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Soft Complete Continuity and Soft Strong Continuity in Soft Topological Spaces

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Abstract: In this paper, we introduce soft complete continuity as a strong form of soft continuity and we introduce soft strong continuity as a strong form of soft complete continuity. Several characterizations, compositions, and restriction theorems are obtained. Moreover, several preservation theorems regarding soft compactness, soft Lindelofness, soft connectedness, soft regularity, soft normality, soft almost regularity, soft mild normality, soft almost compactness, soft almost Lindelofness, soft near compactness, soft near Lindelofness, soft paracompactness, soft near paracompactness, soft almost paracompactness, and soft metacompactness are obtained. In addition to these, the study deals with the correlation between our new concepts in soft topology and their corresponding concepts in general topology; as a result, we show that soft complete continuity (resp. soft strong continuity) in soft topology is an extension of complete continuity (resp. strong continuity) in soft topology.

Keywords: completely continuous function; strongly continuous function; soft weakly continuous functions; soft nearly compact; soft nearly paracompact

MSC: 54A10; 54A40; 54D10

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1. Introduction and Preliminaries

In many fields, such as engineering, the environment, economics, medical science, and social science, classical mathematical ideas have their own difficulties in dealing with uncertainty. Fuzzy sets, rough sets, intuitionistic fuzzy sets, and vague sets are all methods for handling uncertainty [1–4]. According to Molodtsov [5], each of these structures has particular difficulties. These difficulties are mostly due to the limits of the parameterization tool. Molodtsov [5] presented soft sets as a solution to these issues and to handle uncertainty. Many authors have discussed and studied the concepts of soft sets (see [6,7]). The authors [5,8] used soft sets in many different fields, such as operation research, game theory, smoothness of function, probability, and measurement theory.

Soft set theory has been used by several researchers to investigate various mathematical structures. Shabir and Naz [9] introduce soft topology as one of the unique extensions of classical topology. Many classic topological concepts such as generalized open sets, separation axioms, covering properties, etc., [10–18] have been extended and expanded in soft set contexts, but there is still space for substantial contributions. Thus, the study of soft topology is a current trend among topological researchers.

In this paper, we introduce soft complete continuity as a strong form of soft continuity and we introduce soft strong continuity as a strong form of soft complete continuity. Several characterizations, compositions, restrictions, and preservation theorems are obtained. The study deals with the correlation between our new concepts in soft topology and their corresponding concepts in general topology. As a result, we show that soft complete continuity (resp. soft strong continuity) in soft topology is an extension of complete continuity (resp. strong continuity) in soft topology.

The terms STS and TS, which stand for soft topological space and topological space, respectively, will be utilized in this paper. The concepts and phrases from [19,20] will be used throughout this paper.

This paper is organized as follows:

In Section 2, we introduce the notion of "soft completely continuous mappings". We study the correlation between soft completely continuous mappings in soft topology and completely continuous mappings in general topology, and we characterize soft completely continuous mappings. Moreover, we show that this class of soft mappings is strictly contained in the class of soft continuous mappings. Moreover, we study the behavior of soft completely continuous mappings under soft restriction and soft composition. In addition, via soft completely continuous mappings, we obtain several preservation theorems regarding some soft topological properties.

In Section 3, we introduce the notion of "soft strongly continuous mappings". We study the correlation between soft strongly continuous mappings in soft topology and strongly continuous mappings in general topology, and we obtain several characterizations of soft strongly continuous mappings. Moreover, we show that this class of soft mappings is strictly contained in the class of soft completely continuous mappings. Moreover, we study the behavior of soft strongly continuous mappings under soft restriction and soft composition. In addition, via soft strongly continuous mappings, we obtain several preservation theorems regarding some soft topological properties.

Let (T,μ) be a TS, (T,π,B) be a STS, $U \subseteq T$, and $G \in SS(T,\pi)$. Then, the closure of U in (T,μ), the interior of U in (T,μ), the closure of G in (T,π,B), and the soft interior of G in (T,π,B) will be denoted by $Cl_\mu(U)$, $Int_\mu(U)$, $Cl_\pi(G)$, and $Int_\pi(G)$, respectively; the family of all closed sets in (T,μ) (resp. soft closed sets in (T,π,B)) will be denoted by μ^c (resp. π^c); and the family of all clopen sets in (T,μ) (resp. soft clopen sets in (T,π,B)) will be denoted by $CO(T,\mu)$ (resp. $CO(T,\pi,B)$).

Definition 1. *Let (T,μ) be a TS and let $U \subseteq T$. Then*

(a) Ref. [21] U is called a regular open set in (T,μ) if $U = Int_\mu(Cl_\mu(U))$.

(b) Ref. [21] U is called a regular closed set in (T,μ) if $T - U$ is a regular open set in (T,μ).

(c) The family of all regular open sets in (T,μ) will be denoted by $RO(T,\mu)$.

(d) The family of all regular closed sets in (T,μ) will be denoted by $RC(T,\mu)$.

Definition 2. *A function $p : (T,\mu) \longrightarrow (S,\delta)$ between the TSs (T,μ) and (S,δ) is called*

(a) Ref. [22] strongly continuous if $p\big(Cl_\mu(X)\big) \subseteq P(X)$ for every $X \subseteq T$.

(b) Ref. [23] completely continuous if $p^{-1}(U) \in RO(T,\mu)$ for every $U \in \delta$.

Definition 3. *Let (T,π,B) be a STS and let $K \in SS(T,B)$. Then*

(a) Ref. [24] K is called a soft regular open set in (T,π,B) if $U = Int_\pi(Cl_\pi(K))$.

(b) Ref. [24] K is called a regular closed set in (T,π,B) if $1_B - K$ is a soft regular open set in (T,π,B).

(c) The family of all regular open sets in (T,π,B) will be denoted by $RO(T,\pi,B)$.

(d) The family of all regular closed sets in (T,μ) will be denoted by $RC(T,\pi,B)$.

Definition 4. *A soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,v,D)$ is called*

(a) Ref. [25] soft almost open if $f_{pu}(H) \in v$ for every $H \in RO(T,\pi,B)$.

(b) Ref. [26] soft weakly continuous if for every $b_t \in SP(T,B)$ and every $G \in v$ such that $f_{pu}(b_t)\widetilde{\in}G$, there exists $K \in \pi$ such that $b_t\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_v(G)$.

Definition 5. *A STS (T,π,B) is called*

(1) Ref. [27] soft compact (soft Lindelof) if for every $\mathcal{A} \subseteq \pi$ such that $\widetilde{\bigcup}_{A\in\mathcal{A}}A = 1_B$, there exists a finite (resp. countable) subcollection $\mathcal{A} \subseteq \mathcal{A}_1$ such that $\widetilde{\bigcup}_{A\in\mathcal{A}_1} = 1_B$.

(2) Ref. [28] soft connected if $CO(T,\pi,B) = \{0_B, 1_B\}$.

(3) Ref. [29] soft regular if whenever $G \in \pi^c$ and $b_t\widetilde{\in}1_B - G$, then there exists $L, N \in \pi$ such that $b_t\widetilde{\in}L$, $G\widetilde{\subseteq}N$, and $L\widetilde{\cap}N = 0_B$.

(4) Ref. [29] soft normal if whenever $G, H \in \pi^c$ such that $G\widetilde{\cap}H = 0_B$, then there exists $L, N \in \pi$ such that $G\widetilde{\subseteq}L$, $H\widetilde{\subseteq}N$, and $L\widetilde{\cap}N = 0_B$.

(5) Ref. [30] soft almost regular if whenever $G \in RC(T,\pi,B)$ and $b_t \widetilde{\in} 1_B - G$, then there exists $L, N \in \pi$ such that $b_t \widetilde{\in} L$, $G \widetilde{\subseteq} N$, and $L \widetilde{\cap} N = 0_B$.

(6) Ref. [31] soft mildly normal if whenever $G, H \in RC(T,\pi,B)$ such that $G \widetilde{\cap} H = 0_B$, then there exists $L, N \in \pi$ such that $G \widetilde{\subseteq} L$, $H \widetilde{\subseteq} N$, and $L \widetilde{\cap} N = 0_B$.

(7) Ref. [32] soft almost compact (soft almost Lindelof) if for every $\mathcal{A} \subseteq \pi$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists a finite (resp. countable) subcollection $\mathcal{A} \subseteq \mathcal{A}_1$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} Cl_\pi(A) = 1_B$.

(8) Ref. [33] soft nearly compact (soft nearly Lindelof) if for every $\mathcal{A} \subseteq RO(T,\pi,B)$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists a finite (resp. countable) subcollection $\mathcal{A} \subseteq \mathcal{A}_1$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} A = 1_B$.

(9) Ref. [28] soft paracompact if for every $\mathcal{A} \subseteq \pi$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists $\mathcal{K} \subseteq \pi$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K \in \mathcal{K}} K = 1_B$, and for each $K \in \mathcal{K}$ there exists $A \in \mathcal{A}$ such that $A \widetilde{\subseteq} K$.

