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On the Propagation Model of Two-Component Nonlinear Optical Waves

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Abstract: Currently, two-component integrable nonlinear equations from the hierarchies of the vector nonlinear Schrodinger equation and the vector derivative nonlinear Schrödinger equation are being actively investigated. In this paper, we propose a new hierarchy of two-component integrable nonlinear equations, which have an important difference from the already known equations. To construct the hierarchical equations, we use the monodromy matrix method, as first proposed by B.A. Dubrovin. The method we use consists of solving the following sequence of problems. First, using the Lax operator, we find the monodromy matrix, which is a polynomial in the spectral parameter. More precisely, we find a sequence of monodromy matrices dependent on the degree of this polynomial. Each Lax operator has its own sequence of monodromy matrices. Then, using the terms from the decomposition of the monodromy matrix, we construct a sequence of second operators from a Lax pair. A hierarchy of evolutionary integrable nonlinear equations follows from the conditions of compatibility of the sequence of Lax pairs. Also, knowledge of the monodromy matrix allows us to find stationary equations that are analogs of the Novikov equations for the Korteweg–de Vries equation. In addition, the characteristic equation of the monodromy matrix corresponds to the spectral curve equation of the relevant multiphase solution for the integrable nonlinear equation. Since the coefficients of the spectral curve equation are integrals of the hierarchical equations, they can be utilized to find the simplest solutions of the constructed integrable nonlinear equations. In this paper, we demonstrate the operation of this method, starting with the assignment of the Lax operator and ending with the construction of the simplest solutions.

Keywords: spectral curve; derivative NLS equation; Lax pair; vector NLS equation; monodromy matrix

MSC: 35Q51; 37C55; 37K40

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1. Introduction

The transmission of information in optical fibers is carried out by means of modulation of the reference laser signal. The fiber material is selected in such a way that the nonlinear effects resulting from the wave's interaction with the medium compensate for the dispersion. The simplest model for the propagation of a polarized signal in an optical fiber is the focusing nonlinear Schrodinger equation

$$ip_z + p_{tt} + 2|p|^2 p = 0. \tag{1}$$

Here, p is the slowly changing complex amplitude of the modulated signal superimposed on the laser reference wave, z is the coordinate along the direction of the signal propagation, and t is a linear combination of the time and longitudinal coordinates (see, for example, [1] and references therein). It is not difficult to understand that Equation (1) is an equation in the dimensionless variables, i.e., it is obtained from the original equation by replacing variables and functions. This model is obtained from Maxwell's equations by discarding terms that have little effect on the behavior of a nonlinear wave, i.e., it describes the real process with some accuracy [1]. The advantage of Equation (1) is that it refers to integrable nonlinear equations (see, for example, [1–6]), which have solutions in the form of solitary waves (solitons). Solitons are nonlinear waves that propagate indefinitely without the loss of shape and speed. Naturally, in real waveguides, solitons lose energy over time, but amplifiers and repeaters compensate for these losses. When using lasers that generate femtosecond pulses, the dispersion terms of the third, fourth, and fifth orders must be taken into account in the models. These models correspond to the integrable Hirota equations [7–10]

$$ip_z + \alpha(p_{tt} + 2|p|^2 p) - i\beta(p_{ttt} + 6|p|^2 p_t) = 0.$$

and integrable higher nonlinear Schrodinger equations [11–14]. There are also non-integrable models, which we will not discuss in this paper. At the same time, these equations are also actively investigated for the presence of solutions in the form of solitons [15]. To account for other types of interactions between waves and the waveguide medium, derivative variants of nonlinear Schrödinger equations can be used [16–28], including the Kundu–Eckhaus equation [29–31]. Recently, wave models with double polarization have been actively studied, since with the help of appropriate signals, it is possible to transmit twice as much information [32–37].

Nonlinear signals are studied and filtered using a nonlinear Fourier transform [2,4,5,20,28,34,36–42]. In this case, the spectral analysis targets not the nonlinear signal itself, but the first operator from the Lax pair. Every basic integrable nonlinear differential equation can be obtained as a condition for the compatibility of two linear differential equations, called a Lax pair [1–6]. In particular, the Lax pair for Equation (1) has the form [1–6,43,44]

$$\begin{aligned} \partial_t \Psi &= U\Psi, \\ \partial_z \Psi_t &= V_1 \Psi, \end{aligned}$$

where $U = \lambda J + U^0$, $V_1 = \lambda U + V_1^0$,

$$J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & ip \\ -iq & 0 \end{pmatrix}, \quad V_1^0 = \begin{pmatrix} -ipq & -p_t \\ -q_t & ipq \end{pmatrix},$$

$q = -p^*$, $i^2 = -1$.

Note that each basic integrable nonlinear equation is the first equation from an infinite sequence of equations called a hierarchy. Each equation from the hierarchy corresponds to its own second Lax pair operator. In particular, the nonlinear Schrodinger equation is the first equation from the Ablowitz–Kaup–Newell–Sigur hierarchy [43–45]. One of the useful features of hierarchies of integrable nonlinear equations is the fact that there are functions $\widehat{p}(t, z_1, z_2, \dots)$ that satisfy all the equations of the hierarchy simultaneously. Hence, the functions $p(t,z) = \widehat{p}(t, \alpha_1(z), \alpha_2(z), \dots)$ will be solutions of the so-called mixed equations [11–14,43,44]. Therefore, considering mixed equations is one way to increase the number of integrable models for wave propagation in nonlinear media. Another way to construct new integrable models corresponding to the new properties of the studied nonlinear signals is to consider new Lax pairs. In particular, the propagation of bi-polarized waves through nonlinear optical waveguides is characterized by the Manakov system, which is a compatibility condition of linear matrix differential equations with third-order matrices [32–37,46–51]. Also, the compatibility conditions of Lax pairs with third-order matrices lead to two-component derivative nonlinear Schrödinger equations that describe more complex models of bi-polarized waves [52–56].

