axioms

Article

Fixed Point Theorems for Generalized Classes of Operators

Cristiana Ionescu

Department of Mathematics and Informatics, University Politehnica of Bucharest, Splaiul Independenţei 313, 060042 Bucharest, Romania; cristianaionescu58@yahoo.com or cristiana.ionescu@upb.ro

Abstract: In this work, we consider weakly generalized operators, which extend the Geraghty mappings that are studied with regard to the existence and uniqueness of their fixed points, in the setting offered by strong b-metric spaces. Classic results are obtained as corollaries. An example is provided to support these outcomes.

Keywords: generalized metric; generalized class of contractive operators; fixed points

MSC: 47H10; 54H25

1. Introduction

Fixed point theory has been developing extensively, since the ground breaking contraction principle [1], due to the fact that a significant number of problems may be reduced to the determination of such a point associated to adequate operators. From that point onward, scientists focused on generalizations of this result. One of the directions of approach has been related to the class of mappings involved in the fixed point problem. Kannan [2] changed the inequality of Banach so that the mappings in view need not to be continuous anymore—idea followed by Chatterjea [3]. Hardy and Rogers [4] referred to a linear contractive condition, which generalizes the ones previously mentioned. Ćirić [5] introduced the contractive type mappings and related them to the orbitally completness. Berinde [6] replaced the contractive term from the Banach contraction by a sum which involves it and an additional quantity. Another interesting generalization in this direction is due to Geraghty [7], who employed appropriate functions in order to define contractive operators with the fixed point property. Ansari et al. [8] combined further various types of mappings by the use of the operators of class C. Suzuki [9] introduced a new generalized type of contractive condition, and provided conditions related to the domain of definition of the mappings involved, ensuring the existence of their fixed points. These results were even more generalized by García-Falset et al. [10] to the operators with condition (E). Another fruitful idea of developing fixed point results belonged to Wardowski [11], who introduced implicit contractive conditions in this regard.

Another direction of extensions refers to the underlying space used to prove fixed point results. Generalizations had in view of dropping one or more conditions from the metric definition, or enlarge the triangle inequality. One extension in this direction is the b-metric, whose early developments in fixed point theory were presented by Berinde and Păcurar [12]. Shatanawi et al. [13] used comparison functions to prove common fixed point results on b-metric spaces. Amini-Hanandi [14] used this context to develop a theory for quasicontractive operators. Dung and Hang [15] studied features of the Caristi theorem in this background. The same setting was also used by Kamran et al. [16] to develop Feng and Liu type F-contractions, or by Ali et al. [17] to solve integral equations. Real world applications in this context were given by McConnell et al. [18]. Interesting surveys on b-metrics can be found in Berinde and Păcurar [12] or Karapınar [19], also some interesting generalizations can be found in Samreen et al. [20]. The lack of continuity of b-metrics imposed the introduction of strong b-metric spaces by Kirk and Shahzad [21],

Citation: Ionescu, C. Fixed Point Theorems for Generalized Classes of Operators. *Axioms* **2023**, *12*, 69. https://doi.org/10.3390/axioms12010069

Academic Editors: Hari Mohan Srivastava and Adrian Petrusel

Received: 18 December 2022
Revised: 3 January 2023
Accepted: 6 January 2023
Published: 9 January 2023

which allowed for obtaining stronger results. A useful comparison of these metric spaces with other generalized metrics was made by Cobzaş and Czerwik [22].

The paper is organized as follows. Section 2 presents some preliminary aspects used in the sequel. Section 3 contains theorems on the existence of fixed points for weakly contractive contractions. Section 4 presents some uniqueness results on the fixed points of operators introduced in the previous section.

2. Preliminary Issues

Bakhtin [23] and Czerwik [24,25] relaxed the triangle inequality from the metric spaces, moving the focus towards the next definition.

