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Degenerate Fubini-Type Polynomials and Numbers, Degenerate Apostol–Bernoulli Polynomials and Numbers, and Degenerate Apostol–Euler Polynomials and Numbers

Siqintuya Jin [1], Muhammet Cihat Dağli [2] and Feng Qi [3,4,*]

[1] College of Mathematics and Physics, Inner Mongolia Minzu University, Tongliao 028043, China
[2] Department of Mathematics, Akdeniz University, Antalya 07058, Turkey
[3] Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454003, China
[4] Independent Researcher, Dallas, TX 75252-8024, USA
* Correspondence: honest.john.china@gmail.com

Abstract: In this paper, by introducing degenerate Fubini-type polynomials, with the help of the Faà di Bruno formula and some properties of partial Bell polynomials, the authors provide several new explicit formulas and recurrence relations for Fubini-type polynomials and numbers, associate the newly defined degenerate Fubini-type polynomials with degenerate Apostol–Bernoulli polynomials and degenerate Apostol–Euler polynomials of order α. These results enable one to present additional relations for some degenerate special polynomials and numbers.

Keywords: Apostol–Bernoulli polynomials; Apostol–Euler polynomials; degenerate Bernoulli polynomials; explicit formula; generalized Fubini polynomial; generating function; recurrence relation; Stirling number

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1. Motivations

For $z \in \mathbb{C}$, higher-order Bernoulli polynomials $B_n^{(\alpha)}(z)$ and higher-order Euler polynomials $E_n^{(\alpha)}(z)$ of degree n in α are defined in [1] by means of the generating functions

$$\left(\frac{w}{e^w - 1}\right)^\alpha e^{zw} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(z) \frac{w^n}{n!}$$

and

$$\left(\frac{2}{e^w + 1}\right)^\alpha e^{zw} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(z) \frac{w^n}{n!}$$

respectively. For $\alpha = 1$, the quantities $B_n^{(\alpha)}(z)$ and $E_n^{(\alpha)}(z)$ become the classical Bernoulli polynomials $B_n(z)$ and Euler polynomials $E_n(z)$, which are defined by means of the generating functions

$$\frac{e^{zw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{w^n}{n!}, \quad |w| < 2\pi$$

and

$$\frac{2\,e^{zw}}{e^w + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{w^n}{n!}, \quad |w| < \pi$$

respectively, where $z \in \mathbb{C}$. In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n\left(\frac{1}{2}\right)$ are called classical Bernoulli numbers and Euler numbers, respectively.

Generalized Apostol–Bernoulli polynomials $B_n^{(\alpha)}(z,\gamma)$ were defined in [2–4] by Luo and Srivastava by means of the generating function

$$\left(\frac{w}{\gamma\,\mathrm{e}^{w}-1}\right)^{\alpha}\mathrm{e}^{zw}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(z,\gamma)\frac{w^n}{n!}$$

for $z,\gamma\in\mathbb{C}$ and

$$|w|<\begin{cases}2\pi, & \gamma=1;\\ |\ln\gamma|, & \gamma\neq 1.\end{cases}$$

Generalized Apostol–Euler polynomials $E_n^{(\alpha)}(z,\gamma)$ for $z,\gamma\in\mathbb{C}$ were defined in [5] by means of the generating function

$$\left(\frac{2}{\gamma\,\mathrm{e}^{w}+1}\right)^{\alpha}\mathrm{e}^{zw}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(z,\gamma)\frac{w^n}{n!}$$

for $1^{\gamma}=1$ and

$$|w|<\begin{cases}\pi, & \gamma=1;\\ |\ln(-\gamma)|, & \gamma\neq 1.\end{cases}$$

The ideas for these generalizations originated from the paper [6].

