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Homotopes of Quasi-Jordan Algebras

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Abstract: The notion of quasi-Jordan algebras was originally proposed by R. Velasquez and R. Fellipe. Later, M. R. Bremner provided a modification called K-B quasi-Jordan algebras; these include all Jordan algebras and all dialgebras, and hence all associative algebras. Any quasi-Jordan algebra is special if it is isomorphic to a quasi-Jordan subalgebra of some dialgebras. Keeping in view the pivotal role of homotopes in the theory of Jordan algebras, we begin a study of the homotopes of quasi-Jordan algebras; among other related results, we show that the homotopes of any special quasi-Jordan algebra are special quasi-Jordan algebras and that the homotopes of a K-B quasi-Jordan algebra is a quasi-Jordan algebra. In the sequel, we also give some open problems.

Keywords: Jordan algebra; dialgebra; quasi-Jordan algebra; zero part; split quasi-Jordan algebra

MSC: 17C10; 16W10; 55P99

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1. Introduction and Preliminaries

A Jordan algebra is a non-associative algebra with a product $x \circ y$ that satisfies both commutativity, $x \circ y = y \circ x$, and the Jordan identity, $(x^2 \circ y) \circ x = x^2 \circ (y \circ x)$, where $x^2 = x \circ x$. Any associative algebra can be considered a Jordan algebra, sharing the same linear space structure and defined by the Jordan product, $x \circ y := \frac{1}{2}(xy + yx)$. Additionally, it can be regarded as a Lie algebra under the skew–symmetric product $[x, y] := xy - yx$, known as the Lie bracket [1]). Associated with every Jordan algebra $\mathcal{J}$, there exists a corresponding Lie algebra $\mathcal{L}(\mathcal{J})$, such that $\mathcal{J}$ is a linear subspace of $\mathcal{L}(\mathcal{J})$ and the product of $\mathcal{J}$ can be expressed using the Lie bracket in $\mathcal{L}(\mathcal{J})$. Furthermore, the universal enveloping algebra of a Lie algebra exhibits the structure of an associative algebra, as established in the original work by I. Kantor, M. Koecher, and J. Tits in [2–4]. A generalization of Lie algebras, namely Leibniz algebras, was studied by J. Loday [5,6]; he demonstrated that the relationship between Lie algebras and associative algebras is analogous relationship between Leibniz algebras and dialgebras ([7]): a dialgebra over a field K is a K-module $\mathcal{D}$ equipped with associative bilinear products $\dashv, \vdash: \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ such that $x \dashv (y \vdash z) = x \dashv (y \dashv z)$; $(x \vdash y) \dashv z = x \vdash (y \dashv z)$ and $(x \dashv y) \vdash z = (x \vdash y) \vdash z$, $\forall x, y, z \in D$. Every dialgebra $(\mathcal{D}, \dashv, \vdash)$ is a Leibniz algebra with the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$; the universal enveloping algebra of any Leibniz algebra is a dialgebra ([6,7]).

In 2008, R. Velásquez and R. Felipe [8] introduced a new class of non-associative algebras. A vector space $\Im$ over a field of characteristic $\neq 2, 3$ is called a quasi-Jordan algebra if there is a bilinear product $\triangleleft : \Im \times \Im \to \Im$, satisfying $x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y)$ and $(y \triangleleft x) \triangleleft x^2 = (y \triangleleft x^2) \triangleleft x$, called right commutativity and right Jordan identity, respectively; "$\triangleleft$" is called the quasi-Jordan product. If the characteristic of the underlying field of a dialgebra $(\mathcal{D}, \dashv, \vdash)$ is different from 2 and 3, then $\mathcal{D}$ is a quasi-Jordan algebra, under the product $x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x)$; called plus quasi-Jordan algebra, denoted by $\mathcal{D}^+$. A quasi-Jordan algebra is said to be special if it is isomorphic to a subalgebra of some plus quasi-Jordan algebra; otherwise, the quasi-Jordan algebra is called exceptional ([9]).

Later on, M. R. Bremner [10] gave a modification of quasi-Jordan algebras; A quasi-Jordan algebra is called a K-B quasi-Jordan algebra if it satisfies the identity $(z, y, x^2) = 2(zx, y, x)$, where $(.,.,.)$ is the associator. Thus, the class of K-B quasi-Jordan algebras includes all Jordan algebras, all dialgebras and so all associative algebras ([11]). These facts underscore the significance of studying quasi-Jordan algebras. In [12], the present authors introduced the norm structure on quasi-Jordan algebras, called quasi-Jordan Banach algebras, and initiated a study of such algebras from the functional analytic point of view; thus, paving a new research area in the development of non associative algebras. Recently, the authors discussed the involution on quasi-Jordan algebras [13], Opening an other novelty in this area of research. In just a few years time, a reasonable size of stretcher in this research area have been published ([9,14–19]).

From the above definition it is clear that a quasi-Jordan algebra is a Jordan algebra if it is commutative. Every quasi-Jordan algebra $\Im$ contains the following two sets of our interest in the present work, two important sets: $\Im_{ann} := span\{x \triangleleft y - y \triangleleft x : x, y \in \Im\}$ and $\mathcal{Z}(\Im) := \{z \in \Im : x \triangleleft z = 0, \forall x \in \Im\}$, , respectively, called the annihilator and the zero part of $\Im$. From the right commutativity, we obtain $\Im_{ann} \subseteq \mathcal{Z}(\Im)$; and so the quasi-Jordan algebra $\Im$ is a Jordan algebra if $\Im_{ann} = \{0\}$; of course, the converse is always true. If a quasi-Jordan algebra $(\Im, \triangleleft)$ has a left unit e (that is, $e \triangleleft x = x$, for all $x \in \Im$), then $\Im$ is commutative and so it is a Jordan algebra. In this work, unit of a quasi-Jordan algebra would mean a right unit: that is, an element $e \in \Im$ satisfying $x \triangleleft e = x, \forall x \in \Im$ in a quasi-Jordan algebra $\Im$. A quasi-Jordan algebra may have many (right) units ([12]). An element e of dialgebra $(\mathcal{D}, \dashv, \vdash)$ is said to be a bar-unit if $x \dashv e = x = e \vdash x$, for all $x \in \mathcal{D}$. If the dialgebra D has a bar-unit e, $x \triangleleft e = \frac{1}{2}(x \dashv e + e \vdash x) = x, \forall x \in D$, and so e is a unit of D^+.

It is known that if a quasi-Jordan algebra $\Im$ has a unit e then $\Im_{ann}$ and $\mathcal{Z}(\Im)$ are ideals of $\Im$, $\Im_{ann} = \{x \in \Im : e \triangleleft x = 0\} = \mathcal{Z}(\Im)$ and $U(\Im) = \{x + e : x \in \mathcal{Z}(\Im)\}$; here, $U(\Im)$ symbolizes the set of all (right) units in $\Im$ ([20]).

The notion of homotopes is one of the basic tools in the study of Jordan algebras ([1,21–30]). In this article, we initiate a study of homotopes of quasi-Jordan algebras. It is known that the Jordan triple product is an effective tool in the study of Jordan structures ([1,28,31]). We propose a triple product for quasi-Jordan algebras and obtain some of its properties in Section 2. Using this triple product, we prove several theorems on homotopes of quasi-Jordan algebras in Sections 3–5, which are the main results of this work. In Section 3, among other results, we prove that homotope of a plus quasi-Jordan algebra is a plus quasi-Jordan algebra and hence homotopes of special quasi-Jordan algebras are special. We discuss homotopes of K-B quasi-Jordan algebras in Section 4 where we prove that every homotope of any K-B quasi-Jordan algebra is a quasi-Jordan algebra. Section 5 contains discussion of other possible analogues of the Jordan triple product. Some open problems are given in the sequel for further research as outlined in the conclusion.

2. A Quasi Jordan Triple Product

In any Jordan algebra $(J, \circ)$, the binary product "$\circ$" induces an important triple product $\{a, b, c\}$, called the Jordan triple product, defined as follows:

$$\{x, z, y\} = x \circ (z \circ y) - z \circ (y \circ x) + y \circ (x \circ z). \tag{1}$$

As is commonly recognized, the Jordan triple product holds a fundamental position in the examination of broader Jordan systems, often referred to as Jordan triple systems ([1,24,26–29,32,33]). In particular, the Jordan triple product induces certain operators of a fundamental nature; these include the basic operator $U_{(x,y)}$ defined on J by $U_{(x,y)}z := \{x, z, y\}$. The operator $U_{(x,x)}$ is usually written in short as U_x, which being quadratic in x is called the quadratic operator. In fact, the whole of the Jordan algebra theory can be developed on the basis of just the quadratic operators ([1,31]).