(10) Ref. [33] soft nearly paracompact if for every $\mathcal{A} \subseteq RO(T,\pi,B)$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists $\mathcal{K} \subseteq \approx$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K \in \mathcal{K}} K = 1_B$, and for each $K \in \mathcal{K}$ there exists $A \in \mathcal{A}$ such that $A \widetilde{\subseteq} K$.

(11) Ref. [34] soft almost paracompact if for every $\mathcal{A} \subseteq \pi$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists $\mathcal{K} \subseteq \pi$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K \in \mathcal{K}} Cl_\pi(K) = 1_B$, and for each $K \in \mathcal{K}$ there exists $A \in \mathcal{A}$ such that $A \widetilde{\subseteq} K$.

2. Soft Completely Continuous Mappings

In this section, we introduce the notion of "soft completely continuous mappings". We study the correlation between soft completely continuous mappings in soft topology and completely continuous mappings in general topology, and we characterize soft completely continuous mappings. Additionally, we show that this class of soft mappings is strictly contained in the class of soft continuous mappings. Moreover, we study the behavior of soft completely continuous mappings under soft restriction and soft composition. In addition, via soft completely continuous mappings, we obtain several preservation theorems regarding some soft topological properties.

Definition 6. *A soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,\upsilon,D)$ is soft completely continuous if $f_{pu}^{-1}(G) \in RO(T,\pi,B)$ for every $G \in \upsilon$.*

Theorem 1. *For a soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,\upsilon,D)$, the following are equivalent:*

(a) f_{pu} is soft completely continuous.

(b) $f_{pu}^{-1}(H) \in RC(T,\pi,B)$ for every $H \in \upsilon^c$.

Proof. (a) $\longrightarrow$ (b): Let $H \in \upsilon^c$. Then, $1_D - H \in \upsilon$ and by (a), $f_{pu}^{-1}(1_D - H) = 1_B - f_{pu}^{-1}(H) \in RO(T,\pi,B)$. Hence, $f_{pu}^{-1}(H) \in RC(T,\pi,B)$.

(b) $\longrightarrow$ (a): Let $K \in \upsilon$. Then $1_D - K \in \upsilon^c$. Then, by (b), $f_{pu}^{-1}(1_D - K) = 1_B - f_{pu}^{-1}(K) \in RC(T,\pi,B)$. Hence, $f_{pu}^{-1}(H) \in RO(T,\pi,B)$. Therefore, f_{pu} is soft completely continuous. □

Theorem 2. *Let $\{(T,\pi_i) : i \in I\}$ and $\{(S,\upsilon_j) : j \in J\}$ be two families of TSs. Let $p : T \longrightarrow S$ be a function and $u : I \longrightarrow J$ be a bijective function. Then, $f_{pu} : (T, \oplus_{i \in I}\pi_i, I) \longrightarrow (S, \oplus_{j \in J}\upsilon_j, J)$ is soft completely continuous if and only if $p : (T,\pi_i) \longrightarrow \left(S, \upsilon_{u(i)}\right)$ is completely continuous for all $i \in I$.*

Proof. *Necessity*. Suppose that $f_{pu} : (T, \oplus_{i \in I}\pi_i, I) \longrightarrow (S, \oplus_{j \in J}\upsilon_j, J)$ is soft completely continuous. Let $k \in I$ and let $W \in \upsilon_{u(k)}$. Then $(u(k))_W \in \oplus_{j \in J}\upsilon_j$. Since $u : I \longrightarrow J$ is injective, then $f_{pu}^{-1}((u(k))_W) = k_{p^{-1}(W)}$. Since $f_{pu} : (T, \oplus_{i \in I}\pi_i, I) \longrightarrow (S, \oplus_{j \in J}\upsilon_j, J)$ is soft completely continuous, then $f_{pu}^{-1}((u(k))_W) = k_{p^{-1}(W)} \in RO(T,\pi,B)$. Thus, by Proposition 3.28 of [35], $\left(k_{p^{-1}(W)}\right)(k) = p^{-1}(W) \in RO(T,\pi_k)$. Hence, $p : (T,\pi_k) \longrightarrow \left(S, \upsilon_{u(k)}\right)$ is completely continuous.

Sufficiency. Suppose that $p : (T, \pi_i) \longrightarrow \left(S, v_{u(i)}\right)$ is completely continuous for all $i \in I$. Let $G \in \oplus_{j\in J} v_j$. Then, for every $j \in J$, $G(j) \in v_j$. Since $u : I \longrightarrow J$ is bijective, then $p : \left(T, \pi_{u^{-1}(j)}\right) \longrightarrow (S, v_j)$ is completely continuous for all $j \in J$. Thus, $p^{-1}(G(j)) = \left(\left(f^{-1}_{pu}(G)\right)\right)(u^{-1}(j)) \in RO\left(T, \pi_{u^{-1}(j)}\right)$ for all $j \in J$. So, $\left(f^{-1}_{pu}(G)\right)(i) \in RO(T, \pi_i)$ for all $i \in I$. Therefore, by Proposition 3.28 of [35], $f^{-1}_{pu}(G) \in RO(T, \oplus_{i\in I}\pi_i, I)$. It follows that $f_{pu} : (T, \oplus_{i\in I}\pi_i, I) \longrightarrow (S, \oplus_{j\in J} v_j, J)$ is soft completely continuous. □

Corollary 1. *Let $p : (T, \mu) \longrightarrow (S, \delta)$ be a function between two TSs and let $u : I \longrightarrow J$ be a bijective function. Then $p : (T, \mu) \longrightarrow (S, \delta)$ is completely continuous if and only if $f_{pu} : (T, \tau(\mu), I) \longrightarrow (S, \tau(\delta), J)$ is soft completely continuous.*

Proof. For each $i \in I$ and $j \in J$, put $\pi_i = \mu$ and $v_j = \delta$. Then $\tau(\mu) = \oplus_{i\in I}\pi_i$ and $\tau(\delta) = \oplus_{j\in J} v_j$. Thus, by Theorem 2, we obtain the result. □

Theorem 3. *Every soft completely continuous soft mapping is soft continuous.*

Proof. Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be a soft completely continuous mapping. Let $G \in v$. Then, $f^{-1}_{pu}(G) \in RO(T, \pi, B) \subseteq \pi$. Hence, f_{pu} is soft continuous. □

Theorem 3's converse does not necessarily hold in all cases.

Example 1. *Let $T = \{1, 2, 3, 4\}$, $S = \{5, 6\}$, $\mu = \{\emptyset, T, \{1, 2\}\}$, $\delta = \{\emptyset, S, \{5\}\}$, $B = \mathbb{R}$. Define $p : T \longrightarrow S$ and $u : B \longrightarrow B$ as follows: $p(1) = p(2) = 5$, $p(3) = p(4) = 6$, and $u(b) = b$ for all $b \in B$. Since $\{5\} \in \delta$ and $p^{-1}(\{5\}) = \{1, 2\} \in \mu - RO(T, \mu)$, then $p : (T, \mu) \longrightarrow (S, \delta)$ is continuous but not completely continuous. Therefore, by Theorem 5.31 of [20] and Corollary 1, $f_{pu} : (T, \tau(\mu), B) \longrightarrow (S, \tau(\delta), B)$ is soft continuous but not soft completely continuous.*

The following example demonstrates how the soft restriction of a soft completely continuous mapping may not be a soft completely continuous mapping:

Example 2. *Let $T = \{1, 2, 3, 4\}$, $S = \{5, 6, 7\}$, $\mu = \{\emptyset, T, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$, $\delta = \{\emptyset, S, \{5, 6\}\}$, $B = \mathbb{R}$. Define $p : T \longrightarrow S$ and $u : B \longrightarrow B$ as follows: $p(1) = 5$, $p(2) = 6$, $p(3) = p(4) = 7$, and $u(b) = b$ for all $b \in B$. Since $p^{-1}(\{5, 6\}) = \{1, 2\} \in RO(T, \mu)$, then $p : (T, \mu) \longrightarrow (S, \delta)$ is completely continuous and by Corollary 1, $f_{pu} : (T, \tau(\mu), B) \longrightarrow (S, \tau(\delta), D)$ is soft completely continuous. On the other hand, since $C_{\{5,6\}} \in \tau(\delta)$ and $\left(\left(f_{pu}\right)_{|C_{\{1,4\}}}\right)^{-1}\left(C_{\{5.6\}}\right) = \left(f^{-1}_{pu}\left(C_{\{5.6\}}\right)\right)\widetilde{\cap} C_{\{1,4\}} = C_{\{1,2\}}\widetilde{\cap} C_{\{1,4\}} = C_{\{1\}} \notin RO\left(\{1, 4\}, (\tau(\mu))_{\{1,4\}}, B\right)$, then $\left(f_{pu}\right)_{|C_{\{1,4\}}} : \left(\{1, 4\}, (\tau(\mu))_{\{1,4\}}, B\right) \longrightarrow (S, \tau(\delta), D)$ is not soft completely continuous.*

Theorem 4. *If $f_{p_1u_1} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft completely continuous and $f_{p_2u_2} : (S, v, D) \longrightarrow (R, \gamma, E)$ is soft continuous, then $f_{(p_2\circ p_1)(u_2\circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft completely continuous.*