Naturally, all the models are regularly tested in practice when experimenters attempt to detect certain forms of signals obtained theoretically [57–60]. Therefore, one of the goals of theorists is to create new integrable models that could be used to describe nonlinear phenomena. In this paper, we propose a new integrable model, describing the propagation of two interacting nonlinear waves

$$\begin{aligned} ip_{z_2} &= p_{tt} + ip^2 p_t^* + \frac{1}{2}\Big(|p|^4 p + 8|u|^2 p + 4u^2 p^*\Big), \\ iu_{z_2} &= u_{tt} - ip^2 u_t^* - 2i(pu^* + up^*)p_t + \frac{1}{2}(3|p|^4 u + 2|p|^2 p^2 u^* + 4|u|^2 u). \end{aligned}$$

In the absence of one of the waves ($u \equiv 0$), the model is reduced to the Gerdjikov–Ivanov equation [23–25,27,28]. The presented article consists of an introduction, five sections, and concluding remarks. In the first section, we consider various possible variants of the Lax operator in the case of a quadratic spectral bundle. Based on the results of this section, we decided to investigate a model with a more general than usual Lax operator. Section 3 of the paper is devoted to finding the structure of the monodromy matrix and the recurrent relations between its elements.

In Section 4, the stationary equations are derived and equations of spectral curves are considered. In the context of scalar-derivative NLS equations [28], the stationary equations form two groups. But, unlike the scalar case where equations from only one group were applicable, for this model, it's imperative to utilize both sets of equations. Moreover, in the case of standard vector NLS equations [37,61], both components p_1 and p_2 satisfy similar stationary equations. In this paper, components p and u satisfy stationary equations with different structures.

Section 5 defines the sequence of the second equations from the Lax pair and the evolutionary integrable nonlinear equations from the corresponding hierarchy. Note that even hierarchical equations differ from odd ones. In particular, for $n = 3$, we have

$$\begin{aligned} ip_{z_3} &= 2u_{tt} - 2i(p^* u_t - u p_t^*)p - i(6pu^* + 4p^* u)p_t + (|p|^4 - 4|u|^2)u, \\ u_{z_3} &= -p_{ttt} - i(p^* p_{tt} + p p_{tt}^*)p - 6(u^* p_t + p^* u_t)u - (4u^* u_t + 2u u_t^*)p \\ &\quad - 2ip|p_t|^2 - \frac{3}{2}|p|^4 p_t - 2i|pu|^2 p - \frac{i}{2}|p|^6 p. \end{aligned}$$

If we put $u = 0$ in an odd equation, then it ceases to be an evolutionary integrable nonlinear equation. In Section 6, we present the simplest solutions of the second equation from the hierarchy we constructed. In particular, we find a solution in the form of a solitary wave, the components of which are described by different formulas. This is due to the fact that each component satisfies its own nonlinear differential equation.

2. Structure of the Lax Operator for a Quadratic Spectral Bundle

Let the Lax pair be given by the equations

$$i\Psi_t + U\Psi = \mathbf{0}, \tag{2}$$
$$i\Psi_z + V\Psi = \mathbf{0}, \tag{3}$$

where

$$U = \lambda^2 J + \lambda Q_1 + Q_2, \quad V = \lambda^2 K + \lambda S_1 + S_2, \tag{4}$$

J, K are constant matrices, and λ is a spectral parameter.

The condition of compatibility of the Equations (2) and (3) has the form

$$iV_t - iU_z + UV - VU = \mathbf{0}$$

or

$$\begin{aligned} &\lambda^4(JK - KJ) + \lambda^3(JS_1 - S_1 J + Q_1 K - KQ_1) \\ &\qquad + \lambda^2(JS_2 - S_2 J + Q_1 S_1 - S_1 Q_1 + Q_2 K - KQ_2) + \cdots = \mathbf{0}. \end{aligned} \tag{5}$$

Condition (5), and the integrable nonlinear equations, give algebraic constraints on the elements of the matrices U and V. These restrictions have the following form:

$$JK = KJ, \tag{6}$$
$$[J, S_1] = [K, Q_1], \tag{7}$$
$$[J, S_2] + Q_1 S_1 = [K, Q_2] + S_1 Q_1. \tag{8}$$

Usually, solutions of Equation (7) are written using the following relations:

$$S_1 = [K, R], \quad Q_1 = [J, R], \tag{9}$$

where R is a matrix of the same size.

Substituting (9) into (8), and simplifying it, we obtain

$$\begin{aligned} &J(S_2 + RKR - R^2K) - (S_2 + RKR - R^2K)J \\ &= K(Q_2 + RJR - R^2J) - (Q_2 + RJR - R^2J)K. \end{aligned}$$

or

$$[J, S_2 + [RK, R]] = [K, Q_2 + [RJ, R]].$$

Therefore, the following representation of matrices S_2 and Q_2 can be used

$$\begin{aligned} S_2 + [RK, R] &= [K, R_2], \\ Q_2 + [RJ, R] &= [J, R_2] \end{aligned}$$

or

$$\begin{aligned} S_2 &= [K, R_2] + [R, RK], \\ Q_2 &= [J, R_2] + [R, RJ], \end{aligned} \tag{10}$$

where R_2 is a new matrix of the same size.

Note that on the right-hand sides of equalities (9) and (10), it is possible to add linear combinations of matrices commuting with J and K.

Taking into account the remaining constraints from condition (5) leads to the following matrix of operators (2) and (3):

$$\begin{aligned} S_1 &= [K, R], \quad Q_1 = [J, R], \\ S_2 &= [K, R_2] + [R, RK] + s_{20}K, \\ Q_2 &= [J, R_2] + [R, RJ] + q_{20}J, \end{aligned} \tag{11}$$

where s_{20} and q_{20} are certain functions.