In this context, we introduce a counterpart of the Jordan triple product designed for quasi-Jordan algebras, which we term the "quasi-Jordan triple product". Suppose $\Im$

represents a quasi-Jordan algebra. We proceed to define the quasi-Jordan triple product, denoted as $\{.,.,.\}_q : \Im \times \Im \times \Im \to \Im$, in the following manner:

$$\{x,z,y\}_q := x \triangleleft (z \triangleleft y) + (x \triangleleft z) \triangleleft y - (x \triangleleft y) \triangleleft z. \tag{2}$$

$\{a,b,c\}_q$ is called the quasi-Jordan triple product of a,b,c in $\Im$.

One can always attach a (two-sided) unit to any Jordan algebra by following the standard unitization process; this unitization process no longer works for quasi-Jordan algebras ([20], pp. 210–211). The challenge of adding a unit to a quasi-Jordan algebra remains an open problem. As a step towards addressing this unitization problem, Vel'asquez and Felipe [20] introduced a specific class of quasi-Jordan algebras known as "split quasi-Jordan algebras". A quasi-Jordan algebra $\Im$ is designated as "split" over its ideal I if it fulfills the condition $\Im^{ann} \subseteq I \subseteq \mathcal{Z}(\Im)$, where $\Im = J \oplus I$, representing the direct sum of J and I, with J being a subalgebra of $\Im$. In such cases, the subalgebra J functions as a Jordan algebra under the original quasi-Jordan product, denoted as "$\triangleleft$", within J. If x and y belong to J, then both $x \triangleleft y$ and $x \triangleleft y$ are elements of the subalgebra J, indicating that $x \triangleleft y - y \triangleleft x \in J$; but $x \triangleleft y - y \triangleleft x \in \Im^{ann}$, which is entirely contained within I; therefore, $x \triangleleft y - y \triangleleft x \in J \cap I = \{0\}$, and hence $x \triangleleft y = y \triangleleft x$. One can attach a unit to any split quasi-Jordan algebra; specifics regarding this unitization procedure can be found in [20]. If a quasi-Jordan algebra $\Im$ possesses a unit, it follows that $\Im^{ann} = \mathcal{Z}(\Im)$. Consequently, a quasi-Jordan algebra $\Im$ with a unit is a split quasi-Jordan algebra if—and only if—it satisfies the condition $\Im = J \oplus \mathcal{Z}(\Im)$, where J represents a subalgebra of $\Im$. This subalgebra J is referred to as the "Jordan part" of $\Im$. In such instances, each element x in $\Im$ has a unique representation, $x = x_J + x_Z$ with $x_J \in J$ and $x_Z \in \mathcal{Z}(\Im)$. These components are referred to as the "Jordan part" and the "zero part" of x, respectively.

In the case where $\Im = J \oplus I$ represents a split quasi-Jordan algebra over I, the following relations hold for any $x,y,z \in \Im$, we have:

$$\begin{aligned}
\{x,y,z\}_q &= x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y \\
&= x \triangleleft (y_J \triangleleft z_J) + (x \triangleleft y_J) \triangleleft z_J - (x \triangleleft z_J) \triangleleft y_J \\
&= \left\{x, y_J, z_J\right\}_q = \left\{x_J, y_J, z_J\right\}_q + \left\{x_Z, y_J, z_J\right\}_q.
\end{aligned}$$

Here, it is important to note that $\left\{x_Z, y_J, z_J\right\}_q \in I$, which is contained within $\mathcal{Z}(\Im)$ and $\left\{x_J, y_J, z_J\right\}_q \in J$, which is a Jordan algebra. Thus, the quasi-Jordan triple product in the Jordan part coincides with the usual Jordan triple product.

As for Jordan algebras, we may define the operators $U_{(x,y)_q}$ and $U_{(x)_q}$ by $U_{(x,y)_q} z = \{x,z,y\}_q$ and $U_{(x)_q} z = \{x,z,x\}_q$ for all $x,y,z \in \Im$. Thus, $U_{(x)_q}(z) = x \triangleleft (z \triangleleft x) + (x \triangleleft z) \triangleleft x - x^2 \triangleleft z$.

Clearly, $\{x,e,y\}_q = x \triangleleft y$, for all x,y and for any unit e in $\Im$. In general, $\{x,y,z\}_q \neq \{z,y,x\}_q$. Certainly, if e is a (right) unit in a quasi-Jordan algebra $\Im$, then it is evident that $\{x,e,e\}_q = x \neq e \triangleleft x = \{e,e,x\}_q$, for all $x \in \Im$. Moreover, for any $x,y,z \in \Im$ with y *or* $z \in \mathcal{Z}(\Im)$, we have $\{x,y,z\}_q = 0$. Hence, $U_{(z)_q} = U_{(x,z)_q} = 0$, for all $x \in \Im$ and $z \in \mathcal{Z}(\Im)$. Furthermore, note that $U_{(\alpha x)_q} = \alpha^2 U_{x_q}$, for all scalars α and vectors $x \in \Im$.

Proposition 1. *In any plus quasi-Jordan algebra $\mathcal{D}^+$, it is readily apparent*

$$\{x,y,z\}_q = \frac{1}{2}(x \dashv y \dashv z + z \vdash y \vdash x).$$

Therefore,

$$U_{(x)_q}(y) = \frac{1}{2}(x \dashv y \dashv x + x \vdash y \vdash x).$$

Proof.

$$\begin{aligned}
4\{x,y,z\}_q &= 4(x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y) \\
&= (x \dashv y \dashv z + x \dashv z \vdash y + (y \dashv z) \vdash x + z \vdash y \vdash x) \\
&\quad +(x \dashv y \dashv z + y \vdash x \dashv z + z \vdash x \dashv y + z \vdash y \vdash x) \\
&\quad -((x \dashv z) \dashv y + (z \vdash x) \dashv y + y \vdash (x \dashv z) + y \vdash (z \vdash x)) \\
&= (x \dashv y \dashv z + x \dashv y \dashv z) + (x \dashv (z \vdash y) - x \dashv z \dashv y) \\
&\quad +((y \dashv z) \vdash x - y \vdash z \vdash x) + (z \vdash y \vdash x + z \vdash y \vdash x) \\
&= 2(x \dashv y \dashv z + z \vdash y \vdash x).
\end{aligned}$$

This gives the required identities. □

Proposition 2. *In a quasi-Jordan algebra $\Im$ with unit e, the operator $U_{(e)_q}(x) = e \triangleleft x$ holds true for all $x \in \Im$. Additionally, the operator $U_{(x)_q}$ is linear for any $x \in \Im$.*

Proof. For any $x \in \Im$, $U_{(e)_q}(x) = e \triangleleft (x \triangleleft e) + (e \triangleleft x) \triangleleft e - e^2 \triangleleft x = e \triangleleft x + e \triangleleft x - e \triangleleft x = e \triangleleft x$.

Next, for any scalars α, β and vectors $x, y, z \in \Im$, we obtain the linearity of $U_{(x)_q}$, as follows:

$$\begin{aligned}
U_{(x)_q}(\alpha y + \beta z) &= \{x, (\alpha y + \beta z), x\}_q \\
&= x \triangleleft ((\alpha y + \beta z) \triangleleft x) + (x \triangleleft (\alpha y + \beta z)) \triangleleft x \\
&\quad -x^2 \triangleleft (\alpha y + \beta z) \\
&= x \triangleleft (\alpha y \triangleleft x) + (x \triangleleft \alpha y) \triangleleft x - x^2 \triangleleft \alpha y \\
&\quad +x \triangleleft (\beta z \triangleleft x) + (x \triangleleft \beta z) \triangleleft x - x^2 \triangleleft \beta z
\end{aligned}$$

$$\begin{aligned}
&= \alpha(x \triangleleft (y \triangleleft x)) + \alpha((x \triangleleft y) \triangleleft x) - \alpha\left(x^2 \triangleleft y\right) \\
&\quad +\beta(x \triangleleft (z \triangleleft x)) + \beta((x \triangleleft z) \triangleleft x) - \beta\left(x^2 \triangleleft z\right) \\
&= \alpha\{x,y,x\}_q + \beta\{x,z,x\}_q = \alpha U_{(x)_q}(y) + \beta U_{(x)_q}(z).
\end{aligned}$$

□

In a quasi-Jordan algebra $\Im$ with unit e, an element x is deemed invertible with respect to the unit e if there exists an element $y \in \Im$ such that $y \triangleleft x = e + (e \triangleleft x - x)$ and $y \triangleleft x^2 = x + (e \triangleleft x - x) + \left(e \triangleleft x^2 - x^2\right)$. In this context, y is referred to as the inverse of x with respect to the unit e. From the above proposition, $U_{(e)_q}(e') = e \triangleleft e' = e$ for any unit e' in $\Im$; and $U_{(e)_q}(z) = e \triangleleft z = 0$ if—and only if—$z \in \mathcal{Z}(\Im)$. Therefore, $\ker U_{(e)_q} = \mathcal{Z}(\Im)$ and $U_{(e)_q}(\Im) = \Im_e$, where $\Im_e := \{e \triangleleft x : x \in \Im\}$. Furthermore, $U_{(e)_q}$ maps all units to e and all zero elements of $\Im$ to 0. Hence, even though e is invertible with a unique inverse, $U_{(e)_q}$ cannot be invertible as an operator.