Proof. Let $H \in \gamma$. Since $f_{p_2u_2} : (S, v, D) \longrightarrow (R, \gamma, E)$ is soft continuous, then $f^{-1}_{p_2u_2}(H) \in v$. Since $f_{p_1u_1} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft completely continuous, then $f^{-1}_{p_1u_1}\left(f^{-1}_{p_2u_2}(H)\right) = f^{-1}_{(p_2\circ p_1)(u_2\circ u_1)}(H) \in RO(T, \pi, B)$. This ends the proof. □

Corollary 2. *The soft composition of two soft completely continuous mappings is soft completely continuous.*

Theorem 5. *If $f_{p_1u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is surjective, soft almost open, and soft completely continuous, and $f_{p_2u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is a soft mapping such that $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft completely continuous, then $f_{p_2u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is soft continuous.*

Proof. Let $H \in \gamma$. Since $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft completely continuous, then $f^{-1}_{(p_2 \circ p_1)(u_2 \circ u_1)}(H) \in RO(T, \pi, B)$. Since $f_{p_1u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft almost open and $f_{p_1u_1}$ is surjective, then $f_{p_1u_1}\left(f^{-1}_{(p_2 \circ p_1)(u_2 \circ u_1)}(H)\right) = f_{p_1u_1}\left(f^{-1}_{p_1u_1}\left(f^{-1}_{p_2u_2}(H)\right)\right) = f^{-1}_{p_2u_2}(H) \in \upsilon$. This ends the proof. □

Theorem 6. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is surjective and soft completely continuous such that (T, π, B) is soft nearly compact, then (S, υ, D) is soft compact.*

Proof. Let $\mathcal{A} \subseteq \upsilon$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_D$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft completely continuous, then $\left\{f^{-1}_{pu}(A) : A \in \mathcal{A}\right\} \subseteq RO(T, \pi, B)$. Since $\widetilde{\cup}_{A \in \mathcal{A}} f^{-1}_{pu}(A) = f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}} A\right) = 1_B$ and (T, π, B) is soft nearly compact, then there exists a finite subcollection $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} f^{-1}_{pu}(A) = f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}_1} A\right) = 1_B$ and thus, $f_{pu}\left(f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}_1} A\right)\right) = f_{pu}(1_B)$. Since f_{pu} is a surjective, then $f_{pu}(1_B) = 1_D$ and $f_{pu}\left(f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}_1} A\right)\right) = \widetilde{\cup}_{A \in \mathcal{A}_1} A$. Therefore, $\widetilde{\cup}_{A \in \mathcal{A}_1} A = 1_D$. It follows that (S, υ, D) is soft compact. □

Theorem 7. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is surjective and soft completely continuous such that (T, π, B) is soft nearly Lindelof, then (S, υ, D) is soft Lindelof.*

Proof. Let $\mathcal{A} \subseteq \upsilon$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_D$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft completely continuous, then $\left\{f^{-1}_{pu}(A) : A \in \mathcal{A}\right\} \subseteq RO(T, \pi, B)$. Since $\widetilde{\cup}_{A \in \mathcal{A}} f^{-1}_{pu}(A) = f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}} A\right) = 1_B$ and (T, π, B) is soft nearly Lindelof, then there exists a countable subcollection $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} f^{-1}_{pu}(A) = f^{-1}_{pu}\left(\widetilde{\cup}_{A \in \mathcal{A}_1} A\right) = 1_B$. Since f_{pu} is a surjective, then $\widetilde{\cup}_{A \in \mathcal{A}_1} A = 1_D$. Hence, (S, υ, D) is soft Lindelof. □

Theorem 8. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ be surjective, soft completely continuous, and soft closed such that $f^{-1}_{pu}(d_s)$ is a soft compact subset of (T, π, B) for all $d_s \in SP(S, D)$. If (T, π, B) is soft almost regular, then (S, υ, D) is soft regular.*

Proof. Let $K \in \upsilon^c$ and let $d_s \in SP(S, D)$ such that $d_s \widetilde{\in} 1_D - K$. Then, $f^{-1}_{pu}(d_s) \widetilde{\cap} f^{-1}_{pu}(K) = 0_B$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft completely continuous, then by Theorem 1, $f^{-1}_{pu}(K) \in RC(T, \pi, B)$. For each $b_t \widetilde{\in} f^{-1}_{pu}(d_s)$, we have $b_t \widetilde{\in} 1_B - f^{-1}_{pu}(K)$ and by soft regularity of (T, π, B), there exist $H_{b_t}, G_{b_t} \in \pi$ such that $b_t \widetilde{\in} H_{b_t}$, $f^{-1}_{pu}(K) \widetilde{\subseteq} G_{b_t}$, and $H_{b_t} \widetilde{\cap} G_{b_t} = 0_B$. Since $f^{-1}_{pu}(d_s)$ is soft compact and $f^{-1}_{pu}(d_s) \widetilde{\subseteq} \widetilde{\cup}_{b_t \widetilde{\in} f^{-1}_{pu}(d_s)} H_{b_t}$, then there exists a finite subset $\mathcal{M} \subseteq \left\{b_t : b_t \widetilde{\in} f^{-1}_{pu}(d_s)\right\}$ such that $f^{-1}_{pu}(d_s) \widetilde{\subseteq} \widetilde{\cup}_{b_t \in \mathcal{M}} H_{b_t}$. Let $H = \widetilde{\cup}_{b_t \in \mathcal{M}} H_{b_t}$ and $G = \widetilde{\cap}_{b_t \in \mathcal{M}} G_{b_t}$. Then, H, $G \in \pi$ such that $f^{-1}_{pu}(d_s) \widetilde{\subseteq} H$, $f^{-1}_{pu}(K) \widetilde{\subseteq} G$, and $G \widetilde{\cap} H = 0_B$. Let $L = 1_D - f_{pu}(1_B - H)$ and $N = 1_D - f_{pu}(1_B - G)$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft closed, then $f_{pu}(1_B - H), f_{pu}(1_B - G) \in \upsilon^c$ and thus, $L, N \in \upsilon$. □

Claim. *1. $d_s \widetilde{\in} L$.*
2. $K \widetilde{\subseteq} N$.
3. $L \widetilde{\cap} N = 0_D$.

Proof of Claim. 1. Suppose to the contrary that $d_s \widetilde{\in} 1_D - L = f_{pu}(1_B - H)$. Then, there exists $a_x \widetilde{\in} 1_B - H$ such that $d_s = f_{pu}(a_x)$. Thus, $a_x \widetilde{\in} f^{-1}_{pu}(d_s) \widetilde{\subseteq} H$, a contradiction.

2. Suppose to the contrary that there exists $e_y\widetilde{\in}K - N = K\widetilde{\cap}f_{pu}(1_B - G)$. Since $e_y\widetilde{\in}f_{pu}(1_B - G)$, then there exists $a_x\widetilde{\in}1_B - G$ such that $e_y = f_{pu}(a_x)$. However, since $e_y\widetilde{\in}K$, then $a_x\widetilde{\in}f_{pu}^{-1}(K)\widetilde{\subseteq}G$, a contradiction.

3. We will show that $1_D - \left(L\widetilde{\cap}N\right) = 1_D$. Since f_{pu} is surjective, then $f_{pu}(1_B) = 1_D$. So,

$$\begin{array}{rcl} 1_D - \left(L\widetilde{\cap}N\right) & = & (1_D - L)\widetilde{\cup}(1_D - N) \\ & = & f_{pu}(1_B - H)\widetilde{\cup}f_{pu}(1_B - G) \\ & = & f_{pu}((1_B - H)\widetilde{\cup}(1_B - G)) \\ & = & f_{pu}(1_B - \left(H\widetilde{\cap}G\right)) \\ & = & f_{pu}(1_B - 0_B) \\ & = & f_{pu}(1_B) \\ & = & 1_D. \end{array}$$

Therefore, by the above Claim, (S, v, D) is soft regular. □

Theorem 9. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective, soft completely continuous, and soft closed mapping. If (T, π, B) is soft mildly normal, then (S, v, D) is soft normal.*

Proof. Let $M, N \in v^c$ such that $M\widetilde{\cap}N = 0_D$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft completely continuous, then by Theorem 1, $f_{pu}^{-1}(M), f_{pu}^{-1}(N) \in RC(T, \pi, B)$. Since (T, π, B) is soft mildly normal, then there exist $H, G \in v$ such that $f_{pu}^{-1}(M)\widetilde{\subseteq}H$, $f_{pu}^{-1}(N)\widetilde{\subseteq}G$ and $H\widetilde{\cap}G = 0_B$. Let $L = 1_D - f_{pu}(1_B - H)$ and $K = 1_D - f_{pu}(1_B - G)$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft closed, then $f_{pu}(1_B - H)$, $f_{pu}(1_B - G) \in v^c$ and thus, $L, K \in v$. □

Claim. *1. $M\widetilde{\subseteq}L$.*

2. $N\widetilde{\subseteq}K$.