For $z\in\mathbb{C}$ and $\tau\in\mathbb{C}\setminus\{0\}$, Carlitz defined [7] degenerate Bernoulli polynomials $B_n(z,\tau)$ and degenerate Euler polynomials $E_n(z,\tau)$ by

$$\frac{w}{(1+\tau w)^{1/\tau}-1}(1+\tau w)^{z/\tau}=\sum_{n=0}^{\infty}B_n(z,\tau)\frac{w^n}{n!}$$

and

$$\frac{2}{(1+\tau w)^{1/\tau}+1}(1+\tau w)^{z/\tau}=\sum_{n=0}^{\infty}E_n(z,\tau)\frac{w^n}{n!}.$$

When $z=0$, these quantities are respectively called degenerate Bernoulli and Euler numbers.

For $z,\gamma\in\mathbb{C}$ and $\tau\in\mathbb{C}\setminus\{0\}$, the degenerate versions of the Apostol–Bernoulli polynomials and the Apostol–Euler polynomials of order α were introduced in [8] by

$$\left[\frac{w}{\gamma(1+\tau w)^{1/\tau}-1}\right]^{\alpha}(1+\tau w)^{z/\tau}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(z,\tau,\gamma)\frac{w^n}{n!}\tag{1}$$

and

$$\left[\frac{2}{\gamma(1+\tau w)^{1/\tau}+1}\right]^{\alpha}(1+\tau w)^{z/\tau}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(z,\tau,\gamma)\frac{w^n}{n!}\tag{2}$$

respectively. Since $\lim_{\tau\to0}(1+\tau w)^{1/\tau}=\mathrm{e}^{w}$, when $\tau\to0$ and $\alpha=\gamma=1$, the Equations (1) and (2) reduce to the generating functions for classical Bernoulli and Euler polynomials, respectively.

For further and detailed features of the polynomials above, interested readers can consult the studies [5,9–14] and related references therein.

In this paper, we focus on Kılar and Simsek's recent study [15], in which a family of Fubini-type polynomials $a_n^{(\alpha)}(z)$ for $z\in\mathbb{C}$ were introduced as

$$\frac{2^{\alpha}}{(2-\mathrm{e}^{w})^{2\alpha}}\,\mathrm{e}^{zw}=\sum_{n=0}^{\infty}a_n^{(\alpha)}(z)\frac{w^n}{n!},\quad \alpha\in\mathbb{N}_0,\quad |w|<\ln 2.\tag{3}$$

In particular, the quantities $a_n^{(\alpha)}(0)=a_n^{(\alpha)}$ are called the Fubini-type numbers. The two authors connected these polynomials and numbers with other celebrated polynomials and numbers such as the Apostol–Bernoulli numbers, the Frobenius–Euler numbers and the

Stirling numbers via generating function methods and functional equations. Very recently, Srivastava and Kızılateş extended in [16] Fubini-type polynomials $a_n^{(\alpha)}(z)$ to parametric kind families of Fubini-type polynomials by considering two special generating functions and obtained many relations concerning these and other parametric special polynomials and numbers. As emphasized therein, Fubini-type polynomials $a_n^{(\alpha)}(z)$ are special case of generalized Apostol–Euler polynomials $E_n^{(\alpha)}(z,\gamma)$. Concretely speaking, the identity $E_n^{(2\alpha)}\left(z,-\frac{1}{2}\right)=2^{3\alpha}a_n^{(\alpha)}(z)$ is valid. Further investigations on Fubini type polynomials and numbers can be found in [17–21], and plenty of references cited therein.

On the other hand, the last two authors of this paper and several other mathematicians have studied a number of explicit formulas, recursive formulas, and closed-form formulas for some significant polynomials and numbers by applying the Faà di Bruno Formula (6) shown below, employing some properties of partial Bell polynomials (or the Bell polynomials of the second kind), and utilizing a general derivative formula for a ratio of two differentiable functions. See, for example, the papers [22–33] and related references therein.