Proposition 3. *Let $\Im$ be a quasi-Jordan algebra. Then:*

$$U_{(x,y)_q} + U_{(y,x)_q} = U_{(x+y)_q} - U_{(x)_q} - U_{(y)_q}, \ \ for\ all\ x, y \in \Im.$$

Proof. Let z be any element of $\Im$. Then:

$$\begin{aligned} U_{(x+y)_q}(z) &= (x+y) \lhd (z \lhd (x+y)) + ((x+y) \lhd z) \lhd (x+y) \\ &\quad -(x+y)^2 \lhd z \\ &= x \lhd (z \lhd x) + x \lhd (z \lhd y) + y \lhd (z \lhd x) + y \lhd (z \lhd y) \\ &\quad +(x \lhd z) \lhd x + (y \lhd z) \lhd x + (x \lhd z) \lhd y + (y \lhd z) \lhd y \\ &\quad -x^2 \lhd z - (x \lhd y) \lhd z - (y \lhd x) \lhd z - y^2 \lhd z \end{aligned}$$

$$= U_{(x)_q} z + U_{(y)_q} z + \{x,z,y\}_q + \{y,z,x\}_q = U_{(x)_q} z + U_{(y)_q} z + U_{(x,y)_q} z + U_{(y,x)_q} z.$$

□

3. Homotopes

Keeping in view the pivotal role of homotopes in the theory of Jordan algebras ([1,21–29]), We embark on an investigation into the homotopes of quasi-Jordan algebras. Consider any quasi-Jordan algebra $\Im$ and an element $a \in \Im$.We proceed to introduce a new product in $\Im$, as follows:$x \lhd_a y = \{x,a,y\}_q$. This product is clearly a bilinear operator on $\Im$. The linear space $\Im$ equipped with the product "$\lhd_a$" is called the a-homotope of $\Im$, denoted by $\Im_{[a]}$.

If e is a unit in the quasi-Jordan algebra $\Im$, then $x \lhd_e y = \{x,e,y\}_q = x \lhd (e \lhd y) + (x \lhd e) \lhd y - (x \lhd y) \lhd e = x \lhd y$. Hence, the e-homotope $\Im_{[e]}$ coincides with the original algebra $\Im$ itself, for any unit e in $\Im$.

Our primary focus in this section is to demonstrate that homotopes of any special quasi-Jordan algebra also belong to the category of special quasi-Jordan algebras. To establish this, we commence by proving that homotopes of any plus quasi-Jordan algebra, denoted as $(\mathcal{D}^+, \lhd_a)$ fall under the umbrella of quasi-Jordan algebras.

Proposition 4. *Let $\mathcal{D}^+$ be a plus quasi-Jordan algebra and $a \in \mathcal{D}^+$. Then, $(\mathcal{D}^+, \lhd_a)$ is a quasi-Jordan algebra.*

Proof. According to Proposition 1, we establish that $x \lhd_a y = \{x,a,y\}_q = \frac{1}{2}(x \dashv a \dashv y + y \vdash a \vdash x) = \{x,a,y\}_q$ for all $x, y \in \mathcal{D}^+$. This observation leads us to the following:

$$\begin{aligned} x &\dashv (a \dashv (y \vdash (a \vdash z))) = x \dashv (a \dashv (y \dashv (a \vdash z))) \\ &= x \dashv (a \dashv (y \dashv (a \dashv z))) = x \dashv a \dashv y \dashv a \dashv z. \end{aligned}$$

Similarly, $(((z \dashv a) \dashv y) \vdash a) \vdash x = z \vdash a \vdash y \vdash a \vdash x$.

Now,

$$x \lhd_a (y \lhd_a z) = \left\{x, a, \{y,a,z\}_q\right\}_q = \frac{1}{2}(x \dashv (a \dashv \{y,a,z\}_q) + (\{y,a,z\}_q \vdash a) \vdash x)$$

$$\begin{aligned} &= \frac{1}{4}((x \dashv a \dashv y \dashv a \dashv z) + (x \dashv (a \dashv (z \vdash (a \vdash y)))) \\ &\quad +((((y \dashv a) \dashv z) \vdash a) \vdash x) + (z \vdash a \vdash y \vdash a \vdash x)) \\ &= \frac{1}{4}((x \dashv a \dashv y \dashv a \dashv z) + (x \dashv a \dashv z \dashv a \dashv y) \\ &\quad +(y \vdash a \vdash z \vdash a \vdash x) + (z \vdash a \vdash y \vdash a \vdash x)) \end{aligned}$$

and

$$x \lhd_a (z \lhd_a y) = \left\{x, a, \{z,a,y\}_q\right\}_q = \frac{1}{2}(x \dashv (a \dashv \{z,a,y\}_q) + (\{z,a,y\}_q \vdash a) \vdash x)$$

$$= \frac{1}{4}((x \dashv a \dashv z \dashv a \dashv y) + (x \dashv (a \dashv (y \vdash (a \vdash z))))$$

$$
\begin{aligned}
&+((((z \dashv a) \dashv y) \vdash a) \vdash x) + (y \vdash a \vdash z \vdash a \vdash x))\\
&= \frac{1}{4}((x \dashv a \dashv z \dashv a \dashv y) + (x \dashv a \dashv y \dashv a \dashv z)\\
&+(z \vdash a \vdash y \vdash a \vdash x) + (y \vdash a \vdash z \vdash a \vdash x)).
\end{aligned}
$$

So that $x \lhd_a (y \lhd_a z) = x \lhd_a (z \lhd_a y)$.

To prove the Jordan right identity $(y \lhd_a x) \lhd_a x^2 = \left(y \lhd_a x^2\right) \lhd_a x$, we need to show that $\left\{\{y,a,x\}_q, a, \{x,a,x\}_q\right\}_q = \left\{\left\{y,a,\{x,a,x\}_q\right\}_q, a, x\right\}_q$, for all $x, y \in \mathcal{D}^+$. For this, we observe that:

$$
\begin{aligned}
\left\{\{y,a,x\}_q, a, \{x,a,x\}_q\right\}_q &= \frac{1}{2}(\{y,a,x\}_q \dashv \left(a \dashv \{x,a,x\}_q\right)\\
&+\left(\{x,a,x\}_q \vdash a\right) \vdash \{y,a,x\}_q).
\end{aligned}
$$

However,

$$
\begin{aligned}
2\{y,a,x\}_q \dashv (a \dashv \{x,a,x\}_q) &= \{y,a,x\}_q \dashv (a \dashv (x \dashv a \dashv x + x \vdash a \vdash x))\\
&= (\{y,a,x\}_q \dashv (a \dashv (x \dashv (a \dashv x)))) + (\{y,a,x\}_q \dashv (a \dashv (x \vdash (a \vdash x))))\\
&= (\{y,a,x\}_q \dashv (a \dashv x \dashv a \dashv x)) + (\{y,a,x\}_q \dashv (a \dashv x \dashv a \dashv x))\\
&= 2\{y,a,x\}_q \dashv (a \dashv x \dashv a \dashv x)\\
&= (y \dashv a \dashv x \dashv a \dashv x \dashv a \dashv x) + ((x \vdash a \vdash y) \dashv a \dashv x \dashv a \dashv x)
\end{aligned}
$$

and

$$
\begin{aligned}
2(\{x,a,x\}_q \vdash a) \vdash \{y,a,x\}_q &= ((x \dashv a \dashv x + x \vdash a \vdash x) \vdash a) \vdash \{y,a,x\}_q\\
&= (((x \dashv a \dashv x) \vdash a) \vdash \{y,a,x\}_q) + (((x \vdash a \vdash x) \vdash a) \vdash \{y,a,x\}_q)\\
&= (((x \vdash a \vdash x) \vdash a) \vdash \{y,a,x\}_q) + (((x \vdash a \vdash x) \vdash a) \vdash \{y,a,x\}_q)\\
&= 2(x \vdash a \vdash x \vdash a \vdash \{y,a,x\}_q)\\
&= (((x \vdash a \vdash x) \vdash a) \vdash (y \dashv a \dashv x)) + (((x \vdash a \vdash x) \vdash a) \vdash x \vdash a \vdash y)\\
&= ((x \vdash a \vdash x \vdash a) \vdash (y \dashv a \dashv x)) + (x \vdash a \vdash x \vdash a \vdash x \vdash a \vdash y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&4\left\{\{y,a,x\}_q, a, \{x,a,x\}_q\right\}_q\\
= \ &(y \dashv a \dashv x \dashv a \dashv x \dashv a \dashv x) + ((x \vdash a \vdash y) \dashv (a \dashv x \dashv a \dashv x))\\
&+((x \vdash a \vdash x \vdash a) \vdash (y \dashv a \dashv x)) + (x \vdash a \vdash x \vdash a \vdash x \vdash a \vdash y).
\end{aligned}
$$

