

On Unique Determination of Polyhedral Sets

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Abstract: In this paper, we develop in detail the geometric constructions that lead to many uniqueness results for the determination of polyhedral sets, typically scatterers, by a finite minimal number of measurements. We highlight how unique continuation and a suitable reflection principle are enough to proceed with the constructions without any other assumption on the underlying partial differential equation or the boundary condition. We also aim to keep the geometric constructions and their proofs as simple as possible. To illustrate the applicability of this theory, we show how several uniqueness results present in the literature immediately follow from our arguments. Indeed, we believe that this theory may serve as a roadmap for establishing similar uniqueness results for other partial differential equations or boundary conditions.

Keywords: inverse problems; uniqueness; reflection principle; scattering

MSC: 35R30; 35P25

1. Introduction

One of the most classical and important open questions in the inverse scattering problems field is the following: How many (far-field) measurements are needed to uniquely determine a scatterer? In other words, how many experiments corresponding to different incident waves are needed to uniquely determine a scatterer? Here, we limit ourselves to the time-harmonic case and incident waves of the planar type. As a matter of notation, let us point out that a *scatterer* is just a compact set whose complement is connected. Clearly, the answer strongly depends on the kind of waves used, on the nature of the scatterer, i.e., on the boundary condition imposed on its boundary, and on the geometry of the scatterer. Specifically, the answer may differ if the scatterer is an *obstacle*, i.e., it is the closure of its interior, or if the scatterer is a *screen*, i.e., its interior is empty. Just as an example, in \mathbb{R}^3 , typically, an obstacle is a solid object, whereas a screen is a portion of a surface. We recall that a nice introduction to inverse scattering problems, in the acoustic and electromagnetic cases, may be found, for instance, in [1].

Even in the simplest case of acoustic waves with sound-soft obstacles, the answer is not yet complete. By Schiffer's theorem, we know that countably infinitely many incident waves (with the same frequency) are enough. With a given bound on the diameter R of the region containing the unknown obstacle, finitely many incident waves are enough, a fact first noted in [2]; in this case, the number of measurements depends on R and the frequency of the incident waves and it is 1 provided such a frequency is small enough with respect to R . If the scatterer includes screens and obstacles, this result has been extended in [3], where some further although minimal regularity assumptions on the unknown scatterer are required. However, there is long-standing conjecture that one measurement should suffice, independently of the frequency of the incident wave or the size of the scatterer, at least in the obstacle case.

To approach such a conjecture and to possibly find the minimal number of measurements needed in other inverse scattering problems, the unique determination of scatterers belonging to some special classes has been studied. In this respect, one of the most successful special classes has been so far one of the so-called polyhedral scatterers, i.e., scatterers



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whose boundary is the finite union of cells, a cell being the closure of an open subset of a hyperplane. We note that a polyhedral obstacle is just the finite union of polyhedra.

The first breakthrough in this direction was obtained in [4], even if some earlier results may be found in the unpublished work [5]. In [4], it was proved uniqueness for polygonal obstacles satisfying a non-trapping condition, with one measurement in the sound-soft case and two measurements in the sound-hard case. For sound-soft scatterers, this result has been extended in [6] on several aspects. In fact, it was proved that one measurement guarantees uniqueness in any dimension and for general sound-soft polyhedral scatterers, thus removing both the obstacle and the non-trapping assumptions.

After [6], an enormous amount of research has been done in the field, pursuing extensions to other boundary conditions or to other kinds of waves, like electromagnetic waves or elastic waves. We just mention here the earliest and most significant results.

For general polyhedral scatterers, uniqueness was proved with N measurements, N being the dimension of the space, first in [7] in the sound-hard case and then in [8] in the mixed sound-soft and sound-hard case. Uniqueness with one measurement for polyhedral obstacles was proved in the sound-hard case in [9] for $N = 2$ and in [10] for any $N \geq 2$. The two-dimensional result has been extended in [8] to the mixed sound-soft and sound-hard case.

The electromagnetic case has been treated in [11,12], whereas the extension to elastic waves has been developed in [13].

In a different and significant line of research stemming from the breakthrough result of [14], polyhedral structures, in particular corners and edges, play an important role also in the determination of the support of penetrable obstacles by a finite number of measurements.

Despite the different equations or boundary conditions, it is clear that all these uniqueness results for the determination of (non-penetrable) polyhedral scatterers share two crucial common features. Specifically, unique continuation properties for the solutions to the equation and a suitable reflection principle depending on the boundary condition. These are combined in the development of suitable geometric constructions. In this paper, we highlight that, indeed, these two conditions, summarized in Assumption 1, are enough to develop these geometric constructions. Moreover, once Assumption 1 is satisfied, the geometric constructions are completely independent of the other properties of the equation or the boundary condition. Moreover, our effort is to keep these constructions as simple and as general as possible in such a way as to make completely transparent the geometrical procedure involved in these uniqueness results. In Section 2, we state and prove our two main results, Theorem 1 for general polyhedral scatterers and Theorem 2 for polyhedral obstacles. We also aim to keep proofs as simple as possible, and we fully succeeded, at least for Theorem 1. Other important features of our results are the following.

- We can treat at the same time any (finite) number of measurements simply by changing M in the definition of the differential operator A .
- We can treat at the same time different boundary conditions, i.e., we can treat mixed boundary conditions, see Section 2.1, in particular Theorem 3.

To show the power of our general theory, we obtain in a single shot all previously described uniqueness results, possibly with some extensions in the mixed boundary conditions cases. These applications are developed in Section 3, first for the acoustic and then for the electromagnetic and elastic waves. We thus believe that our general constructions will be useful in order to obtain similar uniqueness results for other kinds of equations or boundary conditions.

Finally, we wish to make some comments on the corresponding stability results. The first stability result has been obtained in [15] for sound-soft general polyhedral scatterers with one measurement. Stability has been extended to sound-hard general polyhedral scatterers with N measurements and to sound-hard polyhedral obstacles with 1 measurement in [16]. Finally, stability for the electromagnetic case has been proved in [17] with 2 measurements for general polyhedral scatterers and 1 measurement for polyhedral obsta-

cles. In all these stability results, quantitative versions of Theorem 1 for general scatterers and of Theorem 2 for obstacles are needed. The proof of Theorem 2 is much more difficult than the one of Theorem 1. Likewise, its quantitative version is extremely involved. We refer the interested reader to [17] for details. Here, we limit ourselves to consider quantitative versions of Theorem 1 and to stress some of their common features. In fact, we believe that a general stability result may be obtained independently of the equation and the boundary condition provided the following kinds of properties are satisfied.

- (1) The differential operator A satisfies interior and boundary regularity estimates.
- (2) The differential operator A satisfies quantitative unique continuation results, like a three-spheres inequality with optimal exponents.
- (3) There are uniform bounds on the solutions to the scattering problem, which are independent of the scatterer for scatterers belonging to suitable admissible classes.
- (4) The boundary condition satisfies a quantitative version of the reflection principle, which often reduces to quantitative estimates of solutions with respect to Cauchy data.

As one can easily check, these are the common essential ingredients of all the stability results for general polyhedral scatterers, i.e., are the essential ingredients to obtain a quantitative analog of Theorem 1. We finally note that point (3) above might be challenging for the following reasons. First, dealing with polyhedral structures, we cannot expect to have better than Lipschitz regularity on the scatterers. Moreover, when allowing obstacles and screens or combinations of both, regularity might be even weaker. For the Dirichlet boundary condition in the acoustic case, uniform bounds can be proved in a relatively easy way, even for quite general scatterers; see, for instance, ref. [3]. For the Neumann boundary condition in the acoustic case or the electromagnetic case, uniform bounds for general scatterers are much harder to obtain. In this respect, the strategy developed in [18], and optimized in [19], for Neumann in the acoustic case, might be a good starting point. In fact, the same strategy proved successful in treating the electromagnetic case as well, see [17].

2. The Main Strategies for Uniqueness Results

The integer $N \geq 2$ denotes the dimension of the space. For any $x \in \mathbb{R}^N$ and any $r > 0$, $B_r(x)$ is the open ball with center x and radius r . We also use the notation $B_r = B_r(0)$.

For any $(N - 1)$ -dimensional hyperplane Π , we call T_Π the reflection in Π . Moreover, fixed a unit vector ν orthogonal to Π , we call H_Π^+ and H_Π^- the two connected components, actually half-spaces, of $\mathbb{R}^N \setminus \Pi$, with ν being the exterior unit normal on Π for H_Π^+ and the interior unit normal on Π for H_Π^- .

We consider the following general framework. Let us fix positive integers M, L, L_1, J and J_1 . Let $D \subset \mathbb{R}^N$ be an open set and let $u = (u^1, \dots, u^M) : D \subset \mathbb{R}^N \rightarrow \mathbb{C}^M$ be a complex vector-valued function.

