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Optimized Self-Similar Borel Summation

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Abstract: The method of Fractional Borel Summation is suggested in conjunction with self-similar factor approximants. The method used for extrapolating asymptotic expansions at small variables to large variables, including the variables tending to infinity, is described. The method is based on the combination of optimized perturbation theory, self-similar approximation theory, and Borel-type transformations. General Borel Fractional transformation of the original series is employed. The transformed series is resummed in order to adhere to the asymptotic power laws. The starting point is the formulation of dynamics in the approximations space by employing the notion of self-similarity. The flow in the approximation space is controlled, and "deep" control is incorporated into the definitions of the self-similar approximants. The class of self-similar approximations, satisfying, by design, the power law behavior, such as the use of self-similar factor approximants, is chosen for the reasons of transparency, explicitness, and convenience. A detailed comparison of different methods is performed on a rather large set of examples, employing self-similar factor approximants, self-similar iterated root approximants, as well as the approximation technique of self-similarly modified Padé–Borel approximations.

Keywords: optimized perturbation theory; extrapolation of asymptotic series; fractional Borel-type transforms; factor approximants; critical amplitudes

MSC: 90C59; 81T15; 80M35; 80M50; 40G10; 40G99

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1. Introduction

The problem of extrapolating asymptotic series derived for small variables to finite and even very large values of variables is constantly met in various branches of science, such as physics, chemistry, economics, applied mathematics, etc. Different approaches have been suggested to cure this problem, e.g., Padé summation [1], Borel summation [2], Shanks transformation [3], hypergeometric Meijer approximants [4,5], the renormalization group [6,7], and others [8–14]. These techniques can provide good approximations for finite variables, although they are not applicable for the case of variables tending to infinity.

In the present paper, we describe an original approach based on the transparent physical notions of optimization and self-similarity combined with Borel-type summation. The layout of the paper is as follows. In Section 2, we explain the main ideas: (i) how to make a divergent asymptotic series convergent and how the convergence is optimized by introducing control functions; (ii) how to transform a sequence of approximants into a dynamical system leading to the property of self-similarity; (iii) that the solutions of the self-similar equation of motion extrapolate the given asymptotic series serving as a boundary condition into self-similar approximants that are valid for arbitrary values of variables. In Section 3, we combine the self-similar approximants with Borel-type summation and demonstrate the ways of introducing control functions or control parameters, defined by the minimal difference and minimal derivative optimization conditions. Several admissible

Borel-type transforms are discussed. In Section 4, we discuss the specifics of the critical amplitude calculations. In Section 5, we illustrate the discussed methods by concrete examples. Section 6 concludes the paper.

In our approach to the resummation problem, we are guided by the requirement for the independence of observables from the approximations, transformations, parameters, etc., introduced in the course of the analysis. Nature should not be aware of our difficulties of understanding it. In turn, we should be respectful and introduce minimal necessary assumptions about it. Since the description of the sought function is available to us only by means of truncated series, we need to compensate for the lack of knowledge of the true coefficients by adding some natural assumptions.

First, the initial truncated series is to be transformed so that, instead of the available divergent truncated series, a supposedly better-behaving truncated series is considered. Second, to the transformed series, we apply the so-called approximations with power law behavior at infinity. At this step, we also reconstruct the behavior of the series. Third, the guiding principle of self-similarity leads to self-similar roots and self-similar factor approximants. Fourth, from a technical viewpoint, it is enormously helpful for the expressions for the critical amplitudes to become explicit so that they can be factored into the parts arising from the approximants and from the inverse transformation. Fifth, after an optimization with the minimal difference and (or) minimal derivative conditions, we study the numerical convergence of the sequences of approximations for the sought amplitudes. Besides the numerical convergence, we are also guided, when appropriate, by the rigorous results for the convergence of the diagonal Padé approximants obtained by Gonchar.

2. Optimization and Self-Similarity

In this section, we present the main ideas of the approach in order for the reader to grasp the basis of the particular techniques that are used. The two pivotal points are the notions of convergence optimization and self-similarity.

2.1. Asymptotic Series

The starting point of the consideration concerns the well-known fact that, in practical applications, we often satisfy the necessity of solving problems by applying some kind of perturbation theory, resulting in expansions to powers of a parameter or a variable. Such expansions usually represent asymptotic series diverging for finite values of the expansion variable.

To be specific, let us be interested in a real function $f(x)$ of a real positive variable x, defined by rather complicated equations that, because of their complexity, can be solved only by means of a kind of perturbation theory. The latter would result in an expansion in powers of asymptotically small $x \to 0$:

$$f(x) \simeq f_k(x) \qquad (x \to 0), \tag{1}$$

having the form of a truncated series

$$f_k(x) = f_0(x) \sum_{n=0}^{k} a_n x^n, \tag{2}$$

where $f_0(x)$ is a known function. Our main concern is the summation of the power expansion, because of which we shall concentrate on the expansion of the form

$$f_k(x) = \sum_{n=0}^{k} a_n x^n. \tag{3}$$

As is evident, the more general form (2) can be easily reduced to (3). In practically all interesting problems, expression (3) represents an asymptotic series diverging for a finite value of x. At the same time, the quantities of interest usually correspond to finite and sometimes to quite large values of the variable x. Moreover, in many cases, the most interesting region is the region of very large $x \to \infty$, where the sought function behaves according to the power law

$$f(x) \simeq Bx^{\beta} \qquad (x \to \infty). \tag{4}$$

Thus, the principal problem is how it would be possible to reconstruct the large-variable behavior of the sought function (4), knowing only the small-variable expansion (3)?

2.2. Control Functions

It seems clear that, in order to define an effective sum of the divergent series (3), it is necessary to rearrange the divergent series into a convergent series. When one is interested in rather large values of x, then one cannot resort to Padé approximants, since for asymptotically large x values, the limit of a Padé approximant is proportional to x^{M-N}, depending on which of the approximants from the Padé table is accepted. In that sense, the large-variable behavior of Padé approximants is not defined, since x^{M-N} can tend to either $+\infty$, $-\infty$, or zero, depending on whether $M > N$, $M < N$, or $M = N$. This implies that Padé approximants do not converge for large $x \to \infty$.

To force a sequence to converge, it is necessary to introduce some quantities that regulate the convergence. Such quantities, governing the sequence convergence, can be named *control functions*. The idea of introducing control functions so that the reorganized sequence converges was advanced in Refs. [15,16]. Control functions are to be defined by *optimization conditions* regulating the sequence convergence, so that the initial perturbation theory is reorganized to an *optimized perturbation theory* yielding a converging sequence of optimized approximants [15,16]. All mathematical details can be found in recent reviews [17,18].

The introduction of control functions u_k transforms the initial sequence $\{f_k(x)\}$ into a sequence $\{F_k(x, u_k)\}$. The latter can be symbolically denoted as

$$F_k(x, u_k) = \hat{R}[\, u_k \,] f_k(x). \tag{5}$$

This introduction can be conducted in several ways: by incorporating control functions into an initial approximation of perturbation theory, through a change in variables, or by a direct sequence transformation [17,18]. The derivation of equations defining control functions starts with the Cauchy criterion of convergence. The criterion tells that a sequence converges if and only if, for any $\varepsilon > 0$, there exists a number k_ε such that

$$|\, F_{k+p}(x, u_{k+p}) - F_k(x, u_k) \,| < \varepsilon, \tag{6}$$

for all $k > k_\varepsilon$ and $p > 0$.

From the Cauchy criterion, one can derive (see details in [17,18]) the optimization conditions defining the control functions. One possibility is the *minimal difference* condition

$$F_{k+p}(x, u) - F_k(x, u) = 0 \qquad (p \geq 1), \tag{7}$$

whose simplest variant is

$$F_{k+1}(x, u) - F_k(x, u) = 0, \qquad u = u_k(x). \tag{8}$$

The other possibility is the *minimal derivative* condition

$$\frac{\partial F_k(x, u)}{\partial u} = 0, \qquad u = u_k(x). \tag{9}$$

Several other representations of optimization conditions are admissible [17,18].

In some cases, control functions can be defined by the boundary conditions

$$\hat{R}^{-1}[\, u_k \,] F_k(x, u_k) \simeq f_k(x) \qquad (x \to 0), \tag{10}$$

implying asymptotic equivalence at a small variable of the renormalized and initial terms of the approximations. The latter condition is also called the accuracy-through-order procedure.

After the control functions $u_k(x)$ are found from the optimization conditions, the optimized approximants read as

$$f_k^{opt}(x) = \hat{R}^{-1}[\, u_k \,] F_k(x, u_k(x)). \tag{11}$$

2.3. Self-Similar Relation

Self-similar relations are known to arise in renormalization group theory, where one considers scaling with respect to spatial or momentum variables, as in the real-space decimation procedure [19,20] or in quantum field theory [21,22]. Then, self-similar relations connect characteristic quantities, such as effective Hamiltonians, Lagrangians, or correlation functions, with different spatial or momentum scales. In that sense, renormalization groups in statistical physics or quantum field theory provide relations between the characteristic quantities with scaled variables.

The notion of self-similarity in approximation theory, introduced in Refs. [23,24], does not concern a scaling of variables, but rather, the scaling of approximation orders. The number labeling the approximation order plays the role of discrete time.

It seems natural then to construct a dynamical system in the space of approximations. To this end, it is necessary to introduce an endomorphism into the approximation space. Let us start with an initial approximation

$$f = F_0(x, u_k(x)), \qquad x = x_k(f), \tag{12}$$

defining an expansion function $x_k(f)$. Then, it is possible to define the approximation function

$$y_k(f) = F_k(x_k(f), u_k(x_k(f))). \tag{13}$$

The *approximation space* is given by the closed linear envelope over all admissible approximation functions:

$$\mathcal{A} = \overline{\mathcal{L}}\{y_k(f) :\; k = 0, 1, 2, \ldots\}. \tag{14}$$

This is similar to approximation spaces in approximation theory [25] or to phase spaces in physics [26–28]. Thus, we obtain a dynamical system in discrete time, called the *approximation cascade*:

$$\{y_k(f) :\; \mathbb{Z}_+ \times \mathcal{A} \longmapsto \mathcal{A}\}. \tag{15}$$

The points of the cascade form the cascade trajectory.

The Cauchy criterion (6) now acquires the form

$$|\, y_{k+p}(f) - y_k(f) \,| < \varepsilon. \tag{16}$$

If control functions have been chosen so that the sequence of the optimized approximants is convergent, then, in terms of dynamical theory, this implies the existence of a fixed point, where

$$y_k(y_p^*(f)) = y_p^*(f). \tag{17}$$

Conditions (16) and (17) lead [17,18,23,24] to the *self-similar relation*

$$y_{k+p}(f) = y_k(y_p(f)). \tag{18}$$

Fixed-point solutions $y_k^*(f)$ to this equation, using the relation (13), give the self-similar approximants $f_k^*(x) = \hat{R}^{-1}[u_k] y_k^*(F_0(x, u_k(x)))$. The stability of the method can be checked by investigating map multipliers, similarly to the stability analysis conducted in numerical calculations [29,30].