Continuing, we note the following:

$$
\begin{aligned}
&4\left\{\left\{y,a,\{x,a,x\}_q\right\}_q, a, x\right\}_q\\
&= 2(\left\{\{y,a,(x \dashv a \dashv x)\}_q, a, x\right\}_q + \left\{\{y,a,(x \vdash a \vdash x)\}_q, a, x\right\}_q)\\
&= (\{(y \dashv a \dashv x \dashv a \dashv x), a, x\}_q + \{((x \dashv a \dashv x) \vdash a \vdash y), a, x\}_q\\
&+\{(y \dashv (a \dashv (x \vdash a \vdash x))), a, x\}_q + \{(x \vdash a \vdash x \vdash a \vdash y), a, x\}_q)\\
&= (\{(y \dashv a \dashv x \dashv a \dashv x), a, x\}_q + \{(x \vdash a \vdash x \vdash a \vdash y), a, x\}_q
\end{aligned}
$$

$$
\begin{aligned}
&+\{(y \dashv a \dashv x \dashv a \dashv x), a, x\}_q + \{(x \vdash a \vdash x \vdash a \vdash y), a, x\}_q) \\
&= 2(\{(y \dashv a \dashv x \dashv a \dashv x), a, x\}_q + \{(x \vdash a \vdash x \vdash a \vdash y), a, x\}_q) \\
&= ((y \dashv a \dashv x \dashv a \dashv x \dashv a \dashv x) + ((x \vdash a) \vdash (y \dashv a \dashv x \dashv a \dashv x)) \\
&+((x \vdash a \vdash x \vdash a \vdash y) \dashv (a \dashv x)) + (x \vdash a \vdash x \vdash a \vdash x \vdash a \vdash y)).
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
&4(\left\{\left\{y, a, \{x, a, x\}_q\right\}_q, a, x\right\}_q - \left\{\{y, a, x\}_q, a, \{x, a, x\}_q\right\}_q) \\
&= ((x \vdash a) \vdash (y \dashv a \dashv x \dashv a \dashv x)) - ((x \vdash a \vdash y) \dashv (a \dashv x \dashv a \dashv x)) \\
&+((x \vdash a \vdash x \vdash a \vdash y) \dashv (a \dashv x)) - ((x \vdash a \vdash x \vdash a) \vdash (y \dashv a \dashv x)) \\
&= (((x \vdash a) \vdash y) \dashv (a \dashv x \dashv a \dashv x)) - (((x \vdash a) \vdash y) \dashv (a \dashv x \dashv a \dashv x)) \\
&+((x \vdash a \vdash x \vdash a) \vdash (y \dashv (a \dashv x))) - ((x \vdash a \vdash x \vdash a) \vdash (y \dashv (a \dashv x))) = 0.
\end{aligned}
$$

□

Let $(\mathcal{D}, \dashv, \vdash)$ be a dialgebra, $a \in \mathcal{D}$. Define in $\mathcal{D}$, the new products $\dashv_a$ and $\vdash_a$, as follows: $x \dashv_a y := x \dashv a \dashv y$ and $x \vdash_a y := x \vdash a \vdash y$ for all $x, y \in \mathcal{D}$; here, we dropped the brackets because $\dashv$ and $\vdash$ are associative.

Lemma 1. *Let $\mathcal{D}$ be a dialgebra and $a \in \mathcal{D}$. Then, $(\mathcal{D}, \dashv_a, \vdash_a)$ is a dialgebra over the same field. The dialgebra $(\mathcal{D}, \dashv_a, \vdash_a)$ will be denoted by $\mathcal{D}_{[a]}$; called the a-homotope of $\mathcal{D}$.*

Proof. Clearly, the dialgebra $\mathcal{D}$ is closed under the products $\dashv_a$ and $\vdash_a$. The product $\dashv_a$ is bilinear because $(\alpha x + \beta y) \dashv_a z = (\alpha x + \beta y) \dashv a \dashv z = \alpha x \dashv a \dashv z + \beta y \dashv a \dashv z = \alpha(x \dashv_a z) + \beta(x \dashv_a z)$, for all $x, y, z \in \mathcal{D}$. Similarly, the product $\vdash_a$ is bilinear.

Next, we verify that both $\dashv_a$ and $\vdash_a$ are associative: for this, let $x, y, z \in \mathcal{D}$. Then, by using the associativity of $\dashv$, we obtain $(x \dashv_a y) \dashv_a z = (x \dashv a \dashv y) \dashv a \dashv z = x \dashv a \dashv (y \dashv a \dashv z) = x \dashv_a (y \dashv_a z)$; similarly, $(x \vdash_a y) \vdash_a z = x \vdash_a (y \vdash_a z)$.

We now show that $\dashv_a$ and $\vdash_a$ also satisfy the remaining three defining identities of a dialgebra (see the definition of a dialgebra). For this, let $x, y, z \in \mathcal{D}$. Since both the original multiplications $\dashv$ and $\vdash$ satisfy the three defining identities, we obtain: $(x \vdash_a y) \dashv_a z = (x \vdash a \vdash y) \dashv a \dashv z = ((x \vdash a) \vdash y) \dashv a \dashv z = (x \vdash a) \vdash (y \dashv a \dashv z) = x \vdash a \vdash (y \dashv a \dashv z) = x \vdash_a (y \dashv_a z)$; $(x \dashv_a y) \vdash_a z = (x \dashv a \dashv y) \vdash a \vdash z = ((x \dashv a) \dashv y) \vdash a \vdash z = ((x \dashv a) \vdash y) \vdash a \vdash z = ((x \vdash a) \vdash y) \vdash a \vdash z = (x \vdash_a y) \vdash_a z$; finally, $x \dashv_a (y \vdash_a z) = x \dashv a \dashv (y \vdash a \vdash z) = x \dashv a \dashv (y \vdash (a \vdash z)) = x \dashv a \dashv (y \dashv (a \vdash z)) = x \dashv a \dashv (y \dashv (a \dashv z)) = x \dashv_a (y \dashv_a z)$. □

Remark 1. *Let $\mathcal{D}$ be a dialgebra with a bar-unit e. In this context, we make the following observations:*

1. *Given that $x \dashv e \dashv y = x \dashv y$ and $x \vdash e \vdash y = x \vdash y$ holds true for all $x, y \in \mathcal{D}$, it follows that $\dashv_e = \dashv$ and $\vdash_e = \vdash$. Consequently, the e-homotope $\mathcal{D}_{[e]}$ coincides with the original dialgebra $\mathcal{D}$.*
2. *As previously discussed in [34], an element x in a dialgebra $\mathcal{D}$ is considered regular with respect to a bar-unit e if there exists $y \in \mathcal{D}$ satisfying the conditions $y \dashv x = (e - x) + (e \dashv x)$ and $x \vdash y = (e - x) + (x \vdash e)$; such an element y is called an inverse of x with respect to the bar-unit e. Let x be a regular element in $\mathcal{D}$ with respect to e, and let y be the inverse of x. Then y is a bar unit in the dialgebra $\mathcal{D}_{[x]}$ and x is a bar-unit of $\mathcal{D}_{[y]}$. Indeed, as y is the inverse of x with respect to x, we get $y \dashv x = (e - x) + e \dashv x$ and $x \vdash y = (e - x) + (x \vdash e)$, so that:*

$$
\begin{aligned}
z \dashv_x y &= z \dashv x \dashv y = z \dashv (x \dashv y) = z \dashv (x \vdash y) \\
&= z \dashv ((e - x) + (x \vdash e)) = z \dashv (e - x) + z \dashv (x \vdash e)
\end{aligned}
$$

$$= z \dashv e - z \dashv x + z \dashv x \dashv e = z - z \dashv x + z \dashv x = z$$

and

$$y \vdash_x z = y \vdash x \vdash z = (y \vdash x) \vdash z = (y \dashv x) \vdash z$$
$$= ((e-x) + e \dashv x) \vdash z = e \vdash z - x \vdash z + (e \dashv x) \vdash z$$
$$= e - x \vdash z + (e \vdash x) \vdash z = e - x \vdash z + x \vdash z = z,$$

for all $z \in \mathcal{D}$. Similarly, for all $z \in \mathcal{D}$, we have that:

$$z \dashv_y x = z \dashv y \dashv x = z \dashv ((e-x) + e \dashv x) = z \dashv e - z \dashv x + z \dashv e \dashv x = z$$

and

$$x \vdash_y z = x \vdash y \vdash z = ((e-x) + (x \vdash e)) \vdash z = e \vdash z - x \vdash z + x \vdash e \vdash z = z.$$

Proposition 5. *Let $\mathcal{D}$ be a dialgebra and $a \in \mathcal{D}$. Then, the a-homotope of the plus quasi Jordan algebra $\mathcal{D}^+$ is a plus quasi-Jordan algebra.*