Let $A_k, k = 1, \dots, L$, be the following differential operators with constant coefficients

$$A_k u = \sum_{j=1}^M \sum_{|\alpha| \leq J} a_{\alpha,k}^j D^\alpha u^j$$

and let us define

$$Au = (A_1 u, \dots, A_L u).$$

As usual, α denotes an N -multiindex and $|\alpha|$ is its weight. The constants $a_{\alpha,k}^j$ are complex numbers.

Let $B_k, k = 1, \dots, L_1$, be the following boundary operators with constant coefficients

$$B_k u = \sum_{i=1}^N \sum_{j=1}^M \sum_{|\alpha| \leq J_1} v_i b_{\alpha,i,k}^j D^\alpha u^j + \sum_{j=1}^M \sum_{|\alpha| \leq J_1} b_{\alpha,0,k}^j D^\alpha u^j$$

and let us define

$$Bu = (B_1 u, \dots, B_{L_1} u).$$

As usual, ν denotes the unit exterior normal vector on the boundary of D , where it is well-defined. The constants $b_{\alpha,i,k}^j$ are complex numbers.

On this general structure, we impose the following.

Assumption 1. We assume that the operator A and the boundary condition B satisfy the following.

(A.1) If u satisfies $Au = 0$ in an open set D , then all partial derivatives $D^\alpha u$ appearing in the definitions of A and B are continuous in D .

(A.2) A satisfies the unique continuation property, i.e., if u satisfies $Au = 0$ in an open connected set D and $u = 0$ in some open nonempty $D' \subset D$, then $u = 0$ in D .

(A.3) For any hyperplane Π , with unit normal ν , let $D^+ \subset H_\Pi^+$ be an open set and let u^+ satisfy $Au^+ = 0$ in D^+ . Then there exists an operator $\tilde{T}_{\Pi,\nu}$ such that

$$u^- = \tilde{T}_{\Pi,\nu}(u^+ \circ T_\Pi)$$

satisfies $Au^- = 0$ in $D^- = T_\Pi(D^+)$. Furthermore, the following properties are assumed.

(a) Let $x \in \Pi$ be such that $B_r(x) \cap H_\Pi^+ \subset D^+$. If $Bu^+ = 0$ on $B_r(x) \cap \Pi$, then $Bu^- = 0$ on $B_r(x) \cap \Pi$ as well and, calling

$$u = \begin{cases} u^+ & \text{in } B_r(x) \cap H_\Pi^+ \\ u^- & \text{in } B_r(x) \cap H_\Pi^- \end{cases}$$

we have that $Au = 0$ in $B_r(x)$.

(b) On the other hand, if $Au = 0$ in $B_r(x)$ and

$$u|_{B_r(x) \cap H_\Pi^-} = \tilde{T}_{\Pi,\nu}(u|_{B_r(x) \cap H_\Pi^+} \circ T_\Pi),$$

then $Bu = 0$ on $B_r(x) \cap \Pi$ (on either side of Π).

(c) Let Π_1 be any hyperplane and let $x \in \Pi_1$ be such that, for some ν_1 orthogonal to Π_1 , $B_r(x) \cap H_{\Pi_1}^+ \subset D^+$. If $Bu^+ = 0$ on $B_r(x) \cap \Pi_1$, then $Bu^- = 0$ on $T_\Pi(B_r(x) \cap \Pi_1)$.

Remark 1. Important consequences of (A.3)(a) are the following. First, the condition $Bu = 0$ is independent of the choice of ν , thus excluding Robin-type boundary conditions. Moreover, let Π be a hyperplane and let $x \in \Pi$. If, for some $r > 0$, $Au = 0$ in $B_r(x)$ and $Bu = 0$ on $B_r(x) \cap \Pi$, then, by the unique continuation of (A.2), $u|_{B_r(x) \cap H_\Pi^-} = \tilde{T}_{\Pi,\nu}(u|_{B_r(x) \cap H_\Pi^+} \circ T_\Pi)$ and $u|_{B_r(x) \cap H_\Pi^+} = \tilde{T}_{\Pi,-\nu}(u|_{B_r(x) \cap H_\Pi^-} \circ T_\Pi)$.

Remark 2. Let $Au = 0$ in an open set D . Let $x \in D$ and Π a hyperplane passing through x . If, for some $r > 0$, $B_r(x) \subset D$ and $Bu = 0$ on $B_r(x) \cap \Pi$, then $Bu = 0$ on S , the connected component of $D \cap \Pi$ containing x . In fact, let S_1 be the largest open subset of S where $Bu = 0$. By contradiction, let us assume that S_1 is different from S , i.e., there exists $x_1 \in S$ belonging to the boundary of S_1 . We assume that, for some $r_1 > 0$, $B_{r_1}(x_1) \subset D$, therefore $B_{r_1}(x_1) \cap \Pi \subset S$. Moreover, there exists $x_2 \in S_1$ and $r_2 > 0$ such that $B_{r_2}(x_2) \subset B_{r_1}(x_1)$ and $Bu = 0$ on $B_{r_2}(x_2) \cap \Pi$. By the previous remark, $u|_{B_{r_2}(x_2) \cap H_\Pi^-} = \tilde{T}_{\Pi,\nu}(u|_{B_{r_2}(x_2) \cap H_\Pi^+} \circ T_\Pi)$ and, again, by the unique continuation of (A.2), the same holds on $B_{r_1}(x_1)$. By (A.3)(b), we conclude that $Bu = 0$ on $B_{r_1}(x_1)$ and we obtain a contradiction.

Definition 1. We say that $\Sigma \subset \mathbb{R}^N$ is a scatterer if it is bounded, closed, and $G = \mathbb{R}^N \setminus \Sigma$ is connected. In turn, G is called an exterior domain, i.e., a connected open set whose complement is bounded.

We say that a scatterer Σ is an obstacle if Σ coincides with the closure of its interior.

We call cell the closure of an open subset of an $(N - 1)$ -dimensional hyperplane.

We say that a scatterer Σ is polyhedral if its boundary is the finite union of cells. Without loss of generality, we assume that different cells are internally disjoint.

Remark 3. We note that a cell need not to be an $(N - 1)$ -dimensional polyhedron contained in an $(N - 1)$ -dimensional hyperplane. Moreover, its relative interior can have infinitely many connected components.

However, if Σ is a polyhedral obstacle, then it is just the union of a finite number of pairwise internally disjoint N -dimensional polyhedra. Hence, we can decompose the boundary of polyhedral obstacles into a finite number of $(N - 1)$ -dimensional polyhedra contained in an $(N - 1)$ -dimensional hyperplane. We call these $(N - 1)$ -dimensional polyhedra the $(N - 1)$ -faces of Σ and we denote with \mathcal{F}_{N-1} the set of all $(N - 1)$ -faces of Σ . In turn, the boundary of any $(N - 1)$ -face of Σ is composed by a finite number of $(N - 2)$ -dimensional polyhedra contained in an $(N - 2)$ -dimensional subspace. We call these $(N - 2)$ -dimensional polyhedra the $(N - 2)$ -faces of Σ and we denote with \mathcal{F}_{N-2} the set of all $(N - 2)$ -faces of Σ . Proceeding iteratively, we can define k -faces of Σ for $0 \leq k \leq N - 1$ and we denote with \mathcal{F}_k the set of all k -faces of Σ . We usually call vertices of Σ the 0 -faces of Σ . An important property to recall is that if P is a vertex of Σ , then the span of the normals to all $(N - 1)$ -faces of Σ containing P is the whole \mathbb{R}^N .

Let Σ be a polyhedral scatterer. We assume that u is a solution to $Au = 0$ in $G = \mathbb{R}^N \setminus \Sigma$ and that $Bu = 0$ on flat portions of boundary, i.e., for any $x \in \partial\Sigma$, if x belongs to the interior of a cell C contained in a hyperplane Π and, for some ν orthogonal to Π and some $r > 0$, we have $B_r(x) \cap H_\Pi^+ \subset G$, then $Bu = 0$ on $B_r(x) \cap \Pi$. Please note that if also $B_r(x) \cap H_\Pi^- \subset G$, then $Bu = 0$ on either side of Π . We can also state this condition as follows. For any cell C , contained in a hyperplane Π , and any ν orthogonal to Π such that for any x belonging to the interior of C there exists $r > 0$ with $B_r(x) \cap H_\Pi^+ \subset G$, we have $Bu = 0$ in the interior of the cell C .

Following [6], we consider the following.

Definition 2. A point $x \in G$ is a flat point for u if there exists Π passing through x such that $Bu = 0$ on $\Pi \cap B_r(x)$ for some $r > 0$ with $B_r(x) \subset G$.

