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Controllability Results for First Order Impulsive Fuzzy Differential Systems

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Abstract: In this paper, we investigate the controllability of first-order impulsive fuzzy differential equations. Using the direct construction method, the controllability of first-order linear impulsive fuzzy differential equations is considered with $a < 0$, the $(c1)$ solution, and $a < 0$, the $(c2)$ solution, respectively.In addition, by employing the Banach fixed-point theorem, the controllability of first-order nonlinear impulsive fuzzy differential equations is studied. Finally, examples are presented to illustrate our theoretical results.

Keywords: fuzzy differential equations; controllability; fixed-point theorem; $(c1)$ solution; $(c2)$ solution

MSC: 34A07; 34A37; 39A26; 93C05

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1. Introduction

The study of the controllability is an important area of research and classical differential controlled systems have been discussed in many articles. In 1993, researchers [1] initiated the study of impulsive controls related to impulsive differential equations and obtained some simple results to demonstrate the importance of employing impulsive controls. In [2], the authors demonstrated the controllability of impulsive functional differential equations with nonlocal conditions. In [3], the authors investigated the complete controllability of the control system with impulse effects. In [4], Liu and Willms gave necessary and sufficient conditions for the impulsive controllability of linear dynamical systems. In [5], the main goal was to present a technique suitable for the study of local and global controllability properties for nonlinear systems. In [6], the controllability and the observability of continuous linear time-varying systems with norm-bounded parameter perturbations were analyzed. In [7], the authors studied the existence and uniqueness of solutions and controllability for the semilinear fuzzy integrodifferential equations in n-dimensional fuzzy vector space $(E_N)^n$ using the Banach fixed-point theorem. In [8], the authors used the direct construction method to derive the controllability results for first-order linear fuzzy differential systems, but in practice, the impulsive phenomenon will affect the modeling of the system. Controllability plays an important role in many engineering problems, but little work has been conducted on the controllability of linear and nonlinear impulsive fuzzy differential equations. Since the present results available in the literature cannot model systems with impulsive effects, here we study the controllability of impulsive fuzzy systems.

Motivated by [7–9], we consider the controllability of the following systems.

Firstly, we consider first-order linear impulsive fuzzy differential equations:

$$\begin{cases} y'(t) = ay(t) + b(t) + \tilde{d}u(t), \ t \in J' := J \setminus \{t_k\}_{k \in \mathbb{M}}, \\ \Delta y(t_k) = c_k y(t_k^-), \ 1 + c_k \in \mathbb{R}_-, \ k \in \mathbb{M}, \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases} \tag{1}$$

where $a \in \mathbb{R}$, $y_0 \in \mathbb{R}_F$, $\mathbb{M} = \{1, 2, \cdots, m\}$ and $b : J \to \mathbb{R}_F$, $u : J \to \mathbb{R}_F$, $\tilde{d} \in \mathbb{R}_+$.

Next, we consider the following first-order nonlinear impulsive fuzzy differential equations:

$$\begin{cases} y'(t) = ay(t) + g(t, y(t)) + \tilde{d}u(t), \ \tilde{d} \in \mathbb{R}_+, \ t \in [0, T], \ t \neq t_k, \\ \Delta y(t_k) = c_k y(t_k^-) + g_k, \ g_k \in \mathbb{R}_F, \ 1 + c_k \in \mathbb{R}_-, \ k \in \mathbb{M}, \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases} \tag{2}$$

where $\Delta y(t_k) := y(t_k^+) + (-1)y(t_k^-)$ and $y(t_k^-) = \lim_{\epsilon \to 0^-} y(t_k + \epsilon)$ represents the left limit of $y(t)$ at $t = t_k$, $\mathbb{M} = \{1, 2, \cdots, m\}$. Here $a \in \mathbb{R}_-$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_m < t_{m+1} = T$, and $g : [0, T] \times \mathbb{R}_F \to \mathbb{R}_F$.

In Section 2, we present notations, concepts, and lemmas needed in this paper. In Section 3, we establish some theorems concerning the controllability of impulsive fuzzy differential systems. Finally, in the last section, we provide some examples to illustrate our main results.

2. Preliminaries

Let $J = [0, T]$ and $C(J, \mathbb{R}_F)$ denote the space of all continuous functions from J into $\mathbb{R}_F$. Let $PC(J, \mathbb{R}_F) := \{y : J \to \mathbb{R}_F : y \in C((t_k, t_{k+1}], \mathbb{R}_F), \ k \in \mathbb{M}_0$ and $\exists \ y(t_k^-)$ and $y(t_k^+), k \in \mathbb{M}_0$, with $y(t_k^-) = y(t_k)\}$ with the metric $H_1(u, v) = \sup_{t \in J} D(u(t), v(t))(u, v \in \mathbb{R}_F)$, where $\mathbb{M}_0 := \mathbb{M} \cup \{0\}$, $\mathbb{M} = \{1, 2, \cdots, m\}$, and $t_k < t_{k+1}$ for any $k \in \mathbb{M}_0$; here $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_m < t_{m+1} = T$.

We now collect some concepts which will be used throughout the paper; for more details, see [10,11].

Denote by $\mathbb{R}_F := \{v \mid v : \mathbb{R} \to [0, 1]\}$ the class of the fuzzy subsets of the real axis satisfying the following properties:

(X_1) v is normal (i.e., $\exists \, x_0 \in \mathbb{R}$ s.t. $v(x_0) = 1$).

(X_2) v is a convex fuzzy set (i.e., $v(\xi s_0 + (1 - \xi)s_1) \geq \min\{v(s_0), v(s_1)\}$) for all s_0, $s_1 \in \mathbb{R}$ and $\xi \in [0, 1]$.

(X_3) v is upper semicontinuous on $\mathbb{R}$.

(X_4) $[v]^0 = \overline{\{x \in \mathbb{R} : v(x) > 0\}}$ is compact.

Let $\alpha \in (0, 1]$. Consider the $\alpha-$level set of $v \in \mathbb{R}_F$ by $[v]^\alpha = \{s \in \mathbb{R} \mid v(s) \geq \alpha\}$, which is a nonempty compact interval for all $\alpha \in (0, 1]$. We use $[v]^\alpha = [\underline{v}_\alpha, \overline{v}_\alpha]$ to denote explicitly the $\alpha-$level set of v. We call $\underline{v}_\alpha$ and $\overline{v}_\alpha$ the lower and upper branches of v, respectively. We use the notation $diam([v]^\alpha) = \overline{v}_\alpha - \underline{v}_\alpha$ to denote the length of v.

The support Γ_v of a fuzzy number, v is defined, as a special case of level set, by the following:

$$\Gamma_v = \{s \in \mathbb{R} \mid v(s) > 0\}.$$

Now $\forall \alpha \in [0, 1]$, $u, v \in \mathbb{R}_F$ and $\xi \in \mathbb{R}$; we define the sum $u + v$ and the productξu as $[u + v]^\alpha = [u]^\alpha + [v]^\alpha = [\underline{u}_\alpha + \underline{v}_\alpha, \overline{u}_\alpha + \overline{v}_\alpha]$ and $[\xi u]^\alpha = \xi [u]^\alpha$.

Consider the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$ where $D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha) = \sup_{0 \leq \alpha \leq 1} \max\{|\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha|\}$ (see [12]). Then $(\mathbb{R}_F, D)$ is a complete metric space (see [13]) and (i) $D(u + e, v + e) = D(u, v)$, $\forall \ u, v, e \in \mathbb{R}_F$, (ii) $D(\varsigma u, \varsigma v) = |\varsigma| D(u, v)$, $\forall \ \varsigma \in \mathbb{R}$, $u, v \in \mathbb{R}_F$, (iii) $D(u + e, v + \varrho) \leq D(u, v) + D(e, \varrho)$, $\forall \ u, v, e, \varrho \in \mathbb{R}_F$ are satisfied.

Definition 1. *(see [11], Definition 2.1) Let $f : [a,b] \to \mathbb{R}_F$ be measurable and integrably bounded. The integral of f over $[a,b]$, denoted by $\int_a^b f(t)dt$, is defined levelwise by the expression*

$$\begin{aligned} \left[\int_a^b f(t)dt\right]^\alpha &:= \int_a^b [f(t)]^\alpha dt \\ &= \left\{\int_a^b \tilde{f}(t)dt \mid \tilde{f} : [a,b] \to \mathbb{R}_F \text{ is a measurable selection for } [f(\cdot)]^\alpha\right\}, \end{aligned}$$

for every $\alpha \in [0,1]$.