It is easy to see that the matrix J completely defines the structure of the operator (2). For example, if $J = \mathrm{diag}(1, -1)$, then

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 2r_{12} \\ -2r_{21} & 0 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 0 & 2\tilde{r}_{12} \\ -2\tilde{r}_{21} & 0 \end{pmatrix} + \begin{pmatrix} 2r_{12}r_{21} & -2r_{11}r_{12} \\ 2r_{21}r_{22} & -2r_{12}r_{21} \end{pmatrix} + \begin{pmatrix} q_{20} & 0 \\ 0 & -q_{20} \end{pmatrix}. \end{aligned}$$

In the case of the main variants of the derivative nonlinear Schrödinger equations, we have ($q = -p^*$)

- In the case of the Kaup–Newell equation [16–20,23,28]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \tilde{r}_{12} = r_{11}r_{12}, \quad \tilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = -2r_{12}r_{21};$$

- In the case of the Chen–Lee–Liu equation [22,23,28,62,63]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \widetilde{r}_{12} = r_{11}r_{12}, \quad \widetilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = -pq/4;$$

- in the case of the Gerdjikov–Ivanov equation [23–25,28]

$$r_{12} = ip/2, \quad r_{21} = -iq/2, \quad \widetilde{r}_{12} = r_{11}r_{12}, \quad \widetilde{r}_{21} = r_{21}r_{22}, \quad q_{20} = 0.$$

By choosing other values of q_{20}, other special cases of the generalized derivative nonlinear Schrödinger equation can be obtained [26,28,64–66]. Note that the solutions of these equations are connected by a gauge transformation preserving the amplitude (see [23,27,28,67–69]).

In these models, it is easy to see that matrices Q_1 and Q_2 only depend on functions p and q, and that matrix Q_2 is diagonal. Therefore, adding functions u and v to the non-diagonal terms of matrix Q_2 allows us to explore a new nonlinear integrable model:

$$r_{12} = p/2, \quad r_{21} = q/2, \quad \widetilde{r}_{12} = r_{11}r_{12} + u/2, \quad \widetilde{r}_{21} = r_{21}r_{22} + v/2, \quad q_{20} = 0.$$

This model at $u = v = 0$ is transformed into one of the special cases of the generalized derivative nonlinear Schrödinger equation [28]. In the case of the new model, matrices J, Q_1, and Q_2 are equal:

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & p \\ -q & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{2}\begin{pmatrix} pq & 2u \\ -2v & -pq \end{pmatrix}. \tag{12}$$

3. The Monodromy Matrix

The monodromy matrix is a key object of the spectral analysis of periodic solutions of the integrable nonlinear models. Spectral data in the case of periodic nonlinear signals consist of the spectral curve, its genus, and its parameters. The spectral curve equation is the characteristic equation of the monodromy matrix M, which is a polynomial of the spectral parameter λ

$$M = \sum_{j=0}^{N} m_j(t)\lambda^j,$$

which satisfies equation [70]

$$iM_t + UM - MU = \mathbf{0}. \tag{13}$$

The monodromy matrix also exists in the limiting cases when the solution periods become infinite. In particular, solitary waves are the limiting cases of periodic waves when the period of a nonlinear wave becomes infinitely large.

From Equation (13), where matrix U is determined by equalities (4) and (12), the following structure of matrix M follows

$$M = W_n + \sum_{k=1}^{n-1} c_k W_{n-k} + c_n U + c_{n+1} W_{-1} + c_{n+2} J, \tag{14}$$

where $W_{-1} = \lambda J + Q_1$, $U = \lambda W_{-1} + Q_2$, $W_1 = \lambda U + W_1^0$,

$$W_{k+1} = \lambda V_k + W_{k+1}^0, \quad W_k^0 = \begin{pmatrix} F_k & H_k \\ -G_k & -F_k \end{pmatrix}, \quad k \geq 1,$$

c_k are real constants.

From Equation (6), it also follows the recurrence relations on the elements of the matrix W_k^0:

$$
\begin{aligned}
&H_1 = -\frac{i}{2}p_t, \quad G_1 = \frac{i}{2}q_t, \\
&H_2 = pF_1 - \frac{i}{2}u_t, \quad G_2 = qF_1 + \frac{i}{2}v_t, \\
&H_{k+2} = pF_{k+1} + uF_k - \frac{1}{2}pqH_k - \frac{i}{2}\partial_t H_k, \\
&G_{k+2} = qF_{k+1} + vF_k - \frac{1}{2}pqG_k + \frac{i}{2}\partial_t G_k, \\
&\partial_t F_k = i(vH_k - uG_k + qH_{k+1} - pG_{k+1}).
\end{aligned}
\tag{15}
$$

In particular,

$$
\begin{aligned}
F_1 &= \frac{1}{2}(pv + qu), \\
H_2 &= \frac{1}{2}(pqu + p^2 v - iu_t), \\
G_2 &= \frac{1}{2}(pqv + q^2 u + iv_t), \\
F_2 &= \frac{1}{2}uv - \frac{1}{8}p^2q^2 + \frac{i}{4}(pq_t - qp_t), \\
H_3 &= uvp + \frac{1}{2}u^2 q - \frac{1}{8}p^3q^2 + \frac{i}{4}p^2 q_t - \frac{1}{4}p_{tt}, \\
G_3 &= uvq + \frac{1}{2}v^2 p - \frac{1}{8}p^2q^3 - \frac{i}{4}q^2 p_t - \frac{1}{4}q_{tt}, \\
F_3 &= \frac{1}{4}(uq + pv)pq + \frac{i}{4}(pv_t - vp_t + uq_t - qu_t), \\
H_4 &= \frac{1}{2}u^2 v - \frac{1}{8}p^2q^2u - \frac{i}{4}(2qu + 3pv)p_t + \frac{i}{4}(uq_t - qu_t)p - \frac{1}{4}u_{tt}, \\
G_4 &= \frac{1}{2}uv^2 - \frac{1}{8}p^2q^2v + \frac{i}{4}(3qu + 2pv)q_t + \frac{i}{4}(pv_t - vp_t)q - \frac{1}{4}v_{tt}, \\
F_4 &= pquv - \frac{1}{16}p^3q^3 + \frac{3}{8}(p^2v^2 + q^2u^2) + \frac{1}{8}p_t q_t \\
&\quad + \frac{i}{4}(uv_t - vu_t) - \frac{1}{8}(pq_{tt} + qp_{tt}).
\end{aligned}
$$

It follows from recurrent relations (15) that when the reality conditions

$$
q = \sigma p^*, \quad v = \sigma u^*, \quad \sigma = \pm 1, \tag{16}
$$

are met, the elements of the monodromy matrix satisfy the following relations:

$$
G_k = \sigma H_k^*, \quad F_k^* = F_k. \tag{17}
$$

4. Conservation Laws

Since our work considers the hierarchy of integrable nonlinear equations, we have an infinite set of conservation laws. These conservation laws are divided into two groups. The first group is formed by stationary equations, which are satisfied by multiphase (finite-zone and their degeneration) solutions. In the case of the Korteweg–de Vries equation, these equations are called Novikov equations.