In this paper, for $z \in \mathbb{C}$ and $\tau \in \mathbb{C} \setminus \{0\}$, we introduce a degenerate version of Fubini-type polynomials as follows:

$$\frac{2^{\alpha}}{[2-(1+\tau w)^{1/\tau}]^{2\alpha}}(1+\tau w)^{z/\tau}=\sum_{n=0}^{\infty}a_n^{(\alpha)}(z,\tau)\frac{w^n}{n!}. \tag{4}$$

Note that, for $z=0$, the quantities $a_n^{(\alpha)}(0,\tau)=a_n^{(\alpha)}(\tau)$ are called degenerate Fubini-type numbers. If $\tau \to 0$, then these quantities reduce to Fubini-type polynomials $a_n^{(\alpha)}(z)$, as mentioned above.

In parallel with the conclusion given in [16] (Remark 4), we infer a relation

$$a_n^{(\alpha)}(z,\tau)=\frac{1}{2^{3\alpha}}E_n^{(2\alpha)}\left(z,\tau,-\frac{1}{2}\right) \tag{5}$$

between degenerate Fubini-type polynomials and degenerate Apostol–Euler polynomials of the order α.

In this paper, with the help of the Faà di Bruno Formula (6) and some properties of partial Bell polynomials, we derive some new explicit formulas, closed-form formulas, and recurrence relations for degenerate Fubini-type polynomials and numbers and for Fubini-type polynomials and numbers. Moreover, we provide the relationship between degenerate Fubini-type polynomials and degenerate Apostol–Bernoulli polynomials of the order α.

2. Necessary Lemmamas

In order to prove our main results, we recall several Lemmamas below.

Lemma 1 ([34] (pp. 134 and 139))**.** *The Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $B_{n,k}(w_1,w_2,\dots,w_{n-k+1})$ for $n \ge k \ge 0$, are defined by*

$$B_{n,k}(w_1,w_2,\dots,w_{n-k+1})=\sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1}i\ell_i=n\\ \sum_{i=1}^{n-k+1}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{w_i}{i!}\right)^{\ell_i}$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $B_{n,k}(w_1,w_2,\dots,w_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))B_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{6}$$

Lemma 2 ([34] (p. 135)). *For $n \ge k \ge 0$, we have*

$$B_{n,k}\big(abx_1, ab^2w_2, \dots, ab^{n-k+1}w_{n-k+1}\big) = a^k b^n B_{n,k}(w_1, w_2, \dots, w_{n-k+1}), \tag{7}$$

where a and b are any complex numbers.

Lemma 3 ([34] (pp. 135 and 206)). *For $n \ge k \ge 0$, we have*

$$B_{n,k}(1, 1, \dots, 1) = S(n,k), \tag{8}$$

where $S(n,k)$ stands for the Stirling numbers of the second kind, which can be analytically generated by

$$\frac{(\mathrm{e}^t - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}.$$

Lemma 4 ([35] (Remark 7)). *For $n \ge k \ge 0$, we have*

$$B_{n,k}\left(1, 1-\mu, (1-\mu)(1-2\mu), \dots, \prod_{\ell=0}^{n-k}(1-\ell\mu)\right) = \begin{cases} (-1)^k \dfrac{\mu^n n!}{k!} \displaystyle\sum_{\ell=0}^{k}(-1)^\ell \binom{k}{\ell}\binom{\ell/\mu}{n}, & \mu \ne 0 \\ S(n,k), & \mu = 0 \end{cases} \tag{9}$$

and

$$B_{n,k}(\langle\lambda\rangle_1, \langle\lambda\rangle_2, \dots, \langle\lambda\rangle_{n-k+1}) = (-1)^k \frac{n!}{k!}\sum_{\ell=0}^{k}(-1)^\ell \binom{k}{\ell}\binom{\lambda\ell}{n} \tag{10}$$

for $n \ge k \in \mathbb{N}_0$ and $\lambda, \mu \in \mathbb{C}$, where

$$\langle z\rangle_k = \prod_{\ell=0}^{k-1}(z-\ell) = \begin{cases} z(z-1)\cdots(z-k+1), & k \in \mathbb{N} \\ 1, & k = 0 \end{cases} \tag{11}$$