The self-similar relation (18) describes the motion of a dynamical system (cascade). Generally, a dynamical system does not necessarily tend to a fixed point, but it can display chaotic motion [31]. This is why the incorporation into the approximation cascade of control functions, governing sequence convergence, is so important.

2.4. Self-Similar Root Approximants

Generally, the self-similar relation, depending on additional imposed constraints playing the role of boundary conditions, can lead to different solutions. For instance, when the fixed-point solution is required to satisfy the prescribed asymptotic expansions at small as well as at large variables, we obtain [32] (see also [33,34]) the self-similar root approximants

$$f_k^*(x) = \left(\left((1 + A_1 x)^{n_1} + A_2 x^2 \right)^{n_2} + \ldots + A_k x^k \right)^{n_k}. \tag{19}$$

When the large-variable behavior is in the form of the law

$$f(x) \simeq B x^{\beta} \qquad (x \to \infty), \tag{20}$$

with the known power β, then we have

$$n_j = \frac{j+1}{j} \qquad (j = 1, 2, \ldots, k-1), \qquad n_k = \frac{\beta}{k}, \tag{21}$$

which results in the approximant

$$f_k^*(x) = \left(\left((1 + A_1 x)^2 + A_2 x^2 \right)^{3/2} + \ldots + A_k x^k \right)^{\beta/k}. \tag{22}$$

All control parameters A_j are uniquely defined by the asymptotic boundary condition

$$f_k^*(x) \simeq f_k(x) \qquad (x \to 0). \tag{23}$$

The root approximant (22) at large x values behaves as

$$f_k^*(x) \simeq B_k x^{\beta} \qquad (x \to \infty), \tag{24}$$

which gives the amplitude

$$B_k = \left(\left(\left(A_1^2 + A_2 \right)^{3/2} + A_3 \right)^{4/3} + \ldots + A_k \right)^{\beta/k}. \tag{25}$$

Below, we refer to the root approximants introduced above as iterated root approximants or just as iterated roots.

2.5. Self-Similar Factor Approximants

Accepting the asymptotic expansion (3) as a boundary condition and representing this expansion in the form

$$f_k(x) = \prod_{j=1}^{k} (1 + b_j x)$$

yields [35,36] the fixed-point solution in the form of self-similar factor approximants

$$f_k^*(x) = \prod_{j=1}^{N_k} (1 + A_j x)^{n_j}, \tag{26}$$

where

$$N_k = \begin{cases} k/2, & k = 2, 4, \ldots \\ (k+1)/2, & k = 3, 5, \ldots \end{cases} . \tag{27}$$

For odd k values, scaling relations [17] allow us to set $A_1 = 1$. All other control parameters A_j and n_j are uniquely defined by the boundary condition requiring the asymptotic expansion of $f_k^*(x)$ at small x values to coincide with $f_k(x)$. The factor approximants serve as a generalized representation of the products of functions [37,38]. It has been shown that the self-similar factor approximants provide good accuracy for a wide class of problems [35,36,39,40].

At large x values, the approximant (26) gives

$$f_k^*(x) \simeq B_k x^{\beta_k} \qquad (x \to \infty), \tag{28}$$

with the amplitude

$$B_k = \prod_{j=1}^{N_k} A_j^{n_j} \tag{29}$$

and the large-variable exponent

$$\beta_k = \sum_{j=1}^{N_k} n_j. \tag{30}$$

It is appropriate to notice that Padé approximants are a particular kind of factor approximants, where a portion of the powers equals $+1$ and the other portion is -1, so that

$$f_k^*(x) = \frac{\prod_{j=1}^{M}(1 + A_j x)}{\prod_{j=1}^{N}(1 + A_j x)} = P_{M/N}(x).$$

2.6. Self-Similar Combined Approximants

It is possible to combine different types of approximants. Consider the situation when one type of approximation better suits the small-variable limit, while the other type of approximation is better in the description of the large-variable limit. For this purpose, one can consider the first q terms of the expansion $f_k(x)$ with $q < k$,

$$f_q(x) = \sum_{n=0}^{q} a_n x^n \qquad (q < k). \tag{31}$$

When constructing a self-similar approximant $f_q^*(x)$ on the basis of expansion (31), we define the correcting ratio

$$C_{k/q}(x) \equiv \frac{f_k(x)}{f_q^*(x)} \tag{32}$$

and expand it in powers of x to obtain

$$C_{k/q}(x) \simeq \frac{\sum_{n=0}^{k} a_n x^n}{\sum_{n=0}^{q} a_n x^n} \simeq 1 + \sum_{n=q+1}^{k} a_n x^n. \tag{33}$$

On the basis of the latter expansion, we construct a self-similar approximant $C_{k/q}^*(x)$. The final combined approximant is

$$f_k^*(x) = f_q^*(x) C_{k/q}^*(x). \tag{34}$$

As has been mentioned above, Padé approximants are a particular kind of self-similar approximant. So, for the correcting function $C^*_{k/q}(x)$, it is possible to take a Padé approximant $P_{M/N}(x)$ with $M + N = k - q$. In that case, the final approximant is

$$f_k^*(x) = f_q^*(x) P_{M/N}(x). \tag{35}$$

Of course, the necessary boundary conditions have to be preserved when choosing q, M, and N.

3. Self-Similar Borel-Type Transforms

The convergence of series is known to be improved by resorting to Borel summation [2,41,42]. Borel, or Borel-type summation, can be combined by employing self-similar approximants [43].

3.1. Self-Similar Borel Transform

The Borel transform for expansion (3) is

$$B_k(x) = \sum_{n=0}^{k} \frac{a_n}{n!}\, x^n , \tag{36}$$

with the inverse transformation

$$f_k(x) = \int_0^\infty B_k(xt) e^{-t}\, dt. \tag{37}$$

The series (36) can be summed by employing self-similar approximants, obtaining a self-similar Borel transform $B_k^*(x)$. Then, the inverse transformation (37) gives

$$f_k^*(x) = \int_0^\infty B_k^*(xt) e^{-t}\, dt. \tag{38}$$

The most difficult problem is the study of the large-variable limit. Therefore, we pay more attention to this limiting behavior. In the present case, by substituting the large-variable form of the self-similar approximant,

$$B_k^*(x) \simeq C_k x^{\beta_k} \qquad (x \to \infty), \tag{39}$$

into the integral (37), we obtain

$$f_k^*(x) \simeq B_k x^{\beta_k} \qquad (x \to \infty), \tag{40}$$

where the large-variable amplitude is

$$B_k = C_k \Gamma(\beta_k + 1). \tag{41}$$

3.2. Self-Similar Borel–Leroy Transform

The Borel–Leroy transform reads as

$$B_k(x, \lambda) = \sum_{n=0}^{k} \frac{a_n x^n}{\Gamma(n + \lambda + 1)} , \tag{42}$$

where λ is a control parameter that has to be defined from optimization conditions, providing convergence to the sequence of approximants.

The inverse transformation is

$$f_k(x) = \int_0^\infty B_k(xt, \lambda)\, t^\lambda e^{-t}\, dt. \tag{43}$$

Constructing a self-similar approximation $B_k^*(x,\lambda_k)$ yields the self-similar optimized Borel–Leroy approximant

$$f_k^*(x) = \int_0^\infty B_k^*(xt,\lambda)\, t^\lambda e^{-t}\, dt, \tag{44}$$

where the control parameter $\lambda = \lambda_k$ can be defined, e.g., from either the minimal difference optimization condition (8) or the minimal derivative optimization condition (9) for the large-variable amplitude.

In the large-variable limit, the form

$$B_k^*(x,\lambda) \simeq C_k x^{\beta_k} \qquad (x \to \infty) \tag{45}$$

results in the approximant (40) with the amplitude

$$B_k = C_k \Gamma(\beta_k + \lambda_k + 1). \tag{46}$$

3.3. Self-Similar Iterated Borel Transform

The method used for the Borel and Borel–Leroy transforms can be repeated several times, resulting in iterated transforms [9]. For example, after accomplishing the first step and obtaining the transform

$$B_{k1}(x,\lambda^{(1)}) = \sum_{n=0}^{k} \frac{a_n x^n}{\Gamma(n+\lambda^{(1)}+1)}, \tag{47}$$

it is straightforward to repeat the procedure to obtain the double Borel–Leroy transform

$$B_{k2}(x,\lambda^{(1)},\lambda^{(2)}) = \sum_{n=0}^{k} \frac{a_n x^n}{\Gamma(n+\lambda^{(1)}+1)\Gamma(n+\lambda^{(2)}+1)}. \tag{48}$$

After p iterations, one obtains

$$B_{kp}(x,\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(p)}) = \sum_{n=0}^{k} \frac{a_n x^n}{\prod_{j=1}^{p}\Gamma(n+\lambda^{(j)}+1)}. \tag{49}$$

For simplicity, one assumes that the control parameters at different steps are the same, so that $\lambda^{(j)} = \lambda$, which gives

$$B_{kp}(x,\lambda) = \sum_{n=0}^{k} \frac{a_n x^n}{\Gamma^p(n+\lambda+1)}. \tag{50}$$

The inverse transformation is

$$f_k(x) = \int_0^\infty B_{kp}(xt_1 t_2 t_p,\lambda) \prod_{j=1}^{p} t_j^\lambda\, e^{-t_j}\, dt_j. \tag{51}$$

By constructing a self-similar approximant $B_{kp}^*(x,\lambda_{kp})$ on the basis of the p-iterated transform (50), one obtains [44] the self-similar iterated approximants

$$f_{kp}^*(x) = \int_0^\infty B_{kp}^*(xt_1 t_2 t_p,\lambda_{kp}) \prod_{j=1}^{p} t_j^{\lambda_{kp}}\, e^{-t_j}\, dt_j, \tag{52}$$

where the control parameters λ_{kp} are defined from the optimization conditions. For the large-variable limit of the transform

$$B_{kp}^*(x,\lambda_{kp}) \simeq C_{kp} x^{\beta_{kp}} \qquad (x \to \infty), \tag{53}$$

the large-variable behavior of the sought function becomes

$$f^*_{kp}(x) \simeq B_{kp} x^{\beta_{kp}} \qquad (x \to \infty), \tag{54}$$

with the amplitude

$$B_{kp} = C_{kp} \Gamma^p(\beta_{kp} + \lambda_{kp} + 1). \tag{55}$$

3.4. Self-Similar Mittag–Leffler Transform

A generalization of the Borel summation can be conducted by means of the Mittag–Leffler [45] summation. By introducing the Mittag–Leffler transform

$$M_k(x, \mu) = \sum_{n=0}^{k} \frac{a_n x^n}{\Gamma(n\mu + 1)}, \tag{56}$$

the inverse transformation reads as

$$f_k(x) = \int_0^\infty M_k(xt^\mu, \mu) e^{-t} \, dt. \tag{57}$$

The series (56) can be represented by a self-similar approximant, giving $M^*_k(x, \mu_k)$, where the control parameter μ_k is found from the optimization conditions [46]. Then, the inverse transformation yields the self-similar approximant

$$f^*_k(x) = \int_0^\infty M^*_k(xt^\mu, \mu) e^{-t} \, dt. \tag{58}$$

For large variables, the transform

$$M^*_k(x, \mu) \simeq M_k x^{\beta_k} \qquad (x \to \infty) \tag{59}$$

leads to large-variable behavior of the self-similar approximant

$$f^*_k(x) \simeq B_k x^{\beta_k} \qquad (x \to \infty), \tag{60}$$

with the amplitude

$$B_k = M_k \Gamma(\beta_k \mu_k + 1). \tag{61}$$

For the Mittag–Leffler transform, one can define iterated transforms, similarly to the iterated Borel–Leroy transforms.