For any flat point $x \in G$, let S be the connected component of $G \cap \Pi$ containing x . By Remark 2, we have that $Bu = 0$ on S . Fixing ν orthogonal to Π , we call G^+ the connected component of $G \setminus S$ containing $B_r(x) \cap H_\Pi^+$ and G^- the connected component of $G \setminus S$ containing $B_r(x) \cap H_\Pi^-$. We note that it may happen that $G^+ = G^-$. We call E^+ the connected component of $G^+ \cap T_\Pi(G^-)$ containing $B_r(x) \cap H_\Pi^+$ and E^- the connected component of $G^- \cap T_\Pi(G^+)$ containing $B_r(x) \cap H_\Pi^-$. We note that $E^- = T_\Pi(E^+)$ and that $u|_{E^-} = \tilde{T}_{\Pi,\nu}(u|_{E^+} \circ T_\Pi)$. Let $E = E(x)$ be the set $E = E^+ \cup E^- \cup S$. We note that E is a connected set contained in G , which is symmetric with respect to Π . Moreover, the boundary of E is bounded and contained in $\partial\Sigma \cup T_\Pi(\partial\Sigma)$ and any point $y \in \partial E \setminus \Sigma$ is a flat point.

Finally, any hyperplane Π such that $Bu = 0$ on $\Pi \setminus \overline{B_R}$ for some $R > 0$ will be called a reflection hyperplane.

Remark 4. Let $x \in G$ be a flat point, with respect to the hyperplane Π , and let $E = E(x)$. Then, there are two cases: either E is unbounded or E is bounded. If E is unbounded, then there exists $R > 0$ such that $\Pi \setminus \overline{B_R}$ is contained in E , hence, by (A.3)(b), $Bu = 0$ on $\Pi \setminus \overline{B_R}$. Consequently, if E is unbounded, then Π is a reflection hyperplane.

Moreover, if $\Sigma \subset B_{R_0}$, for some $R_0 > 0$, and $\|x\| > R_0 + 1$, then E is unbounded, thus Π is a reflection hyperplane.

Remark 5. Another important observation is that the reflection in a reflection hyperplane of a reflection hyperplane is still a reflection hyperplane.

In particular, let κ be any $(N - 2)$ -dimensional subspace. We consider the set of all reflection hyperplanes containing κ , assuming such a set is not empty. There are two cases. If the reflection hyperplanes containing κ are finite, let us say $m \geq 1$, then they subdivide the whole space into $2m$ equal sectors. If they are infinite, then, by a simple continuity argument due to the regularity of (A.1), any hyperplane containing κ is a reflection hyperplane.

The key result is the following.

Theorem 1. *Let us assume that u admits a flat point $x \in G$. Then, there exists a reflection hyperplane Π_1 .*

Proof. Let $\gamma : [0, +\infty) \rightarrow \mathbb{R}^N$ be a piecewise smooth curve in G with first endpoint $\gamma(0) = x$ and such that $\|\gamma(t)\| \rightarrow +\infty$ as $t \rightarrow +\infty$.

First of all, we claim that the set of flat points belonging to γ is closed. Analogously, the set of $t \in [0, +\infty)$ such that $x = \gamma(t)$ is a flat point is closed. In fact, let $x_n = \gamma(t_n)$, $n \in \mathbb{N}$, be a sequence of flat points converging to $x = \gamma(t)$ as $n \rightarrow +\infty$. We need to show that x is a flat point as well. Let Π_n be the hyperplane related to x_n and v_n one of its unit normal. Up to a subsequence, we can assume that $v_n \rightarrow v$ as $n \rightarrow +\infty$, and we call Π the hyperplane orthogonal to v passing through x . A simple continuity argument due to the regularity of (A.1) allows us to prove the claim.

If the set $A = \{t \in [0, +\infty) : \gamma(t) \text{ is a flat point}\}$ is unbounded, then the proof is concluded by the last part of Remark 4.

Assume that A is bounded and let $\tilde{t} = \max A$ and $\tilde{x} = \gamma(\tilde{t})$. If the corresponding set $E = E(\tilde{x})$ is unbounded, the proof is concluded. It remains the case in which E is bounded. Then there exists $\tilde{t}_1 > \tilde{t}$ such that $\tilde{x}_1 = \gamma(\tilde{t}_1)$ belongs to $\partial E \cap G = \partial E \setminus \Sigma$. Hence, \tilde{x}_1 is a flat point, and we obtain a contradiction. Therefore, the proof is concluded. \square

When we are dealing with polyhedral obstacles, we can prove something more.

Theorem 2. *Let Σ be a polyhedral obstacle. Let Σ' be a finite union of polyhedra with $\Sigma' \setminus \Sigma \neq \emptyset$. Let \tilde{G} be the unbounded connected component of $\mathbb{R}^N \setminus (\Sigma \cup \Sigma')$. Assume that any $x \in \partial \tilde{G} \cap G$ is a flat point.*

Then there exist N reflection hyperplanes Π_1, \dots, Π_N , whose corresponding normals v_1, \dots, v_N are linearly independent.

To prove this result, we need several preliminary facts and lemmas. First of all, we consider some easy cases.

If Π is a reflection hyperplane such that $\Sigma \subset H_\Pi^+$ for some v orthogonal to Π , then the result is proved by the following argument. Since Σ is an obstacle, we can find $(N-1)$ -faces of Σ , C_1, \dots, C_N , belonging to hyperplanes $\tilde{\Pi}_1, \dots, \tilde{\Pi}_N$ with normals $\tilde{v}_1, \dots, \tilde{v}_N$, respectively, such that $\tilde{v}_1, \dots, \tilde{v}_N$ are linearly independent. By reflecting in Π , it is easy to see that the hyperplanes $\Pi_i = T_\Pi(\tilde{\Pi}_i)$, $i = 1, \dots, N$, satisfy the thesis.

If there exist two different reflection hyperplanes with the same normal v , by iterative reflections, we can find infinitely many equispaced reflection hyperplanes with the same normal. Eventually, for one of these, let us call it Π . We have $\Sigma \subset H_\Pi^+$, and the proof is concluded.

Let $R_0 > 0$ be such that Σ is contained in B_{R_0} . Let κ be an $(N-2)$ -dimensional subspace whose distance from the origin is greater than or equal to $2R_0 + 1$. Let $Q \in \kappa$ be the projection of the origin on κ . Let $v_0 = Q/\|Q\|$. Then there exists a constant α_0 , $0 < \alpha_0 < 1$, depending on R_0 only, such that if Π is a reflection hyperplane containing κ and with normal v such that $v \cdot v_0 \geq \alpha_0$, then $\overline{B_{R_0}}$, hence Σ , is contained in H_Π^+ and the proof is concluded. In particular, assume we have two different reflection hyperplanes Π_1 and Π_2 whose intersection is κ and whose normals v_1 and v_2 satisfy $\alpha_0 \leq v_1 \cdot v_2 < 1$. We have that v_0 belongs to the plane spanned by v_1 and v_2 . By Remark 5, we can find a reflection hyperplane Π containing κ such that its normal v satisfies $v \cdot v_0 \geq \alpha_0$, and the proof is concluded.

From now on, let N_0 , $1 \leq N_0 \leq N$, be the dimension of the span of the normals to reflection hyperplanes.

Lemma 1. *If $N_0 < N$, then the number of reflection hyperplanes is finite.*

Proof. Assume by contradiction that this is not the case. We can find a sequence of different reflection hyperplanes Π_n , with normal v_n , such that v_n converges to v_∞ as $n \rightarrow +\infty$. For

$m > n$, let $\kappa_{n,m} = \Pi_n \cap \Pi_m$, and $\alpha_{n,m} = \nu_n \cdot \nu_m$. Without loss of generality, we can assume that $\alpha_0 \leq \alpha_{n,m} < 1$ for any $m > n$. If for some $m > n$, the distance of $\kappa_{n,m}$ from the origin is greater than or equal to $2R_0 + 1$, we obtain a contradiction by the property described above. Hence, we can assume that the distance of $\kappa_{n,m}$ from the origin is less than $2R_0 + 1$ for any $m > n$.

Let $\mathcal{F} = \bigcup_{k=0}^{N-1} \mathcal{F}_k$ be the set of all faces of Σ . Let $\delta > 0$ be the minimum distance between any two disjoint faces of Σ , i.e.,

$$\delta = \min\{\text{dist}(\sigma, \sigma') : \sigma, \sigma' \in \mathcal{F}, \sigma \cap \sigma' = \emptyset\}.$$

We can find a constant α_1 , $\alpha_0 \leq \alpha_1 < 1$, depending on R_0 and δ only, such that if $\alpha_1 \leq \alpha_{n,m} < 1$ the following holds. For any $P \in \overline{B_{R_0}}$, we can find a reflection hyperplane Π containing $\kappa_{n,m}$, depending on P , such that if P' is the reflection of P in Π , then $0 < \|P' - P\| \leq \delta/2$. Again, without loss of generality, we can assume that $\alpha_1 \leq \alpha_{n,m} < 1$ for any $m > n$.