is called the falling factorial of the number $z \in \mathbb{C}$ and

$$\binom{w}{z} = \begin{cases} \dfrac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)}, & w \notin \mathbb{N}_-, & z, w-z \notin \mathbb{N}_- \\ 0, & w \notin \mathbb{N}_-, & z \in \mathbb{N}_- \text{ or } w-z \in \mathbb{N}_- \\ \dfrac{\langle w\rangle_z}{z!}, & w \in \mathbb{N}_-, & z \in \mathbb{N}_0 \\ \dfrac{\langle w\rangle_{w-z}}{(w-z)!}, & w, z \in \mathbb{N}_-, & w-z \in \mathbb{N}_0 \\ 0, & w, z \in \mathbb{N}_-, & w-z \in \mathbb{N}_- \\ \infty, & w \in \mathbb{N}_-, & z \notin \mathbb{Z} \end{cases} \tag{12}$$

for the classical Euler gamma function

$$\Gamma(w) = \lim_{m\to\infty}\frac{m!m^w}{\prod_{k=0}^{m}(w+k)}, \quad w \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

3. Explicit and Closed-Form Formulas and Recurrence Relations

In this section, among other things, we provide some computational formulas for degenerate Fubini-type numbers, present some explicit formulas and recursive relations for Fubini-type polynomials and numbers, and consequently derive some closed-form formulas and recursive relations for degenerate Apostol–Bernoulli polynomials and degenerate Apostol–Euler polynomials of the order α.

Theorem 1. *For $n \in \mathbb{N}$, degenerate Fubini-type numbers can be computed by*

$$a_n^{(\alpha)}(z) = z(n-1)! \sum_{k=0}^{n} \frac{(2\alpha)_k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \ell \binom{k}{\ell} \binom{z\ell - 1}{n-1}, \tag{13}$$

where the rising factorial

$$(w)_n = \prod_{\ell=0}^{n-1} (w+\ell) = \begin{cases} w(w+1)\cdots(w+n-1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

is also called the Pochhammer symbol or shifted factorial. Consequently, for the special case of degenerate Apostol–Euler polynomials $E_n^{(\alpha)}(z, z, \gamma)$, we have

$$E_n^{(2\alpha)}\left(0, z, -\frac{1}{2}\right) = z(n-1)! 2^{3\alpha} \sum_{k=1}^{n} \frac{\langle 2\alpha \rangle_k}{k!} \sum_{\ell=1}^{k} (-1)^{\ell} \ell \binom{k}{\ell} \binom{z\ell - 1}{n-1}. \tag{14}$$

Proof. Applying $f(u) = (2-u)^{-2\alpha}$ and $u = g(t) = (1+\tau t)^{1/\tau}$ to the Faà di Bruno Formula (6) and making use of the identities (7) and (9) or (10), we find

$$\begin{aligned} \frac{\mathrm{d}^n \left[\left(2 - (1+\tau t)^{1/\tau}\right)^{-2\alpha} \right]}{\mathrm{d}\, t^n} &= \sum_{k=0}^{n} \frac{\mathrm{d}^k (2-u)^{-2\alpha}}{\mathrm{d}\, u^k} B_{n,k}\Big((1+\tau t)^{(1-\tau)/\tau}, \\ (1-\tau)(1+\tau t)^{(1-2\tau)/\tau}, \dots, &(1-\tau)(1-2\tau)\cdots[1-(n-k)\tau](1+\tau t)^{[1-(n-k+1)\tau]/\tau}\Big) \\ &= \sum_{k=0}^{n} \langle -2\alpha \rangle_k (2-u)^{-2\alpha-k} B_{n,k}\Big((1+\tau t)^{(1-\tau)/\tau}, (1-\tau)(1+\tau t)^{(1-2\tau)/\tau}, \\ &\quad \dots, (1-\tau)(1-2\tau)\cdots[1-(n-k)\tau](1+\tau t)^{[1-(n-k+1)\tau]/\tau}\Big) \\ &\to \sum_{k=0}^{n} \langle -2\alpha \rangle_k B_{n,k}(1, 1-\tau, \dots, (1-\tau)(1-2\tau)\cdots[1-(n-k)\tau]) \\ &= \tau(n-1)! \sum_{k=0}^{n} \frac{(2\alpha)_k}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \ell \binom{k}{\ell} \binom{\tau\ell - 1}{n-1} \end{aligned}$$

as $t \to 0$ and $u \to 1$. Considering the generating function for degenerate Fubini-type numbers, that is, for $x = 0$ in Equation (4), completes the proof of (13).