For each hierarchy, there are different solutions with the same number of phases. Each type of multiphase solution has its own type of spectral curve. On the one hand, the coefficients of the spectral curve equation are constant values, and on the other hand, they are functions of the solution and its derivatives. One part of the coefficients of the spectral curve equation is expressed in terms of the coefficients of stationary equations, and the second part consists of additional integrals of the solution and its derivatives. Accordingly, these additional integrals determine the type of multiphase solution with a given number of phases.

The first set of conservation laws is described by the following stationary nonlinear differential equations

$$H_{n+1} + \sum_{j=1}^{n} c_j H_{n+1-j} + c_{n+1} u + c_{n+2} p = 0,$$
$$G_{n+1} + \sum_{j=1}^{n} c_j G_{n+1-j} + c_{n+1} v + c_{n+2} q = 0,$$

and

$$\begin{aligned} &H_{n+2} - pF_{n+1} + \sum_{j=1}^{n-1} c_j \big(H_{n+2-j} - pF_{n+1-j}\big) \\ &\quad - \frac{ic_n}{2} u_t - \frac{c_{n+1}}{2}\left(p^2 q + ip_t\right) + c_{n+2} u = 0, \\ &G_{n+2} - qF_{n+1} + \sum_{j=1}^{n-1} c_j \big(G_{n+2-j} - qF_{n+1-j}\big) \\ &\quad + \frac{ic_n}{2} v_t - \frac{c_{n+1}}{2}\left(pq^2 - iq_t\right) + c_{n+2} v = 0. \end{aligned}$$

These equations also follow from Equation (13). It follows from the stationary equations and reality conditions (16) and (17) that the constants c_j ($j = 1, \dots, n+2$) are real.

Any m-phase solution for $m \le n$ and all values of t and z satisfy these stationary equations. The parameters of the corresponding multiphase solution depend on the constants c_k and coefficients f_k of the spectral curve (see below). It follows from the reality conditions (17) that stationary equations admit reduction (16).

In particular, for $n = 0$, the stationary equations have the form

$$\begin{aligned} &ip_t - 2c_1 u - 2c_2 p = 0, \\ &iq_t + 2c_1 v + 2c_2 q = 0, \\ &iu_t + c_1(p^2 q + ip_t) - 2c_2 u = 0, \\ &iv_t - c_1(pq^2 - iq_t) + 2c_2 v = 0. \end{aligned} \tag{18}$$

It is easy to see that for $c_1 \neq 0$, the components u and v are connected to the components p and q using the following relations

$$u = -\frac{1}{2c_1}(2c_2 p - ip_t), \quad v = -\frac{1}{2c_1}(2c_2 q + iq_t).$$

As will be shown below, in this case, all components are expressed in terms of elliptic functions or their degeneration. Note that for $c_1 = 0$, the system of stationary Equation (18) splits into two separate identical systems, the solutions of which are plane waves. Accordingly, when $c_1 = 0$, additional components can be removed from the model by putting $u = v = 0$.

Recall that the characteristic equation of the monodromy matrix M is the equation of the corresponding spectral curve

$$\Gamma: \quad \mathcal{R}(\mu, \lambda) = \det(M - \mu I) = 0. \tag{19}$$

Here, I is the identity matrix. It follows from Equations (14) and (19) that the equation of the spectral curve has the form

$$\mu^2 = \lambda^{2n+4} + \sum_{k=1}^{2n+4} f_k \lambda^{2n+4-k}, \tag{20}$$

where f_k are constants (integrals). Therefore, the spectral curve Γ is an hyperelliptic curve of genus $g = n + 1$. Naturally, this statement is true only for equations corresponding to non-degenerate connected curves.

For $n = 0$ ($g = 1$), the equation of the spectral curve has the form

$$\mu^2 = \lambda^4 + 2c_1\lambda^3 + (c_1^2 + 2c_2)\lambda^2 + f_3\lambda + f_4,$$

where the integrals f_3 and f_4 are equal

$$\begin{aligned} f_3 &= 2c_1c_2 - c_1pq - uq - pv, \\ f_4 &= c_2^2 + (c_2 - c_1^2)pq + \frac{1}{4}p^2q^2 - uv - c_1(pv + uq). \end{aligned}$$

i.e., for $n = 0$, the non-degenerate spectral curve is elliptic.

For $n = 1$ ($g = 2$), the stationary equations have a more complex form

$$\begin{aligned} &iu_t - (pv + qu)p + ic_1p_t - 2c_2u - 2c_3p = 0, \\ &iv_t + (pv + qu)q + ic_1q_t + 2c_2v + 2c_3q = 0, \\ &p_{tt} - 2(pv + qu)u - ipqp_t + 2ic_1u_t + 2c_2(p^2q + ip_t) - 4c_3u = 0, \\ &q_{tt} - 2(qu + pv)v + ipqq_t - 2ic_1v_t + 2c_2(pq^2 - iq_t) - 4c_3v = 0. \end{aligned} \tag{21}$$

It follows from the first two equations of system (21) that in the case of $g = 2$, the dependence of the components u and v on p and q can be found in the solution of the following linear matrix differential equation

$$i\partial_t\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} pq + 2c_2 & p^2 \\ -q^2 & -pq - 2c_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -ic_1\partial_t\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} 2c_3 & 0 \\ 0 & -2c_3 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}.$$

It is easy to see that for $c_1 = c_3 = 0$, this equation admits solutions of the form $u = v = 0$. In this case, the stationary equations for the components p and q will take a simpler form

$$\begin{aligned} p_{tt} - ipqp_t + 2c_2(p^2q + ip_t) = 0, \\ q_{tt} + ipqq_t + 2c_2(pq^2 - iq_t) = 0. \end{aligned} \tag{22}$$

At the same time, it is not difficult to see that for $c_1 = c_3 = 0$ there are solutions of stationary Equation (21) that satisfy the condition $u^2 + v^2 \not\equiv 0$.