3. $L\widetilde{\cap}N = 0_D$.

Proof of Claim. 1. Since $f_{pu}^{-1}(M)\widetilde{\subseteq}H$, then $1_B - H\widetilde{\subseteq}1_B - f_{pu}^{-1}(M) = f_{pu}^{-1}(1_D - M)$ and so, $f_{pu}(1_B - H)\widetilde{\subseteq}f_{pu}\left(f_{pu}^{-1}(1_D - M)\right) = 1_D - M$. Hence, $M\widetilde{\subseteq}1_D - f_{pu}(1_B - H) = L$.

2. Since $f_{pu}^{-1}(N)\widetilde{\subseteq}G$, then $1_B - G\widetilde{\subseteq}1_B - f_{pu}^{-1}(N) = f_{pu}^{-1}(1_D - N)$ and so, $f_{pu}(1_B - G)\widetilde{\subseteq}f_{pu}\left(f_{pu}^{-1}(1_D - N)\right) = 1_D - N$. Hence, $N\widetilde{\subseteq}1_D - f_{pu}(1_B - G) = K$.

3. We will show that $1_D - \left(L\widetilde{\cap}N\right) = 1_D$. Since f_{pu} is surjective, then $f_{pu}(1_B) = 1_D$. So,

$$\begin{array}{rcl} 1_D - \left(L\widetilde{\cap}N\right) & = & (1_D - L)\widetilde{\cup}(1_D - N) \\ & = & f_{pu}(1_B - H)\widetilde{\cup}f_{pu}(1_B - G) \\ & = & f_{pu}((1_B - H)\widetilde{\cup}(1_B - G)) \\ & = & f_{pu}(1_B - \left(H\widetilde{\cap}G\right)) \\ & = & f_{pu}(1_B - 0_B) \\ & = & f_{pu}(1_B) \\ & = & 1_D. \end{array}$$

Therefore, by the above Claim, (S, v, D) is soft normal. □

Definition 7. *Let (T, π, B) be a STS and let $\mathcal{M} \subseteq SS(T, B)$. Then*

(a) $\mathcal{M}$ is soft point finite in (T, π, B) if for every $b_t \in SP(T, B)$, the set $\{M \in \mathcal{M} : b_t\widetilde{\in}M\}$ is finite.

(b) (T, π, B) is called soft metacompact if for every $\mathcal{K} \subseteq \pi$ such that $\widetilde{\cup}_{K\in\mathcal{K}}K = 1_B$, there exists a soft point finite $\mathcal{H}$ in (T, π, B) such that $\mathcal{H} \subseteq \pi$, $\widetilde{\cup}_{H\in\mathcal{H}}H = 1_B$, and for each $H \in \mathcal{H}$, there exists $K \in \mathcal{K}$ such that $K\widetilde{\subseteq}H$.

Theorem 10. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective, soft completely continuous, and soft open such that $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T, π, B) for each $d_s \in SP(S, D)$. If (T, π, B) is soft nearly paracompact, then (S, v, D) is soft metacompact.*

Proof. Let $\mathcal{H} \subseteq v$ such that $\widetilde{\cup}_{H\in\mathcal{H}}H = 1_D$. Since f_{pu} is soft completely continuous, then $\left\{f_{pu}^{-1}(H) : H \in \mathcal{H}\right\} \subseteq RO(T, \pi, B) \subseteq \pi$. Since (T, π, B) is soft nearly paracompact and

$\widetilde{\cup}_{H\in\mathcal{H}}f_{pu}^{-1}(H) = f_{pu}^{-1}(\widetilde{\cup}_{H\in\mathcal{H}}H) = f_{pu}^{-1}(1_D) = 1_B$, then there exists a collection $\mathcal{K} \subseteq \pi$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K\in\mathcal{K}}K = 1_B$, and for every $K \in \mathcal{K}$ there exists $H \in \mathcal{H}$ such that $K\widetilde{\subseteq}f_{pu}^{-1}(H)$. Let $\mathcal{M} = \{f_{pu}(K) : K \in \mathcal{K}\}$. □

Claim. *1. $\mathcal{M} \subseteq v$.*
2. $\widetilde{\cup}_{M\in\mathcal{M}}M = 1_D$.
3. For each $M \in \mathcal{M}$, there exists $H \in \mathcal{H}$ such that $M\widetilde{\subseteq}H$.
4. $\mathcal{M}$ is soft point finite.

Proof of Claim. 1. Since $\mathcal{K} \subseteq \pi$ and f_{pu} is soft open, then $\mathcal{M} = \{f_{pu}(K) : K \in \mathcal{K}\} \subseteq v$.

2. Since f_{pu} is surjective, then $f_{pu}(1_B) = 1_D$. So, $\widetilde{\cup}_{M\in\mathcal{M}}M = \widetilde{\cup}_{K\in\mathcal{K}}f_{pu}(K) = f_{pu}(\widetilde{\cup}_{K\in\mathcal{K}}K) = f_{pu}(1_B) = 1_D$.

3. Let $M \in \mathcal{M}$. Then, there exists $K \in \mathcal{K}$ such that $f_{pu}(K) = M$. Choose $H \in \mathcal{H}$ such that $K\widetilde{\subseteq}f_{pu}^{-1}(H)$. Then, $M = f_{pu}(K)\widetilde{\subseteq}f_{pu}\left(f_{pu}^{-1}(H)\right)\widetilde{\subseteq}H$.

4. Let $d_s \in SP(S,D)$. Since $\mathcal{K}$ is soft locally finite, then for every $b_t\widetilde{\in}f_{pu}^{-1}(d_s)$, there exists $G_{b_t} \in \pi$ such that $b_t\widetilde{\in}G_{b_t}$ and the collection $\{K \in \mathcal{K} : K\widetilde{\cap}G_{b_t} \neq 0_B\}$ is finite. For each $b_t\widetilde{\in}f_{pu}^{-1}(d_s)$, put $\mathcal{S}_{b_t} = \{K \in \mathcal{K} : K\widetilde{\cap}G_{b_t} \neq 0_B\}$. Since $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T,π,B) and $f_{pu}^{-1}(d_s)\widetilde{\subseteq}\widetilde{\cup}_{b_t\widetilde{\in}f_{pu}^{-1}(d_s)}G_{b_t}$, then there exists a finite subset $\mathcal{A} \subseteq \left\{b_t : b_t\widetilde{\in}f_{pu}^{-1}(d_s)\right\}$ such that $f_{pu}^{-1}(d_s)\widetilde{\subseteq}\widetilde{\cup}_{b_t\in\mathcal{A}}G_{b_t}$. If $d_s\widetilde{\in}f_{pu}(R)$ for some $R \in \mathcal{K}$, then there exists $w_r\widetilde{\in}R\widetilde{\cap}f_{pu}^{-1}(d_s)$. Since $w_r\widetilde{\in}f_{pu}^{-1}(d_s)\widetilde{\subseteq}\widetilde{\cup}_{b_t\in\mathcal{A}}G_{b_t}$, then there exists $b_t \in \mathcal{A}$ such that $w_r\widetilde{\in}G_{b_t}$. Thus, we have $w_r\widetilde{\in}R\widetilde{\cap}G_{b_t}$ and hence $R \in \mathcal{S}_{b_t}$. Therefore, $\{K \in \mathcal{K} : d_s\widetilde{\in}f_{pu}(K)\} \subseteq \{K \in \mathcal{K} : R \in \mathcal{S}_{b_t}, b_t \in \mathcal{A}\}$. Since $\{K \in \mathcal{K} : R \in \mathcal{S}_{b_t}, b_t \in \mathcal{A}\}$ is finite, then $\{K \in \mathcal{K} : d_s\widetilde{\in}f_{pu}(K)\}$ is finite. Hence, $\mathcal{M}$ is soft point finite.

Therefore, by the above Claim, (S,v,D) is soft metacompact. □

3. Soft Strongly Continuous Mappings

In this section, we introduce the notion of "soft strongly continuous mappings". We study the correlation between soft strongly continuous mappings in soft topology and strongly continuous mappings in general topology, and we obtain several characterizations of soft strongly continuous mappings. Moreover, we show that this class of soft mappings is strictly contained in the class of soft completely continuous mappings. Moreover, we study the behavior of soft strongly continuous mappings under soft restriction and soft composition. In addition, via soft strongly continuous mappings, we obtain several preservation theorems regarding some soft topological properties.

Definition 8. *A soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,v,D)$ is soft strongly continuous if for every $M \in SS(T,B)$, $f_{pu}(Cl_\pi(M))\widetilde{\subseteq}f_{pu}(M)$.*

Theorem 11. *For a soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,v,D)$, the following are equivalent:*
(a) f_{pu} is soft strongly continuous.
(b) $f_{pu}^{-1}(H) \in \pi^c$ for every $H \in SS(S,D)$.