From the relation (5), the identity (14) follows. □

Theorem 2. *Fubini-type polynomials $a_n^{(\alpha)}(z)$ possess the explicit formula*

$$a_n^{(\alpha)}(z) = 2^{\alpha} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (2\alpha)_i S(k,i) z^{n-k}, \tag{15}$$

where $S(n,k)$ represents the Stirling numbers of the second kind.

Fubini-type numbers $a_n^{(\alpha)}$ can be computed by

$$a_n^{(\alpha)} = 2^{\alpha} \sum_{i=0}^{n} (2\alpha)_i S(n,i). \tag{16}$$

Generalized Apostol–Euler polynomials $E_n^{(\alpha)}(z,\gamma)$ can be expressed as

$$E_n^{(2\alpha)}\left(z, -\frac{1}{2}\right) = 2^{4\alpha} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{k} (2\alpha)_i S(k,i) z^{n-k}. \tag{17}$$

Proof. From the Formulas (6)–(8), it follows that

$$\begin{aligned}\frac{\mathrm{d}^k\left(2-\mathrm{e}^t\right)^{-2\alpha}}{\mathrm{d}\,t^k} &= \sum_{i=0}^{k}\langle -2\alpha\rangle_i\left(2-\mathrm{e}^t\right)^{-2\alpha-i}B_{k,i}(-\mathrm{e}^t,-\mathrm{e}^t,\dots,-\mathrm{e}^t) \\ &= \sum_{i=0}^{k}\langle -2\alpha\rangle_i\left(2-\mathrm{e}^t\right)^{-2\alpha-i}(-1)^i\,\mathrm{e}^{ti}\,B_{k,i}(1,1,\dots,1) \\ &\to \sum_{i=0}^{k}(2\alpha)_i S(k,i), \quad t\to 0.\end{aligned} \tag{18}$$

It is obvious that $\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}(\mathrm{e}^{zt}) = z^k\,\mathrm{e}^{zt} \to z^k$ as $t\to 0$. Using Leibnitz's formula for the nth derivative of the product of two functions, we obtain

$$\lim_{t\to 0}\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}\left[\frac{2^\alpha}{(2-\mathrm{e}^t)^{2\alpha}}\,\mathrm{e}^{zt}\right] = 2^\alpha\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}(2\alpha)_i S(k,i)z^{n-k}.$$

Considering the generating function in (3), we acquire the Formula (15) for $a_n^{(\alpha)}(z)$.

For $z=0$ in (15), we immediately arrive at the identity (16).

The Equation (17) can be verified from the relation (5) between Fubini-type polynomials $a_n^{(\alpha)}(z)$ and generalized Apostol–Euler polynomials $E_n^{(\alpha)}(z,\gamma)$. The proof is, therefore, complete. □

Theorem 3. *Fubini-type polynomials $a_n^{(\alpha)}(z)$ satisfy the recurrence relation*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-2\alpha)_i S(n-k,i)a_k^{(\alpha)}(z) = 2^\alpha z^n.$$

In particular, Fubini-type numbers $a_n^{(\alpha)}$ satisfy

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-2\alpha)_i S(n-k,i)a_k^{(\alpha)} = 0. \tag{19}$$

Generalized Apostol–Euler polynomials $E_n^{(\alpha)}(z,\gamma)$ possess the recurrence relation

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-2\alpha)_i S(n-k,i)E_k^{(2\alpha)}\left(z,-\frac{1}{2}\right) = 2^{4\alpha}z^n. \tag{20}$$