3.5. Self-Similar Modified Borel–Leroy Transform

The Borel–Leroy transform can be modified [47,48] to the form

$$B_k(x, \lambda, \nu) = a_0 + \sum_{n=1}^{k} \frac{a_n x^n}{\Gamma(n + \lambda + 1) n^\nu}, \tag{62}$$

with the inverse transformation being

$$f_k(x) = \int_0^\infty t^\lambda e^{-t} \left(t \, \frac{\partial}{\partial t} \right)^\nu B_k(xt, \lambda, \nu) \, dt, \tag{63}$$

in which ν is an integer, so that

$$\left(t \, \frac{\partial}{\partial t} \right)^\nu x^n = n^\nu x^n. \tag{64}$$

Constructing a self-similar transform $B_k^*(x, \lambda_k, \nu)$, where λ_k is a control parameter, gives [49] the inverse transformation

$$f_k^*(x) = \int_0^\infty t^{\lambda_k} e^{-t} \left(t \, \frac{\partial}{\partial t} \right)^\nu B_k^*(xt, \lambda_k, \nu) \, dt. \tag{65}$$

With the large-variable behavior of the transform

$$B_k^*(xt, \lambda_k, \nu) \simeq C_k x^{\beta_k} \qquad (x \to \infty), \tag{66}$$

the large-variable limit of the self-similar approximant (65) reads as shown in Equation (60) with the amplitude

$$B_k = C_k \beta_k^\nu \, \Gamma(\beta_k + \lambda_k + 1). \tag{67}$$

As is evident, setting $\lambda_k = 0$ leads to the modified Borel transform.

3.6. Self-Similar Fractional Iterated Transform

The modified Borel–Leroy transform shown in the previous section can be further generalized in two aspects: First, it is straightforward to iterate it. And, second, it is possible to treat the parameters ν as well as the iteration order p, not as integers, but as fractional quantities [44,49].

Starting with the iterated modified Borel–Leroy transform

$$B_{kp}(x, \lambda, u) = a_0 + \sum_{n=1}^k \frac{a_n x^n}{[\Gamma(n + \lambda + 1) n^u]^p}, \tag{68}$$

we have the inverse transformation

$$f_{kp}(x) = \int_0^\infty \prod_{j=1}^p dt_j \, t_j^\lambda e^{-t_j} \left(t_j \, \frac{\partial}{\partial t_j} \right)^u B_{kp}(xt_1 t_2 \ldots t_p, \lambda, u). \tag{69}$$

Then, by constructing a self-similar approximant $B_{kp}^*(x, \lambda_{kp}, u_{kp})$, where λ_{kp} and u_{kp} and $p = p_k$ are control parameters, we obtain a self-similar fractional iterated transform

$$f_{kp}^*(x) = \int_0^\infty \prod_{j=1}^p dt_j \, t_j^{\lambda_{kp}} e^{-t_j} \left(t_j \, \frac{\partial}{\partial t_j} \right)^{u_{kp}} B_{kp}^*(xt_1 t_2 \ldots t_p, \lambda_{kp}, u_{kp}). \tag{70}$$

Strictly speaking, the differential operator in action (64) is defined only for the integer ν. However, it is admissible to formally treat ν as an arbitrary real number u when it acts on power law expressions, considering Equation (64) as a definition.

Then, for the large-variable behavior of the transform

$$B_{kp}^*(x, \lambda_{kp}, u_{kp}) \simeq C_{kp} x^{\beta_{kp}} \qquad (x \to \infty), \tag{71}$$

we find the large-variable limit for the sought function

$$f_{kp}^*(x) \simeq B_{kp} x^{\beta_{kp}} \qquad (x \to \infty), \tag{72}$$

in which the amplitude is

$$B_{kp} = C_{kp} \left[\, \beta_{kp}^{u_{kp}} \, \Gamma(\beta_{kp} + \lambda_{kp} + 1) \, \right]^p. \tag{73}$$

In this, and in the previous sections, the control parameters are defined by the optimization conditions, as discussed in Section 2.2, as the minimal difference or minimal derivative conditions. In particular, when one is interested in the correct evaluation of the large-variable amplitudes, these conditions should be used with respect to the amplitudes

$B_{kp}(u)$ considered as functions of the control parameters u. For brevity, here, we use a single control parameter u, although there can be several of them. It is possible to use the minimal difference condition in the form

$$B_{k+1,p}(u) - B_{kp}(u) = 0, \qquad u = u_{kp}, \tag{74}$$

or in the form

$$B_{k,p+1}(u) - B_{kp}(u) = 0, \qquad u = u_{kp}. \tag{75}$$

Otherwise, one can resort to the minimal derivative condition

$$\frac{\partial B_{kp}(u)}{\partial u} = 0, \qquad u = u_{kp}. \tag{76}$$

For both conditions, the minimal difference and minimal derivative are equivalent.

Previously, in the paper [44], it was suggested that the number of iterations p should be considered a continuous control parameter. The multidimensional integrals are relatively easy to define for the integer p. Introducing a continuous p implies a smooth interpolation between the values of the integrals for discrete p values. The approach is applicable only in the limiting case of large x values.

While setting $u = 0$, the parameter $p = p_k$ can be found from the minimal derivative condition as the unique solution to the equation

$$\frac{\partial B_k(p)}{\partial p} = 0. \tag{77}$$

Alternatively, we can solve the minimal difference equation

$$B_{k+1}(p) - B_k(p) = 0, \tag{78}$$

with a positive integer $k = 1, 2, 3, \ldots$.

For the special singular cases of $\beta = -1, -2\ldots$, one can study the inverse quantity. One of the main advantages of the method involving the combination of the methodology shown in Section 2.6 and the Borel-light summation shown in Section 4.3 is that they do not involve any explicit singular terms. This property allows one to avoid an extra transformation and allows one to work with the original truncations directly. In such cases, one is bound to the optimizations with marginal amplitudes, since the complete expression for the amplitudes involves singularities.

4. Critical Amplitudes from Fractional Iterations

Consider the case where one has to find explicitly a real, sign-definite, positive-valued function $f(x)$ of a real variable x. In addition, the function possesses the power law asymptotic behavior (4). The critical exponent $\beta > 0$ is known. The case of a negative β will be considered by studying the inverse of $f(x)$. Let us find the amplitude B.

We consider the case where it is impossible to find the sought function $f(x)$ explicitly and exactly from some given equations. We show that it is possible to reconstruct the large-variable amplitudes from the given truncated asymptotic expansions given in the form (3).

Several methods of finding effective sums of the truncated series (1)–(3) exist, based on the ideas of Borel, Mittag–Leffler, and Hardy [9,44,46,50–56]. The well-known method is that of Padé [57–59]. The hypergeometric approximants [60–63] can be employed instead of the Padé approximants. More complex are the hypergeometric Meijer approximants [62,63].

Bear in mind that we are interested in accurate analytical calculation of the amplitude B in the expression (4) [64–66]. To this end, we need, of course, to find effective sums for (1)–(3), at least in a general form. The specifics of our techniques and the general knowledge of the large-x asymptotic behavior (4) are sufficient in order to find an analytic expression for the amplitudes and to proceed by applying the minimal difference and minimal derivative conditions given by Equations (74)–(78).

Let us briefly recapitulate the main tenets of Section 3.6 needed for the following applications to concrete problems. First, we set all λ_{kp} values to zero. This simplification brings us back to a manageable number of control parameters. Below, we consider u, which is the order of the operator in (69), as a continuous control parameter. It is possible to define the multidimensional integrals required for calculating the critical amplitudes for integers p. Introducing continuous p values means a smooth interpolation between the values of the integral for discrete p values [44,49]. The number of iterations p can also be regarded as a continuous control parameter [44]. The parameters ν and p ought to be found from the optimization conditions. The general-type optimization conditions are the minimal difference and minimal derivative conditions (74)–(76), expressed as the conditions for the critical amplitudes. The conditions are equivalent and both are applied below.

We expect that the solution to the optimization problem exists and is unique. There are three main realizations of the Fractional Borel Summation with factor approximants described in Section 2.5. Often, we simply use the term "Borel summation with factor approximants", but specify the optimization techniques applied in each particular case. Below we use the following three methods of finding the critical amplitudes.

- The first method is based on introducing the fractional order u of the differentiation operator. The parameter has to be found from the optimization procedure, while p is fixed. The optimization procedure, where only the parameter u is considered and p is fixed, is called Fractional Borel Summation with u-optimization [49].
- The second method is based on the optimization of the other parameter p, where the number of iterations extends to arbitrary real numbers from the original integers (see Section 3.3 and the paper [44]). This has to be found from p-optimization, while u is fixed. The optimization procedure where only the parameter p is considered and u is fixed is called Fractional Borel Summation with p-optimization [49]. The optimization is performed according to formulas analogous to (74)–(76) with straightforward replacement of the parameter u by the parameter p.
- The third method, Fractional Borel-light or self-similar combined approximants is explained in great detail in Sections 2.6 and 4.3. The method was suggested in the paper [49], following the main ideas expressed earlier in [43,46]. It is based on optimization with minimal derivative or minimal difference conditions of the amplitude B_{kp} (or with the marginal amplitudes C_{kp}), either with respect to the fractional u or fractional p, with subsequent correction of the marginal amplitudes with the diagonal Padé approximants [1,43,46,49].

We employ some useful formulas for the factor approximants given above in the general form in Section 3.6. These are adjusted to the calculations of the amplitudes in Section 4.1. We also employ the iterated roots explained in Sections 2.4, and diagonal Padé approximants for odd and even number of the terms k in the truncations [58].

Factor approximants have the advantage of generality, since the case of $\beta = 0$ is included into the consideration automatically, unlike the case of iterated roots, where such a case should be treated individually. On the other hand, iterated roots are very user-friendly and can be treated analytically in rather high orders. The problem of optimization can be formulated with rather high orders of the perturbation theory, while for the factors, only low-order optimizations can be treated analytically.

The diagonal Padé approximants are routinely extendable to very high orders. Fractional Borel Summation with iterated roots was previously considered in detail [49], while the diagonal, odd, and even Padé approximants are discussed in [58].