Let us fix any $m > n$. We have that $\alpha = \alpha_{n,m}$ satisfies $\alpha_1 \leq \alpha < 1$. We call $\kappa = \kappa_{n,m}$. Without loss of generality, up to a rigid change in coordinates, let $\kappa = \tilde{Q} \times \mathbb{R}^{N-2}$, with \tilde{Q} a point in \mathbb{R}^2 , so that $Q = (\tilde{Q}, 0, \dots, 0)$ is the element of κ closest to the origin. Let, for any $P = (P_1, \dots, P_N) \in \mathbb{R}^N$, $s(P)$ denote the distance of P from κ . For any $r > 0$, let $\kappa_r = \{P \in \mathbb{R}^N : s(P) \leq r\}$. We begin with the following remark. Let $r_0 = \min\{r : \Sigma \subset \kappa_r\}$. If $P \in \Sigma$ is such that $s(P) = r_0$, then P cannot belong to the interior of an $(N-1)$ -face of Σ . Instead, it belongs to a k -face of Σ , with $k \leq N-2$, contained in $(P_1, P_2) \times \mathbb{R}^{N-2}$. Please note that such a face may consist of P only. Actually, we can indeed find a vertex P of Σ such that $s(P) = r_0$. Let P' be its reflection in a reflection hyperplane Π as constructed above. We have that $s(P') = r_0$, P' is not contained in $(P_1, P_2) \times \mathbb{R}^{N-2}$ and its distance from P is less than δ . Hence, P' cannot belong to Σ . On the other hand, we can find $(N-1)$ -faces of Σ , C_1, \dots, C_N , belonging to hyperplanes $\tilde{\Pi}_1, \dots, \tilde{\Pi}_N$ with normals $\tilde{\nu}_1, \dots, \tilde{\nu}_N$, respectively, such that $\tilde{\nu}_1, \dots, \tilde{\nu}_N$ are linearly independent and $P \in C_i$ for any $i = 1, \dots, N$. By reflecting in Π , it is easy to see that the hyperplanes $\Pi_i = T_{\Pi}(\tilde{\Pi}_i)$, $i = 1, \dots, N$, are actually reflection hyperplanes passing through P' and thus we obtain a contradiction. The proof is concluded. \square

Before starting the next lemma, we recall the following classical result; see, for details, ([17], Lemma 2.14).

Remark 6. Let D be an open connected set in \mathbb{R}^N , $N \geq 2$. Let C be the union of a finite number of $(N-2)$ -dimensional subspaces. Then $D \setminus C$ is still connected.

Lemma 2. Let $N_0 < N$. Let S be a linear subspace of \mathbb{R}^N of dimension M , with $1 < M \leq N_0$. Then there cannot exist $M+1$ reflection hyperplanes Π_1, \dots, Π_{M+1} , with normals ν_1, \dots, ν_{M+1} belonging to S , and H_0 , an M -dimensional bounded convex polyhedron in S whose boundary is contained in the union of $\tilde{\Pi}_i = \Pi_i \cap S$, $i = 1, \dots, M+1$.

Proof. We let $\mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^{N-M}$ and, without loss of generality, $S = \mathbb{R}^M \times \{0\}$. We identify S with \mathbb{R}^M , H_0 with a bounded convex polyhedron in \mathbb{R}^M and $\tilde{\Pi}_i$, $i = 1, \dots, M+1$, with hyperplanes in \mathbb{R}^M . Let us consider all reflection hyperplanes Π of the kind $\Pi = \tilde{\Pi} \times \mathbb{R}^{N-M}$ with $\tilde{\Pi}$ a hyperplane in \mathbb{R}^M . With a little abuse of notation, we still call $\tilde{\Pi}$ a reflection hyperplane.

Let K be the union of all $(M-2)$ -dimensional subspaces of \mathbb{R}^M obtained by intersecting any pair of different reflection hyperplanes $\tilde{\Pi}$. Let V be the set of all points of \mathbb{R}^M obtained by intersecting any M reflection hyperplanes whose normals are linearly independent. The set K is closed, whereas the set V is finite, hence bounded and closed. Let us fix $R > 0$ such that $V \subset B_R$. Consequently, $H_0 \subset B_R$ as well.

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^M$ be a smooth curve connecting x_0 , a point in the interior of H_0 , to x_1 , a point such that $\|x_1\| \geq 4R + 2$. We can assume that x_1 does not belong to any reflection

hyperplane $\tilde{\Pi}$ and, by Remark 6, that γ does not contain any element of K . Moreover, we can assume that for some $c > 0$, $\|\gamma'(t)\| \geq c$ for any $t \in [0, 1]$.

Let t_0 be the largest $t \in [0, 1]$ such that $\gamma(t) \in H_0$. Let $\tilde{\Pi}^0$ be the (only) reflection hyperplane to which $\gamma(t_0)$ belongs. We call H_1 the reflection of H_0 in $\tilde{\Pi}^0$. It is clear that for some $t > t_0$, $\gamma(t)$ belongs to the interior of H_1 . Moreover, any $(M - 1)$ -dimensional face of H_1 is contained in a reflection hyperplane, and any vertex of H_1 belongs to V . Let $t_1 > t_0$ be the last $t \in (t_0, 1]$ such that $\gamma(t) \in H_1$. Let $\tilde{\Pi}^1$ be the (only) reflection hyperplane to which $\gamma(t_1)$ belongs. We note that $\tilde{\Pi}^1$ is different from $\tilde{\Pi}^0$. We call H_2 the reflection of H_1 in $\tilde{\Pi}^1$. It is clear that for some $t > t_1$, $\gamma(t)$ belongs to the interior of H_2 . Moreover, any $(M - 1)$ -dimensional face of H_2 is contained in a reflection hyperplane, and any vertex of H_2 belongs to V . By induction, with the same procedure, we construct t_1, \dots, t_n until $\|\gamma(t_n)\| > 4R + 1$. If this is the case, $H_n \cap B_R = \emptyset$ but the vertices of H_n must belong to V , and we have a contradiction. Otherwise, we construct a sequence $\{t_n\}_{n \in \mathbb{N}}$, with $t_n \rightarrow t_\infty < 1$. Since $t_{n+1} - t_n \rightarrow 0$ as $n \rightarrow +\infty$, and $\tilde{\Pi}^{n+1}$ is different from $\tilde{\Pi}^n$ for any $n \in \mathbb{N}$, we can find a point $P_n \in H_n \cap K$ such that $\|\gamma(t_n) - P_n\| \rightarrow 0$ as well. Clearly, $P_n \rightarrow \gamma(t_\infty)$ as $n \rightarrow +\infty$. Since K is closed, we conclude that $\gamma(t_\infty) \in K$, thus we obtain a contradiction. \square

Lemma 3. *If $N_0 < N$, then there exists an $(N - N_0)$ -dimensional subspace κ such that κ is contained in all reflection hyperplanes.*

Let Π_1, \dots, Π_m be all the reflection hyperplanes and $W = \{T_{\Pi_1}, \dots, T_{\Pi_m}\}$. Then the set

$$\mathcal{T} = \{T : T \text{ is the finite composition of reflections belonging to } W\}$$

is finite.

Proof. Let Π_1, \dots, Π_{N_0} be reflection hyperplanes whose normals are linearly independent. Then let κ be the intersection of Π_1, \dots, Π_{N_0} . Clearly κ is an $(N - N_0)$ -dimensional subspace. We need to show that all reflection hyperplanes contain κ .

Let $\mathbb{R}^N = \mathbb{R}^{N_0} \times \mathbb{R}^{N-N_0}$. For simplicity and without loss of generality, let us assume that the span of the corresponding normals v_1, \dots, v_{N_0} is $S_{N_0} = \mathbb{R}^{N_0} \times \{0\}$ and that $\kappa = \{0\} \times \mathbb{R}^{N-N_0}$. It is enough to consider the geometry in S_{N_0} . For any reflection hyperplane Π , we call $\tilde{\Pi}$ its intersection with S_{N_0} and we identify it with a hyperplane of \mathbb{R}^{N_0} , which with a little abuse of notation we still call $\tilde{\Pi}$ and still refer to as a reflection hyperplane.

Let us assume, by contradiction, that there exists a reflection hyperplane $\tilde{\Pi}_{N_0+1}$ not passing through the origin. Let v_{N_0+1} be its normal. If any N_0 elements of $\{v_1, \dots, v_{N_0+1}\}$ are linearly independent, then $\tilde{\Pi}_1, \dots, \tilde{\Pi}_{N_0+1}$ bound a convex polyhedron thus contradicting Lemma 2 with $M = N_0$.