Proof. (a) $\longrightarrow$ (b): Let $H \in SS(S,D)$. Then, by (a), $f_{pu}(Cl_\pi(f_{pu}^{-1}(H)))\widetilde{\subseteq}f_{pu}(f_{pu}^{-1}(H))\widetilde{\subseteq}H$ and thus, $Cl_\pi(f_{pu}^{-1}(H))\widetilde{\subseteq}f_{pu}^{-1}\left(f_{pu}(Cl_\pi(f_{pu}^{-1}(H)))\right)\widetilde{\subseteq}f_{pu}^{-1}(H)$. Therefore, $f_{pu}^{-1}(H) \in \pi^c$.

(b) $\longrightarrow$ (a): Let $M \in SS(T,B)$. Then, by (b), $f_{pu}^{-1}(f_{pu}(M)) \in \pi^c$. Since $M\widetilde{\subseteq}f_{pu}^{-1}(f_{pu}(M))$, then $Cl_\pi(M)\widetilde{\subseteq}Cl_\pi\left(f_{pu}^{-1}(f_{pu}(M))\right) = f_{pu}^{-1}(f_{pu}(M))$, and so $f_{pu}(Cl_\pi(M))\widetilde{\subseteq}f_{pu}\left(f_{pu}^{-1}(f_{pu}(M))\right)$ $\widetilde{\subseteq}f_{pu}(M)$. It follows that f_{pu} is soft strongly continuous. □

Theorem 12. *For a soft mapping $f_{pu} : (T,\pi,B) \longrightarrow (S,v,D)$, the following are equivalent:*
(a) f_{pu} is soft strongly continuous.
(b) $f_{pu}^{-1}(H) \in \pi$ for every $H \in SS(S,D)$.
(c) $f_{pu}^{-1}(H) \in CO(T,\pi,B)$ for every $H \in SS(S,D)$.

(d) $f_{pu}^{-1}(d_s) \in \pi$ *for every* $d_s \in SP(S, D)$.
(e) $f_{pu}^{-1}(d_s) \in \pi^c$ *for every* $d_s \in SP(S, D)$.
(f) $f_{pu}^{-1}(d_s) \in CO(T, \pi, B)$ *for every* $d_s \in SP(S, D)$.

Proof. (a) $\longrightarrow$ (b): Let $H \in SS(S, D)$. Then, by (a) and Theorem 11, $f_{pu}^{-1}(1_D - H) = 1_B - f_{pu}^{-1}(H) \in \pi^c$. Hence, $f_{pu}^{-1}(H) \in \pi$.

(b) $\longrightarrow$ (c): Let $H \in SS(S, D)$. Then, by (b), $f_{pu}^{-1}(H) \in \pi$ and $1_B - f_{pu}^{-1}(H) = f_{pu}^{-1}(1_D - H) \in \pi$. Hence, $f_{pu}^{-1}(H) \in CO(T, \pi, B)$.

(c) $\longrightarrow$ (d): Obvious.

(d) $\longrightarrow$ (e): Let $d_s \in SP(S, D)$. Then, by (d), $1_B - f_{pu}^{-1}(d_s) = f_{pu}^{-1}(1_D - d_s) \in \pi$. Hence, $f_{pu}^{-1}(d_s) \in \pi^c$.

(e) $\longrightarrow$ (f): Let $d_s \in SP(S, D)$. Then, by (e), $f_{pu}^{-1}(d_s) \in \pi^c$ and $1_B - f_{pu}^{-1}(d_s) = f_{pu}^{-1}(1_D - d_s) \in \pi^c$. Hence, $f_{pu}^{-1}(d_s) \in CO(T, \pi, B)$.

(f) $\longrightarrow$ (a): Let $H \in SS(S, D)$. We will apply Theorem 11. By (f), $f_{pu}^{-1}(d_s) \in \pi$ for every $d_s \widetilde{\in} 1_D - H$. Thus, $f_{pu}^{-1}(1_D - H) = \widetilde{\bigcup}_{d_s \widetilde{\in} 1_D - H} f_{pu}^{-1}(d_s) \in \pi$. Hence, $1_B - f_{pu}^{-1}(1_D - H) = 1_B - \left(1_B - f_{pu}^{-1}(H)\right) = f_{pu}^{-1}(H) \in \pi^c$. $\square$

Theorem 13. *If* $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ *is soft strongly continuous, then* $p : (T, \pi_b) \longrightarrow \left(S, v_{u(b)}\right)$ *is strongly continuous for all* $b \in B$.

Proof. Suppose that $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft strongly continuous. Let $b \in B$ and let $s \in S$. Then $(u(b))_s \in SP(S, D)$ and part (d) of Theorem 12, we have $f_{pu}^{-1}((u(b))_s) \in \pi$. Thus, $\left(f_{pu}^{-1}((u(b))_s)\right)(b) = p^{-1}(((u(b))_s)(u(b))) = p^{-1}(\{s\}) \in \pi_b$. Hence, by Theorem 1.1 of [23], $p : (T, \pi_b) \longrightarrow \left(S, v_{u(b)}\right)$ is strongly continuous. $\square$

Theorem 14. *Let* $\{(T, \pi_i) : i \in I\}$ *and* $\left\{(S, v_j) : j \in J\right\}$ *be two families of TSs. Let* $p : T \longrightarrow S$ *be a function and* $u : I \longrightarrow J$ *be a bijective function. Then* $f_{pu} : (T, \oplus_{i \in I} \pi_i, I) \longrightarrow (S, \oplus_{j \in J} v_j, J)$ *is soft strongly continuous if and only if* $p : (T, \pi_i) \longrightarrow \left(S, v_{u(i)}\right)$ *is strongly continuous for all* $i \in I$.

Proof. *Necessity*. Suppose that $f_{pu} : (T, \oplus_{i \in I} \pi_i, I) \longrightarrow \left(S, \oplus_{j \in J} v_j, J\right)$ is soft strongly continuous. Let $k \in I$, then by Theorem 13, $p : (T, (\oplus_{i \in I} \pi_i)_k) \longrightarrow \left(S, \left(\oplus_{j \in J} v_j\right)_{u(k)}\right)$ is strongly continuous. However, by Theorem 3.7 of [20], $(\oplus_{i \in I} \pi_i)_k = \pi_k$ and $\left(\oplus_{j \in J} v_j\right)_{u(k)} = v_{u(k)}$. Hence, $p : (T, \pi_i) \longrightarrow \left(S, v_{u(i)}\right)$ is strongly continuous.

Sufficiency. Suppose that $p : (T, \pi_i) \longrightarrow \left(S, v_{u(i)}\right)$ is strongly continuous for all $i \in I$. Let $G \in SS(S, J)$. Then, for every $j \in J$, $G(j) \subseteq S$. Since $u : I \longrightarrow J$ is bijective, then $p : \left(T, \pi_{u^{-1}(j)}\right) \longrightarrow (S, v_j)$ is strongly continuous for all $j \in J$. Thus, $p^{-1}(G(j)) = \left(\left(f_{pu}^{-1}(G)\right)\right)(u^{-1}(j)) \in \pi_{u^{-1}(j)}$ for all $j \in J$. Hence, $\left(f_{pu}^{-1}(G)\right)(i) \in \pi_i$ for all $i \in I$. Therefore, $f_{pu}^{-1}(G) \in \oplus_{i \in I} \pi_i$. It follows that $f_{pu} : (T, \oplus_{i \in I} \pi_i, I) \longrightarrow \left(S, \oplus_{j \in J} v_j, J\right)$ is soft strongly continuous. $\square$

Corollary 3. *Let* $p : (T, \mu) \longrightarrow (S, \delta)$ *be a function between two TSs and let* $u : I \longrightarrow J$ *be a bijective function. Then* $p : (T, \mu) \longrightarrow (S, \delta)$ *is strongly continuous if and only if* $f_{pu} : (T, \tau(\mu), I) \longrightarrow (S, \tau(\delta), J)$ *is soft strongly continuous.*

Proof. For each $i \in I$ and $j \in J$, put $\pi_i = \mu$ and $v_j = \delta$. Then $\tau(\mu) = \oplus_{i \in I} \pi_i$ and $\tau(\delta) = \oplus_{j \in J} v_j$. Thus, by Theorem 14, we obtain the result. $\square$

Theorem 15. *Every soft strongly continuous soft mapping is soft completely continuous.*

Proof. Let $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ be a soft strongly continuous mapping. Let $G \in \upsilon$. Then by part (c) of Theorem 12, $f_{pu}^{-1}(G) \in CO(T, \pi, B) \subseteq RO(T, \pi, B)$. Hence, f_{pu} is soft completely continuous. □

Theorem 15's converse does not necessarily hold in all cases.