For completeness and convenience, below, we give some formulas that are required for actual computations of the critical amplitudes when the index β is known. Two types of approximants are discussed in this context, since the iterated roots, presented in Section 2.4, are well adapted to the calculation of the critical amplitudes and do not require any specific adjustments.

4.1. Critical Amplitudes from Factors and Padé Approximants

It is both convenient and natural to extrapolate the asymptotic series with power law behavior (4) by means of the self-similar factor approximants [35,36]. Let us fix the inner sum of the parameters n_j in the Formula (26) to the known index β, so that

$$\beta = \sum_{j=1}^{N_k} n_j \tag{79}$$

in all orders. Then, the critical amplitudes can be found by extrapolating the series (1)–(3) to the form (26). Here

$$N_k = \begin{cases} k/2+1, & k = 2,4,6,\ldots \\ k/2+1/2, & k = 1,3,5,\ldots \end{cases} . \tag{80}$$

At large x values, the factor approximants behave as

$$f_k^*(x) \simeq B_k x^{\beta} \qquad (x \to \infty), \tag{81}$$

and the amplitude is given by the Formula (29). For the even orders k, one of the A_j values can be set to one. Such a restriction does not change the critical index and only influences the value of the amplitude. From a technical standpoint, it is rather difficult to optimize factor-based methods in high orders, although lower orders can easily be optimized.

4.2. Modified Padé Approximations

Alternatively, in place of the factors, one can apply the well-known Padé approximants $P_{n/m}$ [9]. The Padé approximants can be adapted to calculate the amplitudes at infinity in the expression (4) by means of some transformations. In the case of odd $k = 1,3\ldots$, one should study the following transformed series for the sought function $f(x)$, $T(x) = f(x)^{-1/\beta}$. The following modified Padé approximants

$$P_{n,n+1}(x) = (PadeApproximant[T[x], n, n+1])^{-\beta}, \tag{82}$$

are defined for even cases, where $n = 0,1\ldots,n_{max}$ is a non-negative integer.

In the odd case, $(2n+1)_{max} = k$. The approximants (82) do comply with the power law (4) at $x \to \infty$. One can relatively easily find the sequence of approximations

$$B_n = \lim_{x\to\infty}(xPadeApproximant[T[x], n, n+1])^{-\beta}, \tag{83}$$

for the amplitudes. In the case of even $k = 2,4,\ldots$, the following modified-even Padé approximants

$$P_{n,n}(x) = K(x) \times (PadeApproximant[G[x], n, n]), \tag{84}$$

are defined [58]. Here, the corrector

$$K(x) = (PadeApproximant[T[x], 0, 1])^{-\beta}, \tag{85}$$

was introduced to ensure the correct asymptotic behavior [58]. Also, $G(x) = \frac{f(x)}{K(x)}$ represents the transformed truncated series, which can be approached again with the diagonal Padé approximants. More details on the application of the modified Padé approximants for the Borel summation can be found in [58].

The amplitudes sought at infinity can be found as follows:

$$B_n = B_0 \times \lim_{x\to\infty}(PadeApproximant[G[x], n, n]) \tag{86}$$

with
$$B_0 = \lim_{x \to \infty} \left(K(x) x^{-\beta} \right).$$

Here, $n = 1, 2 \ldots, n_{max}$, is a positive integer. In the even case, $n_{max} = k/2$. It is not impossible but is very difficult from a technical standpoint to optimize Padé-based methods, especially at very high orders.

The methods of the factor, root, and Padé approximants can be applied individually or together in some hybrid form to calculate the marginal amplitude C appearing in the course of the Borel transformation. All of the mentioned approximations can be applied directly to the series (1)–(3) under the asymptotic condition (1) and produce the estimates for the sought amplitude B. Only the iterated roots shown in Section 3.3 are seamlessly defined for all k values. The other two approximations use two different definitions for the odd and even cases.

4.3. Critical Amplitudes from Hybrids of Factors and Padé Approximations

The sought function can be reconstructed from the transform $f_i^*(x)$ directly with the help of corrected Padé approximants, following the general idea of the papers [43,46]. Such a hybrid approach, coined before Borel-light [49], is particularly useful when the integral transformation cannot be applied because of singularities in the transformational Γ-functions or when an explicit formula and not only numerical values are required. In fact, we are dealing with the whole table of approximate values

$$f_{n,i}^*(x) \simeq P_{n/n}(x)\, f_i^*(x), \tag{87}$$

with $i = 1, 2 \ldots k$ [49], while $n = 1, 2 \ldots k/2$ for even k, and $n = 1, 2 \ldots (k-1)/2$ for odd k. Formula (34) is built only on the diagonal sequences in adherence with [67].

Of course, when only the single approximant $f_q^*(x)$ of the order q with $q < k$ is employed, and only the highest possible order diagonal Padé approximant is considered, we return to the particular case of the scheme outlined in Section 2.6. On the other hand, when n is fixed and i is varied in the Formula (87), we are dealing with a different sequence of approximations from that described in Section 2.6. The whole (almost) table (87) was employed in the paper [49].

Since, at $x \to \infty$, $f_i^*(x, u) \simeq C_i(u) x^\beta$, the sought amplitude is approximated by the table of values expressed by means of the hybrid formula

$$B_{n,i}^*(u_i) = P_{n/n}(\infty)\, C_i(u_i)\,. \tag{88}$$

The parameter $u = u_i$ is found from the optimization conditions. Such conditions are analogous to the Equations (74)–(76) [49] but have to be applied properly to the marginal amplitudes $C_i(u_i)$.

5. Examples

In the following sections, we are primarily concerned with the comparison of different factor-based approximations with root-based [49] and Padé-based approaches [58,68].

5.1. Quantum Quartic Oscillator

For the quantum anharmonic oscillator [69], perturbation theory yields a rather long expansion for the ground-state energy

$$E_k(g) = \sum_{n=0}^{k} a_n g^n.$$

Here, the parameter $g \geq 0$ measures a deviation of the anharmonic potential from the quantum harmonic oscillations. The coefficients a_n are rapidly growing in magnitude.

Their concrete values are known for very high orders and can be retrieved from [69]. The strong-coupling limit for $g \to \infty$,

$$E(g) \simeq Bg^{\beta} \tag{89}$$

is also known, and the parameters are $B = 0.667986$ and $\beta = 1/3$.

Factor approximants, when applied to the truncation $E_k(g)$, give the following sequence of approximate amplitudes:

$$B_2 = 0.678929, \; B_3 = 0.750032, \; B_4 = 0.702102, \; B_5 = 0.724883,$$

$$B_6 = 0.706184, \; B_7 = 0.712144, \; B_8 = 0.706593, \; B_9 = 0.704931.$$

Borel factor approximants with $u = 0$ and $p = 1$ give

$$B_2^* = 0.708098, \; B_3^* = 0.742158, \; B_4^* = 0.689432, \; B_5^* = 0.688555,$$

$$B_7^* = 0.672477, \; B_8^* = 0.679915, \; B_9^* = 0.677194.$$

Complex values are omitted here and in what follows.

The results produced by the following Borel-light approximations with a fixed marginal amplitude and varying correcting terms,

$$B_{2,7}^*(1) = \; P_{2/2}(\infty)\, C_7(1) = 0.626631, B_{3,7}^*(1) = \; P_{3/3}(\infty)\, C_7(1) = 0.687644,$$

$$B_{4,7}^*(1) = \; P_{4/4}(\infty)\, C_7(1) = 0.690745, \quad B_{5,7}^*(1) = \; P_{5/5}(\infty)\, C_7(1) = 0.690887,$$

are slightly worse.

The best results are found for rather high orders with the following Borel-light approximations with varying marginal amplitudes and correcting terms of the same order,

$$B_{13,2}^*(1) = \; P_{13/13}(\infty)\, C_2(1) = 0.69341,$$

$$\begin{aligned} B_{13,4}^*(1) &= \; P_{13/13}(\infty)\, C_4(1) = 0.68615, \\ B_{13,5}^*(1) &= \; P_{13/13}(\infty)\, C_5(1) = 0.68766, \\ B_{13,7}^*(1) &= \; P_{13/13}(\infty)\, C_7(1) = 0.67907, \\ B_{13,8}^*(1) &= \; P_{13/13}(\infty)\, C_8(1) = 0.68904, \\ B_{13,9}^*(1) &= \; P_{13/13}(\infty)\, C_9(1) = 0.67397. \end{aligned}$$

Bear in mind that, for the factor approximants, the optimization problems can be solved analytically only for low orders. Only the minimal derivative problem possesses a unique solution. The results of optimization, such as the optimal control parameters, can be used as inputs to construct the sequence of Borel-light approximations. But, even in these cases, the results appear not to be better than those given by the factor approximants by themselves. However, such a strategy can be useful for some other problems and is exploited below.

Various other methods are considered as well. They give even more reasonable results, as shown in Table 1.

The best result is marked in bold in Table 1.

It is achieved in the ninth order of perturbation theory with the Fractional Borel Summation with iterated roots [49]. An even better result, $B = 0.669356$, is found with the same method in the 10th order [49].

The method of Corrected Padé approximants needs more terms to obtain the same accuracy [70] but is able to easily scan very high orders [70]. But, already in the 10th order, it gives a reasonable estimate, $B = 0.655086$.

Table 1. Critical amplitude for the one-dimensional quartic oscillator.

Quartic Oscillator	Amplitude
Factor approximants	0.704931
Borel with factors	0.677194
Borel-light with factors	0.690887
Exact	0.667986
Fractional Borel with roots [49]	**0.670902**
Borel with roots [49]	0.682494
Even Padé–Borel [58]	0.679037
Odd Padé–Borel [58]	0.67926
Even Padé [58]	0.709572
Odd Padé [58]	0.712286

5.2. Schwinger Model: Energy Gap

Let us consider the energy gap between the lowest and second excited states of the scalar boson for the massive Schwinger model in Hamiltonian lattice theory [71,72].

The energy gap $\Delta(z)$ between the two states at small z values can be represented (in low orders), according to the paper [72], as

$$2\Delta(z) \simeq 1 + 6z - 26z^2 + 190.6666666667z^3 - 1756.666666667z^4 + 18048.33650794z^5, \quad (90)$$

with the variable $z = (1/ga)^4$. Here, g stands for a coupling parameter and a is the lattice spacing. The coefficients a_n are known up to the 13th order and can be found in [72].

In the continuous limit, $\Delta(z)$ follows the power law [72]

$$\Delta(z) \simeq Bz^{\beta} \qquad (z \to \infty), \quad (91)$$

where $B = 1.1284$ and $\beta = 1/4$.

Factor approximants, when applied to the original expansion, give the following sequence:

$$B_2 = 1.58276, \; B_3 = 1.574592, \; B_5 = 1.45957, \; B_7 = 1.36643,$$
$$B_8 = 1.05181, \; B_9 = 1.3421,$$

with a reasonable value given by B_8. The complex values are omitted again here and below.