Throughout this paper, we use the symbol $\ominus$ to represent the H-difference. Note that $a_1 \ominus a_2 \neq a_1 + (-1)a_2 := a_1 - a_2$.

Here we simplify the classes of strongly generalized differentiable by considering case (i) and case (ii) as in paper [14].

Definition 2. *(see [11], Definition 2.2) Let $Q : J \to \mathbb{R}_F$ and fix $n_0 \in J$. We say Q is differentiable at n_0, if we have an element $Q'(n_0) \in \mathbb{R}_F$ such that either*

(c1) for all $p > 0$ sufficiently close to 0, the H-differences $Q(n_0 + p) \ominus Q(n_0)$, $Q(n_0) \ominus Q(n_0 - p)$ exist and the limits (in the metric D)

$$\lim_{p\to 0^+} \frac{Q(n_0+p) \ominus Q(n_0)}{p} = \lim_{p\to 0^+} \frac{Q(n_0) \ominus Q(n_0-p)}{p} = Q'(n_0),$$

or

(c2) for all $p > 0$ sufficiently close to 0, the H-differences $Q(n_0) \ominus Q(n_0 + p)$, $Q(n_0 - p) \ominus Q(n_0)$ exist and the limits (in the metric D)

$$\lim_{p\to 0^+} \frac{Q(n_0) \ominus Q(n_0+p)}{-p} = \lim_{p\to 0^+} \frac{Q(n_0-p) \ominus Q(n_0)}{-p} = Q'(n_0).$$

Definition 3. *(See [11], Definition 2.5) Let $Q : J \to \mathbb{R}_F$. We say Q is $(c1)$-differentiable on J if Q is differentiable in the sense (c1) in Definition 2 and its derivative is denoted D_1Q. Similarly, we can define $(c2)$-differentiable and denote it by D_2Q.*

Theorem 4. *(see [11], Theorem 2.6) Let $Q : J \to \mathbb{R}_F$ and put $[Q(t)]^\alpha = [p_\alpha(t), q_\alpha(t)]$ for each $\alpha \in [0,1]$.*

(i) If Q is $(c1)$-differentiable, then p_α and q_α are differentiable functions and $[D_1Q(t)]^\alpha = [p'_\alpha(t), q'_\alpha(t)]$.

(ii) If Q is $(c2)$-differentiable, then p_α and q_α are differentiable functions and we have $[D_2Q(t)]^\alpha = [q'_\alpha(t), p'_\alpha(t)]$.

Theorem 5. *(see [15], Theorem 2.2) Let $K : J \to \mathbb{R}_F$ be a differentiable fuzzy number-valued mapping and we suppose that the derivative K' is integrable over J. Then for each $t \in J$, we have*

(a) if K is $(c1)$-differentiable, then $K(t) = K(b) + \int_b^t K'(s)ds$;

(b) if K is $(c2)$-differentiable, then $K(t) = K(b) \ominus \int_b^t -K'(s)ds$.

Theorem 6. *(see [11], Theorem 2.7) Let Q be (c2)-differentiable on J and assume that the derivative Q' is integrable over J. Then for each $t \in J$ we have*

$$Q(t) = Q(a) \ominus \int_a^t -Q'(\tau)d\tau.$$

Theorem 7. *(see [16], Theorem 2.4) Let $K : J \to \mathbb{R}_F$ be continuous. Define the integral $G(t) := \sigma \ominus \int_v^t -K(s)ds$, $t \in J$, where $\sigma \in \mathbb{R}_F$ is such that the preceding H-difference exists on J. Then, $G(t)$ is $(c2)$-differentiable and $G'(t) = K(t)$.*

Consider the following conditions (here $p : \mathbb{R} \to \mathbb{R}_F$):
($\mathbb{H}1$) For a given $t \in J$, $p(t+h) \ominus p(t)$ and $p(t) \ominus p(t-h)$ exist for $h \to 0^+$;
($\mathbb{H}2$) For a given $t \in J$, $p(t) \ominus p(t+h)$ and $p(t-h) \ominus p(t)$ exist for $h \to 0^+$.

3. Main Results

In this section, we introduce some concepts related to the controllability of our problems.

Definition 8. *(see [7], Definition 4.1) Fuzzy system (1) and (2) is called controllable on $[t_0, T]$ $(T > t_0)$ if, for an arbitrary initial state $y_0 \in \mathbb{R}_F$ at t_0 and final state $y_1 \in \mathbb{R}_F$ at time T_1 (here $t_0 = 0$ and $T_1 = T$), there exists a control $u : J \to \mathbb{R}_F$ such that the system (1) and (2) has a solution y that satisfies $y(T_1) = y_1$(i.e., $[y(T_1)]^\alpha = [y_1]^\alpha$).*

3.1. Controllability of First-Order Linear Impulsive Fuzzy Differential Equations

Consider the following system (see [17]):

$$\begin{cases} y'(t) = ay(t) + b(t) + \tilde{d}u(t),\ t \in J' := J \setminus \{t_k\}_{k \in \mathbb{M}},\ \tilde{d} \in \mathbb{R}_+, \\ \Delta y(t_k) = c_k y(t_k^-),\ 1 + c_k \in \mathbb{R}_-,\ k \in \mathbb{M}, \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases}$$

where $\Delta y(t_k) := y(t_k^+) + (-1)y(t_k^-)$ and $y(t_k^-) = \lim_{\epsilon \to 0^-} y(t_k + \epsilon)$ represents the left limit of $y(t)$ at $t = t_k$, $\mathbb{M} = \{1, 2, \cdots, m\}$.

From ([17], Theorem 3.1, $a < 0$), the $(c1)$-solution of (1) can be written in the form:

$$\begin{aligned} & y(t) \\ = \ & \cosh(a(t - t_k^+))\, \underset{a<0}{p_{1,k}}\, y_0 + \cosh(a(t - t_k^+))\, \underset{a<0}{q_{1,k}}\, y_0 \\ & + \sinh(a(t - t_k^+))\, \underset{a<0}{p_{1,k}}\, y_0 + \sinh(a(t - t_k^+))\, \underset{a<0}{q_{1,k}}\, y_0 \\ & + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \cosh(a(t - t_k^+) - as)(b(s) + \tilde{d}u(s))ds \\ & + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(t - t_k^+) - as)(b(s) + \tilde{d}u(s))ds \\ & + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \sinh(a(t - t_k^+) - as)(b(s) + \tilde{d}u(s))ds \\ & + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(t - t_k^+) - as)(b(s) + \tilde{d}u(s))ds \\ & + \int_{t_{k-1}}^{t_k} (1 + c_k) \cosh(at - as)(b(s) + \tilde{d}u(s))ds \\ & + \int_{t_{k-1}}^{t_k} (1 + c_k) \sinh(at - as)(b(s) + \tilde{d}u(s))ds \\ & + \int_{t_k}^{t} \cosh(at - as)(b(s) + \tilde{d}u(s))ds \\ & + \int_{t_k}^{t} \sinh(at - as)(b(s) + \tilde{d}u(s))ds, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \chi_i &= 1 + c_i\ (1 + c_i \in \mathbb{R}_-), \\ \theta_{\iota i,k} &= \underset{a<0}{p_{i+1,k}}(j)\chi_i \cosh(at_i) + \underset{a<0}{q_{i+1,k}}(j)\chi_i \sinh(at_i)(\iota = 1), \\ \theta_{\iota i,k} &= \underset{a<0}{p_{i+1,k}}(j)\chi_i \sinh(at_i) + \underset{a<0}{q_{i+1,k}}(j)\chi_i \cosh(at_i)(\iota = 2), \end{aligned}$$

$$\underset{a<0}{r_1}(j) = (1+c_j)\sinh(a(t_j - t_{j-1}^+)) + (1+c_j)\cosh(a(t_j - t_{j-1}^+)),$$
$$\underset{a<0}{r_2}(j) = (1+c_j)\sinh(a(t_j - t_{j-1}^+)) - (1+c_j)\cosh(a(t_j - t_{j-1}^+)),$$

$$\underset{a<0}{p_{1,k}}(j) = \frac{\prod_{j=k}^{1} r_1(j) + \prod_{j=k}^{1} r_2(j)}{2}, \quad \underset{a<0}{q_{1,k}}(j) = \frac{\prod_{j=k}^{1} r_1(j) - \prod_{j=k}^{1} r_2(j)}{2},$$
$$\underset{a<0}{p_{i+1,k}}(j) = \frac{\prod_{j=k}^{i+1} r_1(j) + \prod_{j=k}^{i+1} r_2(j)}{2}, \quad \underset{a<0}{q_{i+1,k}}(j) = \frac{\prod_{j=k}^{i+1} r_1(j) - \prod_{j=k}^{i+1} r_2(j)}{2}. \tag{4}$$