Proof. By Lemma 1, $\left(\mathcal{D}_{[a]}, \dashv_a, \vdash_a\right)$ is a dialgebra. Recall that $x \dashv_a y := x \dashv a \dashv y$ and $x \vdash_a y := x \vdash a \vdash y$, for all $x, y \in \mathcal{D}$. Then, $\left(\left(\mathcal{D}_{[a]}\right)^+, \triangleleft\right)$ is a plus quasi-Jordan algebra, where the quasi-Jordan product "$\triangleleft$" is given by:

$$x \triangleleft y = \frac{1}{2}(x \dashv_a y + y \vdash_a x),$$

for all $x, y \in \mathcal{D}_{[a]}$. Which implies,

$$x \triangleleft_a y = \{xay\}_q = \frac{1}{2}(x \dashv a \dashv y + y \vdash a \vdash x),$$

for all $x, y \in \mathcal{D}$. Thus, the quasi-Jordan product "$\triangleleft_a$" in the a-homotope of $\mathcal{D}^+$ is precisely the quasi-Jordan product "$\triangleleft$" in $\left(\mathcal{D}_{[a]}\right)^+$, so that the two structures coincide. □

Corollary 1. *Any homotope of a special quasi-Jordan algebra is a special quasi-Jordan algebra.*

Example 1.
1. *Let $(\mathcal{A}, d)$ be a (non-graded) differential associative algebra. As per our hypothesis, $d(ab) = da\, b + a\, db$ and $d^2 = 0$. Define left and right products on $\mathcal{A}$ with the formulas $x \dashv y := x\, dy$ and $x \vdash y := dx\, y$. Then, $(\mathcal{A}, \dashv, \vdash)$ is a dialgebra ([8]). For an element $a \in \mathcal{A}$ the a-homotope of this dialgebra $\left(\mathcal{A}_{[a]}, \dashv_a, \vdash_a\right)$, has products defined as follows: $x \dashv_a y = x\, da\, dy$ and $x \vdash_a y = dx\, da\, y$. The quasi-Jordan product "$\triangleleft_a$" induced in the plus quasi-Jordan algebra $\left(\mathcal{A}_{[a]}\right)^+$ is defined by:*

   $$x \triangleleft y = \frac{1}{2}(x \dashv_a y + y \vdash_a x) = \frac{1}{2}(x\, da\, dy + dy\, da\, x).$$

   On the other hand, the a-Homotope of the plus quasi-Jordan algebra $\mathcal{A}^+$ is $(\mathcal{A}^+, \triangleleft_a)$ with product $x \triangleleft_a y := \frac{1}{2}(x\, da\, dy + dy\, da\, x)$. This product coincides with the product in the plus quasi-Jordan algebra constructed on $\left(\mathcal{A}_{[a]}, \dashv_a, \vdash_a\right)$; here, the products $\dashv_a, \vdash_a$ are defined as follows: $x \dashv_a y = x\, da\, dy$ and $x \vdash_a y := dx\, da\, y$.
2. *Let $\mathcal{H}$ be a Hilbert space and $e \in \mathcal{H}$ with $\|e\| = 1$. We define two bilinear products in $\mathcal{H}$, as follows:*

   $$x \dashv y := \langle y, e \rangle x \qquad \text{and} \qquad x \vdash y := \langle x, e \rangle y.$$

It is easily seen that $(\mathcal{H}, \dashv, \vdash)$ *is a dialgebra. The plus quasi-Jordan algebra induced by this dialgebra* $(\mathcal{H}, \triangleleft)$, *where* $x \triangleleft y = \langle y, e \rangle x$ *and the a-Homotope of* $\mathcal{H}$ *is* $(\mathcal{H}, \triangleleft_a)$ *with product* $\triangleleft_a$ *defined by:*

$$x \triangleleft_a y = \langle y, e \rangle \langle a, e \rangle x.$$

Which is exactly the plus quasi-Jordan algebra of the dialgebra $\left(\mathcal{H}_{[a]}, \dashv_a, \vdash_a\right)$ *where* $x \dashv_a y = \langle y, e \rangle \langle a, e \rangle x$ *and* $x \vdash_a y = \langle y, e \rangle \langle a, e \rangle x$.

3. *Let* V *be a vector space and fix* $\phi \in V'$ *(the dual space of* V*) with* $\phi \neq 0$. *We define the products* $\dashv$ *and* $\vdash$ *by:*

$$\begin{aligned} x \quad \dashv \quad & y := \phi(y)x; \\ x \quad \vdash \quad & y := \phi(x)y. \end{aligned}$$

Then, $(V, \dashv, \vdash)$ *is a dialgebra that introduces a special quasi-Jordan algebra, namely* $(V, \triangleleft)$*; here, the product is defined by:*

$$x \triangleleft y = \phi(y)x,$$

for all $x, y \in V$. *In this conclusion, for any* $a \in V$ *we find that the a-homotope of this dialgebra is* $\left(V_{[a]}, \dashv_a, \vdash_a\right)$ *with products:*

$$\begin{aligned} x \quad \dashv_a \quad & y := \phi(y)\phi(a)x; \\ x \quad \vdash_a \quad & y := \phi(x)\phi(a)y. \end{aligned}$$

Which implies that the a-homotope of the quasi-Jordan algebra structure of V *has the product defined by* $x \triangleleft_a y = \phi(y)\phi(a)x$, *for all* $x, y \in V$.

4. Homotopes of K-B Quasi-Jordan Algebras

In this context, we delve into the concept of homotopes within the broader framework of K-B quasi-Jordan algebras. Our aim is to demonstrate that homotopes of any K-B quasi-Jordan algebra belong to the category of quasi-Jordan algebras.

Lemma 2. *For any elements* x, y, z, a *in a quasi-Jordan algebra* $\Im$, *we have:* $a \triangleleft \{x, y, z\}_q = a \triangleleft \{z, y, x\}_q$.

Proof. By using the right commutativity of the quasi-Jordan product, we obtain:

$$\begin{aligned} a \triangleleft \{x, y, z\}_q &= a \triangleleft (x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y) \\ &= a \triangleleft (x \triangleleft (y \triangleleft z) + (x \triangleleft y) \triangleleft z - (x \triangleleft z) \triangleleft y) \\ &= a \triangleleft ((y \triangleleft z) \triangleleft x + z \triangleleft (x \triangleleft y) - (z \triangleleft x) \triangleleft y) = a \triangleleft \{zyx\}_q. \end{aligned}$$

□

Lemma 3. *Let* $\Im$ *be a quasi-Jordan algebra and let* $a \in \Im$. *Then,*

$$x \triangleleft_a (y \triangleleft_a z) = x \triangleleft_a (z \triangleleft_a y) \ \ for\ any \ \ x, y, z, a \in \Im.$$

Proof. Let $x, y, z, a \in \Im$, then by using Lemma 2 we obtain:

$$\begin{aligned} x \lhd_a (y \lhd_a z) &= \left\{x, a, \{y, a, z\}_q\right\}_q \\ &= x \lhd \left(a \lhd \{y, a, z\}_q\right) + (x \lhd a) \lhd \{y, a, z\}_q - \left(x \lhd \{y, a, z\}_q\right) \lhd a \\ &= x \lhd \left(a \lhd \{z, a, y\}_q\right) + (x \lhd a) \lhd \{z, a, y\}_q - \left(x \lhd \{z, a, y\}_q\right) \lhd a \\ &= \left\{x, a, \{z, a, y\}_q\right\}_q = x \lhd_a (z \lhd_a y). \end{aligned}$$

□

Lemma 4. *Let $\Im$ be a K-B quasi-Jordan algebra. Then,*

$$\left(y \lhd_a x^2\right) \lhd_a x = (y \lhd_a x) \lhd_a x^2, \tag{3}$$

for all $x, y, a \in \Im$.