Borel factor approximants, calculated with $u = 0$ and $p = 1$, give the following sequence:

$$B_2^* = 2.02627, \; B_3^* = 1.59039, \; B_4^* = 1.55169, \; B_5^* = 1.58126,$$
$$B_6^* = 1.62346, \; B_7^* = 1.59431, \; B_8^* = 1.05511,$$

with a reasonable result that is very similar to the previous result B_8.

The most consistent results are found by applying the Borel-light approximations with a fixed marginal amplitude and varying correcting terms,

$$B_{4,2}^*(1) = \; P_{4/4}(\infty)\; C_2(1) = 1.32253,$$
$$B_{5,2}^*(1) = \; P_{5/5}(\infty)\; C_2(1) = 1.05724,$$
$$B_{6,2}^*(1) = \; P_{6/6}(\infty)\; C_2(1) = 1.15488.$$

The last two terms can be considered lower and upper bounds, respectively.

As mentioned previously, the optimization problem with factor approximants can be solved analytically for low orders only. The optimization results, such as those for the optimal control parameters, can be used to construct the sequence of Borel-light approximations. Such a strategy appears to be useful, as shown below. Indeed, the application of fractional Borel u-optimization with factor approximants for $p = 1, 2$ amounts to solving the equation

$$B_{3,2}(u) - B_{3,1}(u) = 0.$$

It brings a sensible result for the sought amplitude $B = 1.21118$ with the control parameter $u = u_3 = 0.286426$.

The Borel-light approximations with the same marginal amplitude and varying correcting terms give even better results:

$$B^*_{4,3}(u_3) = P_{4/4}(\infty)\, C_3(u_3) = 1.18766,$$

$$B^*_{5,3}(u_3) = P_{5/5}(\infty)\, C_3(u_3) = 1.22814,$$

$$B^*_{6,3}(u_3) = P_{6/6}(\infty)\, C_3(u_3) = 1.18951.$$

Different resummation methods give the results presented in Table 2. The method of Fractional Borel-light summation with iterated roots [49] also gives a rather good result: $B = B_{6,10} = 1.1452$. It is rather close to the value $B = 1.14(3)$ obtained with finite-lattice calculations. Bear in mind that different advanced series methods give $B = 1.25(15)$ [72].

Table 2. Schwinger model—gap.

Schwinger Model	Gap
Factor approximants, 8th order	1.0518
Factor approximants, 9th order	1.3421
Borel with factors	1.0551
Borel-light with factors	1.1549
Borel-light with factors, u-optimal	1.1895
Exact	1.1284
Borel with roots (average) [44]	**1.1224**
Odd Padé, 11th order [58]	1.2266
Corrected Padé approximants [70]	1.2468
Borel-light with roots [49]	1.1452

The best result is marked in bold in Table 2.

5.3. Schwinger Model: Critical Amplitude

The ground-state energy E of the Schwinger model is given by the very short truncated series [72–78],

$$E(x) \simeq 0.5642 - 0.219x + 0.1907x^2 \qquad (x \to 0). \tag{92}$$

The large-x behavior is also known. It is expressed in the form shown in (4),

$$E(x) \simeq Bx^{\beta} \qquad (x \to \infty),$$

with $B = 0.6418$, $\beta = -1/3$. Sometimes, an addition of one more trial term with $a_3 = 0$ may help to improve the results.

Because of the negative β, we work with the inverse truncations when using u-optimization. The problem appears to be quite difficult for the factor approximations, i.e., the factor approximant of the second order gives only $B \approx 0.5456$. The application of fractional

Borel u-optimization with the factor approximants for $k = 3$, $p = 1, 2$ amounts to solving the equation

$$B_{3,2}(u) - B_{3,1}(u) = 0.$$

It brings, after the inversion, a very good result for the sought amplitude

$$B = 0.642257$$

with the uniquely determined control parameter $u = u_3 = 0.0961685$. The result is the best among all of the results represented in Table 3. It is even better than the result found from the self-similar root approximants, which explicitly employs the known subcritical index [79].

A u-optimal Borel-light technique with factors applied to the inverse quantities gives the inverse amplitude

$$B^*_{2,3}(u_k) = P_{2,2}(\infty)\, C_3(u_3),$$

and the total critical amplitude B after inversion equals 0.5827.

For p-optimization with $u = 0$, the parameter $p = p_3$ is found from the minimal derivative condition as the unique solution to the equation

$$\frac{\partial B_3(p)}{\partial p} = 0,$$

with $p = p_3 = 0.897973$ and $B = B_3 = 0.637067$.

As expected, the results found with optimizations and presented above represent an improvement over the non-optimized Borel factor techniques corresponding to $u = 0$ and $p = 1$, which also produce the very reasonable $B \approx 0.6351$.

The results of calculations by various methods are shown in Table 3.

Table 3. Schwinger model: amplitude.

Schwinger	Amplitude
Factor approximant, second order	0.5456
Factor approximant, third order	complex
Fractional Borel with factors, ($p = 1, 2$), u-optimal	**0.6423**
Borel with factors ($u = 0$, $p = 1$)	0.6351
Fractional Borel with factors, p-optimal, min.derivative	0.6371
Fractional Borel with factors, p-optimal, min.difference	0.5639
Borel-light with factors, u-optimal	0.5827
Exact	0.6418
Fractional Borel with roots [49]	0.6672
Odd Padé–Borel [58]	0.6122
Borel with roots [49]	0.6562
Iterated root, second order	0.5523
Odd Padé [58]	0.5344

The best result is marked in bold in Table 3.

It is worth stressing that it is always useful to attack the problem using several methods.

5.4. Anomalous Dimension

Consider the cusp anomalous dimension $\Omega(g)$ of a light-like Wilson loop in the $n = 4$ supersymmetric Yang–Mills theory [79,80]. It depends only on the variable $x = g^2$ expressed though the coupling g. The problem can be written down in terms of the function $f(x) = \frac{\Omega(x)}{x}$. The latter function has the following weak-coupling expansion,

$$f(x) \simeq 4 - 13.1595x + 95.2444x^2 - 937.431x^3, \quad x \to 0.$$

In the strong-coupling limit, $f(x)$ takes the form shown in (4),

$$f(x) \simeq Bx^{\beta}, \quad x \to \infty,$$

with $B = 2$ and $\beta = -1/2$.

Direct application of the factor approximants in the second order gives $B \approx 2.1307$, while in the third order, it gives $B \approx 1.8389$. The Borel summation with factor approximants in the second order gives $B \approx 1.8798$ and is better than the direct factor approximation. In the third order, the Borel summation with factors gives complex results.

Neither of the optimization types brings a unique solution. The best result, $B \approx 2.0233$, is found with p-optimization with $u = 0$,

$$B_3(p) - B_2(p) = 0,$$

with the optimum obtained at $p = p_2 \approx 0.43297$.

The Borel-light technique, when applied to inverse quantities, gives the inverse amplitude

$$B^*_{2,2}(p_2) = \ P_{2/2}(\infty)\ C_2(p_2),$$

and the total critical amplitude after inversion is $B = 1.92018$.

The result $B \approx 2.1115$ is found for the u-optimization

$$B_{3,1}(u) - B_{2,1}(u) = 0,$$

with the optimal value $u = u_2 \approx 0.263034$.

The fractional Borel technique with iterated roots [49] gives a unique solution $B = 1.90291$ in the case of the u-optimization problem.

Some results obtained by different methods are shown in Table 4. The best result, $B = 2.0118$, is obtained from the methodology described above in Section 3.2.

Table 4. Cusp.

Cusp	Amplitude
Factor approximant, second order	2.1307
Factor approximant, third order	1.8389
Borel with factors, second order	1.8798
Borel with factors, third order	complex
Borel-light with factors, p-optimal	1.9202
Exact	2
Fractional Borel with roots [49]	1.9029
Odd Padé [58]	1.7973
Optimal Borel–Leroy [46]	**2.0118**
Borel with roots [49]	2.4416
Iterated Roots	1.6977

The best result is marked in bold in Table 4.

5.5. Two-Dimensional Polymer: Swelling

For a two-dimensional polymer one can theoretically study the so-called swelling factor Y [81]. For Y, perturbation theory yields expansions in the powers of the dimensionless coupling parameter g. The swelling factor can be represented as the following truncation,

$$Y(g) \simeq 1 + \frac{1}{2}\, g - 0.12154525\, g^2 + 0.02663136\, g^3 - 0.13223603\, g^4 \quad (g \to 0). \tag{93}$$

As $g \to \infty$, the swelling factor is expressed in the form of (4), i.e.,

$$Y(g) \simeq B g^{\beta}.$$

Here, the critical index $\beta = 1/2$ is exact [82,83]. As for the amplitude B, one can only say that it is of the order of unity.

Indeed, factor approximants and Borel factor techniques all give results close to unity. For p-optimization with $u = 0$, the parameter p can be found from the minimal derivative condition

$$\frac{\partial B_3(p)}{\partial p} = 0,$$

with $p = p_3 = 0.414668$ and $B = B_3 = 0.982576$. A much lower result, $B \approx 0.82498$, is found with u-optimization,

$$B_{3,2}(u) - B_{3,1}(u) = 0,$$

with the unique optimum located at $u = u_3 \approx 0.443811$. The Borel-light summation technique fails to produce a holomorphic diagonal Padé approximation.

The results are presented in Table 5. Most of the methods give results that are close to the conjectured value of unity. Even Padé approximants, factor approximants, and Borel summation with factors all give values close to one.

Table 5. Two-dimensional polymer.

2*D* Polymer	**Critical Amplitude**
Factor approximant, third order	1.0004
Factor approximant, fourth order	1.00006
Borel with factors, third order, ($p = 1, u = 0$)	1.00734
Borel with factors, fourth order, ($p = 1, u = 0$)	0.97209
Fractional Borel with factors, min. derivative, p-optimal	0.98258
"Exact" conjectured	1
Fractional Borel with roots [49]	0.970678
Even Padé [58]	**1.00002**
Even Padé–Borel [58]	0.977767
Borel with roots [49]	0.9696
Iterated Roots	0.970718

The best result is marked in bold in Table 5.

5.6. Three-Dimensional Polymer: Swelling

For a three-dimensional polymer, one can find the swelling factor $Y(g)$ [81,84], in the form of a truncated series of the type (3),

$$Y(g) \simeq 1 + \frac{4}{3}\, g - 2.075385396\, g^2 + 6.296879676\, g^3 - 25.05725072\, g^4 \quad (g \to 0). \tag{94}$$

The expansion (94) can be extended to the sixth order [81,84]. The strong-coupling behavior of the expansion factor is expressed in the form of (4),

$$\Upsilon(g) \simeq Bg^{\beta} \ (g \to \infty).$$

with $B \approx 1.531$, and $\beta \approx 0.3544$.