From ([17], Theorem 3.2, $a < 0$), the $(c2)$-solution of (1) can be written in the form:

$$y(t) = w(t,0)y_0 \ominus \int_0^t (-1)w(t,s)b(s)ds \ominus \int_0^t (-1)w(t,s)\tilde{d}u(s)ds, \ t \in (t_k, t_{k+1}],$$

provided that H-differences exist. Here, $i(t,0)$ is the number of impulsive points in the interval $(0,t)$,

$$w(t,s) = \begin{cases} e^{a(t-s)}, \ t,s \in (t_k, t_{k+1}], \ k = 0,1,2,\cdots,i(t,0), \\ e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}, t_{k-1} < s \le t_k < t \le t_{k+1}, \ k = 1,2,\cdots,i(t,0)-1, \\ e^{a(t-t_k^+)} \prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}, \ t_{i-1} < s \le t_i < t_k < t \le t_{k+1}, \\ i = 1,2,\cdots,i(t,0)-1, k = 2,\cdots,i(t,0), \end{cases} \tag{5}$$

and

$$\begin{aligned} w(t,t_0) &= w(t,0) = e^{a(t-t_k^+)}(1+c_k)\prod_{j=k}^{1} e^{a(t_j-t_{j-1}^+)}(1+c_{j-1}), \ t_0 = 0, \ c_0 = 0, \\ &= e^{a(t-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}. \end{aligned}$$

Thus, we consider two cases to study the controllability of (1): the $(c1)$ solution and the $(c2)$ solution.

Case 5.1 Consider $a < 0$ via the $(c1)$ solution.

Theorem 9. *In Case 5.1, system* (1) *is controllable, if the control function* $u_7(t)$ *is given by*

$$u_7(t) = \begin{cases} \frac{1}{2}\chi_{i,k}\left[\frac{1}{3\tilde{d}t_{k-1}}y_1\cosh(a(T_1 - t_k^+) - at) \ominus \frac{1}{3\tilde{d}t_{k-1}}y_1\sinh(a(T_1 - t_k^+) - at)\right] \\ \ominus\frac{1}{2}\chi_{i,k}\left[\frac{1}{3\tilde{d}t_{k-1}}\cosh(a(T_1 - t_k^+) - at)\Lambda_{y_0} \ominus \frac{1}{3\tilde{d}t_{k-1}}\sinh(a(T_1 - t_k^+) - at)\Lambda_{y_0}\right] \\ \ominus\frac{1}{\tilde{d}}b(t), \ t_{i-1} < s \le t_i < t_k < t \le t_{k+1}, \\ \frac{1}{(1+c_k)(t_k-t_{k-1})\tilde{d}}\left[\frac{1}{3}y_1\cosh(at_k - at) \ominus \frac{1}{3}y_1\sinh(at_k - at)\right] \\ \ominus\frac{1}{(1+c_k)(t_k-t_{k-1})\tilde{d}}\left[\frac{1}{3}\cosh(at_k - at)\Lambda_{y_0} \ominus \frac{1}{3}\sinh(at_k - at)\Lambda_{y_0}\right] \\ \ominus\frac{1}{\tilde{d}}b(t), \ t_{k-1} < s \le t_k < t \le t_{k+1}, \\ \frac{1}{(T_1-t_k)\tilde{d}}\left[\frac{1}{3}y_1\cosh(at_k^+ - at) \ominus \frac{1}{3}y_1\sinh(at_k^+ - at)\right] \\ \ominus\frac{1}{(T_1-t_k)\tilde{d}}\left[\frac{1}{3}\cosh(at_k^+ - at)\Lambda_{y_0} \ominus \frac{1}{3}\sinh(at_k^+ - at)\Lambda_{y_0}\right] \\ \ominus\frac{1}{\tilde{d}}b(t), \ t,s \in (t_k, t_{k+1}], \end{cases}$$

where the H-differences exist,

$$\theta_{\iota i,k}\chi_{i,k} = 1,$$

and

$$\chi_{i,k} = \begin{cases} \frac{1}{\underset{a<0}{p_{i+1,k}}(j)\chi_i \cosh(at_i) + \underset{a<0}{q_{i+1,k}}(j)\chi_i \sinh(at_i)}, \\ \text{when } \theta_{\iota i,k}(\iota = 1), \\ \frac{1}{\underset{a<0}{p_{i+1,k}}(j)\chi_i \sinh(at_i) + \underset{a<0}{q_{i+1,k}}(j)\chi_i \cosh(at_i)}, \\ \text{when } \theta_{\iota i,k}(\iota = 2), \end{cases}$$

and

$$\begin{aligned} \Lambda_{y_0} \quad = \quad & \cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}\, y_0 + \cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}\, y_0 \\ & + \sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}\, y_0 + \sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}\, y_0. \end{aligned}$$

Proof. Since the H-differences exist in $u_7(t)$, for $T_1 > 0$, we obtain $y(T_1) = y_1$. Thus, the system (1) is controllable in this case. □

Case 5.2 Consider $a < 0$ via the $(c2)$ solution.

Theorem 10. *In Case 5.2, system (1) is controllable, if $W_3^{-1}[0, T_1]$ exists; here*

$$W_3[0, T_1] = \int_0^{T_1} w(T_1, s)\tilde{d}\tilde{d}w(T_1, s)ds.$$

Proof. From $W_3[0, T_1] = \int_0^{T_1} e^{-\int_0^s a(v)dv}\tilde{d}\tilde{d}e^{-\int_0^s a(v)dv}ds$, for $T_1 > 0$, for any final state $y_1 \in \mathbb{R}_F$ we can choose a control function as follows:

$$u_8(t) = -\tilde{d}w(T_1, t)W_3^{-1}[0, T_1]\left(y_0 \ominus \int_0^{T_1}(-b(s))e^{-\int_0^s a(v)dv}ds \ominus e^{-\int_0^{T_1} a(v)dv}y_1\right),\ t \in J,$$

where the H-differences exist. Then we obtain $y(T_1) = y_1$. That means that system (1) is controllable in this case. □

3.2. Controllability of First-Order Nonlinear Impulsive Fuzzy Differential Equations

We consider the following first-order nonlinear impulsive fuzzy differential system:

$$\begin{cases} y'(t) = ay(t) + g(t, y(t)) + \tilde{d}u(t),\ t \in [0, T],\ t \neq t_k,\ \tilde{d} \in \mathbb{R}_+, \\ \Delta y(t_k) = c_k y(t_k^-) + g_k,\ k = 1, 2, \cdots, m, \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases}$$

where $a < 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_m < t_{m+1} = T$, $g \in C([0, T] \times \mathbb{R}_F, \mathbb{R}_F)$ and $\underline{g}^\alpha = -\overline{g}^\alpha$, $1 + c_k < 0$, $g_k \in \mathbb{R}_F$.

