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Fixed-Point Theorems for $\mathcal{L}_\gamma$ Contractions in Branciari Distance Spaces

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Abstract: In this paper, the concepts of Suzuki-type $\mathcal{L}_\gamma$ contractions and Suzuki–Berinde-type $\mathcal{L}_\gamma$ contractions are introduced, and new fixed-point theorems for these two contractions are established.

Keywords: fixed point; $\mathcal{L}$ contraction; $\mathcal{L}_\gamma$ contraction; Suzuki-type $\mathcal{L}_\gamma$ contraction; Suzuki–Berinde-type $\mathcal{L}_\gamma$ contraction; metric space; Branciari distance space

MSC: 47H10; 54H25

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1. Introduction

In 2009, Suzuki [1] generalized the Banach contraction principle to compact metric space by introducing the notion of a contractive map $T : U \to U$, where (U, ϱ) is compact metric space, such that

$$\forall u, v \in U(u \neq v),\ \frac{1}{2}\varrho(u, Tu) < \varrho(u, v) \text{ implies } \varrho(Tu, Tv) < \varrho(u, v).$$

Berinde [2] introduced the notion of almost contractions:

A map $T : U \to U$, where (U, ϱ) is a metric space, is called almost contraction provided that it satisfies

$$\varrho(Tu, Tv) \leq q\varrho(u, v) + K\varrho(v, Tu),$$

where $q \in (0, 1)$ and $K \geq 0$.

Berinde [2] generalized the Banach contraction principle by proving the existence of fixed points for almost contractions defined on complete metric spaces.

On the other hand, Branciari [3] gave a generalization of the notion of metric spaces, which is called Branciari distance spaces, by replacing triangle inequality with trapezoidal inequality, and he gave an extension of Banach contraction principle to Branciari distance spaces. He used the following to obtain the main results:

(1) each open ball is open set;
(2) each Branciari distance is continuous in each the coordinates;
(3) each topology induced by Branciari distance spaces is a Hausdorff topological space.

Sarma et al. showed that (1), (2), and (3) are false (see example 1.1 in [4]), and they extended the Banach contraction principle to a Branciari distance space under the assumption of Hausdorffness of the space (more specifically, the uniqueness of the limits of the converging sequences). Since then, some authors (for example, [5–7]) obtained fixed-point results in Branciari distance spaces under the assumption that the spaces are Hausdorff and/or the Branciari distances are continuous.

In particular, Kadelburg and Radenivić [8] investigated the existence of fixed points in Branciari distance spaces without the two conditions:

· Hausdorffness of Branciari distance spaces;
· Continuity of the Branciari distances.

After that, many authors ([4–6,9–26] and references therein) extended fixed-point results from metric spaces to Branciari distance spaces.

Given function ϑ from (0,∞) into $(1,\infty)$, we consider the following conditions:

(ϑ1) ϑ is non-decreasing;

(ϑ2) for any sequence $\{s_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \vartheta(s_n) = 1 \Leftrightarrow \lim_{n\to\infty} s_n = 0;$$

(ϑ3) there are $q \in (0,1)$ and $k \in (0,\infty)$, such that

$$\lim_{s\to 0^+} \frac{\vartheta(s)-1}{s^q} = k$$

(ϑ4) ϑ is continuous on $(0,\infty)$.

Jleli and Samet [22] obtained a generalization of the Banach contraction principle in Branciari distance spaces by introducing the concept of ϑ contractions, where $\vartheta : (0,\infty) \to (1,\infty)$ satisfies conditions (ϑ1), (ϑ2) and (ϑ3). Ahmad et al. [27] generalized the result of Jleli and Samet [22] to metric spaces by applying conditions (ϑ1), (ϑ2), and (ϑ4), and they introduced the notion of Suzuki–Berinde-type ϑ contractions and investigated the existence of fixed points for such contractions.

Very recently, Cho [24] introduced the concept of $\mathcal{L}$ contractions, which is a more generalized concept than some existing notions of contractions. He proved that every $\mathcal{L}$ contraction mapping defined on complete Branciari distance spaces possesses only one fixed point.

Afterward, the authors [23,28–33] gave generalizations of the result of [24].

In the paper, we introduce the new two concepts of Suzuki-type $\mathcal{L}_\gamma$ contractions and Suzuki–Berinde-type $\mathcal{L}_\gamma$ contractions, which are a generalization of the concept of $\mathcal{L}$ contractions, and we establish two new fixed point theorems for these two contractions in the setting of Branciari distance spaces. We give examples to support main theorem.

Let $\xi : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$ be a function.

Consider the following conditions:

(ξ1) $\xi(1,1) = 1$;

(ξ2) $\xi(t,s) < \frac{s}{t}\ \forall s,t > 1$;

(ξ3) $\xi(t,s) < \frac{\gamma(s)}{\gamma(t)}\ \forall s,t > 1$, where γ is a non-decreasing self-mapping on $[1,\infty)$, satisfying $\gamma^{-1}(\{1\}) = 1$;

(ξ4) for any sequence $\{t_m\}, \{s_m\} \subset (1,\infty)$ with $t_m \le s_m, m = 1,2,3,\cdots$,

$$\lim_{m\to\infty} t_m = \lim_{m\to\infty} s_m > 1 \Rightarrow \lim_{m\to\infty} \sup \xi(t_m, s_m) < 1.$$

A function $\xi : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$ is said to be *$\mathcal{L}$-simulation* [24] whenever the conditions (ξ1), (ξ2), and (ξ4) are satisfied.

Note that $\xi(t,t) < 1\ \forall t > 1$.

We say that $\xi : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$ is an *$\mathcal{L}_\gamma$-simulation* provided that the condition (ξ1), (ξ3) and (ξ4) hold.

Remark 1. *If $\gamma(t) = t\ \forall t \ge 1$, then $\mathcal{L}_\gamma$-simulation is $\mathcal{L}$-simulation.*

Denote $\mathcal{L}$ by the class of all $\mathcal{L}$-simulation functions $\xi : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$, and $\mathcal{L}_\gamma$ by the collection of all $\mathcal{L}_\gamma$-simulation functions $\xi : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$.

Example 1. *Let $\xi_b, \xi_w, \xi_c : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$ be functions defined as follows, respectively:*

(i) $\xi_b(t,s) = \frac{[\gamma(s)]^r}{\gamma(t)}$ *for all $t,s \ge 1$, where $r \in (0,1)$;*

(ii) $\xi_w(t,s) = \frac{\gamma(s)}{\gamma(t)\phi(\gamma(s))}$ $\forall t,s \geq 1$, *where* ϕ *is a non-decreasing and lower semi-continuous self-mapping on* $[1,\infty)$, *satisfying* $\phi^{-1}(\{1\}) = 1$;

(iii) $\xi_c(t,s) = \begin{cases} 1 & \text{if } (s,t) = (1,1), \\ \frac{\gamma(s)}{2\gamma(t)} & \text{if } s < t, \\ \frac{[\gamma(s)]^\lambda}{\gamma(t)} & \text{otherwise}, \end{cases}$

$\forall s,t \geq 1$, *where* $\lambda \in (0,1)$.

Then, $\xi_b, \xi_w, \xi_c \in \mathcal{L}_\gamma$.