Definition 1. *Let S be a nonempty set and $s \geq 1$. A function $\delta\colon S \times S \to [0,\infty)$ is called b-metric if the following hypotheses are fulfilled.*

(i) *$x = y$ if and only if $\delta(x,y) = 0$;*
(ii) *δ is symmetric;*
(iii) *if $x, y, z \in S$, then*

$$\delta(x,z) \leq s(\delta(x,y) + \delta(y,z)).$$

The pair (S,δ) is b-metric space.

The problem of this generalization is the lack of continuity of the distance function, as proved by Husain et al. [26]. Still, in such spaces the uniqueness of limits of convergent sequences is satisfied. A method to overcome this difficulty is given by the next class of metric spaces.

Definition 2 ([21])**.** *Let S be a nonempty set and $s \geq 1$. A function $\delta\colon S \times S \to [0,\infty)$ is called strong b-metric if the following hypotheses are fulfilled.*

(i) *$x = y$ if and only if $\delta(x,y) = 0$;*
(ii) *if $x, y \in S$, $\delta(x,y) = \delta(y,x)$;*
(iii) *if $x, y, z \in S$, then*

$$\delta(x,z) \leq \delta(x,y) + s\delta(y,z).$$

The pair (S,δ) is a strong b-metric space.

The class of metric spaces is strictly included into this one. In this regard, we give the next example inspired by Doan [27].

Example 1. *Let $S = \{0,1,2,3\}$, and*

$$\delta\colon S \times S \to [0,\infty), \quad \delta(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } (x,y) = (0,1) \text{ or } (x,y) = (1,0), \\ 6, & \text{if } (x,y) = (0,2) \text{ or } (x,y) = (2,0), \\ 3, & \text{otherwise}. \end{cases}$$

It can be observed that

$$\delta(0,2) \not\leq \delta(0,1) + \delta(1,2),$$

But all the properties of strong b-metrics are satisfied.

Note that, throughout the paper, (S,δ) designates a strong b-metric space with parameter $s \geq 1$.

For the convergence, Cauchy property and completeness, we have the following definitions.

Definition 3. *Let $\{x_n\} \subseteq S$ be a sequence and $x \in S$. Then:*

(i) *The sequence $\{x_n\}$ is called convergent to x if $\lim\limits_{n\to\infty} \delta(x_n, x) = 0$;*
(ii) *$\{x_n\}$ is called Cauchy sequence in S if $\lim\limits_{n,m\to\infty} \delta(x_n, x_m) = 0$;*

(iii) *The strong b-metric space* (S,δ) *satisfies the completness condition if every Cauchy sequence in S is convergent.*

Proposition 1 ([21]). *Let* $\{x_n\} \subseteq S$. *Then*
(i) *The limit of a convergent sequence in a strong b-metric space is unique.*
(ii) *The strong b-metric is a continuous function.*

Note that the last property means that strong b-metrics are continuous functions, while b-metrics are not. This fact allows us to obtain improved results for the case of strong b-metrics.
Another property of this class of metrics is the next.

Proposition 2 ([21]). *Let* $\{x_n\}$ *be a sequence in a strong b-metric space and suppose*

$$\sum_{n=1}^{\infty} \delta(x_{n+1}, x_n) < \infty.$$

Then, $\{x_n\}$ *is a Cauchy sequence.*

The contractive inequalities we will work with have at the base, the Geraghty mappings.

Definition 4 ([7]). *A function* $\beta\colon [0,\infty) \to [0,1)$, *which satisfies the condition*

$$\lim_{n\to\infty} \beta(t_n) = 1 \textit{ implies } \lim_{n\to\infty} t_n = 0,$$

is called the Geraghty function.

We denote by $\mathcal{F}$ the set of Geraghty functions.
As examples of Geraghty functions, we refer to the following ones.