If y is $(c1)$-differentiable, then

$$\begin{aligned} y(t) \quad = \quad & e^{a(t-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0 \ominus \Bigg\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)} \\ & \times \prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s, y(s))ds \\ & + \int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s, y(s))ds + \int_{t_k}^{t}(-1)e^{a(t-s)}g(s, y(s))ds \\ & + \Bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}u(s)ds \end{aligned}$$

$$+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}u(s)ds+\int_{t_k}^{t}(-1)e^{a(t-s)}\tilde{d}u(s)ds\Big)\Big\}$$
$$+\sum_{i=1}^{k-1}e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(t-t_k^+)}g_k,\ t\in(t_k,t_{k+1}]. \tag{6}$$

If y is $(c2)$-differentiable, then

$$\begin{aligned} y(t) \ =\ & \cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0+\cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0 \\ & +\sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0+\sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0 \\ & +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\int_{t_{k-1}}^{t_k}(1+c_k)\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\int_{t_k}^{t}\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds+\int_{t_k}^{t}\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds \\ & +\sum_{i=1}^{k-1}\Big\{\Big[\cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}(j)+\sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}(j)\Big]g_i \\ & +\Big[\cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}(j)+\sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}(j)\Big]g_i\Big\} \\ & +\cosh(a(t-t_k^+))g_k+\sinh(a(t-t_k^+))g_k,\ t\in(t_k,t_{k+1}], \end{aligned} \tag{7}$$

where χ_i, $\theta_{\iota i,k}$ $(\iota=1,2)$, $\underset{a<0}{r_1}(j)$, $\underset{a<0}{r_2}(j)$, $\underset{a<0}{p_{1,k}}(j)$, $\underset{a<0}{q_{1,k}}(j)$, $\underset{a<0}{p_{i+1,k}}(j)$, $\underset{a<0}{q_{i+1,k}}(j)$ are the same as in (4).

Next, we consider two cases to study the controllability of (2): the $(c1)$-differentiable case and the $(c2)$-differentiable case.

Case 6.1 Consider $a<0$ via the $(c2)$-differentiable case.

Let $t\in(t_k,t_{k+1}]$, $k=0,1,\cdots,m$. Now let the operator $W_{H0k+1}:\widetilde{P(\mathbb{R})}\to\mathbb{R}_F$ be defined by

$$W_{H0k+1}u_0=\begin{cases}\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}u_0(s)ds \\ +\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}u_0(s)ds \\ +\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}\tilde{d}u_0(s)ds,\ u_0\in\overline{\Gamma_u}, \\ 0,\ otherwise,\end{cases}$$

where $\widetilde{P(\mathbb{R})}$ is the set of subsets in $\mathbb{R}$. Then there exist $\underline{W^\alpha_{H0k+1}}$, $\overline{W^\alpha_{H0k+1}}$ such that

$$\underline{W^\alpha_{H0k+1}}(\underline{u_0}) \ =\ \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}\underline{u_0}^{\alpha}(s)ds$$

$$
\begin{aligned}
&+ \int_{t_{k-1}}^{t_k} (-1) e^{a(T_1 - t_k^+)} (1 + c_k) e^{a(t_k - s)} \tilde{d}\underline{u_0}^{\alpha}(s) ds \\
&+ \int_{t_k}^{T_1} e^{a(T_1 - s)} \tilde{d}\underline{u_0}^{\alpha}(s) ds,
\end{aligned}
$$

$$
\begin{aligned}
\overline{W_{H0k+1}^{\alpha}}(\overline{u_0}) \quad = \quad & \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1) e^{a(T_1 - t_k^+)} \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} (1 + c_i) e^{a(t_i - s)} \tilde{d}\overline{u_0}^{\alpha}(s) ds \\
&+ \int_{t_{k-1}}^{t_k} (-1) e^{a(T_1 - t_k^+)} (1 + c_k) e^{a(t_k - s)} \tilde{d}\overline{u_0}^{\alpha}(s) ds \\
&+ \int_{t_k}^{T_1} e^{a(T_1 - s)} \tilde{d}\overline{u_0}^{\alpha}(s) ds.
\end{aligned}
$$

We assume that $\underline{W_{H0k+1}^{\alpha}}$, $\overline{W_{H0k+1}^{\alpha}}$ are bijective mappings and let $\underline{u_0}^{\alpha} = -\overline{u_0}^{\alpha}$ and $\underline{W_{H0k+1}^{\alpha}} = -\overline{W_{H0k+1}^{\alpha}}$.

Hence, the α-level of $u(s)$ given by

$$
\begin{aligned}
[u(s)]^{\alpha} \quad = \quad & [\underline{u}^{\alpha}(s), \overline{u}^{\alpha}(s)] \\
= \quad & \Bigg[(\underline{W_{H0k+1}^{\alpha}})^{-1} \Bigg(e^{a(T_1 - t_k^+)} \prod_{j=k}^{1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} \underline{y_0}^{\alpha} \\
& - \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1) e^{a(T_1 - t_k^+)} \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} (1 + c_i) e^{a(t_i - s)} \underline{g}^{\alpha}(s, y(s)) ds \\
& - \int_{t_{k-1}}^{t_k} (-1) e^{a(T_1 - t_k^+)} (1 + c_k) e^{a(t_k - s)} \underline{g}^{\alpha}(s, y(s)) ds - \int_{t_k}^{T_1} e^{a(T_1 - s)} \underline{g}^{\alpha}(s, y(s)) ds \\
& - \underline{y_1}^{\alpha} + \sum_{i=1}^{k-1} e^{a(T_1 - t_k^+)} \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} \underline{g_i}^{\alpha} + e^{a(T_1 - t_k^+)} \underline{g_k}^{\alpha} \Bigg), \\
& (\overline{W_{H0k+1}^{\alpha}})^{-1} \Bigg(e^{a(T_1 - t_k^+)} \prod_{j=k}^{1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} \overline{y_0}^{\alpha} \\
& - \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1) e^{a(T_1 - t_k^+)} \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} (1 + c_i) e^{a(t_i - s)} \overline{g}^{\alpha}(s, y(s)) ds \\
& - \int_{t_{k-1}}^{t_k} (-1) e^{a(T_1 - t_k^+)} (1 + c_k) e^{a(t_k - s)} \overline{g}^{\alpha}(s, y(s)) ds - \int_{t_k}^{T_1} e^{a(T_1 - s)} \overline{g}^{\alpha}(s, y(s)) ds \\
& - \overline{y_1}^{\alpha} + \sum_{i=1}^{k-1} e^{a(T_1 - t_k^+)} \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} \overline{g_i}^{\alpha} + e^{a(T_1 - t_k^+)} \overline{g_k}^{\alpha} \Bigg) \Bigg],
\end{aligned}
$$

where $\underline{y_0}^{\alpha} = -\overline{y_0}^{\alpha}$, $\underline{y_1}^{\alpha} = -\overline{y_1}^{\alpha}$, $\underline{g_i} = -\overline{g_i}$, $\underline{g_k} = -\overline{g_k}$.

Then, substituting this expression into (6) yields the α-level of $y(T_1)$, i.e.,

$$
\begin{aligned}
& [y(T_1)]^{\alpha} \\
= \quad & \Bigg[e^{a(T_1 - t_k^+)} \prod_{j=k}^{1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} \underline{y_0}^{\alpha} - \Bigg\{ \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1) e^{a(T_1 - t_k^+)} \\
& \times \prod_{j=k}^{i+1} (1 + c_j) e^{a(t_j - t_{j-1}^+)} (1 + c_i) e^{a(t_i - s)} \underline{g}^{\alpha}(s, y(s)) ds \\
& + \int_{t_{k-1}}^{t_k} (-1) e^{a(T_1 - t_k^+)} (1 + c_k) e^{a(t_k - s)} \underline{g}^{\alpha}(s, y(s)) ds + \int_{t_k}^{T_1} e^{a(T_1 - s)} \underline{g}^{\alpha}(s, y(s)) ds
\end{aligned}
$$

$$+\Bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}$$

$$\times(\underline{W_{H0k+1}^{\alpha}})^{-1}\Bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{y_0}^{\alpha}$$

$$-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\underline{g}^{\alpha}(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\underline{y_1}^{\alpha}+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{g_i}^{\alpha}+e^{a(T_1-t_k^+)}\underline{g_k}^{\alpha}\Bigg)ds$$

$$+\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}(\underline{W_{H0k+1}^{\alpha}})^{-1}\Bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{y_0}^{\alpha}$$

$$-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\underline{g}^{\alpha}(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\underline{y_1}^{\alpha}+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{g_i}^{\alpha}+e^{a(T_1-t_k^+)}\underline{g_k}^{\alpha}\Bigg)ds$$

$$+\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}\tilde{d}(\underline{W_{H0k+1}^{\alpha}})^{-1}\Bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{y_0}^{\alpha}$$

$$-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\underline{g}^{\alpha}(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\underline{g}^{\alpha}(s,y(s))ds$$