The equation of the spectral curve for $n = 1$ has the form

$$\mu^2 = \lambda^6 + 2c_1\lambda^5 + (c_1^2 + 2c_2)\lambda^4 + 2(c_1c_2 + c_3)\lambda^3 + f_4\lambda^2 + f_5\lambda + f_6,$$

where integrals f_4, f_5, and f_6 are equal

$$\begin{aligned} f_4 &= c_2^2 + 2c_1c_3 - c_1(pv + qu) - c_2pq + \frac{1}{4}p^2q^2 - uv - \frac{i}{2}(pq_t - qp_t), \\ f_5 &= 2c_2c_3 + (c_3 - c_1c_2)pq - c_1^2(pv + qu) + \frac{1}{2}c_1p^2q^2 - 2c_1uv + \frac{1}{2}pq(pv + qu) \\ &\quad - \frac{i}{2}c_1(pq_t - qp_t) - \frac{i}{2}(uq_t - vp_t), \\ f_6 &= c_3^2 + (c_1c_3 - c_2^2)pq + \frac{1}{4}c_1^2p^2q^2 - c_1^2uv + (c_3 - c_1c_2)(pv + qu) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}c_1pq(pv+qu)+\frac{1}{4}(p^2v^2+q^2u^2)+\frac{1}{2}pquv\\
&-\frac{i}{2}c_2(pq_t-qp_t)-\frac{i}{2}c_1(uq_t-vp_t)-\frac{1}{4}p_tq_t.
\end{aligned}
$$

For $n=2$ $(g=3)$, the stationary equations become even more complicated

$$
\begin{aligned}
&u_{tt}+ip^2v_t+2i(pv+qu)p_t-2u^2v+p^3qv+\frac{3}{2}p^2q^2u\\
&\qquad+c_1(p_{tt}-ipqp_t-2(pv+qu)u)+2ic_2u_t+2c_3(p^2q+ip_t)-4c_4u=0,\\
&v_{tt}-iq^2u_t-2i(pv+qu)q_t-2uv^2+pq^3u+\frac{3}{2}p^2q^2v\\
&\qquad+c_1(q_{tt}+ipqq_t-2(pv+qu)v)-2ic_2v_t+2c_3(pq^2-iq_t)-4c_4v=0,\\
&p_{tt}-ip^2q_t-4uvp-2qu^2+\frac{1}{2}p^3q^2+2c_1(iu_t-(pv+qu)p)\\
&\qquad+2ic_2p_t-4c_3u-4c_4p=0,\\
&q_{tt}+iq^2p_t-4uvq-2pv^2+\frac{1}{2}p^2q^3-2c_1(iv_t+(pv+qu)q)\\
&\qquad-2ic_2q_t-4c_3v-4c_4q=0.
\end{aligned}
$$

It is easy to see that for $n=2$, the dependence between components (u,v) and (p,q) is described by nonlinear equations. At the same time, these equations allow solutions of the form $u=v=0$ for $c_1=c_3=0$. The equations for components p and q in this case have the form

$$
\begin{aligned}
&p_{tt}-ip^2q_t+\frac{1}{2}p^3q^2+2ic_2p_t-4c_4p=0,\\
&q_{tt}+iq^2p_t+\frac{1}{2}p^2q^3-2ic_2q_t-4c_4q=0.
\end{aligned}
\tag{23}
$$

We will omit the values of the coefficients in the equation of the spectral curve for $n=2$, since they are very cumbersome and are not interesting at the moment.

It is worth noting that the nonlinear stationary Equations (22) and (23) are very different, despite the fact that they have the same order.

5. Integrable Nonlinear Evolutionary Equations

Let the second equation of the Lax pair have the form

$$
i\Psi_{t_k}+V_k\Psi=0,
\tag{24}
$$

where $V_k=2^mW_k$, $k=2m$ or $k=2m-1$.

Then, from the compatibility condition of Equations (2) and (24), the following evolutionary equations follow:

$$
\begin{aligned}
ip_{z_k}+2^{m+1}H_{k+1}&=0,\\
-iq_{z_k}+2^{m+1}G_{k+1}&=0,\\
iu_{z_k}+2^{m+1}(H_{k+2}-pF_{k+1})&=0,\\
-iv_{z_k}+2^{m+1}(G_{k+2}-qF_{k+1})&=0
\end{aligned}
\tag{25}
$$

It is not difficult to see that the evolutionary Equation (25), as well as the stationary equations, admit reduction (16). Accordingly, after replacing (16), an integrable nonlinear two-component equation with different dependencies of the components on variables z_k will be obtained from the system of Equations (25). Note that the well-known two-component integrable nonlinear equations [32,33,35–37,46–56] describe models with identical dependencies of components on variables z_k.

We present the first evolutionary integrable equations from the corresponding hierarchy. For $k = 1$, the structures of the first two equations of the system differ from the structures of the last two:

$$\begin{aligned} ip_{z_1} &= 2iu_t - 2(pv + qu)p, \\ -iq_{z_1} &= -2iv_t - 2(pv + qu)q, \\ iu_{z_1} &= p_{tt} - ipqp_t - 2(pv + qu)u, \\ -iv_{z_1} &= q_{tt} + ipqq_t - 2(pv + qu)v. \end{aligned} \tag{26}$$

Note that the last two equations of system (26), when performing reduction (16), are analogous to the derivative nonlinear Schrödinger equation. At the same time, the first two equations describe a fairly simple relationship between the components.

For $k = 2$, the evolutionary equations have the form

$$\begin{aligned} ip_{z_2} &= p_{tt} - ip^2q_t + \frac{1}{2}(p^3q^2 - 8uvp - 4u^2q), \\ -iq_{z_2} &= q_{tt} + iq^2p_t + \frac{1}{2}(p^2q^3 - 8uvq - 4v^2p), \\ iu_{z_2} &= u_{tt} + ip^2v_t + 2i(pv + qu)p_t + \frac{1}{2}(3p^2q^2u + 2p^3qv - 4u^2v), \\ -iv_{z_2} &= v_{tt} - i2q^2u_t - 2i(pv + qu)q_t + \frac{1}{2}(3p^2q^2v + 2pq^3u - 4uv^2). \end{aligned} \tag{27}$$

It is easy to see that Equation (27) is analogous to the two-component coupled derivative nonlinear Schrödinger equation.