Example 2. (1) $\beta\colon [0,\infty) \to [0,1)$, $\beta(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{\sin x}{x}, & \text{if } x > 0, \end{cases}$ *satisfies the conditions of Geraghty.*

(2) *The same conclusion is for* $\beta\colon [0,\infty) \to [0,1)$, $\beta(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{x}{e^x - 1}, & \text{if } x > 0, \end{cases}$

(3) *The function* $\beta\colon [0,\infty) \to [0,1)$, $\beta(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{\ln(1+x)}{x}, & \text{if } x > 0, \end{cases}$ *is a Geraghty operator.*

(4) *The function* $\beta\colon [0,\infty) \to [0,1)$, $\beta(x) = \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{1}{1+x}, & \text{if } x > 0, \end{cases}$ *is a Geraghty operator.*

3. Existence Results

We are now ready to formulate our main results, using Geraghty functions and mappings inspired by works of Samet [28], and Shatanawi and Pitea [29].

Theorem 1. *Let* $G\colon S \to S$ *be a mapping. Suppose that the following conditions are satisfied:*
(1) $\beta \in \mathcal{F}$ *is a Geraghty function,*
(2) *For all* $x, y \in S$, *the next relation is true*

$$\delta(Gy, Gx) \le \beta(\delta(y,x))\delta(y,x) + \Theta(\delta(y,Gy), \delta(y,Gx), \delta(x,Gy), \delta(x,Gx)),$$

where $\Theta\colon [0,\infty)^4 \to \mathbb{R}$ *is continuous, such that* $\Theta(0,y,x,z) = 0$, $\Theta(y,x,0,z) = 0$, $\forall y,x,z \ge 0$.

Then, G has a fixed point $x^* \in S$ *and* $\{G^n x_1\}$, $n \in \mathbb{N}$ *converges to* x^*.

Proof. Define a sequence $\{x_n\}$ by $x_{n+1} = Gx_n$, $\forall n \geq 1$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \geq 1$, then obviously G has a fixed point. Hence, we suppose that $x_n \neq x_{n+1}$, $\forall n \geq 1$. Then, for all $n \geq 1$, we obtain

$$\begin{aligned} \delta(x_{n+2}, x_{n+1}) &\leq \beta(\delta(x_{n+1}, x_n))\delta(x_{n+1}, x_n) \\ &\quad +\Theta(\delta(x_{n+1}, x_n), \delta(x_{n+2}, x_n), \delta(x_{n+1}, x_{n+1}), \delta(x_{n+2}, x_{n+1})) \\ &\leq \beta(\delta(x_{n+1}, x_n))\delta(x_{n+1}, x_n) \\ &\leq \delta(x_{n+1}, x_n), \end{aligned} \tag{1}$$

So $\{\delta(x_{n+1}, x_n)\}$ is decreasing, let us say to $L \geq 0$.

From (1), we have

$$\frac{\delta(x_{n+2}, x_{n+1})}{\delta(x_{n+1}, x_n)} \leq \beta(\delta(x_{n+1}, x_n)), \ \forall \ n \geq 1.$$

If $L > 0$, for $n \to \infty$, we obtain

$$1 \leq \liminf_{n\to\infty} \beta(\delta(x_{n+1}, x_n)) \leq \lim_{n\to\infty} \beta(\delta(x_{n+1}, x_n)) \leq 1.$$

By the condition satisfied by Geraghty functions, it follows that $\delta(x_{n+1}, x_n)$ goes to zero, hence $L = 0$.

Suppose $\{x_n\}$ does not satisfy the Cauchy sequence condition. Let $\epsilon > 0$. From [30], there exists a subsequence $\{x_{n_k}\}$, with $\{\delta(x_{m_k}, x_{n_k})\}$, $\{\delta(x_{n_k}, x_{m_k+1})\}$, $\{\delta(x_{n_k+1}, x_{m_k})\}$ bounded, with

$$\epsilon \leq \liminf_{k\to\infty} \delta(x_{n_k}, x_{m_k}) \leq \limsup_{k\to\infty} \delta(x_{n_k}, x_{m_k}) \leq s\epsilon,$$

