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On Classes of Non-Carathéodory Functions Associated with a Family of Functions Starlike in the Direction of the Real Axis

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Abstract: In this paper, we introduce a new class of analytic functions subordinated by functions which is not Carathéodory. We have obtained some interesting subordination properties, inclusion and integral representation of the defined function class. Several corollaries are presented to highlight the applications of our main results.

Keywords: univalent function; analytic function; convex function; starlike function; Bazilevič function; Fekete–Szegö problem; differential subordination

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1. Introduction

In a study related to analytic functions starlike in one direction, Robertson in [1] defined the following integral

$$g(z)=\int_0^z \frac{1+ie^{-i\alpha}\sin\alpha[p(z)-1]}{1-2\cos\beta z e^{-i\alpha}+e^{-2i\alpha}z^2}\,dz \tag{1}$$

and established that $g(z)$ is univalent in $|z|<1$ if α, β are in $[0,\pi]$ and $\operatorname{Re} p(z)\geq 0$. Here, in this paper, we study the geometrical implications of the integrand defined in (1) and its applications to certain class of analytic functions defined in the unit disc. Let $\mathcal{A}$ denote the class of functions analytic in the unit disc $\mathcal{U}=\{z:\ |z|<1\}$ and having an expansion of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{2}$$

In addition, let $\mathcal{NP}$ denote the class of functions that are analytic in the unit disc and equals 1 at $z=0$. We call $\mathcal{P}$ the class of functions $p\in\mathcal{NP}$ which satisfies $\operatorname{Re}(p(z))>0$, $z\in\mathcal{U}$.

Very well-known subclasses of $\mathcal{A}$ are the so-called family of starlike and convex functions, which we denote here by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. Using the principal of subordination [2], Ma and Minda [3] defined the classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ as follows.

$$\mathcal{S}^*(\psi)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\psi(z)\right\}\quad\text{and}\quad\mathcal{C}(\psi)=\left\{f\in\mathcal{A}:1+\frac{zf''(z)}{f'(z)}\prec\psi(z)\right\},$$

where $\psi(z)\in\mathcal{P}$ maps $\mathcal{U}$ onto a starlike region with respect to 1 with $\psi'(0)>0$ and symmetric with respect to the real axis. The classes $\mathcal{S}^*(\psi)$ and $\mathcal{C}(\psi)$ consolidated the study of several generalizations of starlike and convex functions. Setting ψ to be a conic region, several authors studied the classes of analytic functions associated with the conic regions. Most popular among those studies are $\mathcal{S}^*(\sqrt{1+z})$ defined by Sokół [4] and followed by $\mathcal{S}^*(z+\sqrt{1+z^2})$ defined by Raina and Sokół [5]. For studies related to the conic region, refer to [6–9] and references provided therein.

Convex and Starlike in One Direction

A domain $\mathbb{D}$ is convex in the direction of the line L if each line parallel to L either misses $\mathbb{D}$, or is contained entirely in $\mathbb{D}$, or intersection with $\mathbb{D}$ is either a segment or a ray. Note that such a domain need not be convex or starlike with respect to any point. A function $f \in \mathcal{A}$ is said to be convex in the direction of the line L if it maps the unit disc onto a domain which is convex in the direction of the line L. Here, we denote such a set of functions as $\mathcal{CV}(r)$, if L is the real axis. Similarly, $\mathcal{ST}(r)$ denotes the class of functions starlike in the direction of the real axis, refer to [1] for its formal definition.

Now, we define the function

$$\Lambda[\alpha,\, \beta;\, p(z)] = \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\, p(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha} z^2}, \tag{3}$$

with $\alpha, \beta \in [0, \pi]$ and $p(z) \in \mathcal{P}$. The function $\Lambda[\alpha, \beta; p(z)]$ is related to the class of functions starlike with respect to the real axis (see page 210 in [10]). To be precise, the function $f(z) \in \mathcal{A}$ is said to be in $\mathcal{ST}(r)$ if and only if there is a $\alpha, \beta \in [0, \pi]$ and $p(z) \in \mathcal{P}$ such that

$$\frac{f(z)}{z} = \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\, p(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha} z^2}.$$