Factor approximants give the following sequence:

$$B_2 = 1.50647, \ B_3 = 1.54784, \ B_4 = 1.53523, \ B_5 = 1.53983, \ B_6 = 1.53701.$$

Borel factor approximants give the sequence

$$B_2^* = 1.60365, \ B_3^* = 1.55123, \ B_4^* = 1.51916, \ B_5^* = 1.53117, \ B_6^* = 1.53992 \ .$$

For p-optimization with $u = 0$, the parameter $p = p_3$ can be found from the condition $\frac{\partial B_3(p)}{\partial p} = 0$. The unique solution to the latter equation, $p = p_3 = 0.358042$, is found and the critical amplitude is $B = B_3 = 1.53441$.

The Borel-light technique gives

$$B_{2,3}^*(p_3) = \ P_{2/2}(\infty) \ C_3(p_3) = 1.53998,$$

$$B_{3,3}^*(p_3) = \ P_{3/3}(\infty) \ C_3(p_3) = 1.54002.$$

A much lower result, $B = 1.17626$, is found with u-optimization,

$$B_{3,1}(u) - B_{2,1}(u),$$

with the optimal value $u = u_2 = 0.369247$. However, the Borel-light technique gives

$$B_{2,3}^*(u_2) = \ P_{2/2}(\infty) \ C_3(u_2) = 1.47006,$$

$$B_{3,3}^*(u_2) = \ P_{3/3}(\infty) \ C_3(u_2) = 1.53475,$$

and the latter estimate for the critical amplitude is rather good.

The results are shown in Table 6.

Table 6. Three-dimensional polymer.

3*D* Polymer	**Critical Amplitude**
Factor approximant, fifth order	1.5398
Factor approximant, sixth order	1.537
Borel factors, fifth order	**1.5312**
Borel factors, sixth order	1.5399
Borel-light with factors, sixth order, p-optimal	1.54
Borel-light with factors, sixth order, u-optimal	1.5348
"Exact " numerical [84]	1.5309
Fractional Borel with roots [49]	1.53523
Even Padé, sixth order [58]	1.54022
Even Padé–Borel [58]	1.53296
Odd Padé, fifth order [58]	1.54089
Odd Padé–Borel [58]	1.52996
Iterated Roots, sixth order	1.53611
Borel with roots	1.52718

The best result is marked in bold in Table 6.

The fractional p-optimal Borel technique with roots [49] gives a fairly reasonable value, $B = 1.53523$. Consistent results have also been found with the Padé–Borel techniques shown in the paper [58].

5.7. One-Dimensional Quantum Nonlinear Model

The ground-state energy of the Bose-condensed atoms in a harmonic trap can be expressed in terms of the function $f(g)$, which can be expanded with a dimensionless coupling parameter g. For the fifth order, $f_5(g) = 1 + \sum_{n=1}^{5} a_n z^n$, where all a_n values can be found in the paper [85]. The coefficients rapidly decay by the absolute value. The strong-coupling limit is given in the form (4), $f(g) \simeq \frac{3}{2}\, g^{2/3}$.

Factor approximants give the following, apparently convergent, sequence for the amplitude:

$$B_2 = 1.46572,\ B_3 = 1.49615,\ B_4 = 1.49306,\ B_5 = 1.49188.$$

Borel factor approximants give the sequence

$$B_2^* = 1.52317,\ B_3^* = 1.6084,\ B_4^* = 1.59098,\ B_5^* = 1.42684,$$

which defines the upper and lower bounds.

For p-optimization with $u = 0$, the parameter $p = p_3$ can be found as the unique solution to the equation

$$\frac{\partial B_3(p)}{\partial p} = 0,$$

with $p = p_3 = -0.605018$ and $B = B_3 = 1.47151$. The Borel-light technique gives

$$B_{2,3}^*(p_3) = \ P_{2/2}(\infty)\ C_3(p_3),$$

leading to fairly reasonable numbers, $B = 1.49291$, for the the critical amplitude.

A lower value, $B \approx 1.42983$, is found with u-optimization from the equation

$$B_{3,2}(u) - B_{3,1}(u) = 0,$$

with the optimal value $u = u_3 \approx 0.653582$. The Borel-light technique gives

$$B_{2,3}^*(u_3) = \ P_{2/2}(\infty)\ C_3(u_3),$$

and the critical amplitude,

$$B = B_{2,3}^*(u_3) = 1.50763,$$

is very good.

The results are presented in Table 7. Almost all of them have a good level of accuracy.

Table 7. Non-linear quantum model.

1*d* Non-Linear Model	**Amplitude**
Factor approximant, 4th order	1.4931
Factor approximant, 5th order	1.4919
Borel factors, 4th order	1.591
Borel factors, 5th order	1.4268
Borel-light with factors, 4th order, p-optimal	1.4922
Borel-light with factors, 4th order, u-optimal	**1.5076**
Exact	3/2

Table 7. *Cont.*

1d Non-Linear Model	Amplitude
Fractional Borel with roots [49]	1.4759
Odd Padé [58]	1.4923
Even Padé [58]	1.4918
Borel with roots [49]	1.3851
Iterated roots	1.448

The best result is marked in bold in Table 7.

Particularly good results are obtained with the Fractional Borel-light optimal techniques. The Padé techniques described in the paper [58] also give good results, improving the techniques of iterated roots.

5.8. Three-Dimensional Harmonic Trap

The ground-state energy E of the trapped Bose condensate in the three-dimensional case was investigated in the paper [86]. The energy can be approximated by the following truncation

$$E(c) \simeq \frac{3}{2} + \frac{1}{2}c - \frac{3}{16}c^2 + \frac{9}{64}c^3 - \frac{35}{256}c^4 \qquad (c \to 0), \tag{95}$$

with the "trapping" parameter c.

For a very strong parameter c, the energy behaves as the power law

$$E(c) \simeq Bc^{\beta} \qquad (c \to \infty), \tag{96}$$

And, the critical parameters $B = \frac{5}{4}$, $\beta = 2/5$ [86] are known.

Factor approximation in the second order fails, while in the higher-order factor, the approximants give reasonably good results. However, the Borel summation with the factor approximants in the second order works very well, with $B \approx 1.25983$.

The use of fractional Borel u-optimization with the factor approximants for $k = 3$, $p = 1,2$ amounts to solving the equation

$$B_{3,2}(u) - B_{3,1}(u) = 0.$$

It gives an amplitude of $B = 1.15323$ with the control parameter $u = u_3 = 0.160358$. The corrected Borel-light technique gives the amplitude

$$B^*_{2,3}(u_3) = \; P_{2/2}(\infty)\; C_3(u_3),$$

and the total critical amplitude

$$B = 1.2561,$$

is the best among all estimates.

The result, $B \approx 1.2911$, is found with p-optimization with $u = 0$, which amounts to solving the equation

$$B_3(p) - B_2(p) = 0.$$

The optimum is found for $p = p_2 = 0.705782$. The Borel-light technique gives the amplitude

$$B^*_{2,2}(p_2) = \; P_{2/2}(\infty)\; C_2(p_2),$$

and the total critical amplitude, $B = 1.28598$, is quite reasonable. The results from the calculations using different methods are presented in Table 8. All of them give close and rather accurate values.

Table 8. Trap.

3*d* Trap	Amplitude
Factor approximant, third order	1.30227
Factor approximant, fourth order	1.2848
Borel with factors, second order	1.2598
Borel with factors, third order	1.2916
Borel with factors, fourth order	1.2858
Fractional Borel with factors, min.diff. p-optimal	1.2911
Borel-light with factors, u-optimal	**1.2561**
Borel-light with factors, p-optimal	1.286
Exact	5/4
Fractional Borel with roots [49]	1.2852
Even Padé [58]	1.28211
Even Padé–Borel [58]	1.2855
Borel with roots [49]	1.28492
Iterated roots	1.2739

The best result is marked in bold in Table 8.

In both cases of trapped Bose condensates discussed above, the methods based on the idea of corrected Borel-factor-light approximants work well. They give numbers that are better than those of the Padé techniques or those of methodologies based on iterated roots.

5.9. Bose Temperature Shift

The ideal Bose gas is unstable below the condensation temperature T_0 [86]. Atomic interactions induce the shift $\Delta T_c \equiv T_c - T_0$ to the realistic T_c of a non-ideal Bose system. The shift is characterized by the ratio $\frac{\Delta T_c}{T_0} \simeq c_1 \gamma$, for the asymptotically small gas parameter $\gamma \to 0$.

Monte Carlo simulations [87–89] suggest that

$$c_1 = 1.3. \pm 0.05. \tag{97}$$

In order to calculate c_1 theoretically, it has been suggested that one should first calculate an auxiliary function $c_1(g)$ [90–92]. Then, one can find c_1 as follows:

$$c_1 = \lim_{g \to \infty} c_1(g) \equiv B. \tag{98}$$

The latter limit is found from the expansion over an effective coupling parameter,

$$c_1(g) \simeq = 0.223286g + -0.0661032g^2 + 0.026446g^3 - 0.0129177g^4 + 0.00729073g^5. \tag{99}$$

In the fourth order of factor approximants, we find $c_1 \approx 1.1$, which is much smaller than that expected from the simulations.

In what follows, we work with the original $c_1(g)$. For instance, in p-optimization with $u = 0$, we can set, by analogy with Equation (78),

$$C_3(p) - C_2(p) = 0. \tag{100}$$

The latter equation gives the optimal solution $p = p_2 = 0.41657$. Using the Borel-light technique with the optimal p_2, we have a good result

$$B = B^*_{2,2}(p_2) = P_{2/2}(\infty)\, C_2(p_2) = 1.28421.$$

Even by just setting $p = 1$, $u = 0$, we arrive at

$$B = B^*_{2,2}(1) = P_{2/2}(\infty)\, C_2(1) = 1.17351\ .$$

The application of fractional Borel u-optimization with factor approximants amounts to solving the equation

$$C_{3,1}(u) - C_{2,1}(u) = 0, \tag{101}$$

which is written in analogy with (75). It gives the control parameter $u = u_2 = -0.4351064$. The correction with the Borel-light technique gives the amplitude

$$B^*_{2,2}(u_2) = P_{2/2}(\infty)\, C_2(u_2),$$

where $C_2(u_2) \equiv C_{2,1}(u_2)$, and the sought amplitude is $B = B^*_{2,2} = 1.27224$.

Note that there is another solution to the equation (101), $u_2 = -0.56557$, and it gives $B^*_{2,2} = 1.28553$. The latter result appears to be identical to the results of Modified Even Padé summation [58]. The two solutions are very similar to each other, and the non-uniqueness in such a case does not pose a serious problem.

Various results are shown in Table 9.

Table 9. Shift of the Bose–Einstein condensation temperature and analogous models.

Bose Condensate	Parameter c_1
Factor approximant, third order	1.0248
Factor approximant, fourth order	1.0959
Borel with factors, second order	0.8165
Borel-light with factors, p-optimal	1.2842
Borel-light with factors, u-optimal	1.2722
"Exact" Monte Carlo	1.3 ± 0.05
Fractional Borel-light with roots [49]	1.2498
Modified Even Padé [58]	**1.2885**
Corrected Padé [70]	1.386
Odd Padé [58]	0.985

The best result is marked in bold in Table 9.