From the generalized triangle inequality, we have

$$\begin{aligned} \delta(x_{n_k}, x_{m_k}) &\leq \delta(x_{n_k+1}, x_{m_k}) + s\delta(x_{n_k+1}, x_{n_k}) \\ &\leq \delta(x_{m_k+1}, x_{n_k+1}) + s\delta(x_{m_k+1}, x_{m_k}) + s\delta(x_{n_k+1}, x_{n_k}) \\ &\leq \beta(\delta(x_{m_k}, x_{n_k}))\delta(x_{m_k}, x_{n_k}) \\ &\quad +\Theta(\delta(Gx_{m_k}, x_{m_k}), \delta(x_{n_k+1}, x_{m_k}), \delta(x_{m_k+1}, x_{n_k}), \delta(x_{n_k+1}, x_{n_k})) \\ &\quad +s\delta(x_{m_k+1}, x_{m_k}) + s\delta(x_{n_k+1}, x_{n_k}), \ \forall \ k \geq 1. \end{aligned}$$

Taking lim sup after $k \to \infty$, we have

$$l = \limsup_{k\to\infty} \delta(x_{n_k}, x_{m_k}) \leq l \limsup_{k\to\infty} \beta(\delta(x_{n_k}, x_{m_k})) \leq l,$$

and again, assuming that $l \neq 0$, we obtain $\lim_{k\to\infty} \delta(x_{m_k}, x_{n_k}) = 0$, contradiction.

So $\{x_n\}$ is a Cauchy sequence in S, hence it is convergent, let us say, to x^*.

Now, we have to prove that $\lim_{n\to\infty} Gx_n = Gx^*$ and from the uniqueness of the limit in strong b-metric spaces, it will result that $Gx^* = x^*$.

From the hypotheses, we obtain

$$\begin{aligned} \delta(Gx^*, Gx_n) &\leq \beta(\delta(x^*, x_n))\delta(x^*, x_n) \\ &\quad +\Theta(\delta(Gx_n, x_n), \delta(Gx^*, x_n), \delta(Gx_n, x^*), \delta(Gx^*, x^*)) \\ &= \beta(\delta(x^*, x_n))\delta(x^*, x_n) \\ &\quad +\Theta(\delta(x_{n+1}, x_n), \delta(Gx^*, x_n), \delta(Gx_n, x^*), \delta(Gx^*, x^*)). \end{aligned}$$

Taking $n \to \infty$, and using if necessary lim sup, we have $\lim_{n\to\infty} \delta(Gx^*, x_n) = 0$ and from here we obtain the conclusion. $\square$

Taking particular cases for the Geraghty functions or for the continuous function from the definition of the generalized contraction, we obtain the known results in literature.

Corollary 1. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$:*

$$\delta(Gy, Gx) \leq \sin \delta(y, x) + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$ is continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \geq 0$.
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

Corollary 2. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$,*

$$\delta(Gy, Gx) \leq \frac{\delta^2(x, y)}{e^{\delta(y,x)} - 1} + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$ is continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \geq 0$.
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

Corollary 3. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$:*

$$\delta(Gy, Gx) \leq \ln(\delta(y, x) + 1) + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$ is continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$.
Then G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$ converges to x^*.

Considering now $\Theta \equiv 0$, Theorem 1 gives the next consequence.

Corollary 4. *Let $G\colon S \to S$ be a mapping. Suppose that the following conditions are satisfied:*
(1) *$\beta \in \mathcal{F}$ is a Geraghty function,*
(2) *For all $y, x \in S$, the next relation is true*

$$\delta(Gy, Gx) \leq \beta(\delta(y, x))\delta(y, x).$$

Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

From all these results we can obtain outcomes regarding the classic metric spaces, which are strong b-metric spaces.

We continue with another fixed point existence result. In the following a and b are numbers from the interval $(0, 1)$, so that $a + b \leq 1$.

Theorem 2. *Let $G\colon S \to S$ be an operator. Suppose that the following conditions are satisfied:*
(1) *β is a Geraghty function ($\beta \in \mathcal{F}$),*
(2) *$\delta(Gy, Gx) \leq \beta(\delta(y, x))\big(a\delta(y, Gy) + b\delta(x, Gx)\big) + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx))$, where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$ is continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall x, y, z \in S$.*
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$ converges to x^*.