Therefore, we can assume, up to reordering, that v_2, \dots, v_{N_0+1} are linearly dependent. Let S_{N_0-1} be the span of v_2, \dots, v_{N_0} . Let us call $\tilde{\Pi}_i^1$ the restrictions of Π_i to S_{N_0-1} . The intersection of $\tilde{\Pi}_2^1, \dots, \tilde{\Pi}_{N_0}^1$ is the origin of S_{N_0-1} and $\tilde{\Pi}_{N_0+1}^1$ cannot pass through the origin of S_{N_0-1} , otherwise $\tilde{\Pi}_{N_0+1}$ would pass through the origin of S_{N_0} . If any $N_0 - 1$ elements of $\{v_2, \dots, v_{N_0+1}\}$ are linearly independent, then $\tilde{\Pi}_2^1, \dots, \tilde{\Pi}_{N_0+1}^1$ bound a convex polyhedron thus contradicting Lemma 2 with $M = N_0 - 1$.

Iterating the procedure, and with suitable reordering, we obtain that $v_{N_0-1}, v_{N_0}, v_{N_0+1}$ are linearly dependent. Calling S_2 the span of v_{N_0-1} and v_{N_0} , and $\tilde{\Pi}_i^{N_0-2}$ the restrictions of $\tilde{\Pi}_i$ to S_2 , we obtain that the intersection of $\tilde{\Pi}_{N_0-1}^{N_0-2}$ and $\tilde{\Pi}_{N_0}^{N_0-2}$ is the origin of S_2 and $\tilde{\Pi}_{N_0+1}^{N_0-2}$ cannot pass through the origin of S_2 . If any pair of $\{v_{N_0-1}, v_{N_0}, v_{N_0+1}\}$ are linearly independent, we contradict Lemma 2 with $M = 2$. Hence Π_{N_0+1} is parallel to, and different from, either Π_{N_0-1} or Π_{N_0} . This contradicts the fact that $N_0 < N$ and the first part of the proof is concluded.

Once we have shown that all reflection hyperplanes intersect κ , we have that any element of \mathcal{T} is a rigid change in coordinates, keeping κ fixed. Let us consider $T \in \mathcal{T}$. The

set $\{T(v_1), \dots, T(v_{N_0})\}$ is a base of S_{N_0} and fully characterizes T . Since the image through T of any reflection hyperplane Π is another reflection hyperplane, the set $\{T(v_1), \dots, T(v_{N_0})\}$ is a subset of $\{\pm v_1, \dots, \pm v_m\}$. It immediately follows that \mathcal{T} is finite. \square

Lemma 4. Let $N_0 < N$. Let Σ_0 be a polyhedral scatterer that is symmetric with respect to all reflection hyperplanes. We call $G_0 = \mathbb{R}^N \setminus \Sigma_0$. Let x_0 be any point in G_0 not belonging to any reflection hyperplane. Then there exists a piecewise smooth path $\gamma : [0, +\infty) \rightarrow \mathbb{R}^N$ such that $\|\gamma(t)\| \rightarrow +\infty$ as $t \rightarrow +\infty$ and $\gamma(t)$ belongs to G_0 and does not belong to any reflection hyperplane for any $t \geq 0$.

Proof. Let Π_1, \dots, Π_m be all reflection hyperplanes and κ be their intersection, as in Lemma 3.

We can assume that $0 \in \kappa$ and that, for some $R > 0$, we have $\Sigma \subset B_R$. Let D_0 be the connected component of $\mathbb{R}^N \setminus (\bigcup_{i=1}^m \Pi_i)$ containing x_0 . Please note that for any $s \geq R + 1$, $\partial B_s \cap D_0$ is contained in G_0 and is connected.

Let P be the projection of x_0 on κ . Let l be the half-line starting from P and passing through x_0 . With the exception of P , any element of l belongs to D_0 . Let x_1 be an element of l such that after x_1 , any element of l is outside B_{R+1} . Therefore, the half-line contained in l , which starts from x_1 , is contained in G_0 and does not intersect any reflection hyperplane. To conclude the proof, it is enough to show that $D_0 \cap G_0$ is connected, namely that there exists a piecewise smooth curve γ in $D_0 \cap G_0$ connecting x_0 to x_1 . We denote $s = \|x_1\| \geq R + 1$.

Let K be the union of the intersections of any pair of different reflection hyperplanes. By Remark 6, $G_0 \setminus K$ is still connected. Let $\gamma_0 : [0, 1] \rightarrow \mathbb{R}^N$ be a smooth curve in $G_0 \setminus K$ connecting x_0 to x_1 . We can assume that for some constant $c > 0$ we have $\|\gamma'_0(t)\| \geq c$ for any $t \in [0, 1]$.

If $\gamma_0 \subset \overline{D_0}$, there is nothing to prove. In fact, by a small perturbation argument, we can change γ_0 such that $\gamma_0 \subset G_0 \cap D_0$. Otherwise, calling $t'_0 = 0$, let $t_1 > 0$ be largest t such that $\gamma_0(t_1) \in \partial D_0$ and $\gamma_0(t) \in \overline{D_0}$ for any $0 \leq t < t_1$. We call $\tilde{\Pi}_1$ the reflection hyperplane containing $\gamma_0(t_1)$ and D_1 the reflection of D_0 in $\tilde{\Pi}_1$. Let $t'_1 \geq t_1$ be the largest t such that $\gamma_0(t'_1) \in \tilde{\Pi}_1$ and $\gamma_0(t) \in \overline{D_0} \cup D_1$ for any $t \in (t_1, t'_1)$. By the symmetry of Σ_0 , $T_{\tilde{\Pi}_1}(\gamma_0([t_1, t'_1] \cap D_1))$ is contained in $G_0 \cap D_0$. Hence, by performing such a reflection, we can assume that $\gamma_0(t) \in \overline{D_0}$ for any $t \in [0, t'_1]$. Let us define for any $t \in [0, 1]$

$$\gamma_1(t) = \begin{cases} \gamma_0(t) & \text{if } t \in [0, t'_1] \\ T_{\tilde{\Pi}_1}(\gamma_0(t)) & \text{if } t \in (t'_1, 1]. \end{cases}$$

If $\gamma_1 \subset \overline{D_0}$, by a small perturbation argument we can change γ_1 such that $\gamma_1 \subset G_0 \cap D_0$. Then we connect $\gamma_1(1)$ to x_1 along $\partial B_s \cap D_0$ and the result is proved. Otherwise, let $t_2 > t'_1$ be largest t such that $\gamma_1(t_2) \in \partial D_0$ and $\gamma_1(t) \in \overline{D_0}$ for any $0 \leq t < t_2$. We call $\tilde{\Pi}_2$ the reflection hyperplane containing $\gamma_1(t_2)$ and D_2 the reflection of D_0 in $\tilde{\Pi}_2$. Please note that $\tilde{\Pi}_2$ is different from $\tilde{\Pi}_1$ by construction. Let $t'_2 \geq t_2$ be the largest t such that $\gamma_1(t'_2) \in \tilde{\Pi}_2$ and $\gamma_1(t) \in \overline{D_0} \cup D_2$ for any $t \in (t_2, t'_2)$. By the symmetry of Σ_0 , $T_{\tilde{\Pi}_2}(\gamma_1([t_2, t'_2] \cap D_2))$ is contained in $G_0 \cap D_0$. Hence, by performing such a reflection, we can assume that $\gamma_1(t) \in \overline{D_0}$ for any $t \in [0, t'_2]$. Let us define for any $t \in [0, 1]$

$$\gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t'_2] \\ T_{\tilde{\Pi}_2}(\gamma_1(t)) & \text{if } t \in (t'_2, 1]. \end{cases}$$

We proceed iteratively in the same way. If, for some $n \in \mathbb{N}$, $\gamma_n \subset \overline{D_0}$, by a small perturbation argument we can change γ_n such that $\gamma_n \subset G_0 \cap D_0$. Then we connect $\gamma_n(1)$ to x_1 along $\partial B_s \cap D_0$ and the result is proved. Otherwise, we find a sequence $\{t'_n\}_{n \in \mathbb{N}}$ with $0 < t'_n < t'_{n+1} < 1$ such that $\gamma_{n-1}(t'_n) \in \partial D_0$, with $\tilde{\Pi}_n$ being the reflection hyperplane containing $\gamma_{n-1}(t'_n)$, and $\gamma_{n-1}(t) \in \overline{D_0}$ for any $0 \leq t \leq t'_n$. We note that, by construction, $\tilde{\Pi}_{n+1}$ is different from $\tilde{\Pi}_n$ for any $n \in \mathbb{N}$. Let $t'_\infty = \lim_n t'_n$. We have that $\gamma_0(t'_n) \in \bigcup_{i=1}^m \Pi_i$,

hence $\gamma_0(t'_\infty) \in \bigcup_{i=1}^m \Pi_i$ as well. We conclude that $t'_\infty < 1$. Let $\hat{\Pi}_n$ be the reflection hyperplane to which $\gamma_0(t'_n)$ belongs. We note that $\hat{\Pi}_{n+1}$ is different from $\hat{\Pi}_n$ for any $n \in \mathbb{N}$. Since $t'_{n+1} - t'_n \rightarrow 0$ as $n \rightarrow +\infty$, we can find a point $P_n \in \hat{\Pi}_n \cap \hat{\Pi}_{n+1} \subset K$ such that $\|\gamma_0(t'_n) - P_n\| \rightarrow 0$ as well. Clearly, $P_n \rightarrow \gamma_0(t'_\infty)$ as $n \rightarrow +\infty$. Since K is closed, we conclude that $\gamma_0(t'_\infty) \in K$, thus we obtain a contradiction. \square

Lemma 5. *If $N_0 < N$, then Σ is symmetric with respect to all reflection hyperplanes.*

Proof. Let us consider κ and the set \mathcal{T} defined in Lemma 3. Without loss of generality, we assume that $0 \in \kappa$ and that for some $R > 0$, we have that $\Sigma \subset B_R$.

