded lattices, $S : Y^n \longrightarrow Y$ a positive pseudo t-conorm and consider mappings the $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$ such that, for each $x_0 \in X$, if it holds that*

$$\vartheta(\xi(x)) = x_0 \text{ whenever } x = x_0,$$

then $\xi(x) = x_0$ whenever $x = x_0$.

Proof. Based on considerations similar to [7], if $\vartheta(\xi(x)) = x_0 whenever x = x_0$, so in particular, $\vartheta(\xi(x_0)) = x_0$. Now, if $\xi(x) = \xi(x_0)$ then $\vartheta(\xi(x)) = \vartheta(\xi(x_0)) = x_0$ and, thus $x = x_0$. □

Theorem 10. *Let X and Y be two bounded lattices, $S : Y^n \longrightarrow Y$ a positive pseudo t-conorm, $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$ two decreasing mappings such that:*

(i) $S(\xi(x_1), \ldots, \xi(x_n)) \in Ran(\xi)$, *for $x_i \in X$ and $i = 1, \ldots, n$;*
(ii) $\vartheta(\xi(x)) = 0_X$ *whenever* $x = 0_X$;
(iii) $\vartheta(\xi(x)) = 1_X$ *whenever* $x = 1_X$;
(iv) $S(\xi(x_1), \ldots, \xi(x_n)) = \xi(1_X)$ *whenever $x_i = 1_X$ for every $i = 1, \ldots, n$.*

Then, the n-dimensional function $\mathcal{GP}_{\vartheta,\xi} : X^n \longrightarrow X$ given in Equation (9) is a general pseudo quasi-overlap function.

Proof. $(\mathcal{GP}1)$: Suppose $x_i = 0_X$, for some $i \in \{1, \ldots, n\}$. Since Condition (ii) holds, by Lemma 3, $\xi(x_i) = 0_Y$. Moreover, as Lemma 3 and Condition (ii) hold, by Lemma 2 we have $S(\xi(x_1), \ldots, \xi(x_n)) = \xi(0_X) = 0_Y$. Therefore, by Equation (9) it follows that

$\mathcal{GP}_{\vartheta,\xi}(x_1,\ldots,x_n) = 0_X$.
$(\mathcal{GP}2)$: Suppose $x_i = 1_X$, for every $i \in \{1,\ldots,n\}$. Since Condition **(iv)** holds, by Lemma 3, $\xi(x_i) = 1_Y$. Moreover, as Lemma 3 and Condition **(iii)** hold, by Lemma 2 we have $S(\xi(x_1),\ldots,\xi(x_n)) = \xi(1_X) = 1_Y$. Therefore, by Equation (9) it follows that $\mathcal{GP}_{\vartheta,\xi}(x_1,\ldots,x_n) = 1_X$. Finally, to prove the condition $(\mathcal{GP}3)$, considering $z_i \in X$, with $x_i \leq_X z_i$, for every $i \in \{1,\ldots,n\}$, then $\xi(x_i) \geq_Y \xi(z_i)$, It follows that $\mathcal{GP}_{\vartheta,\xi}(x_1,\ldots,x_n) = \vartheta(S(\xi(x_1),\ldots,\xi(x_n))) \leq_X \vartheta(S(\xi(z_1),\ldots,\xi(z_n))) = \mathcal{GP}_{\vartheta,\xi}(z_1,\ldots,z_n)$ since ϑ and ξ are two decreasing mappings and S is an increasing mapping. □

Corollary 3. *Let X and Y be two bounded lattices, $S : Y^n \longrightarrow Y$ a positive pseudo t-conorm, $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$ two decreasing mappings such that $\mathcal{GP}_{\vartheta,\xi}(x_1,\ldots,x_n) = \vartheta(S(\xi(x_1),\ldots,\xi(x_n)))$*

(i) $\xi(x) = 1_Y$ *whenever* $x = 0_X$;
(ii) $\xi(x) = 0_Y$ *whenever* $x = 1_X$;
(iii) $\vartheta(x) = 1_X$ *whenever* $x = 0_Y$;
(iv) $\vartheta(x) = 0_X$ *whenever* $x = 1_Y$.

Then, the n-dimensional function $\mathcal{GP}_{\vartheta,\xi} : X^n \longrightarrow X$ given in Equation (9) is a general pseudo quasi-overlap function.