Proof. Define a sequence $\{x_n\}$ by $x_{n+1} = Gx_n$, $\forall n \geq 1$.
If $x_{n_0} = x_{n_0+1}$ for some $n_0 \geq 1$, then obviously G has a fixed point.
Hence, we suppose that $x_n \neq x_{n+1}$ $\forall n \geq 1$.

Then, we obtain:

$$\begin{array}{rcl}\delta(x_{n+2},x_{n+1}) & = & \delta(Gx_{n+1},Gx_n) \leq \beta(\delta(Gx_n,x_n))\big(a\delta(x_{n+1},x_n)+b\delta(x_{n+2},x_{n+1})\big) \\ & & +\Theta(\delta(Gx_n,x_n),\delta(Gx_{n+1},x_n),\delta(Gx_n,x_{n+1}),\delta(Gx_{n+1},x_{n+1})) \\ & = & \beta(\delta(Gx_n,x_n))\big(a\delta(x_{n+1},x_n)+b\delta(x_{n+2},x_{n+1})\big) \\ & \leq & a\delta(x_{n+1},x_n)+b\delta(x_{n+2},x_{n+1}),\ \forall n\in\mathbb{N},\end{array}$$

So

$$(1-b)\delta(x_{n+2},x_{n+1}) \leq a\delta(x_{n+1},x_n),\ \forall n\in\mathbb{N}.$$

That is

$$\delta(x_{n+2},x_{n+1}) \leq \frac{a}{1-b}\delta(x_{n+1},x_n) \leq \delta(x_{n+1},x_n),\ \forall n\in\mathbb{N},$$

Since $a+b\leq 1$.

Thus, the sequence $\{(\delta(x_{n+1},x_n)\}$ has positive terms and is decreasing. Therefore, there exists $L\geq 0$ such that $\lim_{n\to\infty}\delta(x_{n+1},x_n)=L$. We will demonstrate that $L=0$.

Suppose, to the contrary, that $L>0$.

Since we have already proved that

$$\delta(x_{n+2},x_{n+1}) \quad\leq\quad \beta(\delta(Gx_n,x_n))\big(a\delta(x_{n+1},x_n)+b\delta(x_{n+2},x_{n+1})\big),\ \forall n\in\mathbb{N},$$

We obtain

$$\delta(x_{n+2},x_{n+1}) \quad\leq\quad a\beta(\delta(Gx_n,x_n))\delta(x_{n+1},x_n)+b\delta(x_{n+2},x_{n+1}),\forall n\in\mathbb{N}.$$

So

$$(1-b)\delta(x_{n+2},x_{n+1}) \leq a\beta(\delta(x_{n+1},x_n))\delta(x_{n+1},x_n),\forall n\in\mathbb{N},$$

which is

$$\frac{\delta(x_{n+2},x_{n+1})}{\delta(x_{n+1},x_n)} \leq \beta(\delta(x_{n+1},x_n)) \leq 1.$$

This implies that $\lim_{n\to\infty}\beta(\delta(x_{n+1},x_n))=L=1$. Since $\beta\in\mathcal{F}$, then $\lim_{n\to\infty}\delta(x_{n+1},x_n)=0$. This is a contradiction. So $L=0$.

Next, we shall show that$\{x_n\}$ is a Cauchy sequence.

Let m, $n>0$, m, $n\in\mathbb{N}$.

From the generalized triangle inequality, we have

$$\begin{array}{rcl}\delta(x_n,x_m) & \leq & \delta(x_{n+1},x_m)+s\delta(x_n,x_{n+1}) \\ & \leq & \delta(x_{m+1},x_{n+1})+s\delta(x_m,x_{m+1})+s\delta(x_n,x_{n+1}) \\ & \leq & \beta(\delta(x_m,x_n))\big(a\delta(x_{n+1},x_n)+b\delta(x_{m+1},x_m)\big) \\ & & +\Theta(\delta(x_{n+1},x_n),\delta(x_{m+1},x_n),\delta(x_{n+1},x_m),\delta(x_{m+1},x_m)) \\ & & +s\delta(x_m,x_{m+1})+s\delta(x_n,x_{n+1}),\ \forall m,n\in\mathbb{N}.\end{array}$$

Taking lim after $m,n\to\infty$, we have

$$\lim_{m,n\to\infty}\delta(x_n,x_m)=0.$$

Thus, $\{x_n\}$ is Cauchy sequence in S, hence it is convergent, let us say, to x^*.