For any $T \in \mathcal{T}$, we can define u_T in $T(G) = \mathbb{R}^N \setminus T(\Sigma)$ such that $Au_T = 0$ in $T(G)$ and $u_T = u$ outside B_{R+1} . Moreover, $Bu_T = 0$ on any flat part of $\partial(T(\Sigma))$.

Let G_0 be the unbounded connected component of $\bigcap_{T \in \mathcal{T}} T(G)$ and $\Sigma_0 = \mathbb{R}^N \setminus G_0$. Let us show that G_0 is symmetric with respect to any reflection hyperplane. Let $x \in G_0$, Π be a reflection hyperplane and $y = T_\Pi(x)$. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^N$ be a continuous curve in G_0 connecting x to a point x_1 outside B_{R+1} . For any $t \in [0, 1]$, $\gamma(t) \in T(G)$ for any $T \in \mathcal{T}$. Since $T_\Pi \circ T$ still belongs to \mathcal{T} , $\gamma(t) \in (T_\Pi \circ T)(G)$ for any $T \in \mathcal{T}$, hence $T_\Pi(\gamma(t)) \in T_\Pi((T_\Pi \circ T)(G)) = T(G)$ for any $T \in \mathcal{T}$. Hence $\gamma_1 = T_\Pi(\gamma) \subset \bigcap_{T \in \mathcal{T}} T(G)$. Since $\mathbb{R}^N \setminus \overline{B_R} \subset G_0$ and $y_1 = T_\Pi(x_1)$ is also outside B_{R+1} , we conclude that $y \in G_0$ as well.

On G_0 , we have that $u = u_T$ for any $T \in \mathcal{T}$. Since $\Sigma_0 = \bigcup_{T \in \mathcal{T}} T(\Sigma)$, we have that Σ_0 is a polyhedral obstacle containing Σ and $Bu = 0$ on any flat portion of $\partial\Sigma_0$. Moreover, by symmetry, no flat portion of $\partial\Sigma_0$ is a subset of any reflection hyperplane.

Let us assume that Σ_0 is different from Σ . Then there exists $x \in \partial\Sigma_0 \cap G$. Without loss of generality, we can assume that x belongs to a flat portion of $\partial\Sigma_0$ and x does not belong to any reflection hyperplane. For some ν normal to the cell of $\partial\Sigma_0$ to which x belongs and some $\varepsilon_0 > 0$, we have that, for any $t \in (0, \varepsilon_0]$, $x + t\nu \in G_0$ and does not belong to any reflection hyperplane. We call $x_0 = x + \varepsilon_0\nu$ and, by Lemma 4, we can find a piecewise smooth curve $\gamma : [0, +\infty) \rightarrow \mathbb{R}^N$ such that $\gamma(0) = x$, $\gamma(t) = x + t\nu$ for any $t \in (0, \varepsilon_0]$ and $\gamma(t)$ belongs to G_0 and does not belong to any reflection hyperplane for any $t \geq \varepsilon_0$. We note that x is a flat point for u . By the same argument used in the proof of Theorem 1, we can find $t_1 \in [0, +\infty)$ such that $x_1 = \gamma(t_1)$ is a flat point belonging to a reflection hyperplane for u . Since no point of γ belongs to any reflection hyperplane, we obtain a contradiction.

We conclude that $\Sigma_0 = \Sigma$ hence Σ is symmetric with respect to any reflection hyperplane. \square

Proof of Theorem 2. By renaming $\Sigma' = \mathbb{R}^N \setminus \tilde{G}$, without loss of generality, we can assume that Σ' is a polyhedral obstacle, with $\Sigma \subset \Sigma'$ and $\Sigma' \setminus \Sigma \neq \emptyset$. Let us assume by contradiction that $N_0 < N$. Let $x_0 \in G$ be such that x_0 does not belong to any reflection hyperplane and x_0 is contained in the interior of Σ' . Let γ be as in Lemma 4 with Σ_0 replaced by Σ . It is clear that there exists $t_0 > 0$ such that $\gamma(t_0) \in \partial\Sigma'$, hence $\gamma(t_0)$ is a flat point for u . Again, by the same argument used in the proof of Theorem 1, we can find $t_1 \geq t_0$ such that $y_1 = \gamma(t_1)$ is a flat point belonging to a reflection hyperplane for u . This contradicts the properties of γ , and the proof is concluded. \square

2.1. The Case of Mixed Boundary Conditions

Now, we briefly consider the case of mixed boundary conditions. Let us assume that we have n different boundary conditions B^1, \dots, B^n satisfying (A.3)(a) and (A.3)(b), with $\tilde{T}_{\Pi, \nu}$ clearly depending on $i \in \{1, \dots, n\}$. We also assume that, for any $i = 1, \dots, n$, the property (A.3)(c) is replaced by

(A.3)(c)₁ Let Π_1 be any hyperplane and let $x \in \Pi_1$ be such that, for some ν_1 orthogonal to Π_1 , $B_r(x) \cap H_{\Pi_1}^+ \subset D^+$. Then, for any $j = 1, \dots, n$, if $B^j u^+ = 0$ on $B_r(x) \cap \Pi_1$, then $B^j u^- = 0$ on $T_\Pi(B_r(x) \cap \Pi_1)$.

Let Σ be a polyhedral scatterer. We assume that u is a solution to $Au = 0$ in $G = \mathbb{R}^N \setminus \Sigma$ and that for any cell C , contained in a hyperplane Π , and any ν orthogonal to Π such that

for any x belonging to the interior of C there exists $r > 0$ with $B_r(x) \cap H_{\Pi}^+ \subset G$, we have $B^i u = 0$ in the interior of the cell C for some $i \in \{1, \dots, n\}$ depending on C .

We note that in particular in the case of polyhedra, on any $(N - 1)$ -face, we can have more than one boundary condition, i.e., we can split such a face into two or more internally pairwise disjoint cells, and on each of them, a different boundary condition is satisfied.

Definition 2 is replaced by the following.

Definition 3. A point $x \in G$ is a flat point for u if there exists Π passing through x and $i \in \{1, \dots, n\}$ such that $B^i u = 0$ on $\Pi \cap B_r(x)$ for some $r > 0$ with $B_r(x) \subset G$.

Furthermore, any hyperplane Π such that, for some $i \in \{1, \dots, n\}$, $B^i u = 0$ on $\Pi \setminus \overline{B_R}$ for some $R > 0$ will be called a reflection hyperplane.

With the same proofs, using the suitable reflection depending on the boundary condition, the two main theorems, Theorems 1 and 2, can be restated in the case of mixed boundary conditions as follows.

Theorem 3. Let us assume that u admits a flat point $x \in G$. Then, there exists a reflection hyperplane Π_1 .

Let Σ be a polyhedral obstacle. Let Σ' be a finite union of polyhedra with $\Sigma' \setminus \Sigma \neq \emptyset$. Let \tilde{G} be the unbounded connected component of $\mathbb{R}^N \setminus (\Sigma \cup \Sigma')$. Assume that any $x \in \partial \tilde{G} \cap G$ is a flat point. Then there exist N reflection hyperplanes Π_1, \dots, Π_N , whose corresponding normals v_1, \dots, v_N are linearly independent.

3. Application to Uniqueness Results for the Determination of Polyhedral Scatterers and Examples

Let Σ and Σ' be two polyhedral scatterers contained in B_{R_0} . Let u satisfy $Au = 0$ in G and be such that $Bu = 0$ on any flat part of $\partial \Sigma$. Let u' satisfy $Au' = 0$ in $G' = \mathbb{R}^N \setminus \Sigma'$ and be such that $Bu' = 0$ on any flat part of $\partial \Sigma'$. Assume that $u = u'$ on an open subset \tilde{D} of \tilde{G} , \tilde{G} being the unbounded connected component of $\mathbb{R}^N \setminus (\Sigma \cup \Sigma')$. For example, \tilde{D} can be any open subset of $\mathbb{R}^N \setminus \overline{B_{R_0}}$. Then, by unique continuation $u = u'$ on \tilde{G} , u and u' have the same reflection hyperplanes. Our aim is to prove a uniqueness result, i.e., to show that $\Sigma = \Sigma'$. In other words, the measurement $u|_{\tilde{D}}$, or any equivalent one, uniquely determines the polyhedral scatterer Σ . The argument is the following. Assume, by contradiction, that $\Sigma \neq \Sigma'$. Up to swapping Σ with Σ' , we can find $x \in \partial \tilde{G} \setminus \Sigma \subset \partial \Sigma' \setminus \Sigma$, hence u admits a flat point. Using Theorem 1, we can conclude that u , and u' as well, admits a reflection hyperplane. If we have suitable conditions guaranteeing that u , and equivalently u' , does not admit any reflection hyperplane, we obtain a contradiction, and the uniqueness result holds true.