Now, let $p(z) = 1 + z/1 - z$ in (3), it can be seen that $\left\{\Lambda\left[\alpha,\, \beta;\, \frac{1+z}{1-z}\right]\right\}_{z=0} = 1$ but $\mathrm{Re}\left\{\Lambda\left[\alpha,\, \beta;\, \frac{1+z}{1-z}\right]\right\} \not> 0$ (see Figure 1). Hence, we observe that in general function $\Lambda[\alpha, \beta; p(z)]$ does not belong to class $\mathcal{P}$, but belongs to $\mathcal{NP}$. Further, to illustrate the fact that impact of $\Lambda[\alpha, \beta; p(z)]$ is not same on all conic region. We let $p(z) = z + \sqrt{1+z^2}$ in (3), then the function $\Lambda\left[\alpha,\, \beta;\, z + \sqrt{1+z^2}\right]$ is convex univalent in $\mathcal{U}$. However, $\mathrm{Re}\,\Lambda\left[\alpha,\, \beta;\, z + \sqrt{1+z^2}\right] \not> 0$ $(z \in \mathcal{U})$, so the function $\Lambda\left[\alpha,\, \beta;\, z + \sqrt{1+z^2}\right]$ which is convex in $\mathcal{U}$ does not belong to $\mathcal{P}$. However, the function $\Lambda\left[\alpha,\, \beta;\, z + \sqrt{1+z^2}\right]$ will be convex and in $\mathcal{P}$ if $|z| < 0.7$ (see Figure 2). From Figures 1 and 2, we can see that $\Lambda[\alpha, \beta; p(z)] \in \mathcal{NP}$ and maps the unit disc onto a domain which is symmetric with respect to the real axis irrespective of the choice of $p(z)$.

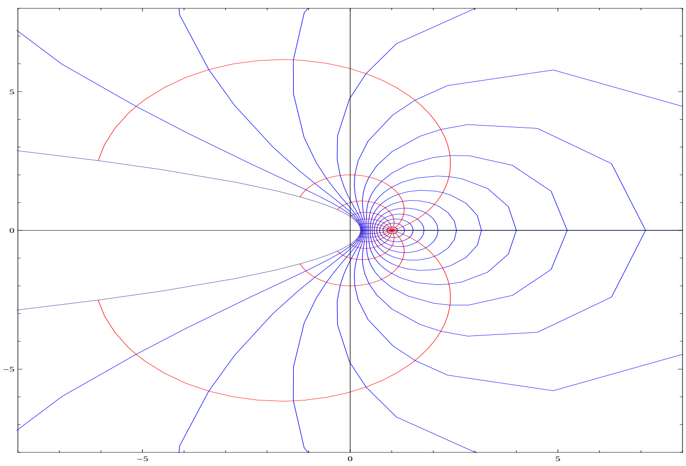

Figure 1. Mapping of the unit disc under $\Lambda\left[\frac{\pi}{2},\, \frac{\pi}{2};\, p(z)\right]$ if $p(z) = 1 + z/1 - z$.

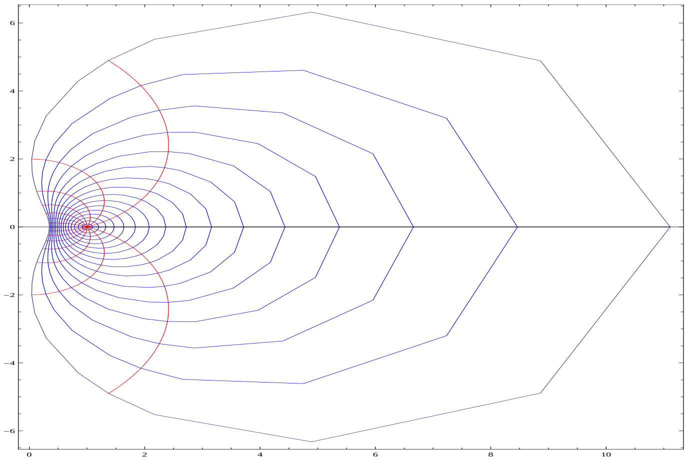

Figure 2. Mapping of $|z| < 0.7$ under $\Lambda\left[\frac{\pi}{2}, \frac{\pi}{2}; p(z)\right]$ if $p(z) = 1 + z/1 - z$.