Proof. It follows from Theorem 10. □

Example 6. *Consider $(Y,\leq,0_Y,1_Y)$ a bounded lattice and let $a \in Y - \{0_Y,1_Y\}$ such that $I_a = \{y \in Y \mid a \parallel y\} = \emptyset$. Then $S_a : Y^2 \longrightarrow Y$ defined for all $x,y \in Y$ by*

$$S_a(x,y) = \begin{cases} x \vee y & \text{if } x \geq a \text{ or } 0_Y \in \{x,y\} \\ a \vee y & \text{otherwise} \end{cases}$$

then S_a is a positive pseudo-t-conorm on Y. Let $\xi : X \longrightarrow Y$ and $\vartheta : Y \longrightarrow X$ two decreasing functions such that

1. $\xi(0_X) = 1_Y$,
2. $\xi(1_X) = 0_Y$,
3. $\vartheta(0_Y) = 1_X$,
4. $\vartheta(1_Y) = 0_X$,
5. $\xi(x) = a$ *if and only if* $x = b$ *for some* $b \in X - \{0_X,1_X\}$ *such that* $I_b = \{y \in X|b \parallel y\} = \emptyset$.
   So, $\mathcal{GP}_{\xi,\vartheta} : X^2 \longrightarrow X$ *given for each* $x,y \in X$ *by*

$$\mathcal{GP}_{\xi,\vartheta}(x,y) = \vartheta(S_a(\xi(x),\xi(y))) = \begin{cases} \vartheta(\xi(x) \vee \xi(y)) & \text{if } x \geq b \text{ or } 0_X \in \{\xi(x),\xi(y)\} \\ \vartheta(a \vee \xi(y)) & \text{otherwise} \end{cases}$$

is a general pseudo quasi-overlap function pseudo-additively generated by the pair (ϑ,ξ).

5. Conclusions

In this paper, we studied the concept of general pseudo quasi-overlap functions on bounded lattices. As discussed extensively in the introduction, these functions generalize, in the bounded lattice setting, the concepts of overlap functions, pseudo-overlap functions, and general quasi-overlap functions and are suitable for use in situations where symmetry and continuity are unnecessary or irrelevant. We have proved a characterization theorem and some construction methods for these functions and used the notions of pseudo t-norms and pseudo-t-conorms to generalize the concepts of additive and multiplicative generators for the context of general pseudo-quasi-overlap functions on lattices and explore some properties related.

One possibility for future works is to explore the dual notion of general pseudo-quasi-overlap functions on bounded lattices, namely general pseudo-quasi-grouping functions on bounded lattices, in order to measure the amount of evidence for or against several alternatives when performing comparisons in multi-criteria decision making or multi-criteria preferences, based on fuzzy preference relations as done in [50] with an n-dimensional t-conorm and t-norm. Moreover, one other possibility is to extend the concepts of pseudo-additive and pseudo-multiplicative generators to the context of general pseudo-quasi-grouping functions and explore how these notions are related.

In another perspective, Dimuro et al. [51] propose some generalizations of the standard form of the Choquet Integral and among these generalizations, one uses a particular type of aggregation function, called overlap functions, which are a particular class of quasi-overlap functions. Likewise, Batista [31] introduced the notion of Pseudo-Choquet Integrals and Absolute Choquet Integrals obtained from the notion of pseudo-overlap functions and Batista et al. in [52] introduced the Quasi-Overlap-based discrete Choquet integral. In this perspective, we propose a generalization of the standard form of the Choquet integral, for future research, using general pseudo-quasi-overlap functions on lattices, in order to obtain applications in decision making and multi-criteria decision making or multi-criteria preferences, especially for the applications of discrete Choquet integrals in fuzzy rule-based classification systems and ensembles of classifiers.

A third research perspective goes in the direction of [26], where quasi-overlap functions on lattices were equipped with a topological space structure, namely, Alexandroff's spaces. From a theoretical point of view, equipping general pseudo-quasi-overlap functions with the property of continuity arising from Alexandroff's spaces introduces the concept of proximity (that is close to but not necessarily identical to), which enables us to extend any operation defined in an algebraic structure. Moreover, the ordering structure equipped with the topological structure carries much more information than only the structure of a poset, which is useful when dealing with intuitions about "continuity", "connectivity" and notions of "far" and "near". On the other hand, from the point of view of potential applications, a focused study on Alexandroff spaces and therefore all the properties of finite spaces can provide an important contribution to image analysis, digital topology, and computer graphics.

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