Under the same assumptions, if Σ and Σ' are different polyhedral obstacles, by using Theorem 2 instead, we can conclude that u , and u' as well, admits N reflection hyperplanes whose normals are linearly independent. If we have suitable conditions guaranteeing that u , and equivalently u' , does not admit N reflection hyperplanes whose normals are linearly independent, the measurement $u|_{\tilde{D}}$, or any equivalent one, uniquely determines the polyhedral obstacle Σ .

For example, let us assume that u and u' can be written as

$$u = u^i + u^s \quad \text{and} \quad u' = u^i + (u')^s$$

where u^i is an entire solution to $Au = 0$. About u^s , we assume that it satisfies the following decay condition at infinity, namely that, for any $\varepsilon > 0$, there exists $R > R_0$ such that, for any unit vector v ,

$$|Bu^s|, |B((u')^s)| \leq \varepsilon \quad \text{outside } B_R. \quad (1)$$

Then on any reflection hyperplane Π of u or of u' we have

$$|Bu^i| \leq \varepsilon \quad \text{on } \Pi \setminus \overline{B_R}. \quad (2)$$

If $u = u'$ on \tilde{D} , or equivalently $u^s = (u')^s$ on \tilde{D} , then, on any reflection hyperplane Π of u and u' , (2) holds. If $\Sigma \neq \Sigma'$, then

$$\text{there exists a hyperplane } \Pi \text{ such that } |Bu^i| \leq \varepsilon \text{ on } \Pi \setminus \overline{B_R}. \quad (3)$$

If we further assume that Σ and Σ' are polyhedral obstacles, then

$$\begin{aligned} &\text{there exist } N \text{ hyperplanes } \Pi_1, \dots, \Pi_N \text{ with linearly independent} \\ &\text{normals such that } |Bu^i| \leq \varepsilon \text{ on } \Pi_i \setminus \overline{B_R} \text{ for any } i = 1, \dots, N. \end{aligned} \quad (4)$$

Hence, proving the corresponding uniqueness results reduces to finding a suitable u^i violating either (3) or (4).

This situation is typical of inverse scattering problems. In this context, u^i is called the *incident field* or *wave*, whereas u^s is called the *scattered field* or *wave*. A suitable condition at infinity, the *radiation condition*, has to be imposed on u^s , and it usually guarantees the validity of (1). The scattered field may be measured either on the set \tilde{D} (*near-field data*) or at infinity (*far-field data*). In fact, typically, some kind of Rellich lemma holds, i.e., if the far-field data of u^s and $(u')^s$ coincide, then u^s and $(u')^s$ coincide on the whole \tilde{G} .

In the next, we collect a few examples to show the applicability of the theory. The list is clearly not exhaustive.

3.1. Acoustic Waves

Let $\omega \in \mathbb{R}$, $\omega > 0$, and

$$Au = \Delta u + \omega^2 u.$$

If u solves $Au = 0$, then u is real-analytic. We use two boundary conditions: $Bu = B_D u = u$ for the Dirichlet case, with $\tilde{T}_\Pi(u) = -u$, or $Bu = B_N u = \nabla u \cdot \nu$ for the Neumann case, with $\tilde{T}_\Pi(u) = u$, in both cases independently of ν . We note that B_D and B_N satisfy (A.3)(c₁) as well.

Please note that by setting $L = L_1 = M$ and $A_k u = Au^k$ and $B_k u = Bu^k$ we can actually consider M different measurements, i.e., we can consider M different experiments using M different incident fields u_k^i .

The radiation condition is the *Sommerfeld radiation condition*. The far-field data corresponds to the so-called *far-field pattern* of the scattered field. The Rellich lemma states that if the far-field pattern is zero, then the scattered wave is identically zero. The Sommerfeld radiation condition indeed implies the validity of (1).

For any *incident direction* $d \in \mathbb{S}^{N-1}$, let the incident wave be given by a planar plane wave

$$u_d^i(x) = e^{i\omega x \cdot d}, \quad x \in \mathbb{R}^N.$$

Since $|u_d^i(x)| = 1$ for any $x \in \mathbb{R}^N$, choosing $0 < \varepsilon < 1$, (3) cannot hold, and we have uniqueness with a single measurement (that is, with $M = 1$ corresponding to using a single incident field) in the Dirichlet case. In the Neumann case, $|B_N u_d^i| = \omega |\nu \cdot d|$. If we obtain N measurements (that is, $M = N$), corresponding to N different incident fields $u_{d_k}^i$, $k = 1, \dots, N$, with d_k linearly independent, then no matter what ν is, $|B_N u_{d_k}^i| = \omega |\nu \cdot d_k| > 0$ for some k and, choosing $0 < \varepsilon < \omega |\nu \cdot d_k|$, (3) cannot hold. Therefore, we have uniqueness with N measurements in the Neumann case or the mixed Dirichlet and Neumann case. For polyhedral obstacles, no matter what d is, $|B_N u_d^i| = \omega |\nu_i \cdot d| > 0$ for some i , where ν_i are the linearly independent normals of reflection hyperplanes. Choosing $0 < \varepsilon < \omega |\nu_i \cdot d|$, (4) can not hold. Therefore, we have uniqueness with 1 measurement in the Neumann case, or in the mixed Dirichlet and Neumann case, for polyhedral obstacles.

3.2. Electromagnetic Waves

Let $N = 3$.

We let $u = (E, H) = (E_1, E_2, E_3, H_1, H_2, H_3) \in \mathbb{C}^6$. Let $\omega \in \mathbb{R}$, $\omega > 0$, and

$$Au = (\nabla \times E - i\omega H, \nabla \times H + i\omega E).$$

If $Au = 0$, then $\Delta E + \omega^2 E = 0$ and $\Delta H + \omega^2 H = 0$, hence any component of u is real-analytic. We consider two boundary conditions: $Bu = B_E = E \times \nu$, for a *perfectly electric conducting* scatterer, and $Bu = B_H = H \times \nu$, for a *perfectly magnetic conducting* scatterer. Then, independently of ν , $\tilde{T}_\Pi(u) = \pm(-T_\Pi(E), T_\Pi(H))$, with $+$ for B_E and $-$ for B_H . We note that B_E and B_H satisfy (A.3)(c₁) as well.

Please note that by setting $L = L_1 = M$ and $A_k u = Au^k$ and $B_k u = Bu^k$ we can actually consider M different measurements, i.e., we can consider M different experiments using M different incident fields u_k^i .

The radiation condition is the *Silver-Müller radiation condition*, which is equivalent to the validity of the Sommerfeld radiation condition for any component of E and H . Therefore, the Silver-Müller radiation condition implies the validity of (1). About near-field data, we can just measure E or H on \tilde{D} . About far-field data, we can just measure the far-field pattern of all components of E or of all components of H .

About incident fields, let $u^i = (E^i, H^i)$ be given by

$$E^i(x) = \frac{i}{\omega} \nabla \times (\nabla \times p e^{i\omega x \cdot d}), \quad H^i(x) = \nabla \times p e^{i\omega x \cdot d}, \quad x \in \mathbb{R}^3, \quad (5)$$

that is,

$$E^i(x) = -\omega i(q \times d) e^{i\omega x \cdot d}, \quad H^i(x) = -\omega i(p \times d) e^{i\omega x \cdot d}, \quad x \in \mathbb{R}^3.$$

Here $u^i = u_{d,p}^i$ is the *normalized electromagnetic plane wave* with incident direction $d \in \mathbb{S}^2$ and polarization vector $p \in \mathbb{R}^3$, $p \neq 0$ orthogonal to d , and $q = p \times d$. Then $|Bu_{d,p}^i| = \omega |\nu \times (q \times d)|$. Arguing as for the Neumann acoustic case, we have uniqueness with two measurements (that is, $M = 2$) corresponding to 2 different incident fields u_{d_1,p_1}^i and u_{d_2,p_2}^i with $q_1 \times d_1$ and $q_2 \times d_2$ linearly independent for B_E and $p_1 \times d_1$ and $p_2 \times d_2$ linearly independent for B_H . If $q_1 \times d_1$ and $q_2 \times d_2$ are linearly independent and $p_1 \times d_1$ and $p_2 \times d_2$ are also linearly independent (for example if $d_1 = d_2$ and p_1 and p_2 are linearly independent) then we have uniqueness with these two measurements also in the mixed electric and magnetic conducting cases. For polyhedral obstacles, instead, we have uniqueness with just 1 measurement in the electric conducting, magnetic conducting, and mixed electric and magnetic conducting cases.