Now, we have to prove that $\lim_{n\to\infty}Gx_n=Gx^*$ and from the uniqueness of the limit in strong b-metric spaces, it will result that $Gx^*=x^*$.

We remark that:

$$\begin{aligned}\delta(Gx^*, Gx_n) &\leq \beta(\delta(x^*, x_n))\big(a\delta(Gx_n, x_n) + b\delta(Gx^*, x^*)\big)\\ &\quad +\Theta(\delta(Gx_n, x_n), \delta(Gx^*, x_n), \delta(Gx_n, x^*), \delta(Gx^*, x^*))\\ &\leq a\delta(x_{n+1}, x_n) + b\delta(Gx^*, x^*)\\ &\quad +\Theta(\delta(x_{n+1}, x_n), \delta(Gx^*, x_n), \delta(x_{n+1}, x^*), \delta(Gx^*, x^*)), \forall n \in \mathbb{N}.\end{aligned}$$

Taking $n \to \infty$ and using the continuity of the strong b-metric, we obtain

$$\delta(Gx^*, x^*) \leq b\delta(Gx^*, x^*),$$

Therefore $Gx^* = x^*$ and we obtain the conclusion. □

Taking $a = b = \frac{1}{2}$ in Theorem 2, we obtain a result in the direction of Kannan.

Corollary 5. *Let $G\colon S \to S$ be an operator. Suppose that the following conditions are satisfied:*
(1) *β is a Geraghty function ($\beta \in \mathcal{F}$),*
(2) $\delta(Gy, Gx) \leq \beta(\delta(y, x))\dfrac{\delta(y, Gy) + \delta(x, Gx)}{2} + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx))$, *where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$ is continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \in S$.*
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$ converges to x^*.

Considering now particular choices for β and Θ in Corollary 5, the next corollaries arise.

Corollary 6. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$:*

$$\delta(Gy, Gx) \leq \sin \delta(y, x) \cdot \frac{\delta(y, Gy) + \delta(x, Gx)}{2\delta(y, x)} + \Theta(\delta(y, Gy), \delta(x, Gy), \delta(y, Gx), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$, continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \geq 0$.
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

Corollary 7. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$:*

$$\delta(Gy, Gx) \leq \frac{\delta(x, y)}{e^{\delta(y,x)} - 1} \frac{\delta(y, Gy) + \delta(x, Gx)}{2\delta(y, x)} + \Theta(\delta(y, Gy), \delta(x, Gy), \delta(y, Gx), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$, continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \geq 0$.
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

Corollary 8. *Let $G\colon S \to S$ be a mapping. Suppose that the following condition is satisfied, for any $x, y \in S$*

$$\delta(Gy, Gx) \leq \ln(\delta(y, x) + 1)\frac{\delta(y, Gy) + \delta(x, Gx)}{2\delta(y, x)} + \Theta(\delta(y, Gy), \delta(x, Gy), \delta(y, Gx), \delta(x, Gx)),$$

where $\Theta\colon [0, \infty)^4 \to \mathbb{R}$, continuous, such that $\Theta(0, y, x, z) = 0$, $\Theta(y, x, 0, z) = 0$, $\forall y, x, z \geq 0$.
Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$ converges to x^*.

Corollary 9. *Let $G\colon S \to S$ be a mapping. Suppose that the following conditions are satisfied:*
(1) *$\beta \in \mathcal{F}$ is a Geraghty function,*

(2) *For all $x, y, z \in S$, the next relation is true*

$$\delta(Gy, Gx) \leq \beta(\delta(y, x)) \frac{\delta(y, Gy) + \delta(x, Gx)}{2}.$$

Then, G has a fixed point $x^* \in S$ and $\{G^n x_1\}$, $n \in \mathbb{N}$, converges to x^*.