Proof. By Proposition 4, the identity (3) is satisfied if $\Im$ is a special quasi-Jordan algebra; hence, by ([9], Theorem 28), it is true if $\Im$ is any K-B quasi-Jordan algebra. □

Proposition 6. *Let $\Im$ be a K-B quasi-Jordan algebra. Then, for any $a \in \Im$, the a-homotope $(\Im, \lhd_a)$ of $\Im$ is a quasi-Jordan algebra.*

Proof. Based on the implications of Lemmas 3 and 4, it is evident that the product $\lhd_a$ qualifies as a quasi-Jordan product. Consequently, $(\Im, \lhd_a)$ takes the form of a quasi-Jordan algebra. □

Remark 2. *From Corollary 1, we know that homotopes of any special quasi-Jordan algebra is a special quasi-Jordan algebra. Recall that special quasi-Jordan algebras comprise a proper subclass of K-B quasi-Jordan algebras. At this stage, we do not know the complete answer to the following question: Is every homotope of a K-B quasi-Jordan algebra itself a K-B quasi-Jordan algebra?*

5. Other Possible Quasi Triple Products

One may attempt to investigate other analogue/s of the Jordan triple product for quasi-Jordan algebras. We define another triple product "$\{.,.,.\}_\circ$" in a quasi-Jordan algebra $(\Im, \lhd)$, as follows:

$$\{x, z, y\}_\circ := (z \lhd y) \lhd x + (z \lhd x) \lhd y - z \lhd (x \lhd y)$$

for all $x, y, z \in \Im$. This triple product is consistent with the quadratic operator studied by Felipe ([35]). Here, we investigate some properties of the triple product "$\{.,.,.\}_\circ$" in the setting of quasi-Jordan algebras $\Im$. We define the operator $U_{(x,y)_\circ}$, as follows:

$$U_{(x,y)_\circ} z = \{x, z, y\}_\circ, \; for\ all\ x, y, z \in \Im. \tag{4}$$

In particular, for any fixed $x \in \Im$, the operator $U_{(x,x)_\circ}$ translates as below: $U_{(x,x)_\circ}(y) = \{x, y, x\}_\circ = 2(y \lhd x) \lhd x - y \lhd x^2$, for all $y \in \Im$.

We note that $\{x, y, z\}_\circ = \{z, y, x\}_\circ$ for all $x, y, z \in \Im$. If e is a unit of $\Im$, then $U_{(e,e)_\circ}(x) = 2(x \lhd e) \lhd e - x \lhd e^2 = 2x \lhd e - x \lhd e = x \lhd e == x$, for all $x \in \Im$; so $U_{(e,e)_\circ}$ is precisely the identity operator on $\Im$. If x or z is in $\mathcal{Z}(\Im)$, then $\{x, y, z\}_\circ = 0$; in particular, the operator $U_{(z,z)_\circ}$ is just the zero operator, for every $z \in \mathcal{Z}(\Im)$.

The operator $U_{(x,x)_\circ}$ is linear because:

$$\begin{aligned} U_{(x,x)_\circ}(\alpha y+\beta z) &= \{x,(\alpha y+\beta z),x\}_\circ \\ &= \alpha\{x,y,x\}_\circ+\beta\{x,z,x\}_\circ \\ &= \alpha U_{(x,x)_\circ}(y)+\beta U_{(x,x)_\circ}(z), \end{aligned}$$

for all scalars α,β and all $x,y,z\in\Im$.

In the case of a plus quasi-Jordan algebra $\mathcal{D}^+$, the triple product $\{.,.,.\}_\circ$ translates as follows:

$$\begin{aligned} 4\{x,y,z\}_\circ &= (y\dashv z\dashv x+z\vdash y\dashv x+x\vdash y\vdash z+x\vdash z\dashv y) \\ &\quad +(y\dashv x\dashv z+x\vdash y\dashv z+z\vdash y\vdash x+z\vdash x\dashv y) \\ &\quad -(y\dashv x\dashv z+y\dashv(z\vdash x)+(x\dashv z)\vdash y+z\vdash x\vdash y) \end{aligned}$$

$$\begin{aligned} &= (x\vdash y\dashv z+x\vdash y\dashv z) \\ &\quad +(x\vdash z\vdash y-(x\dashv z)\vdash y) \\ &\quad +(y\dashv x\dashv z-y\dashv x\dashv z) \\ &\quad +(y\dashv z\dashv x-y\dashv(z\vdash x)) \\ &\quad +(z\vdash x\vdash y-z\vdash x\vdash y) \\ &\quad +(z\vdash y\dashv x+z\vdash y\dashv x) \\ &= 2(x\vdash y\dashv z+z\vdash y\dashv x), \end{aligned}$$

so that:

$$\{x,y,z\}_\circ=\frac{1}{2}(x\vdash y\dashv z+z\vdash y\dashv x)$$

and

$$U_{(x,x)_\circ}(y)=x\vdash y\dashv x.$$

Next, for any $x,y\in\Im$, we define the operator $V_{(x,y)_\circ}$ on quasi-Jordan algebra $\Im$, as follows:

$$V_{(x,y)_\circ}z:=\Big(U_{(x+z,x+z)_\circ}-U_{(x,x)_\circ}-U_{(z,z)_\circ}\Big)y,$$

for all $z\in\Im$. We observe that:

$$\begin{aligned} V_{(x,y)_\circ}z &= 2((y\triangleleft(x+z))\triangleleft(x+z))-y\triangleleft(x+z)^2 \\ &\quad -2(y\triangleleft x)\triangleleft x+y\triangleleft x^2-2(y\triangleleft z)\triangleleft z+y\triangleleft z^2 \\ &= 2(y\triangleleft x)\triangleleft x+2(y\triangleleft x)\triangleleft z+2(y\triangleleft z)\triangleleft x+2(y\triangleleft z)\triangleleft z \\ &\quad -y\triangleleft\Big(x^2+x\triangleleft z+z\triangleleft x+z^2\Big) \\ &\quad -2(y\triangleleft x)\triangleleft x+y\triangleleft x^2-2(y\triangleleft z)\triangleleft z+y\triangleleft z^2 \\ &= 2(y\triangleleft x)\triangleleft x+2(y\triangleleft x)\triangleleft z+2(y\triangleleft z)\triangleleft x+2(y\triangleleft z)\triangleleft z \\ &\quad -y\triangleleft x^2-y\triangleleft(x\triangleleft z)-y\triangleleft(z\triangleleft x)-y\triangleleft z^2 \\ &\quad -y\triangleleft x-y\triangleleft z-2(y\triangleleft x)\triangleleft x+y\triangleleft x^2-2(y\triangleleft z)\triangleleft z+y\triangleleft z^2 \\ &= 2(y\triangleleft x)\triangleleft z+2(y\triangleleft z)\triangleleft x-y\triangleleft(x\triangleleft z)-y\triangleleft(z\triangleleft x) \\ &= 2(y\triangleleft x)\triangleleft z+2(y\triangleleft z)\triangleleft x-2y\triangleleft(x\triangleleft z) \\ &= 2\{x,y,z\}_\circ, \end{aligned}$$

for all $x,y,z\in\Im$.

Proposition 7. *Let $\Im$ be a quasi-Jordan algebra with a unit e. If x is invertible with inverse y with regard to the unit e, then $U_{(x,x)_\circ}y=e\triangleleft x-x^2+U_{(x,x)_\circ}e$.*

Proof. Suppose x has an inverse y with regard to the unit e. Then:

$$\begin{aligned}
U_{(x,x)_\circ}y &= 2(y \lhd x) \lhd x - y \lhd x^2 \\
&= 2(e + e_{\lhd}(x)) \lhd x - x - e_{\lhd}(x) - e_{\lhd}\left(x^2\right) \\
&= 2(e \lhd x + (e \lhd x - x) \lhd x) - x - (e \lhd x - x) - \left(e \lhd x^2 - x^2\right) \\
&= 2e \lhd x + 2(e \lhd x) \lhd x - 2x^2 - x - e \lhd x + x - e \lhd x^2 + x^2 \\
&= e \lhd x + 2(e \lhd x) \lhd x - x^2 - e \lhd x^2 \\
&= e \lhd x - x^2 + U_{(x,x)_\circ}e.
\end{aligned}$$

□

Remark 3. *Since the (right) unit e in a quasi-Jordan algebra $\Im$ may not be a left unit, we may neither obtain $e \lhd x = x$ nor $U_{(x,x)_\circ}e(:= 2(e \lhd x) \lhd x - e \lhd x^2) = x + x^2 - e \lhd x$. Hence, by the above proposition, $U_{(x,x)_\circ}y$ may not equal x; this is inconsistent with a well-known result obtained by N. Jacobson, see ([1], p. 56). Thus, the operator $U_{(x,x)_\circ}$ on the quasi-Jordan algebra $\Im$ is not consistent with the usual quadratic operator defined on a Jordan algebra.*

Next, let us consider homotopes of a plus quasi-Jordan algebra $\mathcal{D}^+$ with respect to the triple product $\{.,.,.\}_\circ$. For any fixed $a \in \Im$, one may attempt to construct the a-homotope of $\mathcal{D}^+$ with respect to the triple product $\{.,.,.\}_0$, as the original vector space $\mathcal{D}$ equipped with the product "$\bullet_a$," defined by: $x \bullet_a y = \{x, a, y\}_0$. It is easy to see that $\{x, a, y\}_0 = \frac{1}{2}(x \vdash a \dashv y + y \vdash a \dashv x)$.