In the case of the Bose temperature shift, the Borel-light method, based on the idea of corrected Borel-type approximants, optimization, and correction with Padé approximations works well. The Padé method modified for an even number of terms in the expansion gives a result that is close to the latter, even without optimization or Borel transformation. But, it should be noted here that the even approximation is close in spirit to the general idea of the corrected approximants, as described in the Section 2.6 and in the paper [70].

Accurate results for the sought parameter are also found with optimal generalized Borel summation with iterated roots, $c_1 = 1.339$ [44]; with the optimal Mittag–Leffler summation with iterated roots with $c_1 = 1.3397$ [46]; and with the corrected iterated roots, $c_1 = 1.3092$ (see [49], and references therein). Kastening [90–92], using the optimized variational perturbation theory, estimated c_1 as 1.27 ± 0.11.

5.10. Fermi Gas: Unitary Limit

The ground-state energy E of a dilute Fermi gas can be obtained from perturbation theory [93], so that

$$E(g) \simeq a_0 + a_1 g + a_2 g^2 + a_3 g^3 + a_4 g^4, \tag{102}$$

with the coefficients

$$a_0 = \frac{3}{10}, \qquad a_1 = -\frac{1}{3\pi}, \qquad a_2 = 0.055661,$$

$$a_3 = -0.00914, \qquad a_4 = -0.018604.$$

The effective coupling parameter $g \equiv |k_F a_s|$ is simply related to the Fermi wave number k_F and the atomic scattering length a_s. The limit of very large effective coupling g is called the unitary Fermi gas [94]. Monte Carlo simulations for the case of $g \to \infty$ [95,96] yield

$$E(\infty) = 0.1116. \tag{103}$$

The experimental value [95,97]

$$E(\infty) = 0.1128,$$

is rather close to the Monte Carlo results.

Factor approximants in the third and fourth orders give rather high estimates for $E(\infty)$, as shown in Table 10, while Borel summation with factor approximants gives rather low estimates.

Table 10. Fermi gas energy in the unitary limit.

Fermi Gas	Unitary Limit
Factor approximant, third order	0.174
Factor approximant, fourth order	0.1644
Borel factors, fourth order	0.0898
Borel-light with factors, p-optimal, min.diff.	**0.1125**
Borel-light with factors, p-optimal, min.deriv.	0.1293
"Exact" Monte Carlo [95,96]	0.1116
Borel-factor-light, Mittag–Leffler [46]	0.1193
Generalized Borel-light with roots [49]	0.1193
Fractional Borel-light with roots [49]	0.1256
Borel with roots [49]	0.1329
Diagonal Padé	0.1705

The best result is marked in bold in Table 10.

None of the methods, including those using optimizations, bring accurate solutions. u-Optimization gives $B \approx 0.178$ with or without correction terms. The result $B \approx 0.272375$ is found with p-optimization by means of the equation

$$B_3(p) - B_2(p) = 0, \;\; u = 0,$$

with the optimum at $p = p_2 = -3.313127$. However, the Borel-light techniques give a holomorphic correcting approximant with the value of the amplitude in the sixth order being

$$B = B^*_{3,2}(p_2) = P_{3/3}(\infty)\; C_2(p_2) = 0.11253.$$

The corresponding approximant

$$f^*_{3,2}(g) \simeq P_{3,3}(g) f^*_2(g)$$

that leads to a very accurate estimate is shown below:

$$f_{3,2}^*(g) = \frac{0.0182215 + 0.226078g + 0.656425g^2 + 0.123943g^3}{0.0607385 + 0.753593g + 2.28882g^2 + g^3} \frac{(1+g)^{0.0438025}}{(1+9.07436g)^{0.0438025}}.$$

The lower-order approximants, $f_{1,2}^*(g), f_{2,2}^*(g)$, appear to be non-holomorphic. From p-optimization with the minimal derivative condition

$$\frac{\partial B_3(p)}{\partial p} = 0,$$

we find $p = p_3 = -3.1325$. The Borel-light technique then gives a reasonable estimate for the amplitude

$$B = B_{3,3}^*(p_3) = \ P_{3/3}(\infty)\ C_3(p_3) = 0.1293.$$

We conclude that the Borel-light methods based on Borel summation with optimization and subsequent correction with Padé approximations work well for the problem of a unitary limit, approaching the quality of Monte Carlo and experimental results.

6. In Lieu of Conclusions

For the first time, Fractional Borel Summation was applied in conjunction with self-similar factor approximants. This is the main distinction from the method developed in [49] in conjunction with the so-called self-similar iterated root approximants. It was found that the technique of Fractional Borel Summation can be most conveniently applied in hybrid form. Such hybrid approximants emerge when the Borel-transformed factor approximations are complemented multiplicatively with Padé approximants to satisfy the original expansions asymptotically. A detailed comparison of different methods is performed on a large set of examples, including the approximation techniques of self-similarly modified Padé-Borel approximations. Such a comparison clearly emphasizes the strong points of each of the techniques.

We confirm that the quality of the analytical reconstructions can match the quality of heavy numerical work. Analytical results are often very similar to the exact numerical data. They follow after a sequence of a few analytical steps, and the most difficult part of solving transcendental equations is performed numerically with any desired accuracy.

The discussed approach to the resummation of the asymptotic series is multi-leveled. General Borel Fractional transformation of the original series was introduced. The found transformed series should be resummed in order to adhere to the asymptotic power laws. One starts with the formulation of dynamics in the approximations space. To this end, self-similarity is employed. The flow in the approximation space should be controlled, and "deep" control is incorporated into the definitions of the self-similar approximants. Certain classes of self-similar approximants satisfying the power law behavior by design, such as root and factor self-similar approximants, are chosen for reasons of transparency, explicitness, and convenience. We also employ properly modified (taking account of power law behavior) Padé approximants by noting that they can be viewed as a particular case of factor approximants. The asymptotic power law properties of the approximations follow directly from the ways in which controls are introduced into the approximation dynamics. The second level of controls concerns the dynamics of Borel-type approximations that emerge after an inverse transformation is accomplished back to the original space of approximations. Self-similarity of the approximants allows us to find the dependencies of the sought critical properties explicitly. The second level of control is then performed by applying the minimal difference and minimal derivative conditions with respect to the parameters that are explicit in the original Borel Fractional transformation.

The main methods compared in the paper each have their own merits. Different methods, based on Padé approximants [58], can be useful as benchmarks for the evaluation of the results. The standard scheme of odd Padé approximants is not competitive with respect to the other considered methods. Of course, this statement concerns only the physical problems considered above. However, the methods based on Padé approximants are indispensable for computations with very long expansions. For shorter expansions, the other methods discussed in the paper should be used. But, in many cases of such expansions, the diagonal Padé approximants reappear in combination with factors and roots. Such hybrid forms of corrected approximants, when the approximations of different types are applied consequentially, are very useful for the practical purpose of accurate summation.

In conclusion, we recapitulate the main steps of the multi-level methodology developed in the present paper.

1. The initial truncated series (2) is transformed into the form of a new, supposedly better behaving series. The chosen transformation is the Fractional Iterated Transform (see Section 3.6). At this stage, the control parameters are introduced. They have to be found at the final optimization step.
2. To arrive to the correct asymptotic behavior at infinity, the transformed series is approximated by the class of approximants with power law behavior (4) at infinity. At this step, we reconstruct the coefficients for an arbitrary n.
3. When the guiding principle of self-similarity, described in Section refself, is applied together with the so-called algebraic transformation of the original series [32], we arrive at two types of approximants with the desired property at infinity. These are the self-similar roots discussed in Section 2.4 and the self-similar factor approximants described in the Sections 2.5 and 4.1.
4. In the limit of a large x, the expressions for the critical amplitudes become explicit and factorize into the parts emerging from the self-similar approximants applied to the transformed series and from the Γ-functional terms emerging from the inverse transformation, being dependent on the type of transformation.
5. The application of self-similar iterated roots was considered previously [49]. Now, after the optimization with the minimal difference and (or) minimal derivative conditions described in Sections 2.2 and 3.6, we resort to the best known (to us) guiding principle for *numerical convergence of the sequences of the approximations* for the sought quantities. The uniqueness of the solution for the given class of approximants is achieved due to careful selection of the transformation to the original truncated series.
6. In the current paper, we employ the self-similar factor approximants together with the most convenient technically and theoretically sound method of so-called hybrid techniques (see Sections 2.6 and 4.3). Typically, in low orders of the perturbation theory, an optimized factor approximant has to be found. To return to the original series, we restore the factors in the form of the diagonal Padé approximants. Besides numerical convergence, we are also guided by the Gonchar results on the convergence of the diagonal Padé approximants [67]. Such a selection leads, by design, to a unique limit that can be found numerically as the approximation for the sought quantity.

The Fractional Borel Summation with iterated roots can be successfully applied for various problems [49], although it is not always the best method. Iterated roots can be exceptionally helpful for problems which appear to be indeterminate and those that are poorly treatable from the standpoint of the Padé approximations [49] and the factor approximations. The cases with fast-growing and rapidly diminishing by magnitude coefficients a_n are better treated by means of Borel summations with iterated roots. The cases with slowly diminishing by magnitude coefficients a_n are best approximated by Borel summations with factors. Fractional Borel Summation with factor approximants in its different realizations can be successfully applied to various problems of the types discussed above. In addition, it is shown to be the best method, or very close to it, for about half of the problems considered in the paper, much more often than the other methods discussed above. Thus, the techniques based on factors are most useful in the context of finding the best method for the particular problems. One can say that the factor approximants are a sharper tool than the iterated roots, but the iterated roots are useful for a wider range of applications.