3.3. Elastic Waves

Let $N \geq 2$. For a \mathbb{C}^N -valued function u , let

$$Au = \mu \Delta u + (\lambda + \mu) \nabla(\operatorname{div}(u)) + \rho \omega^2 u. \quad (6)$$

The equation $Au = 0$ is the *Navier equation*; here λ and μ are the Lamé constants such that $\mu > 0$ and $\lambda + 2\mu > 0$, $\rho > 0$ is the density and $\omega > 0$ is the frequency.

The symmetric gradient of u , $Eu = \frac{1}{2}(\nabla u + (\nabla u)^T)$, is the *strain tensor* while the *stress* $\sigma(u)$ is

$$\sigma(u) = 2\mu Eu + \lambda \operatorname{tr}(Eu) I_N = 2\mu Eu + \lambda \operatorname{div}(u) I_N,$$

where tr denotes the trace and I_N is the identity matrix. The Navier equation $Au = 0$ can be rewritten as

$$\operatorname{div}(\sigma(u)) + \rho \omega^2 u = 0,$$

where the div applies row by row.

By Helmholtz decomposition, any u such that $Au = 0$ satisfies

$$u = u_s + u_p$$

where u_p is the *longitudinal wave* u_p and u_s is the *transversal wave*. We have that both u_p and u_s solve $Au = 0$. Moreover,

$$u_p = -\frac{\nabla \operatorname{div}(u)}{\omega_p^2}, \quad \Delta u_p + \omega_p^2 u_p = 0, \quad \omega_p^2 = \frac{\rho \omega^2}{\lambda + 2\mu}. \quad (7)$$

Finally,

$$u_s = \frac{\nabla \operatorname{div}(u) - \Delta u}{\omega_s^2}, \quad \Delta u_s + \omega_s^2 u_s = 0, \quad \omega_s^2 = \frac{\rho \omega^2}{\mu}. \quad (8)$$

We immediately conclude that u is real-analytic.

About boundary conditions, first of all, we define the *surface traction* $\operatorname{Tr}(u) = \sigma(u)v$. For any v and any vector V , we denote $V_\tau = V - (V \cdot v)v$ the tangential component of V with respect to v . We consider the so-called *third* and *fourth boundary conditions*. Specifically,

$$Bu = B^3 u = (u \cdot v, (\operatorname{Tr}(u))_\tau)$$

and

$$Bu = B^4 u = (u_\tau, \operatorname{Tr}(u) \cdot v)$$

Then, in both cases independently of v , $\tilde{T}_\Pi(u) = \pm T_\Pi(u)$, with $+$ for the third boundary condition and $-$ for the fourth one. We note that B^3 and B^4 satisfy (A.3)(c₁) as well.

Please note that by setting $L = L_1 = M$ and $A_k u = Au^k$ and $B_k u = Bu^k$ we can actually consider M different measurements, i.e., we can consider M different experiments using M different incident fields u_k^i .

The radiation condition is the *Kupradze radiation condition*, which corresponds to both u_s and u_p solving the Sommerfeld radiation condition. Therefore, the Kupradze radiation condition implies the validity of (1). About near-field data, we measure u on \tilde{D} . About far-field data, we measure the far-field pattern of u , which is equivalent to measuring the far-field pattern of both the longitudinal and transversal waves of u .

About incident fields, we let $u_{p,d}^i$ be a *longitudinal plane wave*

$$u_{p,d}^i(x) = d e^{i\omega_p d \cdot x}, \quad x \in \mathbb{R}^N, \quad (9)$$

where $d \in \mathbb{S}^{N-1}$ is the *incident direction*, and let $u_{s,d,q}^i$ be *transversal plane wave*

$$u_{s,d,q}^i(x) = q e^{i\omega_s d \cdot x}, \quad x \in \mathbb{R}^N, \quad (10)$$

where $q \in \mathbb{C}^N \setminus \{0\}$ is a unitary vector orthogonal to d . Then, as incident wave u^i we can choose a linear combination of longitudinal and transversal plane waves, namely

$$u^i = c_p u_{p,d}^i + c_s u_{s,d,q}^i \quad (11)$$

for some $c_p, c_s \in \mathbb{C}$ such that $|c_p|^2 + |c_s|^2 = 1$. In this case $\|u^i(x)\| = 1$ for any $x \in \mathbb{R}^N$. If $c_p = 0$ we call u^i a pure transversal incident wave, and if $c_s = 0$ we call u^i a pure longitudinal incident wave.

Then

$$u^i \cdot v = c_p e^{i\omega_p d \cdot x} d \cdot v + c_s e^{i\omega_s d \cdot x} q \cdot v$$

and

$$u_\tau^i = c_p e^{i\omega_p d \cdot x} d_\tau + c_s e^{i\omega_s d \cdot x} q_\tau.$$

If d and q are row vectors,

$$\sigma(u^i) = c_p i\omega_p e^{i\omega_p d \cdot x} (2\mu d^T d + \lambda I_N) + c_s i\omega_s e^{i\omega_s d \cdot x} \mu (d^T q + q^T d)$$

hence

$$\begin{aligned} \text{Tr}(u^i) &= \sigma(u^i)v = \\ &= c_p i \omega_p e^{i \omega_p d \cdot x} (2\mu(d \cdot v)d^T + \lambda v) + c_s i \omega_s e^{i \omega_s d \cdot x} \mu((q \cdot v)d^T + (d \cdot v)q^T). \end{aligned}$$

Therefore

$$\text{Tr}(u^i) \cdot v = c_p i \omega_p e^{i \omega_p d \cdot x} (2\mu(d \cdot v)^2 + \lambda) + c_s i \omega_s e^{i \omega_s d \cdot x} 2\mu(q \cdot v)(d \cdot v)$$

and

$$(\text{Tr}(u^i))_\tau = c_p i \omega_p e^{i \omega_p d \cdot x} 2\mu(d \cdot v)d_\tau + c_s i \omega_s e^{i \omega_s d \cdot x} \mu((q \cdot v)d_\tau + (d \cdot v)q_\tau).$$

In conclusion,

$$\begin{aligned} B^3(u^i) &= \left(c_p e^{i \omega_p d \cdot x} d \cdot v + c_s e^{i \omega_s d \cdot x} q \cdot v, \right. \\ &\quad \left. c_p i \omega_p e^{i \omega_p d \cdot x} 2\mu(d \cdot v)d_\tau + c_s i \omega_s e^{i \omega_s d \cdot x} \mu((q \cdot v)d_\tau + (d \cdot v)q_\tau) \right) \end{aligned}$$

and

$$\begin{aligned} B^4(u^i) &= \left(c_p e^{i \omega_p d \cdot x} d_\tau + c_s e^{i \omega_s d \cdot x} q_\tau, \right. \\ &\quad \left. c_p i \omega_p e^{i \omega_p d \cdot x} (2\mu(d \cdot v)^2 + \lambda) + c_s i \omega_s e^{i \omega_s d \cdot x} 2\mu(q \cdot v)(d \cdot v) \right). \end{aligned}$$

We conclude that for B^4 , it is enough to use 1 measurement corresponding to any pure longitudinal incident wave with any d (that is, by choosing $c_p = 1$ and $c_s = 0$, for example). In fact, if $d_\tau \neq 0$, then the norm of the first component of $B^4(u^i)$ is $\|d_\tau\| > 0$ and we obtain a contradiction. Otherwise, $|d \cdot v| = 1$ and the modulus of the second component of $B^4(u^i)$ is $\omega_p(2\mu + \lambda) > 0$ and we obtain a contradiction.

Instead, for B^3 , we need to use two measurements to avoid the unfortunate case that both d and q are orthogonal to v . For example, one can use two pure transversal incident waves u_{s,d_1,q_1}^i and u_{s,d_2,q_2}^i with $d_1 = d_2$ and q_1 and q_2 linearly independent. We note that these measurements give uniqueness even in the mixed third and fourth boundary conditions.

Finally, in the case of polyhedral obstacles, one measurement is enough by choosing as an incident wave any pure transversal incident wave or any pure longitudinal incident wave, even in the mixed third and fourth boundary conditions.

4. Conclusions

We have shown that the proof of unique determination of polyhedral scatterers by a finite number of scattering measurements is based on a purely geometric argument, provided some structural assumptions on the equation, and the boundary condition are satisfied. This geometric argument is extremely robust and allows us to treat in the same way different kinds of equations, different boundary conditions, also of mixed type, and any finite number of measurements. Indeed, we have shown that most uniqueness results of this nature available in the literature straightforwardly follow our general procedure. Therefore, the results described here point out the way to obtain these kinds of uniqueness results in other contexts, with the great advantage of just focusing on the physics of the system without having to deal with the challenging geometry of the problem.

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