Assuming $q = -p^*$ and $v = -u^*$, we obtain from Equation (27) a new two-component derivative nonlinear Schrödinger equation

$$\begin{aligned} ip_{z_2} &= p_{tt} + ip^2p_t^* + \frac{1}{2}\Big(|p|^4p + 8|u|^2p + 4u^2p^*\Big), \\ iu_{z_2} &= u_{tt} - ip^2u_t^* - 2i(pu^* + up^*)p_t + \frac{1}{2}(3|p|^4u + 2|p|^2p^2u^* + 4|u|^2u). \end{aligned} \tag{28}$$

Unlike the usual vector derivative nonlinear Schrödinger equation [52–56], in this case, the evolution of components p and u is described by different equations.

For $k = 3$, the evolutionary equations are again divided into two groups. The first two equations are analogs of the nonlinear Schrodinger equation, and the second are analogs of the modified Korteweg–de Vries equation

$$\begin{aligned} ip_{z_3} &= 2u_{tt} + 2i(qu_t - uq_t)p + i(6pv + 4qu)p_t + (p^2q^2 - 4uv)u, \\ -iq_{z_3} &= 2v_{tt} - 2i(pv_t - vp_t)q - i(6qu + 4pv)q_t + (p^2q^2 - 4uv)v, \\ u_{z_3} &= -p_{ttt} + i(qp_{tt} + pq_{tt})p + 6(vp_t + qu_t)u + (4vu_t + 2uv_t)p \\ &\quad + 2ipp_tq_t - \frac{3}{2}p^2q^2p_t - 2ip^2quv + \frac{i}{2}p^4q^3, \\ v_{z_3} &= -q_{ttt} - i4(qp_{tt} + pq_{tt})q + 6(pv_t + uq_t)v + (2vu_t + 4uv_t)q \\ &\quad - 2iqp_tq_t - \frac{3}{2}p^2q^2q_t + 2ipq^2uv - \frac{i}{2}p^3q^4. \end{aligned} \tag{29}$$

Assuming $q = -p^*$ and $v = -u^*$, we obtain from Equation (29) a two-component mixed equation

$$\begin{aligned} ip_{z_3} &= 2u_{tt} - 2i(p^*u_t - up_t^*)p - i(6pu^* + 4p^*u)p_t + (|p|^4 - 4|u|^2)u, \\ u_{z_3} &= -p_{ttt} - i(p^*p_{tt} + pp_{tt}^*)p - 6(u^*p_t + p^*u_t)u - (4u^*u_t + 2uu_t^*)p \\ &\quad - 2ip|p_t|^2 - \frac{3}{2}|p|^4p_t - 2i|pu|^2p - \frac{i}{2}|p|^6p. \end{aligned}$$

The first equation is an analog of the derivative NLS equation and the second equation is an analog of the modified Korteweg–de Vries equation.

Note that furthering the structure of the evolutionary Equations (25) will depend on the parity of the number of equations. For the odd k, the order of derivatives with respect to t will be different, for an even k, it is the same. In particular, for $k = 4$, the system of evolutionary equations will resemble the vector-modified Korteweg–de Vries equation. Note that even equations admit reduction $u = v = 0$, whereas odd ones do not. In this case, Equation (27) under this reduction and under condition (16) passes into the Gerdjikov–Ivanov equation [23–25,27,28,62,63,71]. The model explored in this paper expands upon already established integrable models of nonlinear wave propagation.

6. One-Phase Solutions

To show the differences in component behaviors, we consider examples with $n = 0$ and $c_1 \neq 0$. To find solutions to system (18), we express u and v from the first two equations and substitute them with the rest. After simplification, we have

$$\begin{aligned} &p_{tt} - 2i(c_1^2 - 2c_2)p_t - 2c_1^2 p^2 q - 4c_2^2 p = 0, \\ &q_{tt} + 2i(c_1^2 - 2c_2)q_t - 2c_1^2 p q^2 - 4c_2^2 q = 0. \end{aligned} \tag{30}$$

It is worth noting that Equation (30) differs from Equations (22) and (23).

Following [72], we will make a replacement in these equations

$$p = \sqrt{r}\exp\left\{-\int \frac{w}{2r}dt\right\}, \quad q = \sqrt{r}\exp\left\{\int \frac{w}{2r}dt\right\}, \tag{31}$$

where

$$r = pq, \quad w = pq_t - qp_t.$$

After simplification, we have

$$w = -2i(c_1^2 - 2c_2)r + ic_3, \tag{32}$$

$$2rr_{tt} - (r_t)^2 - 8c_1^2 r^3 + 4c_1^2(c_1^2 - 4c_2)r^2 - c_3^2 = 0. \tag{33}$$

Here, c_3 is an integration constant.

Additional relations follow from the equation of the spectral curve. Converting expressions for constants f_3 and f_4 using substitutions (31) and (32), we obtain

$$\begin{aligned} &f_3 = 2c_1c_2 - \frac{c_3}{2c_1}, \\ &(r_t)^2 = 4c_1^2 r^3 - 4c_1^2(c_1^2 - 4c_2)r^2 + 4c_1^2(4c_2^2 - c_3 - 4f_4)r - c_3^2. \end{aligned} \tag{34}$$

It is not difficult to check the compatibility of Equations (33) and (34). It follows from these equations that the function $r(t)$ is an elliptic function or its degeneration.

In particular, if the spectral curve is given by the equation

$$\mu^2 = \left((\lambda - a)^2 + b^2\right)^2,$$

then the constants c_k and f_4 have the following values

$$c_1 = -2a, \quad c_2 = a^2 + b^2, \quad c_3 = 0, \quad f_4 = \left(a^2 + b^2\right)^2.$$