Note that $\{x, a, y\}_0 = \{y, a, x\}_0$, and so $x \bullet_a y = y \bullet_a x$, for all $x, y \in \Im$. However, $(y \bullet_a x) \bullet_a x^2 \neq \left(y \bullet_a x^2\right) \bullet_a x$. For instance, if $x = y = e$ (a bar-unit in $\mathcal{D}$), then:

$$(y \bullet_a x) \bullet_a x^2 = \left(e \bullet_a e^2\right) \bullet_a e = \frac{1}{2}(a \dashv a \dashv a + a \vdash a \vdash a)$$

$$\neq a \vdash a \dashv a = (e \bullet_a e) \bullet_a e^2 = (y \bullet_a x) \bullet_a x^2.$$

This proves the following result:

Proposition 8. *The a-homotope of $\mathcal{D}^+$ with respect to the triple product $\{.,.,.\}_\circ$ (as defined above) is not a quasi-Jordan algebra.*

One may ask for other possible quasi-Jordan algebra analogues of classical Jordan triple product. By definition of the Jordan triple product, we have that:

$$U_{x,y}z = \{x, z, y\} = (x \circ z) \circ y + x \circ (z \circ y) - z \circ (x \circ y) = T_x T_y z + T_y T_x z - T_{x \circ y} z,$$

for all $x, y, z \in \Im$, where the operator T_a is the multiplication by a. In case of a quasi Jordan algebra $(\Im, \lhd)$, we have right multiplication and left multiplication operators, R_x and L_x, respectively; each of the two operators coincides with T_x if $(\Im, \lhd)$ is a Jordan algebra. Therefore, in attempting to construct a possible triple product in a quasi-Jordan algebra, we may replace T_x by L_x or R_x: so in the above construction of the Jordan triple product, the expression $T_x T_y$ may be replaced by any one of the four expressions $R_x R_y, R_x L_y, L_x R_y, L_x L_y$; similarly, the expression $T_y T_x$ may be replaced by any one of the four expressions $R_y R_x, R_y L_x, L_y R_x, L_y L_x$. Of course, we have $L_x R_y = L_x Ly$ by the right commutativity of quasi-Jordan product "$\lhd$". This reduces the number of possible replacements for $T_x T_y$ to 3 and 3 for $T_y T_x$, too. Next, the expression $T_{x \circ y}$ may be replaced by any one of the expressions $R_{x \lhd y}, R_{y \lhd x}, L_{x \lhd y}, L_{y \lhd x}$; again, by the right commutativity, we have $R_{x \lhd y} = R_{y \lhd x}$ and so for $T_{x \circ y}$ we have three possible replacements. Thus, there are $3^3 = 27$ possible combinations of the left multiplication and right multiplication operators that are all possible analogues of the Jordan triple product for quasi-Jordan algebras. These 27 possible constructions are listed below; notice that the above two triple quasi-Jordan triple products $\{.,.,.\}_\circ$ and $\{.,.,.\}_q$ are induced by the following constructions $U_{(x,y)_1}$ and $U_{(x,y)_{23}}$, respectively:

1. $U_{(x,y)_1} := R_x R_y + R_y R_x - R_{x\triangleleft y}$;
2. $U_{(x,y)_2} := R_x R_y + R_y R_x - L_{x\triangleleft y}$;
3. $U_{(x,y)_3} := R_x R_y + R_y R_x - L_{y\triangleleft x}$;
4. $U_{(x,y)_4} := R_x R_y + R_y L_x - R_{x\triangleleft y}$;
5. $U_{(x,y)_5} := R_x R_y + R_y L_x - L_{x\triangleleft y}$;
6. $U_{(x,y)_6} := R_x R_y + R_y L_x - L_{y\triangleleft x}$;
7. $U_{(x,y)_7} := R_x R_y + L_y L_x - R_{x\triangleleft y}$;
8. $U_{(x,y)_8} := R_x R_y + L_y L_x - L_{x\triangleleft y}$;
9. $U_{(x,y)_9} := R_x R_y + L_y L_x - L_{y\triangleleft x}$;
10. $U_{(x,y)_{10}} := R_x L_y + R_y R_x - R_{x\triangleleft y}$;
11. $U_{(x,y)_{11}} := R_x L_y + R_y R_x - L_{x\triangleleft y}$;
12. $U_{(x,y)_{12}} := R_x L_y + R_y R_x - L_{y\triangleleft x}$;
13. $U_{(x,y)_{13}} := R_x L_y + R_y L_x - R_{x\triangleleft y}$;
14. $U_{(x,y)_{14}} := R_x L_y + R_y L_x - L_{x\triangleleft y}$;
15. $U_{(x,y)_{15}} := R_x L_y + R_y L_x - L_{y\triangleleft x}$;
16. $U_{(x,y)_{16}} := R_x L_y + L_y L_x - R_{x\triangleleft y}$;
17. $U_{(x,y)_{17}} := R_x L_y + L_y L_x - L_{x\triangleleft y}$;
18. $U_{(x,y)_{18}} := R_x L_y + L_y L_x - L_{y\triangleleft x}$;
19. $U_{(x,y)_{19}} := L_x L_y + R_y R_x - R_{x\triangleleft y}$;
20. $U_{(x,y)_{20}} := L_x L_y + R_y R_x - L_{x\triangleleft y}$;
21. $U_{(x,y)_{21}} := L_x L_y + R_y R_x - L_{y\triangleleft x}$;
22. $U_{(x,y)_{22}} := L_x L_y + R_y L_x - R_{x\triangleleft y}$;
23. $U_{(x,y)_{23}} := L_x L_y + R_y L_x - L_{x\triangleleft y}$;
24. $U_{(x,y)_{24}} := L_x L_y + R_y L_x - L_{y\triangleleft x}$;
25. $U_{(x,y)_{25}} := L_x L_y + L_y L_x - R_{x\triangleleft y}$;
26. $U_{(x,y)_{26}} := L_x L_y + L_y L_x - L_{x\triangleleft y}$;
27. $U_{(x,y)_{27}} := L_x L_y + L_y L_x - L_{y\triangleleft x}$.

For $y = x$, the above list of operators reduces to:

$U_{(x,x)_1} = 2R_x^2 - R_{x^2}$;
$U_{(x,x)_2} = U_{(x,x)_3} = 2R_x^2 - L_{x^2}$;
$U_{(x,x)_7} = U_{(x,x)_{19}} = R_x^2 + L_x R_x - R_{x^2} = R_x^2 + L_x^2 - R_{x^2}$;
$U_{(x,x)_8} = U_{(x,x)_{20}} = U_{(x,x)_{21}} = R_x^2 + L_x R_x - L_{x^2} = R_x^2 + L_x^2 - L_{x^2}$;
$U_{(x,x)_4} = R_x^2 + R_x L_x - R_{x^2}$;
$U_{(x,x)_5} = U_{(x,x)_6} = U_{(x,x)_{12}} = R_x^2 + R_x L_x - L_{x^2}$;
$U_{(x,x)_9} = U_{(x,x)_{25}} = 2L_x R_x - R_{x^2} = 2L_x^2 - R_{x^2}$;
$U_{(x,x)_{10}} = U_{(x,x)_{11}} = U_{(x,x)_{26}} = U_{(x,x)_{27}} = 2L_x R_x - L_{x^2} = 2L_x^2 - L_{x^2}$;
$U_{(x,x)_{13}} = 2R_x L_x - R_{x^2}$;
$U_{(x,x)_{14}} = U_{(x,x)_{15}} = 2R_x L_x - L_{x^2}$;
$U_{(x,x)_{16}} = U_{(x,x)_{22}} = R_x L_x + L_x R_x - R_{x^2} = R_x L_x + L_x^2 - R_{x^2}$;
$U_{(x,x)_{17}} = U_{(x,x)_{18}} = U_{(x,x)_{23}} = U_{(x,x)_{24}}$
$= R_x L_x + L_x R_x - L_{x^2} = R_x L_x + L_x^2 - L_{x^2}$.

If the quasi-Jordan product "$\triangleleft$" is commutative, then it is a Jordan product on $\Im$ and $L_x = R_x = T_x$, for all $x \in \Im$; hence, $U_{(x,y)_k} z = T_x T_y z + T_y T_x z - T_{x\triangleleft y} z = U_{x,y} z$ (the usual Jordan triple product) and so $U_{(x,x)_k} = 2T_x T_x - T_{x^2} = U_x$ (the usual quadratic operator), for all $x, y, z \in \Im$ and for all $k \in \{1, \ldots, 27\}$.