We provide now an example of a mapping, which fulfills the conditions in Corollary 4.

Example 3. *Let $S = \{0, 1, 2, 3\}$, and*

$$\delta \colon S \times S \to [0, \infty), \begin{cases} \delta(x, x) = 0, x \in S; \\ \delta(0, 1) = \delta(0, 2) = \frac{2}{5}; \\ \delta(0, 3) = \delta(2, 3) = \frac{2}{3}; \\ \delta(1, 2) = \delta(1, 3) = \frac{3}{4}; \\ \delta(x, y) = \delta(y, x), \forall x, y \in S, \end{cases}$$

which is a strong b-metric, but not a classic metric.

Consider $G \colon S \to S$, $G0 = 0$, $G1 = 0$, $G2 = 0$, and $G3 = 1$, and $\beta(t) = \frac{1}{1+t}$, $t \geq 0$. The mapping G satisfies the conditions in Corollary 4, and its fixed point is zero.

4. Uniqueness Results

We focus now on the uniqueness of fixed points of mappings which satisfy the conditions of the previous theorems.

Theorem 3. *Suppose that (S, δ) is a strong b-metric space and that the conditions in Theorem 1 are satisfied. Then, the fixed point of the mapping G is unique.*

Proof. Presume that x and $y \in S$ are fixed points of the operator G.

If $y \neq x$, from the contraction inequality, we have

$$\begin{aligned} \delta(y, x) &= \delta(Gy, Gx) \leq \beta(\delta(y, x))\delta(y, x) + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx)) \\ &= \beta(\delta(y, x))\delta(y, x) \\ &< \delta(y, x), \end{aligned}$$

contradiction.

Hence, the fixed point of G is unique. □

Theorem 4. *Suppose that (S, δ) is a strong b-metric space and that the conditions in Theorem 2 are satisfied. Then, the fixed point of the mapping G is unique.*

Proof. Presume that y and $x \in S$ are fixed points of the operator G. From the contraction inequality, we have

$$\begin{aligned} & \delta(y, x) = \delta(Gy, Gx) \\ \leq \ & \beta(\delta(y, x))\big(a\delta(y, Gy) + b\delta(x, Gx)\big) + \Theta(\delta(y, Gy), \delta(y, Gx), \delta(x, Gy), \delta(x, Gx)) = 0, \end{aligned}$$

so $y = x$.

Hence, the fixed point of G is unique. □

Corollary 10. *Suppose that hypotheses from Corollaries 1–4 are satisfied. Then the fixed point of the operator G is unique.*

Corollary 11. *Suppose that hypotheses from Corollaries 5–8, or Corollary 9 are satisfied. Then, the fixed point of the operator G is unique.*