For these values of constants, the function $r(t)$ satisfies the equation

$$(r_t)^2 = 16a^2 r^3 + 64a^2 b^2 r^2.$$

Solving this equation, we have

$$r(t) = -4b^2 \operatorname{sech}^2(4abt + \alpha), \quad \alpha = const.$$

In this case,

$$w(t) = 16ib^2(a^2 - b^2) \operatorname{sech}^2(4abt + \alpha).$$

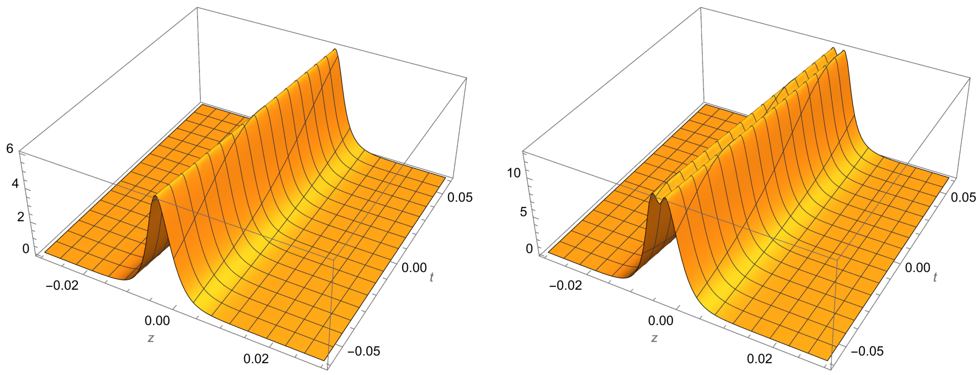
Substituting this value of functions $r(t)$ and $w(t)$ into Formulas (31) and (27), we obtain the solution to Equation (27)

$$\begin{aligned} p &= 2ib \operatorname{sech}(\phi_1(t, z_2))e^{i\phi_2(t,z_2)}, \\ q &= 2ib \operatorname{sech}(\phi_1(t, z_2))e^{-i\phi_2(t,z_2)}, \\ u &= 2b \operatorname{sech}(\phi_1(t, z_2))(ia - b \tanh(\phi_1(t, z_2)))e^{i\phi_2(t,z_2)}, \\ v &= 2b \operatorname{sech}(\phi_1(t, z_2))(ia + b \tanh(\phi_1(t, z_2)))e^{-i\phi_2(t,z_2)}, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \phi_1(t, z_2) &= 4abt - 16ab(a^2 - b^2)z_2, \\ \phi_2(t, z_2) &= 2(a^2 - b^2)t + 4(a^4 - 6a^2b^2 + b^4)z_2. \end{aligned}$$

It is easy to see that the components of solution (35) satisfy the reductions $q = -p^*$ and $v = -u^*$. The amplitudes of the components p and u for $a = 2$ and $b = 3$ are shown in Figure 1.



$|p(t, z_2)|$ $|u(t, z_2)|$

Figure 1. The amplitudes of the solution (35) for $a = 2$ and $b = 3$.

It is not difficult to see that the shape of the component u is quite different from the shape of component p. Component p is a classical soliton. At the same time, component u is defined using a completely new expression.

Assuming $c_3 = 0$, $0 < k < 1$, it is possible to construct three different solutions in elliptic Jacobi functions [73,74].

If $c_2 = (c_1^4 - 1 - k^2)/(4c_1^2)$, then the function r(t) is expressed in terms of $\operatorname{sn}(t; k)$:

$$r(t) = \frac{k^2}{c_1^2} \operatorname{sn}^2(t; k).$$

The spectral curve of this solution is determined by the equation

$$\mu^2 = \prod_{j=1}^{4} (\lambda - \lambda_j), \tag{36}$$

where

$$\lambda_{1,2} = -\frac{1+c_1^2 \pm k}{2c_1}, \quad \lambda_{3,4} = \frac{1-c_1^2 \pm k}{2c_1}.$$

In this case,

$$\begin{aligned} p(t,z_2) &= \frac{k}{c_1}\,\mathrm{sn}(\phi_1(t,z_2);k)e^{i\phi_2(t,z_2)}, \\ u(t,z_2) &= -\frac{k}{2c_1^2}\Big(c_1^2\,\mathrm{sn}(\phi_1(t,z_2);k) - i\,\mathrm{sn}'(\phi_1(t,z_2);k)\Big)e^{i\phi_2(t,z_2)}, \\ q &= p^*, \quad v = u^*, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \phi_1(t,z_2) &= t + \frac{c_1^4+1+k^2}{c_1^2}z_2, \\ \phi_2(t,z_2) &= \frac{c_1^4+1+k^2}{2c_1^2}t + \frac{c_1^8+6(k^2+1)c_1^4+1+4k^2+k^4}{4c_1^4}z_2. \end{aligned}$$

Since the solution (37) to Equation (27) satisfies the conditions $q = p^*$, $v = u^*$, it is not suitable for describing the propagation of nonlinear optical signals.

When $c_2 = (c_1^4+2-k^2)/(4c_1^2)$, function $r(t)$ has the form

$$r(t) = -\frac{1}{c_1^2}\,\mathrm{dn}^2(t;k).$$

The spectral curve is again determined by Equation (36). Only the branching points in this case are not real, but complex conjugates

$$\lambda_{1,2} = -\frac{c_1}{2} \pm i\frac{1+\sqrt{1-k^2}}{2c_1}, \quad \lambda_{3,4} = -\frac{c_1}{2} \pm i\frac{1-\sqrt{1-k^2}}{2c_1}.$$

The following periodic solution to Equation (27) corresponds to this curve:

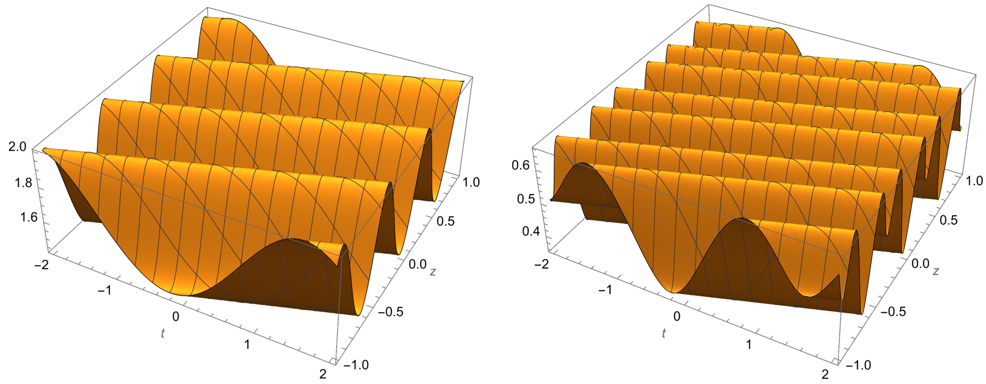
$$\begin{aligned} p(t,z_2) &= \frac{i}{c_1}\,\mathrm{dn}(\phi_1(t,z_2);k)e^{i\phi_2(t,z_2)}, \\ u(t,z_2) &= -\frac{i}{2c_1^2}\Big(c_1^2\,\mathrm{dn}(\phi_1(t,z_2);k) - i\,\mathrm{dn}'(\phi_1(t,z_2);k)\Big)e^{i\phi_2(t,z_2)}, \\ q &= -p^*, \quad v = -u^*, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \phi_1(t,z_2) &= t + \frac{c_1^4-2+k^2}{c_1^2}z_2, \\ \phi_2(t,z_2) &= \frac{c_1^4-2+k^2}{2c_1^2}t + \frac{c_1^8+6(k^2-2)c_1^4+6-6k^2+k^4}{4c_1^4}z_2. \end{aligned}$$