Note that if $\gamma(t) = t$ $\forall t \geq 1$, *then* $\xi_b, \xi_w, \xi_c \in \mathcal{L}$*(see [24]).*

Example 2. *Let* $\xi_1, \xi_2, \xi_3 : [1,\infty) \times [1,\infty) \to (-\infty,\infty)$ *be functions defined as follows:*

(i) $\xi_1(t,s) = \frac{\gamma(\psi(s))}{\gamma(\varphi(t))}$, $\forall t,s \geq 1$, *where* ψ *and* φ *are continuous self-mappings on* $[1,\infty)$, *satisfying* $\psi(t) = \varphi(t) = 1 \Leftrightarrow t = 1$, $\psi(t) < t \leq \varphi(t)$, $\forall t > 1$ *and* φ *is an increasing mapping;*

(ii) $\xi_2(t,s) = \frac{\gamma(\eta(s))}{\gamma(t)}$, $\forall s,t \geq 1$, *where* η *is a upper semi-continuous self-mapping on* $[1,\infty)$, *satisfying* $\eta(t) < t$, $\forall t > 1$ *and* $\eta(t) = 1 \Leftrightarrow t = 1$;

(iii) $\xi_3(t,s) = \frac{\gamma(s)}{\gamma(\int_0^t \phi(u)du)}$, $\forall s,t \geq 1$, *where* ϕ *is a self-mapping on* $[0,\infty)$, *satisfying* $\forall t \geq 1$, $\int_0^t \phi(s)ds$ *exists and* $\int_0^t \phi(s)ds > t$, *and* $\int_0^1 \phi(s)ds = 1$.

Then, $\xi_1, \xi_2, \xi_3 \in \mathcal{L}_\gamma$.

Note that if $\gamma(t) = t$ $\forall t \geq 1$, *then* $\xi_1, \xi_2, \xi_3 \in \mathcal{L}$ *(see [30]).*

The following definitions are in [3].

A map $\varrho : U \times U \to [0,\infty)$, where U is a non-empty set, is said to be *Branciari distance* on U if the following conditions are satisfied:

for all $u, v \in U$ and for $z, w \in U - \{u, v\}$

(ϱ1) $\varrho(u,v) = 0 \Leftrightarrow u = v$;
(ϱ2) $\varrho(u,v) = \varrho(v,u)$;
(ϱ3) $\varrho(u,v) \leq \varrho(u,z) + \varrho(z,w) + \varrho(w,v)$ (trapezoidal inequality).

The pairs (U,ϱ) is said to be a *Branciari distance space*.

Note that Branciari distance space (U,ϱ) can not reduce the standard metric space and it does not have a topology which is compatible with ϱ (e.g., [34] and Remark 4 (3)). For such reasons, we call (U,ϱ) a Branciari distance space, not a rectangular metric space or a generalized metric space.

Remark 2. *If the triangle inequality is satisfied, the trapezoidal inequality is satisfied. However, the converse is not true. Thus, the class of Branciari distance spaces includes metric spaces.*

The notion of convergence in Branciari distance spaces is similar to that of metric spaces (e.g., [3]).

Let (U,ϱ) be a Branciari distance space and $\{u_n\} \subset U$ be a sequence and $u \in U$. Then, we say that

- $(\cdot)$ $\{u_n\}$ converges to u, whenever $\lim_{n\to\infty} \varrho(u_n,u) = 0$;
- $(\cdot)$ $\{u_n\}$ is a Cauchy sequence, when $\lim_{n,m\to\infty} \varrho(u_n,u_m) = 0$;
- $(\cdot)$ (U,ϱ) is complete if every Cauchy sequence in U converges to some point in U.

Let (U,ϱ) be a Branciari distance space, and let τ_ϱ be the topology on U, such that

$$U - C \in \tau_\varrho \Longleftrightarrow \forall \{u_n\} \subset C, \lim_{n\to\infty} \varrho(u_n,u) = 0 \text{ implies } u \in C. \tag{1}$$

The topology τ_ϱ induced by (1) is called a sequential topology.

A map $T : U \to U$ is said to be continuous at $u \in U$ if, and only if, $\forall V \in \tau_\varrho$ contains Tu, and there exists $W \in \tau_\varrho$ containing u, such that $TW \subset V$ (see [24]).

We say that T is continuous, whenever it is continuous at each point $u \in U$.

Remark 3. *A map $T : U \to U$, where (U, ϱ) is a Branciari distance space, is continuous if, and only if, the following condition holds:*

$$\lim_{n\to\infty} \varrho(Tu_n, Tu) = 0, \text{ whenever } \lim_{n\to\infty} \varrho(u_n, u) = 0 \text{ for any sequence } \{u_n\} \subset U.$$

Let us recall the following example in [4] where we can understand the characteristics of the branchiari distance spaces.

Example 3. *Let $U = \{0, 2\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, and define a map $\varrho : U \times U \to [0, \infty)$ by*

$$\varrho(u,v) = \begin{cases} 0, (u = v), \\ 1, (u, v \in \{0, 2\}), \\ 1, (u, v \in \{\frac{1}{n} : n \in \mathbb{N}\}), \\ \frac{1}{n}, (u \in \{0, 2\} \text{ and } v \in \{\frac{1}{n} : n \in \mathbb{N}\}). \end{cases}$$

Then, (U, ϱ) is a Branciari distance space.

We have the following.

(i) *Limit is not unique.*
We infer that

$$\lim_{n\to\infty} \varrho(\frac{1}{n}, 0) = \lim_{n\to\infty} \frac{1}{n} = 0 \text{ and } \lim_{n\to\infty} \varrho(\frac{1}{n}, 2) = \lim_{n\to\infty} \frac{1}{n} = 0. \quad (2)$$

Hence, the sequence $\{\frac{1}{n}\}$ is convergent to 0 and 2, and the limit is not unique.

(ii) *The convergent sequence $\{\frac{1}{n}\}$ is not a Cauchy sequence.*

$$\lim_{n,m\to\infty} \varrho(\frac{1}{n}, \frac{1}{m}) \neq 0, \text{ because } \varrho(\frac{1}{n}, \frac{1}{m}) = 1.$$

Hence, $\{\frac{1}{n}\}$ is not a Cauchy sequence.

(iii) $\lim_{n\to\infty} \varrho(\frac{1}{n}, \frac{1}{2}) \neq \varrho(0, \frac{1}{2})$.

(iv) *The open ball with center $\frac{1}{3}$ and radius $\frac{2}{3}$ is the set $B(\frac{1}{3}, \frac{2}{3}) = \{\frac{1}{3}, 0, 2\}$. There is no $r > 0$, such that*

$$B(0, r) \subset B(\frac{1}{3}, \frac{2}{3}). \quad (3)$$

Remark 4. *(i)* *It folows from (2) that the sequential topology on U is not a Hausdorff space.*

(ii) *The Branciari distance ϱ is not continuous with respect to the sequential topology on U. In fact, let $y \in U$ be a fixed point, such that $y \neq 0$ and $y \neq 2$.*
We show that

$$\lim_{n\to\infty} \varrho(\frac{1}{n}, 2) = 0.$$

However,

$$\lim_{n\to\infty} \varrho(\frac{1}{n}, y) \neq \varrho(2, y).$$

Hence, $\varrho(\cdot, y)$ is not continuous with respect to the sequential topology on U.

(iii) *From (3) the family $\{B(u, r) : u, r > 0\}$, where $B(u, r) = \{v : \varrho(u, v) < r\}$, is not a basis for any topology on (U, ϱ), and so there is no topology which is compatible with the Branciari distance ϱ.*

(iv) *It is known that the sequential topology is not compatible with the Branciari distance ϱ.*

(v) *There is no Cauchy sequence, so it is a complete Branciari distance space.*

Note that Example 3 shows that the Branciari distance space is much weaker in mathematical structure than the metric space. As we have seen in the example above and Remark 4, there are some mathematical drawbacks to the Branciari distance. Nevertheless, it is attractive for researchers to study the existence of fixed points in this space without additional conditions such as the uniqueness of the limit of the converging sequence in Branciari distance spaces or/and continuity of a Branciari distance with respect to the sequential topology on a Branciari distance space.