Motivated by [11–17], we now define a generalized class of Bazilevič functions.

Definition 1. *For* $0 \le \alpha, \beta < \pi$, $\nu \ge 0$ *and* $\gamma \in \mathbb{C}$ *such that* $\mathrm{Re}(\gamma) > 0$, *a function* f *belongs to the class* $\mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$ *if it satisfies*

$$\left\{(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma f'(z)\left(\frac{f(z)}{z}\right)^{\nu-1}\right\} \prec \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha}z^2}, \tag{4}$$

where $\psi(z) \in \mathcal{NP}$ *has a power series representation of the*

$$\psi(z) = 1 + R_1 z + R_2 z^2 + R_3 z^3 \cdots. \tag{5}$$

Setting $\alpha = \beta = 0$ and $p(z) = 1 - z^2$ in Definition 1, we get

$$\mathcal{MS}^{\nu}(0, 0; \gamma; 1 - z^2) = \left\{f \in \mathcal{A} :\ \mathrm{Re}\left[(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\nu}\right] > 0\right\}.$$

For different choices of the parameters, the class $\mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$ reduces to those classes which have been studied in [18–21]. In particular $\mathcal{MS}^{\nu}(0, 0; 1; 1 - z^2)$ is the well-known class of Bazilevič functions. For other studies closely related to this present study, refer to [22–24].

2. Inclusion Relations and Integral Representations

Now, we state some results which we use to establish our main results.

Lemma 1 ([25])**.** *Let* g *be convex in* $\mathcal{U}$, *with* $g(0) = a$, $\gamma \ne 0$ *and* $\mathrm{Re}(\gamma) > 0$. *Suppose that* $\vartheta(z)$ *is analytic* $\mathcal{U}$, *which is given by*

$$\vartheta(z) = a + \vartheta_n z^n + \vartheta_{n+1} z^{n+1} + \cdots, \quad z \in \mathcal{U}. \tag{6}$$

If

$$\vartheta(z) + \frac{z\vartheta'(z)}{\delta} \prec g(z),$$

then

$$\vartheta(z) \prec q(z) \prec g(z),$$

where

$$q(z) = \frac{\delta}{n\, z^{\delta/n}} \int_0^z g(t)\, t^{(\delta/n)-1} dt.$$

The function q *is convex and is the best* (a, n)*-dominant.*

In order to further broaden our study, we drop the necessity of $p(z)$ in (3) to satisfy the condition $\operatorname{Re} p(z) > 0$. So, hereafter, throughout this paper, we denote

$$\Lambda[\alpha,\ \beta;\ \psi(z)] = \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha}z^2}, \tag{7}$$

where $\psi(z) \in \mathcal{NP}$ is defined as in (5).

Theorem 1. *Let the function $\Lambda[\alpha,\ \beta;\ \psi(z)]$ defined as in (7) be convex univalent in $\mathbb{U}$. Let $f \in \mathcal{MS}^{\nu}(\alpha,\ \beta;\ \gamma;\ \psi(z))$ with $\operatorname{Re}(\gamma) > 0$ and $\nu \neq 0$, then*

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec q(z) = \frac{\nu}{\gamma} z^{\frac{-\nu}{\gamma}} \int_0^z t^{\frac{\nu}{\gamma}-1}\left(\frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(t)]}{1 - 2\cos\beta t e^{-i\alpha} + e^{-2i\alpha}t^2}\right)dt \prec \Lambda[\alpha,\ \beta;\ \psi(z)]. \tag{8}$$

and $q(z)$ is the best dominant.