$$-\underline{y_1}^{\alpha}+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{g_i}^{\alpha}+e^{a(T_1-t_k^+)}\underline{g_k}^{\alpha}\Bigg)ds\Bigg)\Bigg\}$$

$$+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\underline{g_i}^{\alpha}+e^{a(T_1-t_k^+)}\underline{g_k}^{\alpha},$$

$$e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{y_0}^{\alpha}-\Bigg\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}$$

$$\times\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\overline{g}^{\alpha}(s,y(s))ds$$

$$+\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\overline{g}^{\alpha}(s,y(s))ds+\int_{t_k}^{T_1}e^{a(T_1-s)}\overline{g}^{\alpha}(s,y(s))ds$$

$$+\Bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}$$

$$\times(\overline{W_{H0k+1}^{\alpha}})^{-1}\Bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{y_0}^{\alpha}$$

$$-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\overline{g}^{\alpha}(s,y(s))ds$$

$$
\begin{aligned}
&-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\overline{g}^\alpha(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\overline{g}^\alpha(s,y(s))ds\\
&-\overline{y_1}^\alpha+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{g_i}^\alpha+e^{a(T_1-t_k^+)}\overline{g_k}^\alpha\Big)ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}(\overline{W_{H0k+1}^\alpha})^{-1}\Big(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{y_0}^\alpha\\
&-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\overline{g}^\alpha(s,y(s))ds\\
&-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\overline{g}^\alpha(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\overline{g}^\alpha(s,y(s))ds\\
&-\overline{y_1}^\alpha+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{g_i}^\alpha+e^{a(T_1-t_k^+)}\overline{g_k}^\alpha\Big)ds\\
&+\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}\tilde{d}(\overline{W_{H0k+1}^\alpha})^{-1}\Big(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{y_0}^\alpha\\
&-\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\overline{g}^\alpha(s,y(s))ds\\
&-\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\overline{g}^\alpha(s,y(s))ds-\int_{t_k}^{T_1}e^{a(T_1-s)}\overline{g}^\alpha(s,y(s))ds\\
&-\overline{y_1}^\alpha+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{g_i}^\alpha+e^{a(T_1-t_k^+)}\overline{g_k}^\alpha\Big)ds\Big)\Big\}\\
&+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}\overline{g_i}^\alpha+e^{a(T_1-t_k^+)}\overline{g_k}^\alpha\Big]\\
= \;& [\underline{y_1}^\alpha,\overline{y_1}^\alpha]\\
= \;& [y_1]^\alpha.
\end{aligned}
$$

Using the control above we may consider the operator $\Phi : C((t_k,t_{k+1}],\mathbb{R}_F) \to C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$, where

$$
\begin{aligned}
(\Phi y)(t) \;=\;& e^{a(t-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\ominus\Big\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\\
&\times\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds+\int_{t_k}^{t}(-1)e^{a(t-s)}g(s,y(s))ds\\
&+\Big(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}\\
&\times W_{H0k+1}^{-1}\Big(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\Big)ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}W_{H0k+1}^{-1}\Big(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds\\
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\Big)ds\\
&+\int_{t_k}^{t}(-1)e^{a(t-s)}\tilde{d}W_{H0k+1}^{-1}\Big(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds\\
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\Big)ds\Big)\Big\}\\
&+\sum_{i=1}^{k-1}e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(t-t_k^+)}g_k,
\end{aligned}
$$

where the fuzzy mappings W_{H0k+1}^{-1} satisfies the above statements.

Note that $\Phi y(T_1)=y_1$, which means that the u steers (6) from $(\Phi y)(0)$ to y_1 in finite time T_1 for $t\in(t_k,t_{k+1}]$, $k=0,1,\cdots,m$, provided that we obtain a fixed point of the nonlinear operator Φ.

Assume the following hypotheses:

(H_1) for $x(\cdot)\in\mathbb{R}_F$ and $y(\cdot)\in\mathbb{R}_F$, and assumethere is a positive number C such that

$$
d_H\Big([g(s,y(\cdot))]^\alpha,[g(s,x(\cdot))]^\alpha\Big)\le Cd_H\Big([y(\cdot)]^\alpha,[x(\cdot)]^\alpha\Big),\ s\in J.
$$

Then we have

$$
H_1(g(s,y),g(s,x))\le CH_1(y,x),\ s\in J.
$$

(H_2) $2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\mid\prod_{j=k}^{i+1}(1+c_j)(1+c_i)\mid Cds+2\mid 1+c_k\mid(t_k-t_{k-1})C+2(T_1-t_k)C<1.$

Theorem 11. *In Case 6.1, suppose that (H_1), (H_2) are satisfied. Then system (2) is controllable.*

Proof. We can easily check that Φ is continuous function from $C((t_k,t_{k+1}],\mathbb{R}_F)$ to $C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$. For $x,y\in C((t_k,t_{k+1}],\mathbb{R}_F)$, we obtain

$$
\begin{aligned}
&d_H\Big([\Phi y(t)]^\alpha,[\Phi x(t)]^\alpha\Big)\\
=\ &d_H\Big(\Big[e^{a(t-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\ominus\Big\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds+\int_{t_k}^{t}(-1)e^{a(t-s)}g(s,y(s))ds\\
&+\bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}\\
&\times W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds\\
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds\\
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds\\
&+\int_{t_k}^{t}(-1)e^{a(t-s)}\tilde{d}W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\\
&\ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,y(s))ds\\
&\ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,y(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,y(s))ds\\
&\ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds\bigg)\bigg\}\\
&+\sum_{i=1}^{k-1}e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(t-t_k^+)}g_k\bigg]^{\alpha},\\
&\bigg[e^{a(t-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0\ominus\bigg\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\\
&\times\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,x(s))ds\\
&+\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,x(s))ds+\int_{t_k}^{t}(-1)e^{a(t-s)}g(s,x(s))ds\\
&+\bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\tilde{d}
\end{aligned}
$$

$$
\begin{aligned}
& \times W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0 \\
& \ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,x(s))ds \\
& \ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,x(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,x(s))ds \\
& \ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds \\
& +\int_{t_{k-1}}^{t_k}(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}\tilde{d}W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0 \\
& \ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,x(s))ds \\
& \ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,x(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,x(s))ds \\
& \ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds \\
& +\int_{t_k}^{t}(-1)e^{a(t-s)}\tilde{d}W_{H0k+1}^{-1}\bigg(e^{a(T_1-t_k^+)}\prod_{j=k}^{1}(1+c_j)e^{a(t_j-t_{j-1}^+)}y_0 \\
& \ominus\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}g(s,x(s))ds \\
& \ominus\int_{t_{k-1}}^{t_k}(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}g(s,x(s))ds\ominus\int_{t_k}^{T_1}(-1)e^{a(T_1-s)}g(s,x(s))ds \\
& \ominus y_1+\sum_{i=1}^{k-1}e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(T_1-t_k^+)}g_k\bigg)ds\bigg)\bigg\} \\
& +\sum_{i=1}^{k-1}e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}g_i+e^{a(t-t_k^+)}g_k\bigg]^{\alpha}\bigg) \\
\leq\ & \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\mid(-1)e^{a(t-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
& +\int_{t_{k-1}}^{t_k}\mid(-1)e^{a(t-t_k^+)}(1+c_k)e^{a(t_k-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
& +\int_{t_k}^{t}\mid(-1)e^{a(t-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
& +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\mid(-1)e^{a(T_1-t_k^+)}\prod_{j=k}^{i+1}(1+c_j)e^{a(t_j-t_{j-1}^+)}(1+c_i)e^{a(t_i-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
& +\int_{t_{k-1}}^{t_k}\mid(-1)e^{a(T_1-t_k^+)}(1+c_k)e^{a(t_k-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
& +\int_{t_k}^{T_1}\mid(-1)e^{a(T_1-s)}\mid d_H\Big([g(s,y(s))]^{\alpha},[g(s,x(s))]^{\alpha}\Big)ds \\
\leq\ & \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\mid\prod_{j=k}^{i+1}(1+c_j)(1+c_i)\mid Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds \\
& +\int_{t_{k-1}}^{t_k}\mid 1+c_k\mid Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds+\int_{t_k}^{t}Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds
\end{aligned}
$$

$$+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds$$
$$+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds+\int_{t_k}^{T_1} Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds,$$