Proof. Since

$$\left[\left(2-\mathrm{e}^t\right)^{2\alpha}\right]\left[\frac{2^\alpha}{(2-\mathrm{e}^t)^{2\alpha}}\,\mathrm{e}^{zt}\right] = 2^\alpha\,\mathrm{e}^{zt},$$

by remembering the generating function of Fubini-type polynomials (3) and by proceeding as in the proof of (18), differentiating n times with respect to t on both sides deduces

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{n-k}}{\mathrm{d}\,t^{n-k}}\left[\left(2-\mathrm{e}^t\right)^{2\alpha}\right]\frac{\mathrm{d}^k}{\mathrm{d}\,t^k}\left[\frac{2^\alpha}{(2-\mathrm{e}^t)^{2\alpha}}\,\mathrm{e}^{zt}\right] \\ &= \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}\langle 2\alpha\rangle_i\left(2-\mathrm{e}^t\right)^{2\alpha-i}(-1)^i\,\mathrm{e}^{ti}\,S(n-k,i)\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left[\frac{2^\alpha}{(2-\mathrm{e}^t)^{2\alpha}}\,\mathrm{e}^{zt}\right] \\ &\to \sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{n-k}(-2\alpha)_i S(n-k,i)a_k^{(\alpha)}(z), \quad t\to 0 \\ &= 2^\alpha z^n.\end{aligned}$$

The first required result is thus proved.

Further setting $z = 0$ in the first result immediately yields Equation (19).

The Formula (20) can be straightforwardly derived by the same manipulation, as in proofs of previous theorems. $\square$

Theorem 4. *The relation*

$$a^{(\alpha)}_{n-2\alpha}(z,\tau) = \frac{B^{(2\alpha)}_n(z,\tau,\frac{1}{2})}{2^\alpha \langle n\rangle_{2\alpha}} \tag{21}$$

holds true, where $B^{(\alpha)}_n(z,\tau,\gamma)$ stands for degenerate Apostol–Bernoulli polynomials of order α defined by (1).

Proof. When putting $\gamma = \frac{1}{2}$ and replacing α by 2α in (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(2\alpha)}_n\left(z,\tau,\frac{1}{2}\right)\frac{t^n}{n!} &= \left[\frac{t}{\frac{1}{2}(1+\tau t)^{1/\tau}-1}\right]^{2\alpha}(1+\tau t)^{z/\tau}\\ &= 2^\alpha t^{2\alpha}\sum_{n=0}^{\infty} a^{(\alpha)}_n(z,\tau)\frac{t^n}{n!}\\ &= 2^\alpha \sum_{n=2\alpha}^{\infty} a^{(\alpha)}_{n-2\alpha}(z,\tau)\frac{t^n}{(n-2\alpha)!}\\ &= 2^\alpha \sum_{n=0}^{\infty}\langle n\rangle_{2\alpha} a^{(\alpha)}_{n-2\alpha}(z,\tau)\frac{t^n}{n!},\end{aligned}$$

which completes the proof. $\square$

We now are in a position to conclude our study in this paper with the following two recurrence relations for degenerate Fubini-type polynomials by applying the generating function methods.

Theorem 5. *For $n \geq 0$, we have*

$$a^{(\alpha)}_n(z+1,\tau) = 2a^{(\alpha)}_n(z,\tau) - \sqrt{2}\,a^{(\alpha-1/2)}_n(z,\tau).$$

Proof. From the generating function in (4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left[a^{(\alpha)}_n(z+1;\tau) - a^{(\alpha)}_n(z,\tau)\right]\frac{t^n}{n!} &= \frac{2^\alpha(1+\tau t)^{z/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}}\left[(1+\tau t)^{1/\tau}-1\right]\\ &= \frac{2^\alpha(1+\tau t)^{z/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha-1}}\left[-1+\frac{1}{2-(1+\tau t)^{1/\tau}}\right]\\ &= \frac{2^\alpha(1+\tau t)^{z/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}} - \sqrt{2}\,\frac{2^{\alpha-1/2}(1+\tau t)^{z/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2(\alpha-1/2)}}\\ &= \sum_{n=0}^{\infty} a^{(\alpha)}_n(z,\tau)\frac{t^n}{n!} - \sqrt{2}\sum_{n=0}^{\infty} a^{(\alpha-1/2)}_n(z,\tau)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left[a^{(\alpha)}_n(z,\tau) - \sqrt{2}\,a^{(\alpha-1/2)}_n(z,\tau)\right]\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of the terms $\frac{t^n}{n!}$ completes the proof. $\square$