In case of the plus quasi-Jordan algebra $\mathcal{D}^+$ of a dialgebra $(\mathcal{D}, \dashv, \vdash)$, we observe that:

$$R_x R_y z = \frac{1}{4}((z \dashv y \dashv x) + (y \vdash z \dashv x) + (x \vdash z \dashv y) + (x \vdash y \vdash z));$$

$$R_y R_x z = \frac{1}{4}((z \dashv x \dashv y) + (x \vdash z \dashv y) + (y \vdash z \dashv x) + (z \vdash y \vdash x));$$

$$R_x L_y z = \frac{1}{4}((y \dashv z \dashv x) + (z \vdash y \dashv x) + (x \vdash y \dashv z) + (x \vdash z \vdash y));$$

$$R_y L_x z = \frac{1}{4}((x \dashv z \dashv y) + (z \vdash x \dashv y) + (y \vdash x \dashv z) + (y \vdash z \vdash x));$$

$$L_x L_y z = \frac{1}{4}((x \dashv y \dashv z) + (x \dashv z \dashv y) + (y \vdash z \vdash x) + (z \vdash y \vdash x));$$

$$L_y L_x z = \frac{1}{4}((y \dashv x \dashv z) + (y \dashv z \dashv x) + (x \vdash z \vdash y) + (z \vdash x \vdash y));$$

$$R_{x \triangleleft y} z = \frac{1}{4}((z \dashv x \dashv y) + (z \dashv y \dashv x) + (x \vdash y \vdash z) + (y \vdash x \vdash z));$$

$$L_{x \triangleleft y} z = \frac{1}{4}((x \dashv y \dashv z) + (y \vdash x \dashv z) + (z \vdash x \dashv y) + (z \vdash y \vdash x));$$

$$L_{y \triangleleft x} z = \frac{1}{4}((y \dashv x \dashv z) + (x \vdash y \dashv z) + (z \vdash y \dashv x) + (z \vdash x \vdash y)).$$

$$\begin{aligned}
R_{x^2} y &= \frac{1}{4}(2(y \dashv x \dashv x) + 2(x \vdash x \vdash y)); \\
L_{x^2} y &= \frac{1}{4}((x \dashv x \dashv y) + (y \vdash x \dashv x) + (x \vdash x \dashv y) + (y \vdash x \vdash x)); \\
R_x^2 y &= \frac{1}{4}((y \dashv x \dashv x) + 2(x \vdash y \dashv x) + (x \vdash x \vdash y)); \\
L_x^2 y &= \frac{1}{4}((x \dashv x \dashv y) + (x \dashv y \dashv x) + (x \vdash y \vdash x) + (y \vdash x \vdash x)); \\
R_x L_x y &= \frac{1}{4}((x \dashv y \dashv x) + (y \vdash x \dashv x) + (x \vdash x \dashv y) + (x \vdash y \vdash x)).
\end{aligned}$$

Hence, in case of plus quasi-Jordan algebra $\mathcal{D}^+$, the above list of 27 possible constructions specializes as follows:

1. $U_{(x,x)_1} y = (x \vdash y \dashv x)$.
2. $U_{(x,x)_2} y = U_{(x,x)_3} y = \frac{1}{4}(2(y \dashv x \dashv x) + 4(x \vdash y \dashv x) + 2(x \vdash x \vdash y)$ $- ((x \dashv x \dashv y) + (y \vdash x \dashv x) + (x \vdash x \dashv y) + (y \vdash x \vdash x)));$
3. $U_{(x,x)_4} y = \frac{1}{4}(2(x \vdash y \dashv x) + (x \dashv x \dashv y) + (x \dashv y \dashv x) + (x \vdash y \vdash x)$ $+ (y \vdash x \vdash x) - (y \dashv x \dashv x) - (x \vdash x \vdash y));$
4. $U_{(x,x)_5} y = U_{(x,x)_6} y = \frac{1}{4}((y \dashv x \dashv x) + 2(x \vdash y \dashv x) + (x \vdash x \vdash y)$ $+ (x \dashv y \dashv x) + (x \vdash y \vdash x) - ((y \vdash x \dashv x) + (x \vdash x \dashv y)));$
5. $U_{(x,x)_7} y = \frac{1}{4}(2(x \vdash y \dashv x) + (x \dashv y \dashv x) + (y \vdash x \dashv x) + (x \vdash x \dashv y)$ $+ (x \vdash y \vdash x) - (y \dashv x \dashv x) - (x \vdash x \vdash y));$
6. $U_{(x,x)_8} y = \frac{1}{4}((y \dashv x \dashv x) + 2(x \vdash y \dashv x) + (x \vdash x \vdash y) + (y \vdash x \dashv x)$ $+ (x \vdash x \dashv y) - (x \dashv x \dashv y) - (y \vdash x \vdash x));$
7. $U_{(x,x)_9} y = \frac{1}{2}(((x \dashv x \dashv y) + (x \dashv y \dashv x) + (x \vdash y \vdash x)$ $+ (y \vdash x \vdash x)) - (y \dashv x \dashv x) - (x \vdash x \vdash y));$
8. $U_{(x,x)_{10}} y = U_{(x,x)_{11}} y = \frac{1}{4}((x \dashv x \dashv y) + 2(x \dashv y \dashv x)$ $+ 2(x \vdash y \vdash x) + (y \vdash x \vdash x) - (y \vdash x \dashv x) - (x \vdash x \dashv y));$
9. $U_{(x,x)_{13}} y = \frac{1}{2}((x \dashv y \dashv x) + (y \vdash x \dashv x) + (x \vdash x \dashv y)$ $+ (x \vdash y \vdash x) - (y \dashv x \dashv x) - (x \vdash x \vdash y));$
10. $U_{(x,x)_{14}} y = U_{(x,x)_{15}} y = \frac{1}{4}((x \dashv y \dashv x) + 2(y \vdash x \dashv x) + 2(x \vdash x \dashv y)$ $+ (x \vdash y \vdash x) - ((x \dashv x \dashv y) + (y \vdash x \vdash x)));$

11. $U_{(x,x)_{16}}y = \frac{1}{4}(2(x \dashv y \dashv x) + (y \vdash x \dashv x) + (x \vdash x \dashv y) + 2(x \vdash y \vdash x)$
$+ (x \dashv x \dashv y) + (y \vdash x \vdash x) - 2(y \dashv x \dashv x) - 2(x \vdash x \vdash y));$
12. $U_{(x,x)_{17}}y = \frac{1}{2}((x \dashv y \dashv x) + (x \vdash y \vdash x)).$

6. Conclusions

The Jordan binary product "$\circ$" induces the Jordan triple product, $\{a, b, c\}$, defined by $\{a, b, c\} = a \circ (b \circ c) - b \circ (c \circ a) + c \circ (a \circ b)$. The Jordan triple product, as highlighted, plays a central role in the study of Jordan algebras and more extensive Jordan systems known as Jordan triple systems. Notably, this product gives rise to essential operators, such as the fundamental operators $U_{(a,b)}$ and $V_{(a,b)}$ which are defined on J, as follows: $U_{(a,b)}x := \{a, x, b\}$ and $V_{(a,b)}x := \{a, b, x\}$. The operator $U_{(a,a)}$ is usually written in short as U_a, which being quadratic in a is called the quadratic operator. It is a well-established fact that the entire theory of Jordan algebras can be developed solely based on the quadratic operators.

In this article, we explored the potential for finding suitable analogues to the Jordan triple product and its associated operators $U_{(a,b)}$ and $V_{(a,b)}$ within the realm of quasi-Jordan algebras. We embarked on an investigation involving all 27 possible combinations of the quasi-Jordan product "$\triangleleft$", aiming to identify quasi-Jordan triple products that exhibit behaviors akin to the conventional Jordan triple product found in Jordan algebras. In the context of Jordan algebras, all 27 combinations coincide with the Jordan triple product. Out of these potential combinations, we specifically examined two:: $\{x, z, y\}_\circ := U_{(x,y)_1}z = (z \triangleleft y) \triangleleft x + (z \triangleleft x) \triangleleft y - z \triangleleft (x \triangleleft y)$ and $\{x, z, y\}_q := U_{(x,y)_{23}}z = x \triangleleft (z \triangleleft y) + (x \triangleleft z) \triangleleft y - (x \triangleleft y) \triangleleft z$. Our primary objective was to examine these products in the context of quasi-Jordan algebras, leading to the discovery of some intriguing properties. Notably, we established the following key findings:

1. The homotope of a plus quasi-Jordan algebra, induced by the triple product $\{x, z, y\}_\circ$ may not be a quasi-Jordan algebra.
2. Any homotope of a special quasi-Jordan algebra, induced by the triple product $\{x, z, y\}_q$, is itself a special quasi-Jordan algebra.
3. Any homotope of a K-B quasi-Jordan algebra, induced by the triple product $\{x, z, y\}_q$, transforms into a quasi-Jordan algebra.

These results provide valuable insights into the behavior of these products in various types of quasi-Jordan algebras. The question is whether such homotopes of an arbitrary K-B quasi-Jordan algebra is again a K-B quasi-Jordan algebra is still an open problem. The main difficulty in getting some positive answer to this question is due to non-availability of the Macdonald's theorems for quasi-Jordan algebras, even for K-B quasi-Jordan algebras. It is known, due to V. Voronin, that straightforward generalizations of the Shirshov and Macdonald theorems do not hold for general quasi-Jordan algebras. However, certain appropriate analogues of these theorems in the setting of K-B quasi-Jordan algebras are known, again due to Voronin. In the sequel, we intend to go further investigating the homotopes and isotopes induced by the above picked two as well as other possible triple products; particularly, in the setting of quasi-Jordan Banach algebras.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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