Lemma 1 ([35])**.** *Let (U, ϱ) be a Branciari distance space, $\{u_n\} \subset U$ be a Cauchy sequence and $u, v \in U$. If there is a positive integer n_0, such that*

(i) $u_n \neq u_m \; \forall n, m > n_0$;
(ii) $u_n \neq u \; \forall n > n_0$;
(iii) $u_n \neq v \; \forall n > n_0$;
(iv) $\lim_{n\to\infty} \varrho(u_n, u) = \lim_{n\to\infty} \varrho(u_n, v)$,
then, $u = v$.

From now on, let φ be a function from $[0, \infty) \times [0, \infty)$ into $(-\infty, \infty)$, such that

$$\varphi(s, t) \leq \frac{1}{2}s - t, \; \forall s, t \in [0, \infty).$$

Note that if $\frac{1}{2}s < t \;\; \forall s, t \in [0, \infty)$, then the following conditions are satisfied.

(i) $\varphi(s, t) < 0$;
(ii) $\varphi(\min\{s, u\}, t) < 0$.

2. Fixed-Point Results

2.1. Fixed Points for Suzuki-Type $\mathcal{L}_\gamma$ Contractions

Let (U, ϱ) be a Branciari distance space.

A map T from U into itself is *Suzuk-type $\mathcal{L}_\gamma$ contraction* with respect to $\xi \in \mathcal{L}_\gamma$ provided that it satisfies the condition:

$\forall u, v \in U$ with $\varrho(Tu, Tv) > 0$

$$\begin{aligned} &\varphi(\min\{\varrho(u, Tu), \varrho(v, Tu)\}, \varrho(u, v)) < 0 \\ &\Rightarrow \xi(\vartheta((Tu, Tv)), \vartheta(\varrho(u, v))) \geq 1 \end{aligned} \tag{4}$$

where $\vartheta : (0, \infty) \to (1, \infty)$ is a function.

Lemma 2. *Let $l > 0$, and let $\{t_n\} \subset (l, \infty)$ be a sequence, such that*

$$t_n \leq t_{n-1} \; \forall n = 1, 2, 3, \cdots, \textit{ and } \lim_{n\to\infty} t_n = l.$$

If $\vartheta : (0, \infty) \to (1, \infty)$ is non-decreasing, then we show that

$$\lim_{n\to\infty} \vartheta(t_n) = \lim_{n\to\infty} \vartheta(t_{n-1}) = \lim_{t\to l^+} \vartheta(t) > 1.$$

Proof. Since ϑ is non-decreasing and $\{t_n\}$ is non-increasing,

$$\lim_{t\to l^+} \vartheta(t) = \lim_{n\to\infty} \vartheta(t_n) \leq \lim_{n\to\infty} \vartheta(t_{n-1}) \leq \lim_{t\to l^+} \vartheta(t).$$

Thus, we established that $\lim_{n\to\infty} \vartheta(t_n) = \lim_{n\to\infty} \vartheta(t_{n-1}) = \lim_{t\to l^+} \vartheta(t) > \vartheta(l) > 1$. □

We now establish main theorem.

Theorem 1. *Let (U, ϱ) be a complete Branciari distance space. Suppose that mapping T from U into itself is a Suzuki-type $\mathcal{L}_\gamma$ contraction with respect to $\xi \in \mathcal{L}_\gamma$. If ϑ is non-decreasing, then T possesses only one fixed point, and for every initial point $u_0 \in U$, the Picard sequence $\{T^n u_0\}$ is convergent to the fixed point.*

Proof. Firstly, when a fixed point exists, let us show that it is unique.

Assume that $w = Tw$ and $u = Tu$, such that $u \neq w$.

Then, $\varrho(w, u) > 0$ and $\varphi(\min\{\varrho(w, Tw), \varrho(u, Tw)\}, \varrho(w, u))$
$= \varphi(\min\{0, \varrho(u, w)\}, \varrho(u, w)) \leq \frac{1}{2}\min\{0, \varrho(u, w)\} - \varrho(w, u) < 0$.

From (4), we have

$$\begin{aligned}1 \leq &\xi(\vartheta(\varrho(Tw, Tu)), \vartheta(\varrho(w, u)))\\ =&\xi(\vartheta(\varrho(w, u)), \vartheta(\varrho(w, u))) < 1\end{aligned}$$

which is a contradiction.

Hence, $w = u$, and the fixed point of T is unique.

Secondly, let us show the existence of fixed points.

Let $u_0 \in U$, and let $\{u_n\} \subset U$ be a sequence defined by $u_n = Tu_{n-1} = T^n u_0$, $\forall n \in \mathbb{N}$.

If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $u_{n_0} = Tu_{n_0}$, and the proof is completed.

Assume that

$$u_{n-1} \neq u_n \ \forall n \in \mathbb{N}. \tag{5}$$

We infer that

$$\begin{aligned}&\varphi(\min\{\varrho(u_{n-1}, Tu_{n-1}), \varrho(u_n, Tu_{n-1})\}, \varrho(u_{n-1}, u_n))\\ =&\varphi(\min\{\varrho(u_{n-1}, u_n), \varrho(u_n, u_n)\}, \varrho(u_{n-1}, u_n))\\ =&\frac{1}{2}\min\{\varrho(u_{n-1}, u_n), 0\} - \varrho(u_{n-1}, u_n) < 0.\end{aligned} \tag{6}$$

It follows from (4), (5), and (6) that for all $n \in \mathbb{N}$

$$\begin{aligned}1 \leq &\xi(\vartheta(\varrho(Tu_{n-1}, Tu_n)), \vartheta(\varrho(u_{n-1}, u_n)))\\ =&\xi(\vartheta(\varrho(u_n, u_{n+1})), \vartheta(\varrho(u_{n-1}, u_n)))\\ <&\frac{\gamma(\vartheta(\varrho(u_{n-1}, u_n)))}{\gamma(\vartheta(\varrho(u_n, u_{n+1})))}.\end{aligned} \tag{7}$$

Consequently, we show that

$$\gamma(\vartheta(\varrho(u_n, u_{n+1}))) < \gamma(\vartheta(\varrho(u_{n-1}, u_n))) \ \forall n \in \mathbb{N}$$

which yields

$$\vartheta(\varrho(u_n, u_{n+1})) < \vartheta(\varrho(u_{n-1}, u_n)) \ \forall n \in \mathbb{N}.$$

Thus,

$$\varrho(u_n, u_{n+1}) < \varrho(u_{n-1}, u_n) \ \forall n \in \mathbb{N}. \tag{8}$$

So, the sequence $\{\varrho(u_{n-1}, u_n)\}$ is decreasing, and hence there is an $l \geq 0$, such that

$$\lim_{n\to\infty} \varrho(u_{n-1}, u_n) = l.$$

We prove that $l = 0$.

Assume that $l > 0$.

Let $t_{n-1} = \vartheta(\varrho(u_{n-1}, u_n))$ and $t_n = \vartheta(\varrho(u_n, u_{n+1})) \ \forall n \in \mathbb{N}$.

Then, $t_n < t_{n-1} \ \forall n \in \mathbb{N}$.

By applying Lemma 2,

$$\lim_{n\to\infty} t_{n-1} = \lim_{n\to\infty} t_n = \lim_{t\to l^+} \theta(t) > 1.$$

By applying ($\xi 3$), we have

$$1 \leq \lim_{n\to\infty} \sup \xi(t_n, t_{n-1}) < 1.$$

This is a contradiction.