Proof. Let $h(z)$ be defined by

$$h(z) = \left(\frac{f(z)}{z}\right)^{\nu}, \quad z \in \mathcal{U}. \tag{9}$$

Then the function $h(z)$ is of the form $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ and is analytic in $\mathcal{U}$. Differentiating both sides of (9) and by simplifying, we have

$$(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma f'(z)\left(\frac{f(z)}{z}\right)^{\nu-1} = h(z) + \frac{\gamma}{\nu} z h'(z). \tag{10}$$

By hypothesis $f \in \mathcal{MS}^{\nu}(\alpha,\ \beta;\ \gamma;\ \psi(z))$, so from Definition 1, we have

$$h(z) + \frac{\gamma}{\nu} z h'(z) \prec \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha}z^2}.$$

Applying Lemma 1 to (10) with $\delta = \frac{\nu}{\gamma}$ and $n = 1$, we get

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec \frac{\nu}{\gamma} z^{\frac{-\nu}{\gamma}} \int_0^z t^{\frac{\nu}{\gamma}-1}\left(\frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(t)]}{1 - 2\cos\beta t e^{-i\alpha} + e^{-2i\alpha}t^2}\right)dt \prec \Lambda[\alpha,\ \beta;\ \psi(z)]. \tag{11}$$

Hence, the proof of the Theorem 1 □

Remark 1. *From (10), it can be easily seen that if $\gamma = 0$, we can get*

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec \frac{e^{-i\alpha}[\cos\alpha + i\sin\alpha\,\psi(z)]}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha}z^2}.$$

Corollary 1. *Let $f \in \mathcal{MS}^{\nu}(\frac{\pi}{2},\ 0;\ \gamma;\ 1+z^2)$ with $\operatorname{Re}(\gamma) > 0$, then for $\nu \neq 0$, we have*

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec q(z) = \frac{\nu}{\gamma} z^{\frac{-\nu}{\gamma}} \int_0^z t^{\frac{\nu}{\gamma}-1}\left(\frac{1-it}{1+it}\right)dt \prec \frac{1+z}{1-z}.$$

and $q(z)$ is the best dominant.

Proof. Let $\psi(z) = 1 + z^2$ in (7). Since $\psi(z) = 1 + z^2$ maps unit disc onto convex domain in the right half plane, the choice of $p(z) = 1 + z^2$ is admissible as per the Definition 1. Replacing $\alpha = \frac{\pi}{2}$, $\beta = 0$ and $\psi(z) = 1 + z^2$ in (7), we get

$$\Lambda\left(\frac{\pi}{2},\ 0,\ 1+z^2\right) = \frac{1+z^2}{1+2iz-z^2} = \frac{1-iz}{1+iz}.$$

Clearly,the function $\Lambda\left(\frac{\pi}{2}, 0, 1+z^2\right)$ maps the unit disc on to convex region which is symmetric with respect to the real axis (see Figure 3).

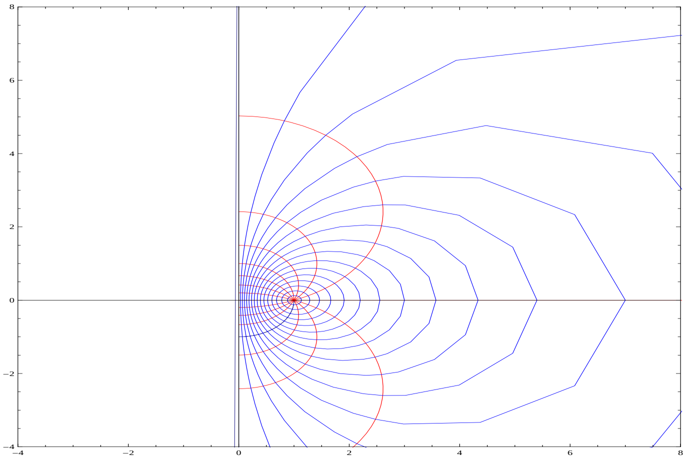

Figure 3. Mapping of the unit disc under $\Lambda\left(\frac{\pi}{2}, 0, \psi(z)\right)$ if $\psi(z) = 1 + z^2$.