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cacciopoli, R. Un teorema generale sull' esistenza di elementi uniti in una transformazione funzionale. *Rend. Acad. Nazionale Lincei* **1930**, *1*, 794–799.
2. Kannan, R. Some results on fixed points II. *Am. Math. Mon.* **1969**, *76*, 405–408.
3. Chatterjea, S.K. Fixed point theorems. *C.R. Acad. Bulgaire Sci.* **1972**, *25*, 727–730. [CrossRef]
4. Hardy, G.E.; Rogers, T.D. A generalization of a fixed point theorem of Reich. *Can. Math. Bull.* **1973**, *16*, 201–206. [CrossRef]
5. Ćirić, LB. Generalized contractions and fixed point theorems. *Publ. l'Institut Mathématique* **1971**, *12*, 19–26.
6. Berinde, V. Approximating fixed points of weak contractions using the Picard iteration. *Nonlinear Anal. Forum* **2004**, *9*, 43–53.
7. Geraghty, M. On contractive mappings. *Proc. Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
8. Ansari, A.A.; Shatanawi, W.; Kurdi, A.; Maniu, G. Best proximity points in complete metric spaces with (P)-property via C-class functions. *J. Math. Anal.* **2016**, *7*, 54–67.
9. Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095. [CrossRef]
10. García-Falset, J.; Llorens-Fuster, E.; Suzuki, T. Fixed point theory for a class of generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2011**, *375*, 185–195. [CrossRef]
11. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
12. Berinde, V.; Păcurar, M. The early developments in fixed point theory on *b*-metric spaces: A brief survey and some important related aspects. *Carpathian J. Math.* **2022**, *38*, 523–538. [CrossRef]
13. Shatanawi, W.; Pitea, A.; Lazović, R. Contraction conditions using comparison functions on *b*-metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 135. [CrossRef]
14. Amini-Harandi, A. Fixed point theory for quasi-contraction maps in *b*-metric spaces. *Fixed Point Theory* **2014**, *15*, 351–358.
15. Dung, N.V.; Hang, V.T.L. On relaxations of contraction constants and Caristi theorem in *b*-metric spaces. *J. Fixed Point Theory Appl.* **2016**, *8*, 267–284. [CrossRef]
16. Kamran, T.; Postolache, M.; Ali, M.U.; Kiran, Q. Feng and Liu type F-contraction in *b*-metric spaces with application to integral equations. *J. Math. Anal.* **2016**, *7*, 18–27.
17. Ali, M.U.; Kamran, T.; Postolache, M. Solution of Volterra integral inclusion in *b*-metric spaces via new fixed point theorem. *Nonlinear Anal. Model. Control* **2017**, *22*, 389–400. [CrossRef]
18. McConnell, R.; Kwok, R.; Curlander, J.C.; Kober, W.; Pang, S.S. Psi correlation and dynamic time warping: Two methods for tracking ice floes in SAR images. *IEEE Trans. Geoscie. Remote Sens.* **1991**, *29*, 1004–1012. [CrossRef]
19. Karapınar, E. A short survey on the recent fixed point results on *b*-metric spaces. *Constr. Math. Anal.* **2018**, *1*, 15–44. [CrossRef]
20. Samreen, M.; Kamran, T.; Postolache, M. Extended *b*-metric space, extended *b*-comparison function and nonlinear contractions. *U. Politeh. Buch. Ser. A* **2018**, *8*, 21–28.
21. Kirk, W.; Shahzad, N. *Fixed Point Theory in Distance Spaces*; Springer: Cham, Switzerland, 2014.
22. Cobzaş, Ş.; Czerwik, S. The completion of generalized *b*-metric spaces and fixed points. *Fixed Point Theory* **2020**, *21*, 133–150. [CrossRef]
23. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal. Gos. Ped. Inst. Unianowsk* **1989**, *30*, 26–37.
24. Czerwik, S. Contraction mappings in *b*-metric spaces. *Acta Math. Inform. Univ. Ostrav.* **1993**, *1*, 5–11.
25. Czerwik, S. Nonlinear set-valued contraction mappings in *b*-metric spaces. *Atti Semin. Mat. Fis. Univ. Modena* **1998**, *46*, 263–276.
26. Hussain, N.; Dorić, D.; Kadelburg, Z.; Radenović, S. Suzuki-type fixed point results in metric type spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 126. [CrossRef]
27. Doan, H. A new type of Kannan's fixed point theorem in strong *b*-metric spaces. *AIMS Math.* **2021**, *6*, 7895–7908. [CrossRef]
28. Samet, B. Some results on best proximity points. *J. Optim. Theory Appl.* **2013**, *159*, 281–291. [CrossRef]

29. Shatanawi, W.; Pitea, A. Best proximity point and best proximity coupled point in a complete metric space with (P)-property. *Filomat* **2015**, *29*, 65–76. [CrossRef]
30. Roshan, J.R.; Parvaneh, V.; Kadelberg, Z. Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces. *J. Nonlinear Sci. Appl.* **2014**, *7*, 229–245. [CrossRef]