Example 3. *Let $T = \{1,2,3,4\}$, $S = \{5,6,7\}$, $\mu = \{\emptyset, T, \{1,2\}, \{3\}, \{1,2,3\}\}$, $\delta = \{\emptyset, S, \{5\}, \{6\}, \{5,6\}\}$, and $B = \mathbb{R}$. Define $p : T \longrightarrow S$ and $u : B \longrightarrow B$ as follows: $p(1) = p(2) = 5$, $p(3) = p(4) = 7$, and $u(b) = b$ for all $b \in B$. Since $p^{-1}(\{6\}) = \emptyset \in RO(T, \mu)$ and $p^{-1}(\{5\}) = p^{-1}(\{5,6\}) = \{1,2\} \in RO(T, \mu) - CO(T, \mu)$, then $p : (T, \mu) \longrightarrow (S, \delta)$ is completely continuous but not strongly continuous. Therefore, by Corollaries 1 and 3, $f_{pu} : (T, \tau(\mu), B) \longrightarrow (S, \tau(\delta), B)$ is soft completely continuous but not soft strongly continuous.*

Theorem 16. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft weakly continuous such that (S, υ, D) is soft discrete, then f_{pu} is soft strongly continuous.*

Proof. Suppose that $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft weakly continuous such that (S, υ, D) is soft discrete. Let $d_s \in SP(S, D)$. To see that $f_{pu}^{-1}(d_s) \in \pi$. Let $b_t \widetilde{\in} f_{pu}^{-1}(d_s)$. Then, $f_{pu}(b_t) \widetilde{\in} d_s \in \upsilon$ and by soft weak continuity of f_{pu}, there exists $G \in \pi$ such that $b_t \widetilde{\in} G$ and $f_{pu}(G) \widetilde{\subseteq} Cl_{\upsilon}(d_s) = d_s$. Thus, we have $b_t \widetilde{\in} G \widetilde{\subseteq} f_{pu}^{-1}(f_{pu}(G)) \widetilde{\subseteq} f_{pu}^{-1}(d_s)$. Hence, $f_{pu}^{-1}(d_s) \in \pi$. □

Corollary 4. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft continuous such that (S, υ, D) is soft discrete, then f_{pu} is soft strongly continuous.*

Theorem 17. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is a soft mapping such that (T, π, B) is soft discrete, then f_{pu} is soft strongly continuous.*

Proof. Obvious. □

Theorem 18. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ be an injective soft mapping. Then f_{pu} is soft strongly continuous if and only if (T, π, B) is soft discrete.*

Proof. *Necessity.* Suppose that f_{pu} is soft strongly continuous. We will show that $SP(T, B) \subseteq \pi$. Let $b_t \in SP(T, B)$. Then by soft strong continuity of f_{pu} we have $f_{pu}^{-1}(f_{pu}(b_t)) \in \pi$. Since f_{pu} is injective, then $f_{pu}^{-1}(f_{pu}(b_t)) = b_t$. Therefore, $b_t \in \pi$.

Sufficiency. Follows from Theorem 17. □

Theorem 19. *A soft homeomorphism $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous if and only if (T, π, B) and (S, υ, D) are soft discrete STSs.*

Proof. *Necessity.* Suppose that f_{pu} is soft homeomorphism and soft strongly continuous. Then f_{pu} is injective and by Theorem 18, (T, π, B) is soft discrete. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft homeomorphism and (T, π, B) is soft discrete, then (S, υ, D) is soft discrete.

Sufficiency. Follows from Theorem 17. □

Theorem 20. *For a STS $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$, the following are equivalent:*

(a) $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous.

(b) $f_{pu} : (T, \pi, B) \longrightarrow (S, \gamma, D)$ is soft continuous for any soft topology γ on S relative to D.

Proof. (a) $\longrightarrow$ (b): Let γ be a soft topology on S relative to D. To see that $f_{pu} : (T, \pi, B) \longrightarrow (S, \gamma, D)$ is soft continuous, let $G \in \gamma$. Then $G \in SS(S, D)$ and by (a), $f_{pu}^{-1}(G) \in \pi$.

(b) $\longrightarrow$ (a): By (b), we have $f_{pu} : (T, \pi, B) \longrightarrow (S, SS(S, D), D)$ is soft continuous. Thus, Corollary 4 ends the proof. $\square$

Theorem 21. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ be soft strongly continuous and X be a non-empty subset of T. If (X, π_X, B) is soft connected, then $f_{pu}(C_X)$ is a single soft point.*

Proof. Suppose to the contrary that (X, π_X, B) is soft connected and $f_{pu}(C_X)$ is not a single soft point. Choose $d_s \tilde{\in} f_{pu}(C_X)$. Then, by Theorem 12 (f), $f_{pu}^{-1}(d_s) \in CO(T, \pi, B)$. Therefore, we have $f_{pu}^{-1}(d_s) \tilde{\cap} C_X \in CO(X, \pi_X, B) - \{0_B, 1_B\}$. Hence, (X, π_X, B) is not soft connected, a contradiction. $\square$

Theorem 22. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous and X is any non-empty subset of T. Then, $\left(f_{pu}\right)_{|C_X} : (X, \pi_X, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous.*

Proof. Let $d_s \in SP(S, D)$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous, then $f_{pu}^{-1}(d_s) \in \pi$, and so $\left(\left(f_{pu}\right)_{|C_X}\right)^{-1}(d_s) = f_{pu}^{-1}(d_s) \tilde{\cap} C_X \in \pi_X$. Hence, $\left(f_{pu}\right)_{|C_X} : (X, \pi_X, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous. $\square$

Theorem 23. *If $f_{p_1 u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous and $f_{p_2 u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is any soft mapping, then $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft strongly continuous.*

Proof. Let $H \in SS(R, E)$. Then, $f_{p_2 u_2}^{-1}(H) \in SS(S, D)$. Since $f_{p_1 u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft strongly continuous, then $f_{p_1 u_1}^{-1}\left(f_{p_2 u_2}^{-1}(H)\right) = f_{(p_2 \circ p_1)(u_2 \circ u_1)}^{-1}(H) \in \pi$. This ends the proof. $\square$

Corollary 5. *The soft composition of two strongly continuous functions is strongly continuous.*

The example shown below shows how Theorem 23's theorem is not necessarily true for soft continuous functions.

Example 4. *Let $T = \mathbb{R}$, $B = \mathbb{N}$, $\pi = \{0_B, 1_B\}$, and $\upsilon = SS(T, B)$. Consider the identities functions $p : T \longrightarrow T$ and $u : B \longrightarrow B$. Consider the soft mappings $f_{pu} : (T, \pi, B) \longrightarrow (T, \pi, B)$ and $f_{pu} : (T, \pi, B) \longrightarrow (T, \upsilon, B)$. Then, $f_{pu} : (T, \pi, B) \longrightarrow (T, \pi, B)$ is soft continuous but $f_{(p \circ p)(u \circ u)} : (T, \pi, B) \longrightarrow (T, \upsilon, B)$ is not soft continuous.*

Theorem 24. *If $f_{p_1 u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is a soft weakly continuous mapping and $f_{p_2 u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is soft strongly continuous, then $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft strongly continuous.*

Proof. Let $H \in SS(R, E)$. Since $f_{p_2 u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is soft strongly continuous, then $f_{p_2 u_2}^{-1}(H) \in CO(S, \upsilon, D)$. Since $f_{p_1 u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is soft weakly continuous, then by Theorem 5.1 of [26],

$$\begin{array}{rcl} f_{(p_2 \circ p_1)(u_2 \circ u_1)}^{-1}(H) & = & f_{p_1 u_1}^{-1}\left(f_{p_2 u_2}^{-1}(H)\right) \\ & \tilde{\subseteq} & Int_\pi\left(f_{p_1 u_1}^{-1}\left(Cl_\upsilon\left(f_{p_2 u_2}^{-1}(H)\right)\right)\right) \\ & = & Int_\pi\left(f_{p_1 u_1}^{-1}\left(f_{p_2 u_2}^{-1}(H)\right)\right) \\ & = & Int_\pi\left(f_{(p_2 \circ p_1)(u_2 \circ u_1)}^{-1}(H)\right). \end{array}$$

This ends the proof. $\square$

Corollary 6. *If $f_{p_1 u_1} : (T, \pi, B) \longrightarrow (S, \upsilon, D)$ is a soft continuous mapping and $f_{p_2 u_2} : (S, \upsilon, D) \longrightarrow (R, \gamma, E)$ is soft strongly continuous, then $f_{(p_2 \circ p_1)(u_2 \circ u_1)} : (T, \pi, B) \longrightarrow (R, \gamma, E)$ is soft strongly continuous.*

Theorem 25. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be a soft strongly continuous such that (T, π, B) is soft compact. Then, $f_{pu}^{-1}(H)$ is a soft compact subset of (T, π, B) for every $H \in SS(S, D)$.*

Proof. Let $H \in SS(S, D)$. Since $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is soft strongly continuous, then $f_{pu}^{-1}(H) \in \pi^c$. Since (T, π, B) is soft compact, then $f_{pu}^{-1}(H)$ is a soft compact subset of (T, π, B). □

Definition 9. *A STS (T, π, B) is said to be a soft C-C space if the soft closed sets in (T, π, B) coincide with soft compact sets of (T, π, B).*

Theorem 26. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be a soft mapping such that (T, π, B) is a soft C-C space and (S, v, D) is a hereditarily soft compact. Then, the following are equivalent:*
(a) f_{pu} is soft strongly continuous.
(b) $f_{pu}^{-1}(H)$ is a soft compact subset of (T, π, B) for every soft compact subset H of (S, v, D).