then

$$\begin{aligned}
D(\Phi y(t),\Phi x(t)) &= \sup_{0<\alpha\leq 1} d_H\Big([\Phi y(t)]^\alpha,[\Phi x(t)]^\alpha\Big)\\
&\leq \sup_{0<\alpha\leq 1}\Big(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds+\int_{t_k}^{t} Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds\\
&\quad+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds+\int_{t_k}^{T_1} Cd_H\Big([y(s)]^\alpha,[x(s)]^\alpha\Big)ds\Big)\\
&= \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid CD(y(s),x(s))ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid CD(y(s),x(s))ds+\int_{t_k}^{t} CD(y(s),x(s))ds\\
&\quad+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid CD(y(s),x(s))ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid CD(y(s),x(s))ds+\int_{t_k}^{T_1} CD(y(s),x(s))ds;
\end{aligned}$$

thus,

$$\begin{aligned}
H_1(\Phi y,\Phi x) &= \sup_{0\leq t\leq T_1} D(\Phi y(t),\Phi x(t))\\
&\leq \sup_{0\leq t\leq T_1}\Big(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid CD(y(s),x(s))ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid CD(y(s),x(s))ds+\int_{t_k}^{t} CD(y(s),x(s))ds\\
&\quad+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid CD(y(s),x(s))ds\\
&\quad+\int_{t_{k-1}}^{t_k} \mid 1+c_k \mid CD(y(s),x(s))ds+\int_{t_k}^{T_1} CD(y(s),x(s))ds\Big)\\
&= \Big(2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \mid \prod_{j=k}^{i+1}(1+c_j)(1+c_i) \mid Cds+2 \mid 1+c_k \mid (t_k-t_{k-1})C\\
&\quad+2(T_1-t_k)C\Big)H_1(y,x).
\end{aligned}$$

According to hypothesis $(H2)$, Φ is a contraction mapping. According to the Banach fixed-point theorem, Φ has a fixed point $y \in C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$.

In summary, the proof is completed. □

Case 6.2 Consider $a<0$ via the $(c1)$-differentiable case.

Let $t \in (t_k, t_{k+1}]$, $k = 0, 1, \cdots, m$. Now let the operator $W_{S0k+1} : \widetilde{P(\mathbb{R})} \to \mathbb{R}_F$ be defined by

$$W_{S0k+1}u_0 = \begin{cases} \sum\limits_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}u_0(s)ds \\ + \sum\limits_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}u_0(s)ds \\ + \sum\limits_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}u_0(s)ds \\ + \sum\limits_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}u_0(s)ds \\ + \int_{t_{k-1}}^{t_k} (1 + c_k) \cosh(aT_1 - as) \tilde{d}u_0(s)ds + \int_{t_{k-1}}^{t_k} (1 + c_k) \sinh(aT_1 - as) \tilde{d}u_0(s)ds \\ + \int_{t_k}^{T_1} \cosh(aT_1 - as) \tilde{d}u_0(s)ds + \int_{t_k}^{T_1} \sinh(aT_1 - as) \tilde{d}u_0(s)ds, \ u_0 \in \overline{\Gamma_u}, \\ 0, \ otherwise, \end{cases}$$

where $\widetilde{P(\mathbb{R})}$ is the set of subsets in $\mathbb{R}$. Then there exist $\underline{W^{\alpha}_{S0k+1}}$, $\overline{W^{\alpha}_{S0k+1}}$ such that

$$\begin{aligned} \underline{W^{\alpha}_{S0k+1}}(\underline{u_0}) &= \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\, \theta_{1i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\, \theta_{2i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_{k-1}}^{t_k} (-1)(1 + c_k) \cosh(aT_1 - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_{k-1}}^{t_k} (1 + c_k) \sinh(aT_1 - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds + \int_{t_k}^{T_1} \cosh(aT_1 - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_k}^{T_1} (-1) \sinh(aT_1 - as) \tilde{d}\underline{u_0}^{\alpha}(s)ds, \ \underline{u_0}^{\alpha}(s) \in [\underline{u}^{\alpha}(s), u^1(s)], \\ \overline{W^{\alpha}_{S0k+1}}(\overline{u_0}) &= \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\, \theta_{1i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(T_1 - t_k^+) - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\, \theta_{2i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(T_1 - t_k^+) - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_{k-1}}^{t_k} (-1)(1 + c_k) \cosh(aT_1 - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_{k-1}}^{t_k} (1 + c_k) \sinh(aT_1 - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds + \int_{t_k}^{T_1} \cosh(aT_1 - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds \\ &\quad + \int_{t_k}^{T_1} (-1) \sinh(aT_1 - as) \tilde{d}\overline{u_0}^{\alpha}(s)ds, \ \overline{u_0}^{\alpha}(s) \in [u^1(s), \overline{u}^{\alpha}(s)]. \end{aligned}$$

We assume that $\underline{W^{\alpha}_{S0k+1}}$, $\overline{W^{\alpha}_{S0k+1}}$ are bijective mappings and let $\underline{u_0}^{\alpha} = -\overline{u_0}^{\alpha}$ and $\underline{W^{\alpha}_{S0k+1}} = -\overline{W^{\alpha}_{S0k+1}}$.

Hence, the α-level of $u(s)$ given by

$$\begin{aligned}
[u(s)]^{\alpha} &= [\underline{u}^{\alpha}(s), \overline{u}^{\alpha}(s)] \\
&= \Bigg[(\underline{W^{\alpha}_{S0k+1}})^{-1} \Bigg(\cosh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}} \underline{y_0}^{\alpha} + (-1)\cosh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}} \underline{y_0}^{\alpha} \\
&\quad + (-1)\sinh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}} \underline{y_0}^{\alpha} + \sinh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}} \underline{y_0}^{\alpha} \\
&\quad - \Bigg\{ \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\,\theta_{1i,k} \cosh(a(T_1 - t_k^+) - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(T_1 - t_k^+) - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(T_1 - t_k^+) - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\,\theta_{2i,k} \sinh(a(T_1 - t_k^+) - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \int_{t_{k-1}}^{t_k} (-1)(1 + c_k)\cosh(aT_1 - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \int_{t_{k-1}}^{t_k} (1 + c_k)\sinh(aT_1 - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \int_{t_k}^{T_1} \cosh(aT_1 - as)\underline{g}^{\alpha}(s, y(s))ds \\
&\quad + \int_{t_k}^{T_1} (-1)\sinh(aT_1 - as)\underline{g}^{\alpha}(s, y(s))ds \Bigg\} - \underline{y_1}^{\alpha} \\
&\quad + \sum_{i=1}^{k-1} \Bigg\{ \Bigg[\cosh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}}(j) + \sinh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}}(j) \Bigg] \underline{g_i}^{\alpha} \\
&\quad + (-1)\Bigg[\cosh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}}(j) + \sinh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}}(j) \Bigg] \underline{g_i}^{\alpha} \Bigg\} \\
&\quad + \cosh(a(T_1 - t_k^+))\underline{g_k}^{\alpha} + (-1)\sinh(a(T_1 - t_k^+))\underline{g_k}^{\alpha} \Bigg), \\
&\quad (\overline{W^{\alpha}_{S0k+1}})^{-1} \Bigg(\cosh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}} \overline{y_0}^{\alpha} + (-1)\cosh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}} \overline{y_0}^{\alpha} \\
&\quad + (-1)\sinh(a(T_1 - t_k^+)) \underset{a<0}{p_{1,k}} \overline{y_0}^{\alpha} + \sinh(a(T_1 - t_k^+)) \underset{a<0}{q_{1,k}} \overline{y_0}^{\alpha} \\
&\quad - \Bigg\{ \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\,\theta_{1i,k} \cosh(a(T_1 - t_k^+) - as)\overline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{2i,k} \cosh(a(T_1 - t_k^+) - as)\overline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} \theta_{1i,k} \sinh(a(T_1 - t_k^+) - as)\overline{g}^{\alpha}(s, y(s))ds \\
&\quad + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (-1)\,\theta_{2i,k} \sinh(a(T_1 - t_k^+) - as)\overline{g}^{\alpha}(s, y(s))ds \\
&\quad + \int_{t_{k-1}}^{t_k} (-1)(1 + c_k)\cosh(aT_1 - as)\overline{g}^{\alpha}(s, y(s))ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(aT_1-as)\overline{g}^{\alpha}(s,y(s))ds \\
&+\int_{t_k}^{T_1}\cosh(aT_1-as)\overline{g}^{\alpha}(s,y(s))ds \\
&+\int_{t_k}^{T_1}(-1)\sinh(aT_1-as)\overline{g}^{\alpha}(s,y(s))ds\Bigg\}-\overline{y_1}^{\alpha} \\
&+\sum_{i=1}^{k-1}\Bigg\{\Bigg[\cosh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}(j)+\sinh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}(j)\Bigg]\overline{g_i}^{\alpha} \\
&+(-1)\Bigg[\cosh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}(j)+\sinh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}(j)\Bigg]\overline{g_i}^{\alpha}\Bigg\} \\
&+\cosh(a(T_1-t_k^+))\overline{g_k}^{\alpha}+(-1)\sinh(a(T_1-t_k^+))\overline{g_k}^{\alpha}\Bigg)\Bigg],
\end{aligned}
$$

where $\underline{y_0}^{\alpha}=-\overline{y_0}^{\alpha}$, $\underline{y_1}^{\alpha}=-\overline{y_1}^{\alpha}$, $\underline{g_i}=-\overline{g_i}$, $\underline{g_k}=-\overline{g_k}$.