Theorem 6. *For $n \geq 0$, degenerate Fubini-type polynomials $a^{(\alpha)}_n(z,\tau)$ satisfy the recurrence relation*

$$a^{(\alpha)}_{n+1}(z_1+z_2+\tau,\tau) = (z_1+z_2+\tau)a^{(\alpha)}_n(z_1+z_2,\tau) + \sqrt{2}\,\alpha a^{(\alpha+1/2)}_n(z_1+z_2+1,\tau). \tag{22}$$

In particular, we have

$$a_{n+1}^{(\alpha)}(w) = wa_n^{(\alpha)}(w) + \sqrt{2}\,\alpha a_n^{(\alpha+1/2)}(w+1). \tag{23}$$

Proof. Differentiating on both sides of (4) with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{2^{\alpha}(1+\tau t)^{z/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}}\right) = 2^{\alpha}\left(\frac{z(1+\tau t)^{(z-\tau)/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}} + \frac{2\alpha(1+\tau t)^{(z-\tau+1)/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha+1}}\right). \tag{24}$$

Replacing z by $z_1 + z_2 + \tau$ and evaluating the terms on both sides of (24) separately lead to

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{2^{\alpha}(1+\tau t)^{(z_1+z_2+\tau)/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}}\right) = \sum_{n=0}^{\infty} a_{n+1}^{(\alpha)}(z_1+z_2+\tau,\tau)\frac{t^n}{n!}, \tag{25}$$

$$(z_1+z_2+\tau)\frac{2^{\alpha}(1+\tau t)^{(z_1+z_2)/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2\alpha}} = (z_1+z_2+\tau)\sum_{n=0}^{\infty} a_n^{(\alpha)}(z_1+z_2,\tau)\frac{t^n}{n!}, \tag{26}$$

and

$$\sqrt{2}\,\alpha\frac{2^{\alpha+1/2}(1+\tau t)^{(z_1+z_2+1)/\tau}}{[2-(1+\tau t)^{1/\tau}]^{2(\alpha+1/2)}} = \sqrt{2}\,\alpha\sum_{n=0}^{\infty} a_n^{(\alpha+1/2)}(z_1+z_2+1,\tau)\frac{t^n}{n!}. \tag{27}$$

Substituting (25)–(27) into (24) gives the Formula (22).

Letting $\tau \to 0$ and taking $z_1 + z_2 = w$ in (22) enables us to derive the Formula (23) for Fubini-type polynomials. The proof is complete. □

Remark 1. *From the relations* (5) *and* (21)*, the counterpart identities in Theorems 5 and 6 can be presented for degenerate Apostol–Bernoulli polynomials and degenerate Apostol–Euler polynomials of the order* α*.*

Remark 2. *We note that, in recent years, the last two authors and their coauthors have investigated several other degenerate polynomials and numbers in the papers [36,37].*

4. Conclusions

In our recent study, we introduced and dealt with degenerate versions of Fubini-type polynomials. Utilizing the Faà di Bruno Formula (6), employing some properties of partial Bell polynomials, such as (7)–(10), and using generating function methods, we derived several new explicit formulas, closed-form formulas, and recurrence relations for degenerate Fubini-type polynomials and numbers and for Fubini-type polynomials and numbers, defined by Kılar and Simsek in [15]. Furthermore, by associating degenerate Fubini-type polynomials with degenerate Apostol–Bernoulli polynomials and degenerate Apostol–Euler polynomials of the order α, we presented some identities for these polynomials and numbers.

In the future, a relation involving degenerate Fubini-type polynomials and degenerate Apostol–Genocchi polynomials of order α, defined by [8] (Equation 2.6), could be given and further relations could be obtained by similar methods used in this paper.

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