On replacing the superordinate function in Theorem 1, we get the desired result. $\square$

Corollary 2. *Let* $f \in \mathcal{MS}^{\nu}\left(\frac{\pi}{2}, \frac{\pi}{2}; \gamma; (1+z)^2\right)$ *with* $\mathrm{Re}(\gamma) > 0$ *, then for* $\nu \neq 0$*, we have*

$$\left(\frac{f(z)}{z}\right)^{\nu} \prec q(z) = \frac{\nu}{\gamma} z^{\frac{-\nu}{\gamma}} \int_0^z t^{\frac{\nu}{\gamma}-1}\left(\frac{1+t}{1-t}\right) dt \prec \frac{1+z}{1-z}.$$

and $q(z)$ *is the best dominant.*

Remark 2. *Notice that* $\psi(z) = (1+z)^2$ *in the Corollary 2 does not belong to* $\mathcal{P}$ *(see Figure 4). However,* $\psi(z) = (1+z)^2$ *is admissible as per the definition of the function class* $\mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$*, as* $\psi(0) = 1$ *and* $\psi(z) \in \mathcal{NP}$ *.*

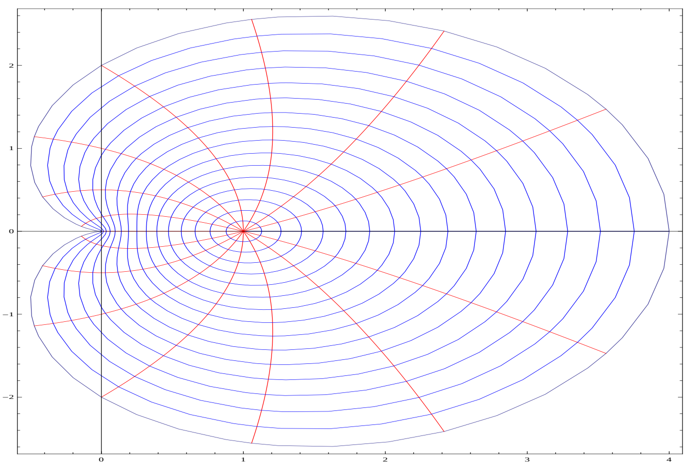

Figure 4. Mapping of the unit disc under $\Lambda\left(\frac{\pi}{2}, \frac{\pi}{2}, \psi(z)\right)$ if $\psi(z) = (1+z)^2$.

If we let $\gamma = 1 = \nu$ in Corollary 1, we get

Corollary 3. *Let* $f \in \mathcal{MS}^{\nu}\left(\frac{\pi}{2}, 0; 1; 1+z^2\right)$ *with* $\mathrm{Re}(\gamma) > 0$*, then*

$$f(z) \prec q(z) = -z - i\log(1+z^2).$$

and $q(z)$ *is the best dominant.*

If we let $\gamma = 1 = \nu$ in Corollary 2, we get

Corollary 4. *Let* $f \in \mathcal{MS}^{\nu}(\frac{\pi}{2}, \frac{\pi}{2}; 1; (1+z)^2)$ *with* $\mathrm{Re}(\gamma) > 0$, *then*

$$f(z) \prec q(z) = -z - 2\log(1-z).$$

and $q(z)$ *is the best dominant.*

As a consequence of Theorem 1, we have the following integral representation of the class $\mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$.

Theorem 2. *Let the function* $\Lambda[\alpha, \beta; \psi(z)]$ *defined as in* (7) *be convex in* $\mathcal{U}$. *Let* $f \in \mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$ *with* $0 \le \gamma \le 1$, *then for* $\nu \neq 0$, *we have*

(i) *for* $0 < \gamma \le 1$,

$$f(z) = \left\{ \frac{\nu}{\gamma} z^{-\nu\left(\frac{1}{\gamma}-1\right)} \int_0^z t^{\frac{\nu}{\gamma}-1} \left(\frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(t)])}{1 - 2\cos\beta w(t) e^{-i\alpha} + e^{-2i\alpha}[w(t)]^2} \right) dt \right\}^{\frac{1}{\nu}},$$

(ii) *for* $\gamma = 0$,

$$f(z) = z \left\{ \frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(z)])}{1 - 2\cos\beta w(z) e^{-i\alpha} + e^{-2i\alpha}[w(z)]^2} \right\}^{\frac{1}{\nu}},$$

where w *is analytic in* $\mathcal{U}$ *with* $w(0) = 0$ *and* $|w(z)| < 1$.