Proof. (a) $\longrightarrow$ (b): Let H be any soft compact subset of (S, v, D). Then, by (a), $f_{pu}^{-1}(H) \in \pi^c$. Since (T, π, B) is a soft C-C space, then $f_{pu}^{-1}(H)$ is a soft compact subset of (T, π, B).

(b) $\longrightarrow$ (a): Let $H \in SS(S, D)$. Since (S, v, D) is a hereditarily soft compact, then H is a soft compact subset of (S, v, D). Thus, by (b), $f_{pu}^{-1}(H)$ is a soft compact subset of (T, π, B). Since (T, π, B) is a soft C-C space, then $f_{pu}^{-1}(H) \in \pi^c$. □

Theorem 27. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is a soft strongly continuous mapping, then for any soft compact subset K of (T, π, B), $f_{pu}(K)$ is a finite soft set.*

Proof. Let K be any soft compact subset of (T, π, B). Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(d_s) : d_s \in SP(S, D) \right\} \subseteq \pi$. Since $K \widetilde{\subseteq} \widetilde{\cup}_{d_s \in SP(S,D)} f_{pu}^{-1}(d_s)$, then there exists a finite subset $\mathcal{M} \subseteq SP(S, D)$ such that $K \widetilde{\subseteq} \widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}^{-1}(d_s)$. Thus, $f_{pu}(K) \widetilde{\subseteq} f_{pu}\left(\widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}^{-1}(d_s) \right) = \widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}\left(f_{pu}^{-1}(d_s) \right) \widetilde{\subseteq} \widetilde{\cup}_{d_s \in \mathcal{M}} d_s$. Since $\widetilde{\cup}_{d_s \in \mathcal{M}} d_s$ is a finite soft set, then $f_{pu}(K)$ is a finite soft set. □

Theorem 28. *If $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ is a soft strongly continuous mapping, then for any soft Lindelof subset K of (T, π, B), $f_{pu}(K)$ is a countable soft set.*

Proof. Let K be any soft Lindelof subset of (T, π, B). Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(d_s) : d_s \in SP(S, D) \right\} \subseteq \pi$. Since $K \widetilde{\subseteq} \widetilde{\cup}_{d_s \in SP(S,D)} f_{pu}^{-1}(d_s)$, then there exists a countable subset $\mathcal{M} \subseteq SP(S, D)$ such that $K \widetilde{\subseteq} \widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}^{-1}(d_s)$. Thus, $f_{pu}(K) \widetilde{\subseteq} f_{pu}\left(\widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}^{-1}(d_s) \right) = \widetilde{\cup}_{d_s \in \mathcal{M}} f_{pu}\left(f_{pu}^{-1}(d_s) \right) \widetilde{\subseteq} \widetilde{\cup}_{d_s \in \mathcal{M}} d_s$. Since $\widetilde{\cup}_{d_s \in \mathcal{M}} d_s$ is a countable soft set, then $f_{pu}(K)$ is a countable soft set. □

Theorem 29. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective and soft strongly continuous such that (T, π, B) is soft almost compact. Then, (S, v, D) is soft compact.*

Proof. Let $\mathcal{A} \subseteq v$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_D$. Then, $\widetilde{\cup}_{A \in \mathcal{A}} f_{pu}^{-1}(A) = 1_B$. Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(A) : A \in \mathcal{A} \right\} \subseteq CO(T, \pi, B) \subseteq \pi$. Since (T, π, B) is soft almost compact, then there exists a finite subfamily $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} Cl_{\pi}\left(f_{pu}^{-1}(A) \right) = \widetilde{\cup}_{A \in \mathcal{A}_1} f_{pu}^{-1}(A) = f_{pu}^{-1}(\widetilde{\cup}_{A \in \mathcal{A}_1} A) = 1_B$ and thus, $f_{pu}\left(f_{pu}^{-1}(\widetilde{\cup}_{A \in \mathcal{A}_1} A) \right) = f_{pu}(1_B)$. Since f_{pu} is surjective, then $f_{pu}\left(f_{pu}^{-1}(\widetilde{\cup}_{A \in \mathcal{A}_1} A) \right) = \widetilde{\cup}_{A \in \mathcal{A}_1} A$ and $f_{pu}(1_B) = 1_D$. Therefore, $\widetilde{\cup}_{A \in \mathcal{A}_1} A = 1_D$. It follows that (S, v, D) is soft compact. □

Theorem 30. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective and soft strongly continuous such that (T, π, B) is soft almost Lindelof. Then, (S, v, D) is soft Lindelof.*

Proof. Let $\mathcal{A} \subseteq v$ such that $\widetilde{\cup}_{A\in\mathcal{A}} A = 1_D$. Then, $\widetilde{\cup}_{A\in\mathcal{A}} f_{pu}^{-1}(A) = 1_B$. Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(A) : A \in \mathcal{A} \right\} \subseteq CO(T, \pi, B) \subseteq \pi$. Since (T, π, B) is soft almost Lindelof, then there exists a countable subfamily $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\widetilde{\cup}_{A\in\mathcal{A}_1} Cl_\pi\left(f_{pu}^{-1}(A)\right) = \widetilde{\cup}_{A\in\mathcal{A}_1} f_{pu}^{-1}(A) = f_{pu}^{-1}\left(\widetilde{\cup}_{A\in\mathcal{A}_1} A\right) = 1_B$ and thus, $f_{pu}\left(f_{pu}^{-1}\left(\widetilde{\cup}_{A\in\mathcal{A}_1} A\right)\right) = f_{pu}(1_B)$. Since f_{pu} is surjective, then $f_{pu}\left(f_{pu}^{-1}\left(\widetilde{\cup}_{A\in\mathcal{A}_1} A\right)\right) = \widetilde{\cup}_{A\in\mathcal{A}_1} A$ and $f_{pu}(1_B) = 1_D$. Therefore, $\widetilde{\cup}_{A\in\mathcal{A}_1} A = 1_D$. It follows that (S, v, D) is soft Lindelof. □

Theorem 31. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective, soft strongly continuous, and soft open mapping such that $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T, π, B) for each $d_s \in SP(S, D)$. If (T, π, B) is soft almost paracompact, then (S, v, D) is soft metacompact.*

Proof. Let $\mathcal{H} \subseteq v$ such that $\widetilde{\cup}_{H\in\mathcal{H}} H = 1_D$. Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(H) : H \in \mathcal{H} \right\} \subseteq CO(T, \pi, B) \subseteq \pi$. Since (T, π, B) is soft almost paracompact and $\widetilde{\cup}_{H\in\mathcal{H}} f_{pu}^{-1}(H) = f_{pu}^{-1}\left(\widetilde{\cup}_{H\in\mathcal{H}} H\right) = f_{pu}^{-1}(1_D) = 1_B$, then there exists a collection $\mathcal{K} \subseteq \pi$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K\in\mathcal{K}} Cl_\pi(K) = 1_B$, and for every $K \in \mathcal{K}$ there exists $H \in \mathcal{H}$ such that $K \widetilde{\subseteq} f_{pu}^{-1}(H)$. Let $\mathcal{M} = \left\{ f_{pu}(K) : K \in \mathcal{K} \right\}$. □

Claim. *1. $\mathcal{M} \subseteq v$.*
2. $\widetilde{\cup}_{M\in\mathcal{M}} M = 1_D$.
3. For each $M \in \mathcal{M}$, there exists $H \in \mathcal{H}$ such that $M \widetilde{\subseteq} H$.
4. $\mathcal{M}$ is soft point finite.

Proof of Claim. 1. Since $\mathcal{K} \subseteq \pi$ and f_{pu} is soft open, then $\mathcal{M} = \left\{ f_{pu}(K) : K \in \mathcal{K} \right\} \subseteq v$.

2. Since f_{pu} is surjective, then $f_{pu}(1_B) = 1_D$. Since f_{pu} is soft strongly continuous, then for every $K \in \mathcal{K}$, $f_{pu}(K) = f_{pu}(Cl_\pi(K))$. Thus, $\widetilde{\cup}_{M\in\mathcal{M}} M = \widetilde{\cup}_{K\in\mathcal{K}} f_{pu}(K) = \widetilde{\cup}_{K\in\mathcal{K}} f_{pu}(Cl_\pi(K)) = f_{pu}\left(\widetilde{\cup}_{K\in\mathcal{K}} Cl_\pi(K)\right) = f_{pu}(1_B) = 1_D$.

3. Let $M \in \mathcal{M}$. Then, there exists $K \in \mathcal{K}$ such that $f_{pu}(K) = M$. Choose $H \in \mathcal{H}$ such that $K \widetilde{\subseteq} f_{pu}^{-1}(H)$. So, $M = f_{pu}(K) \widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(H)\right) \widetilde{\subseteq} H$.