Since reductions $q = -p^*$, $v = -u^*$ are performed, solution (38) can be used to describe the propagation of a periodic nonlinear two-component wave. The amplitudes of the components p and u for $c_1 = 0.5$ and $k = 0.7$ are shown in Figure 2.

It is easy to see that component p is an ordinary "dnoidal" wave, while component u has a non-standard shape.



$|p(t,z_2)|$ $|u(t,z_2)|$

Figure 2. The amplitudes of the solution (38) for $c_1 = 0.5$, $k = 0.7$.

The third elliptic solution to Equation (27) for $c_2 = (c_1^4 - 1 + 2k^2)/(4c_1^4)$ is expressed in terms of the $\mathrm{cn}(t;k)$ [73,74]:

$$r(t) = -\frac{k^2}{c_1^2}\,\mathrm{cn}^2(t;k),$$

and

$$\begin{aligned} p(t,z_2) &= \frac{ik}{c_1}\,\mathrm{cn}(\phi_1(t,z_2);k)e^{i\phi_2(t,z_2)}, \\ u(t,z_2) &= -\frac{ik}{2c_1^2}\Big(c_1^2\,\mathrm{cn}(\phi_1(t,z_2);k) - i\,\mathrm{cn}'(\phi_1(t,z_2);k)\Big)e^{i\phi_2(t,z_2)}, \\ q &= -p^*, \quad v = -u^*, \end{aligned} \tag{39}$$

where

$$\begin{aligned} \phi_1(t,z_2) &= t + \frac{c_1^4 + 1 - 2k^2}{c_1^2} z_2, \\ \phi_2(t,z_2) &= \frac{c_1^4 + 1 - 2k^2}{2c_1^2} t + \frac{c_1^8 + 6(1-2k^2)c_1^4 + 1 - 6k^2 + 6k^4}{4c_1^4} z_2. \end{aligned}$$

The spectral curve of the solution (39) is given by Equation (36), where

$$\lambda_{1,2} = -\frac{c_1}{2} + \frac{\sqrt{1-k^2} \pm ik}{2c_1}, \quad \lambda_{3,4} = -\frac{c_1}{2} - \frac{\sqrt{1-k^2} \pm ik}{2c_1}.$$

The examples we considered have shown that, as in the case of the standard nonlinear Schrödinger equation, the locations of the branching points of the spectral curve correspond to the sign of reduction. If $q = p^*$, then the branching points are on the real axis. If $q = -p^*$, then the branching points form complex conjugate pairs.

7. Concluding Remarks

Equation (28), from our point of view, is the most useful for practical applications. For $u = 0$ and $z_2 = -z$, this equation becomes the Gerdjikov–Ivanov equation [24,25]

$$ip_z + p_{tt} + ip^2 p_t^* + \frac{1}{2}|p|^4 p = 0, \tag{40}$$

This equation, along with its generalizations, finds application in nonlinear optics [75–80]. It is worth noting that in nonlinear optics, only the perturbed Gerdjikov–Ivanov equations are considered. This is due to the fact that the unperturbed equation does not adequately describe the propagation of waves in nonlinear optics. In our paper, the first equation from system (28)

$$ip_z + p_{tt} + ip^2 p_t^* + \frac{1}{2}\left(|p|^4 p + 8|u|^2 p + 4u^2 p^*\right) = 0, \quad z = -z_2,$$

is also a perturbed Gerdjikov–Ivanov equation. The perturbation u is related to the main signal p by means of stationary equations. In the case of soliton propagation, the corresponding link is given by Equation (18). It should be noted that due to the integrability of the equation, the perturbation u satisfies the second equation of the system (28). Since the authors are mathematicians, the physical meaning of perturbation u is not fully understood by them. The authors hope that this physical meaning will be found.

Another possibility of constructing a perturbed equation involves the use of mixed equations. In particular, the function $p(t, \alpha_1(z), \alpha_2(z))$ satisfies to the equation

$$ip_z - \alpha_2'(z)\left(p_{tt} + ip^2 p_t^* + \frac{1}{2}\left(|p|^4 p + 8|u|^2 p + 4u^2 p^*\right)\right)$$
$$-\alpha_1'(z)(2iu_t - 2(pu^* + up^*)p) = 0.$$

From (15) and (25), the following equality

$$\partial_t F_{k+1} = 2^{-m} \partial_{z_k} F_1 \tag{41}$$

follows; thus, there exists the function Φ, such that

$$F_1 = \partial_t \Phi, \quad F_{k+1} = 2^{-m} \partial_{z_k} \Phi.$$

Note that the same statement is also true for other hierarchies (see [37,61,81,82]).

As shown in [61], using Equation (41), it is possible to construct a new perturbed Gerdjikov–Ivanov equation, which will be an analog of the Kundu–Eckhaus equation.

To summarize, we note that the monodromy matrix method is a very useful method. Using this method, we investigated completely different hierarchies of integrable nonlinear equations [20,28,37,61,82–84]. We considered Lax operators with matrices of the second [20,28,84], third [28,37,61], and fifth [82,83] orders. In all our works, we found hierarchies of integrable nonlinear equations, the simplest solutions of these equations, and spectral curves corresponding to these solutions. In this paper, we show the operation of this method using the example of a new Lax operator and invite other researchers to use the monodromy method in their work.

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