Remark 3. *Theorems 1 and 2 are not valid for* $\nu = 0$. *Let us suppose that* $\nu = 0$, *then* (4) *can be equivalently written as*

$$\frac{d}{dz} \log\left[\frac{f(z)}{z}\right] = \frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(z)])}{\gamma z[1 - 2\cos\beta w(z) e^{-i\alpha} + e^{-2i\alpha}[w(z)]^2]} - \frac{1-2\gamma}{\gamma z}.$$

Integrating the above expression, we get

$$f(z) = z\, \exp\left\{ \int_0^z \left(\frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(t)])}{\gamma t[1 - 2\cos\beta w(t) e^{-i\alpha} + e^{-2i\alpha}[w(t)]^2]} - \frac{1-2\gamma}{\gamma t} \right) dt \right\} \quad (\gamma \neq 0).$$

Unlike in Theorem 1, $\Lambda[\alpha, \beta; \psi(z)]$ *needs not be convex if* $\nu = 0$.

3. Initial Coefficients' Bounds

The Fekete–Szegö problem possesses various geometric quantities which are helpful in establishing univalence and norm estimates. Most of all recent papers establish the Fekete–Szegö inequalities for the defined function classes.

We need the following well-known coefficient estimates for functions belonging to the class $\mathcal{P}$.

Lemma 2 ([3])**.** *Let* $p \in \mathcal{P}$ *and also let* v *be a complex number, then*

$$|p_2 - v p_1^2| \le 2\,\max\{1, |2v - 1|\}. \tag{12}$$

The result is sharp for functions given by

$$p(z) = p_2(z) = \frac{1+z^2}{1-z^2}, \qquad p(z) = p_1(z) = \frac{1+z}{1-z}.$$

Lemma 3 ([26])**.** *If* $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, *then* $|p_k| \le 2$ *for all* $k \ge 1$, *and the inequality is sharp for* $p(z) = p_1(z) = \frac{1+z}{1-z}$.

Theorem 3. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{MS}^{\nu}(\alpha,\, \beta;\, \gamma;\, \psi(z))$ for $z \in \mathcal{U}$. Also, let $\alpha,\, \beta \in [0,\, \pi]$ and $p(z) \in \mathcal{NP}$ satisfy the condition for all $z \in \mathcal{U}$*

$$\left|\mathrm{Im}\left(\frac{i\sin\alpha\, zp'(z)}{e^{-i\alpha}[\cos\alpha + i\sin\alpha\, p(z)]} - \frac{e^{-2i\alpha}z^2 - 2\cos\beta z e^{-i\alpha}}{1 - 2\cos\beta z e^{-i\alpha} + e^{-2i\alpha}z^2}\right)\right| < 1. \tag{13}$$

Then, the bounds of the initial coefficients of f are given by

$$|a_2| \le \frac{\sqrt{4\cos^2\beta + \sin^2\alpha R_1^2}}{|\nu+\gamma|} \tag{14}$$

and

$$|a_3| \le \frac{2\cos\beta}{|\nu+2\gamma|}\max\left\{1,\, \left|\frac{\sec\beta - 4\cos\beta}{2}\right|\right\} + \frac{\sin\alpha|R_1|}{|\nu+2\gamma|}$$
$$\max\left\{1,\, \left|\frac{R_2}{R_1} + 2e^{-i\alpha}\cos\beta - \frac{(\nu-1)e^{-i\alpha}(2\cos\beta + i\sin\alpha\, R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2 R_1}\right|\right\}. \tag{15}$$

Further, the Fekete–Szegö inequality for $\mu \in \mathbb{C}$ is given by

$$\left|a_3 - \mu a_2^2\right| \le \frac{2\cos\beta}{|\nu+2\gamma|}\max\left\{1,\, \left|\frac{e^{-i\alpha}\sec\beta - 4e^{-i\alpha}\cos\beta}{2}\right|\right\} + \frac{\sin\alpha|R_1|}{|\nu+2\gamma|}$$
$$\max\left\{1,\, \left|\frac{R_2}{R_1} + 2e^{-i\alpha}\cos\beta - \frac{(\nu+2\mu-1)e^{-i\alpha}(2\cos\beta + i\sin\alpha\, R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2 R_1}\right|\right\}.$$