4. Let $d_s \in SP(S, D)$. Since $\mathcal{K}$ is soft locally finite, then for every $b_t \widetilde{\in} f_{pu}^{-1}(d_s)$, there exists $G_{b_t} \in \pi$ such that $b_t \widetilde{\in} G_{b_t}$ and the collection $\left\{ K \in \mathcal{K} : K \widetilde{\cap} G_{b_t} \neq 0_B \right\}$ is finite. For each $b_t \widetilde{\in} f_{pu}^{-1}(d_s)$, put $\mathcal{S}_{b_t} = \left\{ K \in \mathcal{K} : K \widetilde{\cap} G_{b_t} \neq 0_B \right\}$. Since $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T, π, B) and $f_{pu}^{-1}(d_s) \widetilde{\subseteq} \widetilde{\cup}_{b_t \widetilde{\in} f_{pu}^{-1}(d_s)} G_{b_t}$, then there exists a finite subset $\mathcal{A} \subseteq \left\{ b_t : b_t \widetilde{\in} f_{pu}^{-1}(d_s) \right\}$ such that $f_{pu}^{-1}(d_s) \widetilde{\subseteq} \widetilde{\cup}_{b_t\in\mathcal{A}} G_{b_t}$. If $d_s \widetilde{\in} f_{pu}(R)$ for some $R \in \mathcal{K}$, then there exists $w_r \widetilde{\in} R \widetilde{\cap} f_{pu}^{-1}(d_s)$. Since $w_r \widetilde{\in} f_{pu}^{-1}(d_s) \widetilde{\subseteq} \widetilde{\cup}_{b_t\in\mathcal{A}} G_{b_t}$, then there exists $b_t \in \mathcal{A}$ such that $w_r \widetilde{\in} G_{b_t}$. Thus, we have $w_r \widetilde{\in} R \widetilde{\cap} G_{b_t}$ and hence $R \in \mathcal{S}_{b_t}$. Therefore, $\left\{ K \in \mathcal{K} : d_s \widetilde{\in} f_{pu}(K) \right\} \subseteq \left\{ K \in \mathcal{K} : R \in \mathcal{S}_{b_t}, b_t \in \mathcal{A} \right\}$. Since $\left\{ K \in \mathcal{K} : R \in \mathcal{S}_{b_t}, b_t \in \mathcal{A} \right\}$ is finite, then $\left\{ K \in \mathcal{K} : d_s \widetilde{\in} f_{pu}(K) \right\}$ is finite. Hence, $\mathcal{M}$ is soft point finite.

Therefore, by the above Claim, (S, v, D) is soft metacompact. □

Theorem 32. *Let $f_{pu} : (T, \pi, B) \longrightarrow (S, v, D)$ be surjective, soft strongly continuous, soft closed, and soft almost open mapping such that $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T, π, B) for each $d_s \in SP(S, D)$. If (T, π, B) is soft nearly paracompact, then (S, v, D) is soft paracompact.*

Proof. Let $\mathcal{H} \subseteq v$ such that $\widetilde{\cup}_{H\in\mathcal{H}} H = 1_D$. Since f_{pu} is soft strongly continuous, then $\left\{ f_{pu}^{-1}(H) : H \in \mathcal{H} \right\} \subseteq CO(T, \pi, B) \subseteq \pi$. Since (T, π, B) is soft nearly paracompact and $\widetilde{\cup}_{H\in\mathcal{H}} f_{pu}^{-1}(H) = f_{pu}^{-1}\left(\widetilde{\cup}_{H\in\mathcal{H}} H\right) = f_{pu}^{-1}(1_D) = 1_B$, then there exists a collection $\mathcal{K} \subseteq \pi$ such that $\mathcal{K}$ is soft locally finite, $\widetilde{\cup}_{K\in\mathcal{K}}Int_\pi(Cl_\pi(K)) = 1_B$, and for every $K \in \mathcal{K}$ there exists $H \in \mathcal{H}$ such that $K\widetilde{\subseteq}f_{pu}^{-1}(H)$. Let $\mathcal{M} = \{Int_\upsilon(f_{pu}(K)) : K \in \mathcal{K}\}$. Then, $\mathcal{M} \subseteq \upsilon$. □

Claim. *1.* $\widetilde{\cup}_{M\in\mathcal{M}}M = 1_D$.

2. For each $M \in \mathcal{M}$, *there exists* $H \in \mathcal{H}$ *such that* $K\widetilde{\subseteq}H$.

3. $\mathcal{M}$ *is soft locally finite.*

Proof of Claim. 1. Since f_{pu} is surjective, then $f_{pu}(1_B) = 1_D$. Since $\mathcal{K} \subseteq \pi$, then for every $K \in \mathcal{K}$, $Int_\pi(Cl_\pi(K)) \in RO\ (T, \pi, B)$. Since f_{pu} is soft almost open, then for every $K \in \mathcal{K}$, $f_{pu}(Int_\pi(Cl_\pi(K))) \in \upsilon$. Since f_{pu} is soft strongly continuous, then for every $K \in \mathcal{K}$, $f_{pu}(Int_\pi(Cl_\pi(K)))\widetilde{\subseteq}f_{pu}(Cl_\pi(K)) = f_{pu}(K)$ and thus, $f_{pu}(Int_\pi(Cl_\pi(K)))\widetilde{\subseteq}Int_\upsilon(f_{pu}(K))$. Therefore,

$$\begin{aligned} 1_D &= f_{pu}(1_B) \\ &= f_{pu}\left(\widetilde{\cup}_{k\in\mathcal{K}}(Int_\pi(Cl_\pi(K)))\right) \\ &= \widetilde{\cup}_{k\in\mathcal{K}}f_{pu}(Int_\pi(Cl_\pi(K))) \\ &\widetilde{\subseteq} \widetilde{\cup}_{k\in\mathcal{K}}Int_\upsilon(f_{pu}(K)) \\ &= \widetilde{\cup}_{M\in\mathcal{M}}M. \end{aligned}$$

2. Let $M \in \mathcal{M}$. Then, there exists $K \in \mathcal{K}$ such that $Int_\pi\left(f_{pu}(K)\right) = M$. Choose $H \in \mathcal{H}$ such that $K\widetilde{\subseteq}f_{pu}^{-1}(H)$. Thus, $M = Int_\pi\left(f_{pu}(K)\right)\widetilde{\subseteq}f_{pu}(K)\widetilde{\subseteq}f_{pu}\left(f_{pu}^{-1}(H)\right)\widetilde{\subseteq}H$.

3. Let $d_s \in SP(S, D)$. Since $\mathcal{K}$ is soft locally finite, then for every $b_t\widetilde{\in}f_{pu}^{-1}(d_s)$, there exists $G_{b_t} \in \pi$ such that $b_t\widetilde{\in}G_{b_t}$ and the collection $\{K \in \mathcal{K} : K\widetilde{\cap}G_{b_t} \neq 0_B\}$ is finite. For each $b_t\widetilde{\in}f_{pu}^{-1}(d_s)$, put $\mathcal{S}_{b_t} = \{K \in \mathcal{K} : K\widetilde{\cap}G_{b_t} \neq 0_B\}$. Since $f_{pu}^{-1}(d_s)$ is a soft compact subset of (T, π, B) and $f_{pu}^{-1}(d_s)\widetilde{\subseteq}\widetilde{\cup}_{b_t\widetilde{\in}f_{pu}^{-1}(d_s)}G_{b_t}$, then there exists a finite subset $\mathcal{A} \subseteq \left\{b_t : b_t\widetilde{\in}f_{pu}^{-1}(d_s)\right\}$ such that $f_{pu}^{-1}(d_s)\widetilde{\subseteq}\widetilde{\cup}_{b_t\in\mathcal{A}}G_{b_t}$. Let $G = \widetilde{\cup}_{b_t\in\mathcal{A}}G_{b_t}$. Then, the collection $\{K \in \mathcal{K} : K\widetilde{\cap}G \neq 0_B\}$ is finite. Let $S = 1_D - f_{pu}(1_D - G)$. Since f_{pu} is soft closed, then $f_{pu}(1_D - G) \in \upsilon^c$. Thus, we have and $d_s\widetilde{\in}S \in \upsilon$ and the collection $\{M \in \mathcal{M} : M\widetilde{\cap}S \neq 0_D\}$ is finite. Hence, $\mathcal{M}$ is soft locally finite.

Therefore, by the above Claim, (S, υ, D) is soft paracompact. □

4. Conclusions

Numerous facets of our daily existence are uncertain. The soft set theory is one of the ideas put forth to deal with uncertainty. This study focuses on soft topology, a novel mathematical framework developed by topologists using soft sets.

In this paper, soft complete continuity and soft strong continuity as stronger forms of soft continuity are introduced. Several characterizations and relationships related to them are given. Moreover, several soft mapping theorems regarding soft compactness, soft Lindelofness, soft connectedness, soft regularity, soft normality, soft almost regularity, soft mild normality, soft almost compactness, soft almost Lindelofness, soft near compactness, soft near Lindelofness, soft paracompactness, soft near paracompactness, soft almost paracompactness, and soft metacompactness are obtained. The link between our novel concepts in soft topological spaces and their topologically corresponding notions have been investigated.

The following topics could be considered in future studies: (1) investigating soft metacompactness; (2) investigating soft *C-C* spaces.

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