Thus,

$$\lim_{n\to\infty} \varrho(u_{n-1}, u_n) = 0. \tag{9}$$

Now, we show that $\{u_n\}$ is a Cauchy sequence.

On the contrary, assume that $\{u_n\}$ is not a Cauchy sequence.

Then, there is an $\epsilon > 0$ for which we can find subsequences $\{u_{m(j)}\}$ and $\{u_{n(j)}\}$ of $\{u_n\}$, such that $m(j)$ is the smallest index for which

$$m(j) > n(j) > j,\ \varrho(u_{m(j)}, u_{n(j)}) \geq \epsilon \text{ and } \varrho(u_{m(j)-1}, u_{n(j)}) < \epsilon. \tag{10}$$

From (10), we infer that

$$\begin{aligned} &\epsilon \\ \leq &\varrho(u_{m(j)}, u_{n(j)}) \\ \leq &\varrho(u_{n(j)}, u_{m(j)-2}) + \varrho(u_{m(j)-2}, u_{m(j)-1}) + \varrho(u_{m(j)-1}, u_{m(j)}) \\ < &\epsilon + \varrho(u_{m(j)-2}, u_{m(j)-1}) + \varrho(u_{m(j)-1}, x_{m(j)}). \end{aligned} \tag{11}$$

By letting $j \to \infty$ in (11), we have

$$\lim_{n\to\infty} \varrho(u_{m(j)}, u_{n(j)}) = \epsilon.$$

On the other hand, we obtain

$$\varrho(u_{m(j)}, u_{n(j)}) \leq \varrho(u_{n(j)}, u_{n(j)+1}) + \varrho(u_{n(j)+1}, u_{m(j)+1}) + \varrho(u_{m(j)+1}, u_{m(j)})$$

and

$$\varrho(u_{n(j)+1}, u_{m(j)+1}) \leq \varrho(u_{n(j)+1}, u_{n(j)}) + \varrho(u_{n(j)}, u_{m(j)}) + \varrho(u_{m(j)}, u_{m(j)+1}).$$

Thus,

$$\lim_{j\to\infty} \varrho(u_{n(j)+1}, u_{m(j)+1}) = \epsilon.$$

It follows from (9) that there exists $N_1 \in \mathbb{N}$, such that

$$\varrho(u_{n(j)}, u_{n(j)+1}) < \epsilon,\ \forall j > N_1.$$

Hence, we infer that $\forall k > N_1$

$$\begin{aligned} &\frac{1}{2}\min\{\varrho(u_{n(j)}, Tu_{n(j)}), \varrho(u_{m(j)}, Tu_{n(j)})\} \\ = &\frac{1}{2}\min\{\varrho(u_{n(j)}, u_{n(j)+1}), \varrho(u_{m(j)}, u_{n(j)+1}) \\ < &\epsilon \\ \leq &d(u_{n(j)}, u_{m(j)}) \end{aligned}$$

which implies

$$\varphi(\min\{\varrho(u_{n(j)}, Tu_{n(j)}), \varrho(u_{m(j)}, Tu_{n(j)+1})\}, \varrho(u_{n(j)}, u_{m(j)})) < 0,\ \forall j > N_1.$$

It follows from (4) that $\forall j > N_1$

$$\begin{aligned}1 \leq &\xi(\vartheta(\varrho(Tu_{n(j)}, Tu_{m(j)})), \vartheta(\varrho(u_{n(j)}, u_{m(j)})))\\ =&\xi(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})), \vartheta(\varrho(u_{n(j)}, u_{m(j)}))\\ <&\frac{\gamma(\vartheta(\varrho(u_{n(j)}, u_{m(j)})))}{\gamma(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})))}\end{aligned}$$

which implies

$$\gamma(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1}))) < \gamma(\vartheta(\varrho(u_{n(j)}, u_{m(j)})))$$

and so

$$\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})) < \vartheta(\varrho(u_{n(j)}, u_{m(j)}))\ \forall j > N_1.$$

Put

$$t_j = \vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})) \text{ and } t_{j-1} = \vartheta(\varrho(u_{n(j)}, u_{m(j)}))$$

Then, we have

$$t_j < t_{j-1}\ \forall j > N_1$$

and

$$\lim_{j\to\infty} \varrho(u_{n(j)+1}, u_{m(j)+1}) = \lim_{j\to\infty} \varrho(u_{n(j)}, u_{m(j)}) = \epsilon.$$

By Lemma 2,

$$\lim_{j\to\infty} t_j = \lim_{j\to\infty} t_{j-1} = \lim_{t\to\epsilon^+} \vartheta(t) > 1.$$

From (ξ3), we have

$$1 \leq \limsup_{j\to\infty} \xi(t_j, t_{j-1}) < 1$$

which leads to a contradiction.

Thus, $\{u_n\}$ is a Cauchy sequence.

It follows from completeness of U that there is $u \in U$, such that

$$\lim_{n\to\infty} \varrho(u_n, u) = 0. \tag{12}$$

We may assume that there is $m_0 \in \mathbb{N}$, such that

$$\varrho(u_{n+1}, u) < \varrho(u_n, u)\ , \forall n > m_0.$$

We infer that $\forall n > m_0$

$$\begin{aligned}&\varphi(\min\{\varrho(u_n, Tu_n), \varrho(u, Tu_n)\}, \varrho(u_n, u))\\ =&\varphi(\min\{\varrho(u_n, u_{n+1}), \varrho(u, u_{n+1})\}, \varrho(u_n, u))\\ \leq&\frac{1}{2}\min\{\varrho(u_n, u_{n+1}), \varrho(u, u_{n+1})\} - \varrho(u_n, u)\\ <&0.\end{aligned}$$

Applying (4), we establish that

$$1 \leq \xi(\vartheta(\varrho(Tu_n, Tu)), \vartheta(\varrho(u_n, u))) < \frac{\gamma(\vartheta(\varrho(u_n, u)))}{\gamma(\vartheta(\varrho(Tu_n, Tu)))},\ \forall n \geq m_0.$$

which implies

$$\gamma(\theta(\varrho(Tu_n, Tu))) < \gamma(\theta(\varrho(u_n, u))),\ \forall n \geq m_0.$$

Hence,

$$\varrho(Tu_n, Tu) < \varrho(u_n, u),\ \forall n \geq m_0$$

and hence,

$$\lim_{n\to\infty} \varrho(u_{n+1}, Tu) = 0. \tag{13}$$

Applying Lemma 1 with (12) and (13), we have $u = Tu$. $\square$

The following example interprets Theorem 1.

Example 4. *Let $U = \{1, 2, 3, 4\}$, and let us define $\varrho : U \times U \to [0, \infty)$ as follows:*

$$\begin{aligned}
&\varrho(1,2) = \varrho(2,1) = 3,\\
&\varrho(2,3) = \varrho(3,2) = \varrho(1,3) = \varrho(3,1) = 1,\\
&\varrho(1,4) = \varrho(4,1) = \varrho(2,4) = \varrho(4,2) = \varrho(3,4) = \varrho(4,3) = 4,\\
&\varrho(u,u) = 0 \ \forall u \in U.
\end{aligned}$$

Then, (U, ϱ) is a complete Branciari distance space, but not a metric space (see [6]).
Define a map $T : U \to U$ by

$$Tu = \begin{cases} 2 & (u = 1, 2),\\ 4 & (u = 3),\\ 3 & (u = 4). \end{cases}$$

Let $\vartheta : (0, \infty) \to (1, \infty)$ be a function defined by

$$\vartheta(t) = \begin{cases} e^{\sqrt{t}} & (0 < t \le 1),\\ 3 & (t > 1). \end{cases}$$