Then, substituting this expression into (7) yields the α-level of $y(T_1)$, i.e.,

$$
\begin{aligned}
[y(T_1)]^{\alpha} \quad = \quad &\Bigg[\cosh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}\underline{y_0}^{\alpha}+(-1)\cosh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}\underline{y_0}^{\alpha} \\
&+(-1)\sinh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}\underline{y_0}^{\alpha}+\sinh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}\underline{y_0}^{\alpha} \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)\theta_{1i,k}\cosh(a(T_1-t_k^+)-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(T_1-t_k^+)-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)\theta_{2i,k}\sinh(a(T_1-t_k^+)-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(T_1-t_k^+)-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\int_{t_{k-1}}^{t_k}(-1)(1+c_k)\cosh(aT_1-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(aT_1-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\int_{t_k}^{T_1}\cosh(aT_1-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds \\
&+\int_{t_k}^{T_1}(-)\sinh(aT_1-as)(\underline{g}^{\alpha}(s,y(s))+\tilde{d}\underline{u}^{\alpha}(s))ds, \\
&\cosh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}\overline{y_0}^{\alpha}+(-1)\cosh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}\overline{y_0}^{\alpha} \\
&+(-1)\sinh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}\overline{y_0}^{\alpha}+\sinh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}\overline{y_0}^{\alpha} \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)\theta_{1i,k}\cosh(a(T_1-t_k^+)-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(T_1-t_k^+)-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds \\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-1)\theta_{2i,k}\sinh(a(T_1-t_k^+)-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(T_1-t_k^+)-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds\\
&+\int_{t_{k-1}}^{t_k}(-1)(1+c_k)\cosh(aT_1-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(aT_1-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds\\
&+\int_{t_k}^{T_1}\cosh(aT_1-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds\\
&+\int_{t_k}^{T_1}(-)\sinh(aT_1-as)(\overline{g}^{\alpha}(s,y(s))+\tilde{d}\overline{u}^{\alpha}(s))ds\Big]\\
=\;&[\underline{y_1}^{\alpha},\overline{y_1}^{\alpha}]\\
=\;&[y_1]^{\alpha}.
\end{aligned}
$$

Using the control above, we may consider the operator $\Phi : C((t_k,t_{k+1}],\mathbb{R}_F) \to C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$, where

$$
\begin{aligned}
&(\Phi y)(t)\\
=\;&\cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0+\cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0\\
&+\sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0+\sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_k}^{t}\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_k}^{t}\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds,
\end{aligned}
$$

where

$$
\begin{aligned}
u(t)\;=\;&\Big[W_{S0k+1}^{-1}\Big(\cosh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}y_0+\cosh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}y_0\\
&+\sinh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}}y_0+\sinh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}}y_0\\
&\ominus\Big\{\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\cosh(a(T_1-t_k^+)-as)g(s,y(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(T_1-t_k^+)-as)g(s,y(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\sinh(a(T_1-t_k^+)-as)g(s,y(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(T_1-t_k^+)-as)g(s,y(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\cosh(aT_1-as)g(s,y(s))ds+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(aT_1-as)g(s,y(s))ds\\
&+\int_{t_k}^{T_1}\cosh(aT_1-as)g(s,y(s))ds+\int_{t_k}^{T_1}\sinh(aT_1-as)g(s,y(s))ds\Bigg\}\ominus y_1\\
&+\sum_{i=1}^{k-1}\Bigg\{\Bigg[\cosh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}(j)}+\sinh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}(j)}\Bigg]g_i\\
&+\Bigg[\cosh(a(T_1-t_k^+))\underset{a<0}{q_{1,k}(j)}+\sinh(a(T_1-t_k^+))\underset{a<0}{p_{1,k}(j)}\Bigg]g_i\Bigg\}\\
&+\cosh(a(T_1-t_k^+))g_k+\sinh(a(T_1-t_k^+))g_k\Bigg)\Bigg](t),
\end{aligned}
$$

where the fuzzy mapping W_{S0k+1}^{-1} satisfies the above statements.

Note that $\Phi y(T_1)=y_1$, which means that the u steers (7) from $(\Phi y)(0)$ to y_1 in finite time T_1 for $t\in(t_k,t_{k+1}]$, $k=0,1,\cdots,m$, provided that we obtain a fixed point of the nonlinear operator Φ.

Assume the following hypotheses:

(S_1) for $x(\cdot)\in\mathbb{R}_F$ and $y(\cdot)\in\mathbb{R}_F$, and assume there is a positive number C such that

$$
d_H\Big([g(s,y(\cdot))]^\alpha,[g(s,x(\cdot))]^\alpha\Big)\le Cd_H\Big([y(\cdot)]^\alpha,[x(\cdot)]^\alpha\Big),\ s\in J.
$$

Then we have

$$
H_1(g(s,y),g(s,x))\le CH_1(y,x),\ s\in J.
$$

(S_2) $2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}-\theta_{1i,k}e^{-(a(T_1-t_k^+)-as)}Cds+2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}e^{-(a(T_1-t_k^+)-as)}Cds$
$+2\int_{t_{k-1}}^{t_k}-(1+c_k)e^{-(aT_1-as)}Cds+2\int_{t_k}^{T_1}e^{-(aT_1-as)}Cds<1.$

Theorem 12. *In Case 6.2, suppose that (S_1), (S_2) are satisfied. Then system (2) is controllable.*

Proof. We can easily check that Φ is a continuous function from $C((t_k,t_{k+1}],\mathbb{R}_F)$ to $C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$. For $x,y\in C((t_k,t_{k+1}],\mathbb{R}_F)$, we obtain

$$
\begin{aligned}
&d_H\Big([\Phi y(t)]^\alpha,[\Phi x(t)]^\alpha\Big)\\
=\ &d_H\Bigg(\Bigg[\cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0+\cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0+\sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}y_0\\
&+\sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}y_0+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_{k-1}}^{t_k}(1+c_k)\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_k}^{t}\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds+\int_{t_k}^{t}\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\Big]^{\alpha},\\
&\Big[\cosh(a(t-t_k^+))\underset{a<0}{p_{1,k}}\,y_0+\cosh(a(t-t_k^+))\underset{a<0}{q_{1,k}}\,y_0+\sinh(a(t-t_k^+))\underset{a<0}{p_{1,k}}\,y_0\\
&+\sinh(a(t-t_k^+))\underset{a<0}{q_{1,k}}\,y_0+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\cosh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{1i,k}\sinh(a(t-t_k^+)-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_{k-1}}^{t_k}(1+c_k)\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\\
&+\int_{t_k}^{t}\cosh(at-as)(g(s,y(s))+\tilde{d}u(s))ds+\int_{t_k}^{t}\sinh(at-as)(g(s,y(s))+\tilde{d}u(s))ds\Big]^{\alpha}\Big)\\
\leq\;&\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-\theta_{1i,k}\cosh(a(t-t_k^+)-as)+\theta_{1i,k}\sinh(a(t-t_k^+)-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(\theta_{2i,k}\cosh(a(t-t_k^+)-as)-\theta_{2i,k}\sinh(a(t-t_k^+)-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\int_{t_{k-1}}^{t_k}((1+c_k)\sinh(at-as)-(1+c_k)\cosh(at-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\int_{t_k}^{t}(\cosh(at-as)-\sinh(at-as))d_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(-\theta_{1i,k}\cosh(a(T_1-t_k^+)-as)+\theta_{1i,k}\sinh(a(T_1-t_k^+)-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}(\theta_{2i,k}\cosh(a(T_1-t_k^+)-as)+\theta_{2i,k}\sinh(a(T_1-t_k^+)-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\int_{t_{k-1}}^{t_k}((1+c_k)\sinh(aT_1-as)-(1+c_k)\cosh(aT_1-as))Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\int_{t_k}^{T_1}(\cos(aT_1-as)-\sinh(aT_1-as))d_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
\leq\;&\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}-\theta_{1i,k}e^{-(a(t-t_k^+)-as)}Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i}\theta_{2i,k}e^{-(a(t-t_k^+)-as)}Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds\\
&+\int_{t_{k-1}}^{t_k}-(1+c_k)e^{-(at-as)}Cd_H\Big([y(s)]^{\alpha},[x(s)]^{\alpha}\Big)ds
\end{aligned}
$$