Proof. The function $\Lambda[\alpha,\, \beta;\, \psi(z)]$ defined in (7) belongs to $\mathcal{NP}$. The hypothesis (13) is equivalent to $\left|\mathrm{Im}\left(\frac{z\Lambda'[\alpha,\beta;\psi(z)]}{\Lambda[\alpha,\beta;\psi(z)]}\right)\right| < 1$, which implies the function $\Lambda[\alpha,\, \beta;\, \psi(z)] \in \mathcal{P}$ (see Theorem 2 in [27]). Now, $f \in \mathcal{MS}^{\nu}(\alpha,\, \beta;\, \gamma;\, \psi(z))$ $(z \in \mathcal{U})$ implies that there is a Schwarz function $w(z)$ such that

$$\left\{(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu} + \gamma f'(z)\left(\frac{f(z)}{z}\right)^{\nu-1}\right\} = \frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(z)])}{1 - 2\cos\beta w(z)e^{-i\alpha} + e^{-2i\alpha}[w(z)]^2}. \tag{16}$$

Define the function $h(z)$ by

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots = \frac{1+w(z)}{1-w(z)} \prec \frac{1+z}{1-z}, \quad z \in \mathcal{U}. \tag{17}$$

We can note that $h(0) = 1$ and $h \in \mathcal{P}$ (see Lemma 3). Using (17), it is easy to see that

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{1}{2}\left[h_1 z + \left(h_2 - \frac{h_1^2}{2}\right)z^2 + \left(h_3 - h_1 h_2 + \frac{h_1^3}{4}\right)z^3 + \cdots\right].$$

On applying the above expression in (16), after a long and tedious computation, we get

$$\frac{e^{-i\alpha}(\cos\alpha + i\sin\alpha\, \psi[w(z)])}{1 - 2\cos\beta w(z)e^{-i\alpha} + e^{-2i\alpha}[w(z)]^2} = 1 + \frac{e^{-i\alpha}\, h_1}{2}(2\cos\beta + i\sin\alpha\, R_1)z$$
$$+\left\{e^{-i\alpha}\cos\beta\left[h_2 - \frac{h_1^2}{4}\left(e^{-i\alpha}\sec\beta - 4e^{-i\alpha}\cos\beta + 2\right)\right]\right.$$
$$\left. + \frac{ie^{-i\alpha}\sin\alpha R_1}{2}\left[h_2 - \frac{h_1^2}{2}\left(1 - \frac{R_2}{R_1} - 2e^{-i\alpha}\cos\beta\right)\right]\right\}z^2 + \cdots. \tag{18}$$

The left-hand side of (16) is equivalent to

$$\left\{(1-\gamma)\left(\frac{f(z)}{z}\right)^{\nu}+\gamma f'(z)\left(\frac{f(z)}{z}\right)^{\nu-1}\right\}$$
$$=1+(\nu+\gamma)a_2z+(\nu+2\gamma)\left[a_3+\frac{(\nu-1)a_2^2}{2}\right]z^2+\cdots. \tag{19}$$

From (18) and (19), we have

$$a_2=\frac{e^{-i\alpha}\,h_1}{2(\nu+\gamma)}(2\cos\beta+i\sin\alpha\,R_1) \tag{20}$$

and

$$a_3=\frac{e^{-i\alpha}\cos\beta}{(\nu+2\gamma)}\left[h_2-\frac{h_1^2}{4}\left(e^{-i\alpha}\sec\beta-4e^{-i\alpha}\cos\beta+2\right)\right]+\frac{ie^{-i\alpha}\sin\alpha R_1}{2(\nu+2\gamma)}$$
$$\left[h_2-\frac{h_1^2}{2}\left(1-\frac{R_2}{R_1}-2e^{-i\alpha}\cos\beta+\frac{(\nu-1)e^{-i\alpha}(2\cos\beta+i\sin\alpha\,R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2R_1}\right)\right] \tag{21}$$

Hence, applying Lemma 3 in (20), we get (14). To obtain (15), we apply Lemma 2 in (21).