Then, ϑ is non-decreasing.
We prove that T is a $\mathcal{L}_\gamma$ contraction with respect to ξ_2, where $\xi_2(t,s) = \frac{\gamma(\eta(s))}{\gamma(t)}$, $\eta(s) = \frac{3}{2}s - \frac{1}{2}\ \forall s \ge 1, \gamma(t) = \frac{1}{2}t + \frac{1}{2}\ \forall t \ge 1.$
We have

$$\varrho(Tu, Tv) = \begin{cases} \varrho(2,2) = 0 & (u = 1, v = 2),\\ \varrho(2,4) = 4 & (u = 1, v = 3),\\ \varrho(2,3) = 1 & (u = 1, v = 4),\\ \varrho(2,4) = 4 & (u = 2, v = 3),\\ \varrho(2,3) = 1 & (u = 2, v = 4),\\ \varrho(3,4) = 4 & (u = 3, v = 4) \end{cases}$$

so

$$\varrho(Tu, Tv) > 0 \Leftrightarrow (u = 1, v = 3), (u = 1, v = 4), (u = 2, v = 3), (u = 2, v = 4), (u = 3, v = 4).$$

We establish that

$$\varrho(u, v) = \begin{cases} 1 & (u = 1, v = 3),\\ 4 & (u = 1, v = 4),\\ 1 & (u = 2, v = 3),\\ 4 & (u = 2, v = 4),\\ 4 & (u = 3, v = 4) \end{cases}$$

and

$$m(u,v)=\begin{cases}1 & (u=1,v=3),\\ 3 & (u=1,v=4),\\ 0 & (u=2,v=3),\\ 0 & (u=2,v=4),\\ 1 & (u=3,v=4).\end{cases}$$

We infer that for all $u,v\in U$ *with* $\varrho(Tu,Tv)>0$,

$$\varphi(m(u,v),\varrho(u,v))\leq\frac{1}{2}m(u,v)-\varrho(u,v)<0.$$

Thus, we have

$$\begin{aligned}&\xi_2(\vartheta(d(Tu,Tv)),\vartheta(\varrho(u,v))\\ =&\frac{\gamma(\eta(\vartheta(\varrho(u,v)))}{\gamma(\vartheta(\varrho(Tu,Tv)))}\\ =&\begin{cases}\frac{\gamma(\eta(\vartheta(1)))}{\gamma(\vartheta(4))} & (u=1,v=3),\\ \frac{\gamma(\eta(\vartheta(4)))}{\gamma(\vartheta(1))} & (u=1,v=4),\\ \frac{\gamma(\eta(\vartheta(1)))}{\gamma(\vartheta(4))} & (u=2,v=3),\\ \frac{\gamma(\eta(\vartheta(4)))}{\gamma(\vartheta(1))} & (u=2,v=4),\\ \frac{\gamma(\eta(\vartheta(4)))}{\gamma(\vartheta(4))} & (u=3,v=4)\end{cases}\end{aligned}$$

which yields

$$\begin{aligned}&\xi_2(\vartheta(\varrho(Tu,Tv)),\vartheta(\varrho(u,v))\\ \geq&\frac{\gamma(\eta(\vartheta(1)))}{\gamma(\vartheta(4))}\\ =&\frac{\frac{1}{2}(\frac{3}{2}e-\frac{1}{2})+\frac{1}{2}}{\frac{1}{2}3+\frac{1}{2}}>1,\ \textit{because}\ \frac{3}{2}e-\frac{1}{2}-3=\frac{3}{2}(e-\frac{7}{3})>0.\end{aligned}$$

Hence, T *is a* $\mathcal{L}_\gamma$ *contraction with respect to* ξ_2*. Thus, all hypotheses of Theorem 1 are satisfied, and* T *possesses a unique fixed point* $u=2$*.*

Note that T *is not* $\mathcal{L}$ *contraction [24] with respect to* $\xi_2(t,s)=\frac{\eta(s)}{t}$*. In fact, for* $u=3,v=4$*, we establish that*

$$\xi_2(\vartheta(\varrho(Tu,Tv)),\vartheta(\varrho(u,v))=\frac{\eta(\vartheta(4))}{\vartheta(4)}<\frac{\vartheta(4)}{\vartheta(4)}=1.$$

Note that Banach condition principle is not satisfied. In fact, if $u=3,v=1$*, then*

$$\varrho(T3,T2)\leq k\varrho(3,2),k\in(0,1)$$

which implies

$$k\geq 4.$$

Furthermore, the ϑ *contraction condition [22] is not satisfied.*
Note that ϑ *satisfies conditions* $(\vartheta1)$*,* $(\vartheta2)$ *and* $(\vartheta3)$*.*
If for $u=3,v=2$

$$\vartheta(\varrho(T3,T2))\leq[\vartheta(\varrho(3,2))]^k,k\in(0,1)$$

then

$$\vartheta(4)\leq[\vartheta(1)]^k<\vartheta(1)$$

which is a contradiction.

Hence, T is not a ϑ contraction.

The following example shows that in Theorem 1, the condition that the function ϑ is non-decreasing cannot be dropped.

Example 5. *Let $U = \{0,2\} \cup \{\frac{1}{n} : n = 3,4,5,\cdots\}$, and let $\varrho : U \times U \to [0,\infty)$ be a map defined by*

$$\varrho(u,v) = \begin{cases} 0 \ (u = v), \\ 0 \ (u, v \in \{0,2\}), \\ 0 \ (u, v \in \{\frac{1}{n} : n = 3,4,5,\cdots\}), \\ \frac{1}{n} \ (u \in \{0,2\} \text{ and } v \in \{\frac{1}{n} : n = 3,4,5,\cdots\}) \\ \frac{1}{n} \ (u \in \{\frac{1}{n} : n = 3,4,5,\cdots\} \text{ and } v \in \{0,2\}). \end{cases}$$

Then, (U,ϱ) is a complete Branciari distance space.

Define a mapping $T : U \to U$ by

$$Tu = \begin{cases} 2 & (u = 0), \\ 0 & (u = 2), \\ \frac{1}{n+1} & (u = \frac{1}{n}, n = 3,4,5,\cdots). \end{cases}$$

Let $\vartheta : (0,\infty) \to (1,\infty)$ be a function defined by

$$\vartheta(t) = \frac{1}{t} + 1$$

and let

$$\eta(s) = \frac{3}{2}s - \frac{1}{2} \ \forall s \geq 1, \gamma(t) = \frac{1}{2}t + \frac{1}{2} \ \forall t \geq 1.$$

We infer that

$$\varphi(\min\{\varrho(u,Tu),\varrho(v,Tu)\},\varrho(u,v)) = \begin{cases} 0 & (u = 0, v = 2), \\ \frac{1}{2}\frac{1}{n+1} - \frac{1}{n} & (u = 0, v = \frac{1}{n}, n = 3,4,5,\cdots), \\ \frac{1}{2}\frac{1}{n+1} - \frac{1}{n} & (u = 2, v = \frac{1}{n}, n = 3,4,5,\cdots), \\ 0 & (u = \frac{1}{n}, v = \frac{1}{m}, m > n, n = 3,4,5,\cdots), \end{cases}$$

and we show that

$$\varrho(Tu,Tv) = \begin{cases} \frac{1}{n} & (u = 0, v = \frac{1}{n}, n = 3,4,5,\cdots), \\ \frac{1}{n} & (u = 2, v = \frac{1}{n}, n = 3,4,5,\cdots). \end{cases}$$

Thus, the following is satisfied:

$$\varrho(Tu,Tv) > 0 \text{ and } \varphi(\min\{\varrho(u,Tu),\varrho(v,Tu)\},\varrho(u,v)) < 0 \Leftrightarrow (u = 0, v = \frac{1}{n}) \text{ and } (u = 2, v = \frac{1}{n})$$

where $n = 3,4,5,\cdots$.