$$+\int_{t_k}^{t} e^{-(at-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds$$
$$+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k} e^{-(a(T_1-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds$$
$$+\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k} e^{-(a(T_1-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds$$
$$+\int_{t_{k-1}}^{t_k} -(1+c_k) e^{-(aT_1-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds$$
$$+\int_{t_k}^{T_1} e^{-(aT_1-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds,$$

then

$$\begin{aligned}
D(\Phi y(t), \Phi x(t)) &= \sup_{0<\alpha\le 1} d_H\Big([\Phi y(t)]^\alpha, [\Phi x(t)]^\alpha\Big) \\
&\le \sup_{0<\alpha\le 1}\Bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k} e^{-(a(t-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k} e^{-(a(t-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k) e^{-(at-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\int_{t_k}^{t} e^{-(at-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k} e^{-(a(T_1-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k} e^{-(a(T_1-t_k^+)-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k) e^{-(aT_1-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds \\
&\quad +\int_{t_k}^{T_1} e^{-(aT_1-as)} C d_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big) ds\Bigg) \\
&\le \sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k} e^{-(a(t-t_k^+)-as)} C D(y(s), x(s)) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k} e^{-(a(t-t_k^+)-as)} C D(y(s), x(s)) ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k) e^{-(at-as)} C D(y(s), x(s)) ds \\
&\quad +\int_{t_k}^{t} e^{-(at-as)} C D(y(s), x(s)) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k} e^{-(a(T_1-t_k^+)-as)} C D(y(s), x(s)) ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k} e^{-(a(T_1-t_k^+)-as)} C D(y(s), x(s)) ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k) e^{-(aT_1-as)} C D(y(s), x(s)) ds
\end{aligned}$$

$$+\int_{t_k}^{T_1} e^{-(aT_1-as)}CD(y(s),x(s))ds;$$

thus,

$$\begin{aligned}
H_1(\Phi y,\Phi x) &= \sup_{0\le t\le T_1} D(\Phi y(t),\Phi x(t)) \\
&\le \sup_{0\le t\le T_1}\Bigg(\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k}e^{-(a(t-t_k^+)-as)}CD(y(s),x(s))ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k}e^{-(a(t-t_k^+)-as)}CD(y(s),x(s))ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k)e^{-(at-as)}CD(y(s),x(s))ds \\
&\quad +\int_{t_k}^{t} e^{-(at-as)}CD(y(s),x(s))ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k}e^{-(a(T_1-t_k^+)-as)}CD(y(s),x(s))ds \\
&\quad +\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k}e^{-(a(T_1-t_k^+)-as)}CD(y(s),x(s))ds \\
&\quad +\int_{t_{k-1}}^{t_k} -(1+c_k)e^{-(aT_1-as)}CD(y(s),x(s))ds \\
&\quad +\int_{t_k}^{T_1} e^{-(aT_1-as)}CD(y(s),x(s))ds\Bigg) \\
&\le \Bigg[2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} -\theta_{1i,k}e^{-(a(T_1-t_k^+)-as)}Cds+2\sum_{i=1}^{k-1}\int_{t_{i-1}}^{t_i} \theta_{2i,k}e^{-(a(T_1-t_k^+)-as)}Cds \\
&\quad +2\int_{t_{k-1}}^{t_k} -(1+c_k)e^{-(aT_1-as)}Cds+2\int_{t_k}^{T_1} e^{-(aT_1-as)}Cds\Bigg]H_1(y,x).
\end{aligned}$$

Based on the hypothesis $(S2)$, Φ is a contraction mapping. Based on the Banach fixed-point theorem, Φ has a fixed point $y\in C((t_k,t_{k+1}],\mathbb{R}_F)$, $k=0,1,\cdots,m$.

In summary, the proof is completed. □

4. An Example

In this section, we provide the following examples to prove our theorems.

Example 13. *Consider the following nonlinear impulsive fuzzy differential equations:*

$$\begin{cases} y'(t)=ay(t)+g(t,y(t))+\tilde{d}u(t),\ t\in[0,t_3],\ t\ne t_k,\ k=1,2, \\ \Delta y(t_k)=c_k y(t_k^-)+g_k,\ k=1,2, \\ y(0)=\gamma\in\mathbb{R}_F, \end{cases} \tag{8}$$

where $a=-1$, $g(t,y(t))=t\sin(y(t))\gamma$, $[\gamma]^\alpha=[\alpha-1,1-\alpha]$, $t_1=0.1$, $t_2=0.2$, $t_3=0.3$, $c_1=-1.1$, $c_2=-1.2$, $g_1=\gamma$ *and* $g_2=2\gamma$, $y_1=0.6\gamma$, $\tilde{d}=3$.

From what we know above, note that $g:[0,0.3]\times\mathbb{R}_F\to\mathbb{R}_F$ *is continuous and for* $x,y\in\mathbb{R}_F$, *we have*

$$\begin{aligned}
d_H\Big([g(s,y)]^\alpha,[g(s,x)]^\alpha\Big) &= d_H\Big([s\cdot\sin(y(s))\gamma]^\alpha,[s\cdot\sin(x(s))\gamma]^\alpha\Big) \\
&\le s d_H\Big([y(s)\gamma]^\alpha,[x(s)\gamma]^\alpha\Big) \\
&= s\max\{|\,(\underline{y}(s))^\alpha-(\underline{x}(s))^\alpha\,|,|\,(\overline{y}(s))^\alpha-(\overline{x}(s))^\alpha\,|\}
\end{aligned}$$

$$= \quad sd_H\Big([y(s)]^\alpha, [x(s)]^\alpha\Big).$$

Then $C = s = 0.3$. *When* $a < 0$, *via the* $(c2)$ *solution, for* $t \in (0.2, 0.3]$, *we obtain*

$$2\int_0^{0.1} \mid (1+c_1)(1+c_2) \mid Cds + 2 \mid 1+c_2 \mid (t_2 - t_1)C + 2(T_1 - t_2)C = 0.0192 < 1.$$

When $a < 0$, *via the* $(c1)$ *solution, for* $t \in (0.2, 0.3]$, *we obtain*

$$\begin{aligned} & 2\int_0^{t_1} -\theta_{1i,k}e^{-(a(T_1-t_2)-as)}Cds + 2\int_0^{t_1} \theta_{2i,k}e^{-(a(T_1-t_2)-as)}Cds \\ & +2\int_{t_1}^{t_2} -(1+c_2)e^{-(aT_1-as)}Cds + 2\int_{t_2}^{T_1} e^{-(aT_1-as)}Cds = 0.0756 < 1. \end{aligned}$$

Thus, according to Theorems 11 and 12, we can deduce that the system is controllable.

5. Conclusions

In this paper, we have mainly provided the controllability results of the first-order linear and nonlinear impulsive fuzzy differential equations. Firstly, we demonstrated the controllability of first-order linear impulsive fuzzy differential equations by employing the direct construction method. Driven by the controllability of linear impulsive fuzzy differential equations, the controllability results of nonlinear impulsive fuzzy differential equations were then provided using a fixed-point theorem. In future works, we will study the problem of optimal control.

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