In view of the Equations (20) and (21), for $\mu\in\mathbb{C}$, we have

$$\left|a_3-\mu a_2^2\right|=\left|\frac{e^{-i\alpha}\cos\beta}{(\nu+2\gamma)}\left[h_2-\frac{h_1^2}{4}\left(e^{-i\alpha}\sec\beta-4e^{-i\alpha}\cos\beta+2\right)\right]+\frac{ie^{-i\alpha}\sin\alpha R_1}{2(\nu+2\gamma)}\right.$$
$$\left[h_2-\frac{h_1^2}{2}\left(1-\frac{R_2}{R_1}-2e^{-i\alpha}\cos\beta+\frac{(\nu-1)e^{-i\alpha}(2\cos\beta+i\sin\alpha\,R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2R_1}\right)\right]$$
$$\left.-\frac{\mu e^{-2i\alpha}\,h_1^2(2\cos\beta+i\sin\alpha\,R_1)^2}{4(\nu+\gamma)^2}\right|$$

$$=\left|\frac{e^{-i\alpha}\cos\beta}{(\nu+2\gamma)}\left[h_2-\frac{h_1^2}{4}\left(e^{-i\alpha}\sec\beta-4e^{-i\alpha}\cos\beta+2\right)\right]+\frac{ie^{-i\alpha}\sin\alpha R_1}{2(\nu+2\gamma)}\left[h_2-\frac{h_1^2}{2}\right.\right.$$
$$\left.\left.\left(1-\frac{R_2}{R_1}-2e^{-i\alpha}\cos\beta+\frac{(\nu+2\mu-1)e^{-i\alpha}(2\cos\beta+i\sin\alpha\,R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2R_1}\right)\right]\right|$$

$$\leq\frac{2|e^{-i\alpha}\cos\beta|}{|\nu+2\gamma|}\max\left\{1,\left|\frac{e^{-i\alpha}\sec\beta-4e^{-i\alpha}\cos\beta}{2}\right|\right\}+\frac{|ie^{-i\alpha}\sin\alpha R_1|}{|\nu+2\gamma|}$$
$$\max\left\{1,\left|\frac{R_2}{R_1}+2e^{-i\alpha}\cos\beta-\frac{(\nu+2\mu-1)e^{-i\alpha}(2\cos\beta+i\sin\alpha\,R_1)^2(\nu+2\gamma)}{2i\sin\alpha(\nu+\gamma)^2R_1}\right|\right\}. \tag{22}$$

On simplifying (22), we get (16). Hence, the proof of Theorem 3 is completed. □

Letting $\nu=0$, $\gamma=1$, $\alpha=\frac{\pi}{2}=\beta$ and $\psi(z)=(1+z)^2$ in Theorem 3, we get

Corollary 5. *If $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{A}$ satisfy the inequality*

$$\frac{zf'(z)}{f(z)}\prec\frac{1+z}{1-z}.$$

Then, the bounds of the initial coefficients of f are given by

$$|a_2| \leq 2, \qquad |a_3| \leq 3.$$

and the Fekete–Szegö inequality for $\mu \in \mathbb{C}$ is given by

$$\left|a_3 - \mu a_2^2\right| \leq \max\{1, |4\mu - 3|\}.$$

4. Conclusions

The main derivation we have provided here is that a certain differential characterization subordinate to a function which is not Carathéodory. Apart from the function being not Carathéodory, it was challenging as it involved a long computation when it came to find the coefficient estimate. Further, we have discussed some geometrical and analytic properties of the function $\Lambda[\alpha, \beta; \psi(z)]$ in detail. However, in the defined function class, the left-hand side of differential characterization in (4) is closely related to the well-known studies conducted by various authors (see [11,28,29]). Some subordination properties and initial coefficient estimates are our main results.

The further scope of this study is that it can be extended by taking special functions such as exponential function, Legendre polynomial, q-Hermite polynomial, Chebyshev polynomial, or Fibonacci sequence instead of considering $\psi(z)$ as in (4). We also note that the extremal function in the defined function class $\mathcal{MS}^{\nu}(\alpha, \beta; \gamma; \psi(z))$ could not be established here.

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