Thus, we obtain that for $(u = 0, v = \frac{1}{n}), (u = 2, v = \frac{1}{n})$

$$\begin{aligned} &\xi_2(\vartheta(\varrho(Tu,Tv)),\vartheta(\varrho(u,v)) \\ =&\frac{\gamma(\eta(\vartheta(\varrho(u,v))))}{\gamma(\vartheta(\varrho(Tu,Tv)))} \\ =&\frac{\gamma(\eta(\vartheta(\frac{1}{n})))}{\gamma(\vartheta(\frac{1}{n+1}))} \\ =&\frac{3n+4}{2n+6} > 1, \text{ where } n = 3,4,5,\cdots. \end{aligned}$$

Thus, T is a $\mathcal{L}_\gamma$ contraction with respect to ξ_2. However, T has no fixed point. Note that $\vartheta(t) = \frac{1}{t} + 1$, $\forall t > 0$ is not a non-decreasing function.

The following Corollary 1 is obtained from Theorem 1.

Corollary 1. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $\xi \in \mathcal{L}_\gamma$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v))) \geq 1.$$

If ϑ is non-decreasing, then T possesses only one fixed point.

Corollary 2. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $\xi \in \mathcal{L}$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(\varrho(u, Tu), \varrho(v, Tu)) < 0 \Rightarrow \xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v))) \geq 1.$$

If ϑ is non-decreasing, then T possesses only one fixed point.

Corollary 3. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $\xi \in \mathcal{L}$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v))) \geq 1.$$

If ϑ is non-decreasing, then T possesses only one fixed point.

Remark 5. *(1) It does not take continuity of ϑ to obtain Corollary 3, and continuity of ϑ is not required to prove Theorem 2.1 of [24].*

(2) Corollary 2 is a generalization of Theorem 2.1 of [24].

2.2. Fixed Points for Suzuki–Berinde-Type $\mathcal{L}_\gamma$ Contractions

Let (U, ϱ) be a Branciari distance space.

A map $T : U \to U$ is a *Suzuk–Berinde-type $\mathcal{L}_\gamma$ contraction* with respect to $\xi \in \mathcal{L}_\gamma$, provided that the condition is satisfied:

$\forall u, v \in U$ with $\varrho(Tu, Tv) > 0$

$$\varphi(m(u, v), \varrho(u, v)) < 0$$

$$\Rightarrow \xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v) + Km(u, v))) \geq 1 \tag{14}$$

where $\theta : (0, \infty) \to (1, \infty), K \in (0, \infty)$, and $m(u, v) = \min\{\varrho(u, Tu), \varrho(v, Tu)\}$.

Theorem 2. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a Suzuki–Berinde-type $\mathcal{L}_\gamma$ contraction with respect to $\xi \in \mathcal{L}_\gamma$. If ϑ is non-decreasing and continuous, then T possesses only one fixed point, and for every initial point $u_0 \in U$, the Picard sequence $\{T^n u_0\}$ is convergent to the fixed point.*

Proof. Let $u_0 \in U$ and let $\{u_n = T^n u_0\} \subset U$ be a sequence, such that

$$u_{n-1} \neq u_n \ \forall n = 1, 2, 3 \cdots. \tag{15}$$

We infer that

$$m(u_{n-1}, u_n) = \min\{\varrho(u_{n-1}, u_n), \varrho(u_n, u_n)\} = 0 \tag{16}$$

and

$$\varphi(m(u_{n-1}, u_n), \varrho(u_{n-1}, u_n)) \leq -\varrho(u_{n-1}, u_n) < 0. \tag{17}$$

It follows from (14), (15), (16) and (17) that $\forall n \in \mathbb{N}$

$$\begin{aligned} 1 \leq & \xi(\vartheta(\varrho(Tu_{n-1}, Tu_n)), \vartheta(\varrho(u_{n-1}, u_n) + Km(u_{n-1}, u_n))) \\ = & \xi(\vartheta(\varrho(u_n, u_{n+1})), \vartheta(\varrho(u_{n-1}, u_n))) \\ < & \frac{\gamma(\vartheta(\varrho(u_{n-1}, u_n)))}{\gamma(\vartheta(\varrho(u_n, u_{n+1})))}, \end{aligned} \tag{18}$$

which shows that $\{\varrho(u_{n-1}, u_n)\}$ is decreasing, because ϑ and γ are non-decreasing.

Hence,

$$\lim_{n\to\infty} \varrho(u_{n-1}, u_n) = l$$

where $l \geq 0$.

We prove that $l = 0$.

Assume that $l > 0$.

Then, since ϑ is continuous, we have

$$\lim_{n\to\infty} \vartheta(\varrho(u_{n-1}, u_n)) = \vartheta(l) > 1.$$

Let $t_{n-1} = \vartheta(\varrho(u_{n-1}, u_n))$ and $t_n = \vartheta(\varrho(u_n, u_{n+1}))$ $\forall n \in \mathbb{N}$.

Then,

$$t_n < t_{n-1}\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} t_{n-1} = \lim_{n\to\infty} t_n = \theta(l) > 1.$$

By applying (ξ3), we show that

$$1 \leq \lim_{n\to\infty} \sup \xi(t_n, t_{n-1}) < 1,$$

which leads to a contradiction.

Thus,

$$\lim_{n\to\infty} \varrho(u_{n-1}, u_n) = 0. \tag{19}$$

We shall show that $\{u_n\}$ is Cauchy.

On the contrary, assume that $\{u_n\}$ is not a Cauchy sequence.

Then, there is an $\epsilon > 0$ for which we can find subsequences $\{u_{m(j)}\}$ and $\{u_{n(j)}\}$ of $\{u_n\}$, such that $m(j)$ is the smallest index for which

$$m(j) > n(j) > j,\ \varrho(u_{m(j)}, u_{n(j)}) \geq \epsilon \text{ and } \varrho(u_{m(j)-1}, u_{n(j)}) < \epsilon. \tag{20}$$

As demonstrated in the proof of Theorem 1, we show that

$$\lim_{j\to\infty} \varrho(u_{m(j)}, u_{n(j)}) = \epsilon \text{ and } \lim_{j\to\infty} \varrho(u_{n(j)+1}, u_{m(j)+1}) = \epsilon. \tag{21}$$

From (19), there is an $N \in \mathbb{N}$, such that

$$\varrho(u_{n(j)}, u_{n(j)+1}) < \epsilon,\ \forall j > N.$$

Thus, we infer that $\forall j > N$

$$\varrho(u_{n(j)}, Tu_{n(j)}) = \varrho(u_{n(j)}, u_{n(j)+1}) < \epsilon \leq \varrho(u_{n(j)}, u_{m(j)}) \tag{22}$$

and

$$m(u_{n(j)}, u_{m(j)}) = \min\{\varrho(u_{n(j)}, u_{n(j)+1}), \varrho(u_{m(j)}, u_{n(j)+1})\}. \tag{23}$$

From (22) and (23), we obtain that

$$\varphi(m(u_{n(j)}, u_{m(j)}), \varrho(u_{n(j)}, u_{m(j)})) \leq \frac{1}{2} m(u_{n(j)}, u_{m(j)}) - \varrho(u_{n(j)}, u_{m(j)}) < 0.$$

By applying (14), we have

$$\begin{aligned}1 \leq & \xi(\vartheta(\varrho(Tu_{n(j)}, Tu_{m(j)})), \vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)}))) \\ = & \xi(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})), \vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)}))) \\ < & \frac{\gamma(\vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)})))}{\gamma(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})))}\end{aligned}$$

which implies

$$\gamma(\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1}))) < \gamma(\vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)})))$$

and so

$$\vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})) < \vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)})).$$

Let

$$t_j = \vartheta(\varrho(u_{n(j)+1}, u_{m(j)+1})) \text{ and } s_j = \vartheta(\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)}))$$

Then,

$$t_j < s_j \ \forall j \in \mathbb{N}.$$

Applying (21) and (22), we obtain that

$$\lim_{j\to\infty} \varrho(u_{n(j)+1}, u_{m(j)+1}) = \epsilon$$

and

$$\lim_{j\to\infty} [\varrho(u_{n(j)}, u_{m(j)}) + Km(u_{n(j)}, u_{m(j)})] = \epsilon.$$

By continuity of ϑ, we have

$$\lim_{j\to\infty} t_j = \lim_{j\to\infty} s_j = \vartheta(\epsilon) > 1.$$

From (ξ3), we have

$$1 \leq \lim_{k\to\infty} \sup \xi(t_k, s_k) < 1$$

which leads to a contradiction.

Thus, $\{u_n\}$ is a Cauchy sequence.

It follows from the completeness of U that

$$\lim_{n\to\infty} \varrho(u_n, u) = 0 \text{ for some } u \in U. \tag{24}$$

We may assume that

$$\varrho(u_{n+1}, u) < \varrho(u_n, u) \ \forall n \in \mathbb{N}. \tag{25}$$

We infer that

$$m(u_n, u) = \min\{\varrho(u_n, u_{n+1}), \varrho(u, u_{n+1})\} \ \forall n \in \mathbb{N}. \tag{26}$$

From (25) and (26), we show that $\forall n \in \mathbb{N}$

$$\varphi(m(u_n, u), \varrho(u_n, u)) \leq \frac{1}{2} m(u_n, u) - \varrho(u_n, u) < 0.$$

It follows from (14) that $\forall n \in \mathbb{N}$,

$$1 \leq \xi(\vartheta(\varrho(Tu_n, Tu)), \vartheta(\varrho(u_n, u) + Km(u_n, u))) < \frac{\gamma(\vartheta((u_n, u) + Km(u_n, u)))}{\gamma(\vartheta(\varrho(Tu_n, Tu)))}$$

which implies

$$\gamma(\vartheta(\varrho(Tu_n, Tu))) < \gamma(\vartheta(\varrho(u_n, u) + Km(u_n, u))), \ \forall n \in \mathbb{N}.$$

Hence,

$$\varrho(Tu_n, Tu))) < \varrho(u_n, u) + Km(u_n, u) \ \forall n \in \mathbb{N},$$

and hence

$$\lim_{n\to\infty} \varrho(u_{n+1}, Tu) = 0. \tag{27}$$

By applying Lemma 1 with (24) and (27), we have $z = Tz$.
Now, we prove the uniqueness of the fixed points.
Let u and v be fixed points of T, such that

$$u \neq v.$$

Then, $\varrho(u, v) > 0$ and $m(w, u) = 0$. Hence, we have

$$\varphi(m(u, v), \varrho(u, v)) \leq -d(u, v) < 0.$$

Thus, from (14), we infer that

$$\begin{aligned} 1 \leq & \xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v) + Km(u, v))) \\ = & \xi(\vartheta(\varrho(u, v)), \vartheta(\varrho(u, v))) < 1. \end{aligned}$$

This is a contradiction. Thus, T possesses only one fixed point. □

The following example illustrates Theorem 2.

Example 6. *Let $U = \{0, 2\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ and let $\varrho : U \times U \to [0, \infty)$ be a map defined as follows:*

$$\varrho(u, v) = \begin{cases} 0 & (u = v), \\ 1 & (u, v \in \{0, 2\}) \text{ or } (u, v \in \{\frac{1}{n} : n \in \mathbb{N}\}), \\ v & (u \in \{0, 2\} \text{ and } v \in \{\frac{1}{n} : n \in \mathbb{N}\}), \\ u & (u \in \{\frac{1}{n} : n = 1, 2, 3, \cdots\} \text{ and } v \in \{0, 2\}). \end{cases}$$

Then, (U, ϱ) is a complete Branciari distance space (see [4]).
Let $T : U \to U$ be a map defined by

$$Tu = \begin{cases} 0 & (u = 0 \text{ or } 2), \\ \frac{1}{n+1} & (u = \frac{1}{n}, n \in \mathbb{N}). \end{cases}$$

Let $\vartheta(t) = e^t \ \forall t \in (0, \infty)$ and $K = 3$.
We show that (14) is satisfied with the $\mathcal{L}_\gamma$ simulation ξ_b, where $\xi_b(t, s) = \frac{[\gamma(s)]^k}{\gamma(t)} \ \forall t, s \in (1, \infty)$, $k = \frac{1}{2}$ and $\gamma(t) = 1 + \ln t$, $\forall t \in (1, \infty)$.
We infer that

$$\varrho(Tu, Tv) > 0 \Leftrightarrow (u = \frac{1}{n}, v = 0), (u = \frac{1}{n}, v = 2), \text{or } (u = \frac{1}{n}, v = \frac{1}{m}, n \neq m).$$

We consider the following two cases.
Case 1: Let $u = \frac{1}{n}$ and $v = 0$ (or $u = \frac{1}{n}$ and $v = 2$).

Then, we show that

$$m(u,v) = \frac{1}{n+1} \text{ and } d(u,v) = \frac{1}{n},$$

and

$$\varphi(m(u,v), \varrho(u,v)) = \varphi(\frac{1}{n+1}, \frac{1}{n}) < 0.$$

It follows from (14) that

$$\begin{aligned}
&\xi(\vartheta(\varrho(Tu,Tv)), \vartheta(\varrho(u,v) + Km(u,v))) \\
=&\xi(\vartheta(\frac{1}{n+1}), \vartheta(\frac{1}{n} + \frac{3}{n+1})) \\
=&\frac{[\gamma(\vartheta(\frac{1}{n} + \frac{3}{n+1}))]^{\frac{1}{2}}}{\gamma(\vartheta(\frac{1}{n+1}))} \\
\geq&\frac{\sqrt{1+\frac{4}{1+n}}}{1+\frac{1}{1+n}} > 1
\end{aligned}$$

because that

$$\left(\sqrt{1+\frac{4}{1+n}}\right)^2 - \left(1+\frac{1}{1+n}\right)^2 = \frac{2n+1}{(1+n)^2} > 0.$$

Case 2: Let $u = \frac{1}{n}$ and $v = \frac{1}{m}, n \neq m$.
Then, we infer that

$$m(u,v) = 1 \text{ and } \varrho(u,v) = 1,$$

and so

$$\varphi(m(u,v), \varrho(u,v)) < 0.$$

Thus, we have

$$\begin{aligned}
&\xi(\vartheta(\varrho(Tu,Tv)), \vartheta(\varrho(u,v) + Lm(u,v))) \\
=&\frac{[\gamma(\vartheta(\varrho(u,v) + Km(u,v)))]^{\frac{1}{2}}}{\vartheta(\varrho(Tu,Tv))} \\
=&\frac{[1+\ln e^4]^{\frac{1}{2}}}{1+\ln e} \\
=&\frac{\sqrt{5}}{2} > 1.
\end{aligned}$$

Hence, all assumptions of Theorem 2 hold, and T possesses only one fixed point $u = 0$.
Notice that the almost contraction condition is not satisfied. In fact, let $u = \frac{1}{n}, v = \frac{1}{n+1}$.
Then,

$$\varrho(T\frac{1}{n}, T\frac{1}{n+1}) \leq k\varrho(\frac{1}{n}, \frac{1}{n+1}) + K\varrho(\frac{1}{n+1}, T\frac{1}{n}), k \in (0,1), L \geq 0$$

so

$$\varrho(\frac{1}{n+1}, \frac{1}{n+2}) \leq k\varrho(\frac{1}{n}, \frac{1}{n+1}) + K\varrho(\frac{1}{n+1}, \frac{1}{n+1})$$

which yields

$$k \geq 1.$$

Furthermore, note that the Suzuki–Berinde-type ϑ contraction condition [27] is not satisfied. Let $\vartheta(t)$ satisfy conditions (ϑ1),(ϑ2), and (ϑ4).

For $u = \frac{1}{n}, v = \frac{1}{n+1}$*, we infer that*

$$\frac{1}{2}\varrho(\frac{1}{n}, T\frac{1}{n}) = \frac{1}{2} < \varrho(\frac{1}{n}, \frac{1}{n+1})$$

and

$$n(\frac{1}{n}, \frac{1}{n+1}) = \min\{\varrho(\frac{1}{n}, T\frac{1}{n}), \varrho(\frac{1}{n}, T\frac{1}{n+1}), \varrho(\frac{1}{n+1}, T\frac{1}{n})\} = 0.$$

If

$$\vartheta(\varrho(T\frac{1}{n}, T\frac{1}{n+1})) \leq [\vartheta(\varrho(\frac{1}{n}, \frac{1}{n+1}))]^k + Kn(\frac{1}{n}, \frac{1}{n+1}), \text{ where } k \in (0,1), K \geq 0$$

then

$$\vartheta(1) \leq [\vartheta(1)]^k < \vartheta(1)$$

which leads to a contradiction. Hence, T is not a Suzuki–Berinde-type ϑ contraction map.

The following Corollary 4 is obtained from the Theorem 2.

Corollary 4. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there are $\xi \in \mathcal{L}_\gamma$ and $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v) + Km(u, v))) \geq 1$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

By taking $\gamma(t) = t, \forall t \geq 1$ in Theorem 2 (resp. Corollary 4), we have the following Corollary 5 (resp. Corollary 6).

Corollary 5. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there are $\xi \in \mathcal{L}$ and $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(m(u, v), \varrho(u, v)) < 0 \Rightarrow \xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v) + Km(u, v))) \geq 1$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Corollary 6. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there are $\xi \in \mathcal{L}$ and $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\xi(\vartheta(\varrho(Tu, Tv)), \vartheta(\varrho(u, v) + Km(u, v))) \geq 1$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

3. Consequence

By pplying simulation functions given in Examples 1 and 2, we have some fixed point results.

The following Corollary 7 is obtained by letting $\xi = \xi_b$ in Theorem 1.

Corollary 7. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $k \in (0, 1)$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(m(u, v), \varrho(u, v)) < 0 \Rightarrow \gamma(\vartheta(\varrho(Tu, Tv)) \leq [\gamma(\vartheta(\varrho(u, v)))]^k.$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Corollary 8. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $k \in (0, 1)$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\gamma(\vartheta(\varrho(Tu, Tv)) \leq [\gamma(\vartheta(\varrho(u, v)))]^k.$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

The following Corollary 9 is obtained by taking $\xi = \xi_b$ in Theorem 2.

Corollary 9. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there are $k \in (0, 1)$ and $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(m(u, v), \varrho(u, v)) < 0 \Rightarrow \gamma(\vartheta(\varrho(Tu, Tv)) \leq [\gamma(\vartheta(\varrho(u, v) + Km(u, v)))]^k.$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Corollary 10. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there are $k \in (0, 1)$ and $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\gamma(\vartheta(\varrho(Tu, Tv))) \leq [\gamma(\vartheta(\varrho(u, v) + Km(u, v)))]^k.$$

If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Remark 6. *(1) Corollary 8 is a generalization of Theorem 2.1 of [22] and Theorem 2.1 of [27], respectively. By taking $\gamma(t) = t, \forall t \geq 1$ in Corollary 8, Corollary 8 reduces Theorem 2.1 of [22] without condition $(\theta 2)$ and $(\theta 3)$ and reduces Theorem 2.1 of [27] without condition $(\theta 2)$ and $(\theta 4)$, respectively.*
(2) *Corollary 9 is a generalization of Theorem 3.2 of [27] to Branciari distance space without condition $(\theta 2)$.*

By taking $\xi = \xi_w$ in Theorem 1, the following result is obtained.

Corollary 11. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(m(u, v), \varrho(u, v)) < 0 \Rightarrow \gamma(\vartheta(\varrho(Tu, Tv))) \leq \frac{\gamma(\vartheta(\varrho(u, v)))}{\phi(\gamma(\vartheta(\varrho(u, v))))}$$

where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing, then T possesses only one fixed point.

Corollary 12. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\gamma(\vartheta(\varrho(Tu, Tv))) \leq \frac{\gamma(\vartheta(\varrho(u, v)))}{\phi(\gamma(\vartheta(\varrho(u, v))))}$$

where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing, then T possesses only one fixed point.

By taking $\xi = \xi_w$ in Theorem 2, the following Corollary 13 is obtained.

Corollary 13. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\varphi(m(u, v), \varrho(u, v)) < 0 \Rightarrow \gamma(\vartheta(\varrho(Tu, Tv))) \leq \frac{\gamma(\vartheta(\varrho((u, v) + Lm(u, v)))}{\phi(\gamma(\vartheta(\varrho((u, v) + Km(u, v))))}$$

where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Corollary 14. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that there is $K \geq 0$, such that for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$*

$$\gamma(\vartheta(\varrho(Tu, Tv))) \leq \frac{\gamma(\vartheta(\varrho((u, v) + Lm(u, v)))}{\phi(\gamma(\vartheta(\varrho((u, v) + Km(u, v))))}$$

where ϕ is a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing and continuous, then T possesses only one fixed point.

Remark 7. *Corollary 12 is a generalization of Corollary 8 of [24]. In fact, if $\gamma(t) = t$, $\forall t \geq 1$ Corollary 12 reduces Corollary 8 of [24].*

Taking $\gamma(t) = t\ \forall t \geq 1$ and $\theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^{\alpha}})\ \forall t > 0$ in Corollary 14, the following result is obtained.

Corollary 15. *Let (U, ϱ) be a complete Branciari distance space and $T : U \to U$ be a map. Suppose that the condition holds:*

for all $u, v \in U$ with $\varrho(Tu, Tv) > 0$

$$\varphi(m(u, v), \varrho(u, v)) < 0$$

$$\Rightarrow 2 - \frac{2}{\pi}\arctan(\frac{1}{\varrho(Tu, Tv)^r}) \leq \frac{2 - \frac{2}{\pi}\arctan(\frac{1}{\varrho(u,v)^r})}{\phi(2 - \frac{2}{\pi}\arctan(\frac{1}{\varrho(u,v)^r}))}$$

where $r \in (0, 1)$ and ϕ denote a non-decreasing and lower semi-continuous self-mapping on $[1, \infty)$, satisfying $\phi^{-1}(\{1\}) = 1$.

Then, T possesses only one fixed point.

4. Conclusions

One can use $\mathcal{L}_\gamma$ simulation functions to consolidate and merge some existing fixed-point results in Branciari distance spaces. By applying $\mathcal{L}_\gamma$ simulation functions to the main theorem, one can obtain some fixed-point results. Moreover, fixed-point theorems in the paper can be derived in the setting metric spaces, and by using $\mathcal{L}_\gamma$ simulation functions, the existing fixed-point theorem in the setting metric spaces can be interpreted.

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