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References

1. Baker, G.A.; Graves-Morris, P. *Padé Approximants*; Cambridge University: Cambridge, UK, 1996.
2. Borel, E. Mémoire sur les séries divergentes. *Ann. Sci. Ecole Norm. Super.* **1899**, *16*, 9–131. [CrossRef]
3. Shanks, D. Non-linear transformations of divergent and slowly convergent sequences. *J. Math. Phys.* **1955**, *34*, 1–42. [CrossRef]
4. Shalaby, A.M. Precise critical exponents of the $O(n)$-symmetric quantum field model using hypergeometric-Meijer resummation. *Phys. Rev. D* **2020**, *101*, 105006. [CrossRef]
5. Shalaby, A.M. Critical exponents of the $O(N)$-symmetric ϕ^4 model from the ϵ^7 hypergeometric-Meijer resummation. *Eur. Phys. J. C* **2021**, *81*, 87 [CrossRef]
6. Wilson, K.G.; Kogut, J.B. The renormalization group and the epsilon expansion. *Phys. Rep.* **1974**, *12*, 75–199. [CrossRef]
7. Dupuis, N.; Canet, L.; Eichhorn, A.; Metzner, W.; Pawlowski, J.M.; Tissier, M.; Wschebor, N. The nonperturbative functional renormalization group and its applications. *Phys. Rep.* **2021**, *910*, 1–114. [CrossRef]
8. Hardy, G.H. *Divergent Series*; Clarendon Press: Oxford, UK, 1949.
9. Bender, C.M.; Orszag, S.A. *Advanced Mathematical Methods for Scientists and Engineers*; Springer: New York, NY, USA, 1999.
10. Kleinert, H.; Schulte-Frohlinde, V. *Critical Properties of ϕ^4-Theories*; World Scientific: New York, NY, USA, 2001.
11. Kardar, M. *Statistical Physics of Fields*; Cambridge University Press: Cambridge, UK, 2007.
12. Grigoriev, A.A.; Leinartas, E.K.; Lyapin, A.P. Summation of functions and polynomial solutions to a multidimensional difference Eequation. *J. Sib. Fed. Univ. Math. Phys.* **2023**, *16*, 153–161.
13. Leinartas, E.K.; Petrochenko, M.E. Multidimensional analogues of the Euler-Maclaurin summation formula and the Borel transform of power series. *Sib. Electron. Math. Rep.* **2022**, *19*, 91–100. [CrossRef]
14. Leinartas, E.K. Multidimensional Hadamard composition and sums with linear constraints on the summation indices. *Sib. Math. J.* **1989**, *30*, 250–255. [CrossRef]
15. Yukalov, V.I. Theory of perturbations with a strong interaction. *Moscow Univ. Phys. Bull.* **1976**, *31*, 10–15.
16. Yukalov, V.I. Model of a hybrid crystal. *Theor. Math. Phys.* **1976**, *28*, 652–660. [CrossRef]
17. Yukalov, V.I. Interplay between approximation theory and renormalization group. *Phys. Part. Nucl.* **2019**, *50*, 141–209. [CrossRef]
18. Yukalov, V.I.; Yukalova, E.P. From asymptotic series to self-similar approximants. *Physics* **2021**, *3*, 829–878. [CrossRef]
19. Kadanoff, L.P. *Statistical Physics: Statics, Dynamics and Renormalization*; World Scientific: Singapore, 2000.
20. Efrati, E.; Wang, Z.; Kolan, A.; Kadanoff, L.P. Real-space renormalization in statistical mechanics. *Rev. Mod. Phys.* **2014**, *86*, 647–669. [CrossRef]
21. Bogolubov, N.N.; Shirkov, D.V. *Introduction to the Theory of Quantized Fields*; Wiley: New York, NY, USA, 1980.
22. Bogolubov, N.N.; Shirkov, D.V. *Quantum Fields*; Benjamin: London, UK, 1983.
23. Yukalov, V.I. Method of self-similar approximations. *J. Math. Phys.* **1991**, *32*, 1235–1239. [CrossRef]
24. Yukalov, V.I. Stability conditions for method of self-similar approximations. *J. Math. Phys.* **1992**, *33*, 3994–4001. [CrossRef]
25. Pietsch, A. Approximation spaces. *J. Approx. Theory* **1981**, *32*, 115–134. [CrossRef]
26. Huang, K. *Statistical Mechanics*; Wiley: New York, NY, USA, 1963.
27. Kubo, R. *Statistical Mechanics*; North-Holland: Amsterdam, The Netherlands, 1965.
28. Bogolubov, N.N. *Quantum Statistical Mechanics*; World Scientific: Singapore, 2015.
29. Richtmyer, R.D.; Morton, K.W. *Difference Methods for Initial-Value Problems*; Interscience: New York, NY, USA, 1967.
30. Golub, G.H.; Ortega, J.M. *Scientific Computing and Differential Equations*; Academic: Boston, MA, USA, 1992.
31. Zaslavsky, G.M. *Chaos in Dynamic Systems*; Harwood: Chur, NY, USA, 1985.
32. Gluzman, S.; Yukalov, V.I. Unified approach to crossover phenomena. *Phys. Rev. E* **1998**, *58*, 4197–4209. [CrossRef]
33. Yukalov, V.I.; Yukalova, E.P.; Gluzman, S. Self-similar interpolation in quantum mechanics. *Phys. Rev. A* **1998**, *58*, 96–115. [CrossRef]
34. Yukalov, V.I.; Gluzman, S. Self-similar crossover in statistical physics. *Phys. A* **1999**, *273*, 401–415. [CrossRef]
35. Gluzman, S.; Yukalov, V.I.; Sornette, D. Self-similar factor approximants. *Phys. Rev. E* **2003**, *67*, 026109. [CrossRef] [PubMed]
36. Yukalov, V.I.; Gluzman, S.; Sornette, D. Summation of power series by self-similar factor approximants. *Phys. A* **2003**, *328*, 409–438. [CrossRef]
37. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Gordon and Breach: New York, NY, USA, 1986.
38. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*; Elsevier: Amsterdam, The Netherlands, 2007
39. Yukalov, V.I.; Gluzman, S. Extrapolation of power series by self-similar factor and root approximants. *Int. J. Mod. Phys. B* **2004**, *18*, 3027–3046. [CrossRef]

40. Yukalova, E.P.; Yukalov, V.I.; Gluzman, S. Self-similar factor approximants for evolution equations and boundary-value problems. *Ann. Phys.* **2008**, *323*, 3074–3090. [CrossRef]
41. Borel, E. *Leçons sur les Series Divergentes*; Gauthier-Villars: Paris, France, 1928.
42. Glimm, J.; Jaffe, A. *Quantum Physics*; Springer: Berlin, Germany, 1987.
43. Yukalov, V.I.; Gluzman, S. Methods of retrieving large-variable exponents. *Symmetry* **2022**, *14*, 332. [CrossRef]
44. Gluzman, S. Iterative Borel summation with self-similar iterated roots. *Symmetry* **2022**, *14*, 2094. [CrossRef]
45. Mittag-Leffler, G. Sur la répresentation analytique d'une branche uniforme d'une fonction monogene. *Acta Math.* **1905**, *29*, 101–181. [CrossRef]
46. Gluzman, S. Optimal Mittag-Leffler summation. *Axioms* **2022**, *11*, 202. [CrossRef]
47. Kazakov, D.I.; Tarasov, O.V.; Shirkov, D.V. Analytic continuation of the results of perturbation theory for the model $g\phi^4$ to the region $g \gtrsim 1$. *Theor. Math. Phys.* **1979**, *38*, 15–25. [CrossRef]
48. Kompaniets, M.V. Prediction of the higher-order terms based on Borel resummation with conformal mapping. *J. Phys. Conf. Ser.* **2016**, *762*, 012075. [CrossRef]
49. Gluzman, S. Borel transform and scale-invariant fractional derivatives united. *Symmetry* **2023**, *15*, 1266. [CrossRef]
50. Suslov, I.M. Divergent Perturbation Series. *J. Exp. Theor. Phys.* **2005**, *100*, 1188–1233. [CrossRef]
51. Sidi, S. *Practical Extrapolation Methods*; Cambridge University Press: Cambridge, UK, 2003.
52. Graffi, S.; Grecchi, V.; Simon, B. Borel summability: Application to the anharmonic oscillator. *Phys. Lett. B* **1970**, *32*, 631–634. [CrossRef]
53. Simon, B. Twelve tales in mathematical physics: An expanded Heineman prize lecture. *J. Math. Phys.* **2022**, *63*, 021101. [CrossRef]
54. Antonenko, S.A.; Sokolov, A.I. Critical exponents for a three-dimensional $O(n)$-symmetric model with $n > 3$. *Phys. Rev. E* **1995**, *51*, 1894–1898. [CrossRef]
55. Sanders, S.; Holthau, M. Hypergeometric continuation of divergent perturbation series: I. Critical exponents of the Bose-Hubbard model. *New J. Phys.* **2017**, *19*, 103036. [CrossRef]
56. Sanders, S.; Holthau, M. Hypergeometric continuation of divergent perturbation series: II. Comparison with Shanks transformation and Padé approximation. *J. Phys. A Math. Theor.* **2017**, *50*, 465302. [CrossRef]
57. Abhignan, V.; Sankaranarayanan, R. Continued functions and perturbation series: Simple tools for convergence of diverging series in $O(n)$-symmetric ϕ^4 field theory at weak coupling limit. *J. Stat. Phys.* **2021**, *183*, 4. [CrossRef]
58. Gluzman, S. Modified Padé-Borel summation. *Axioms* **2023**, *12*, 50. [CrossRef]
59. Abhignan, V. Extrapolation from hypergeometric functions, continued functions and Borel-Leroy transformation; Resummation of perturbative renormalization functions from field theories. *J. Stat. Phys.* **2023**, *190*, 95. [CrossRef]
60. Mera, H.; Pedersen, T.G.; Nikolic, B.K. Nonperturbative quantum physics from low-order perturbation theory. *Phys. Rev. Lett.* **2015**, *115*, 143001. [CrossRef] [PubMed]
61. Alvarez, G.; Silverston, H.J. A new method to sum divergent power series: Educated match. *J. Phys. Commun.* **2017**, *1*, 025005. [CrossRef]
62. Mera, H.; Pedersen, T.G.; Nikolic, B.K. Fast summation of divergent series and resurgent transseries in quantum field theories from Meijer-G approximants. *Phys. Rev. D* **2018**, *97*, 105027. [CrossRef]
63. Shalaby, A.M. Weak-coupling, strong-coupling and large-order parametrization of the hypergeometric-Meijer approximants. *Results Phys.* **2020**, *19*, 103376. [CrossRef]
64. Bender, C.M.; Boettcher, S. Determination of $f(\infty)$ from the asymptotic series for $f(x)$ about $x = 0$. *J. Math. Phys.* **1994**, *35*, 1914–1921. [CrossRef]
65. Dhatt, S.; Bhattacharyya, K. Asymptotic response of observables from divergent weak-coupling expansions: A fractional-calculus-assisted Padé technique. *Phys. Rev. E* **2012**, *86*, 026711. [CrossRef] [PubMed]
66. Dhatt, S.; Bhattacharyya, K. Accurate estimates of asymptotic indices via fractional calculus. *J. Math. Chem.* **2014**, *52*, 231–239. [CrossRef]
67. Gonchar, A.A. Rational approximation of analytic functions. *Proc. Steklov Inst. Math.* **2011**, 272 (Suppl. 2), S44–S57. [CrossRef]
68. Andrianov, I.V.; Manevitch, L.I. *Asymptology: Ideas, Methods, and Applications*; Kluwer: Dordrecht, The Netherlands, 2002.
69. Hioe, F.T.; McMillen, D.; Montroll, E.W. Quantum theory of anharmonic oscillators: Energy levels of a single and a pair of coupled oscillators with quartic coupling. *Phys. Rep.* **1978**, *43*, 305–335. [CrossRef]
70. Gluzman, S.; Yukalov, V.I. Self-Similarly corrected Padé approximants for indeterminate problem. *Eur. Phys. J. Plus* **2016**, *131*, 340–361. [CrossRef]
71. Schwinger, J. Gauge invariance and mass. *Phys. Rev.* **1962**, *128*, 2425–2428. [CrossRef]
72. Hamer, C.J.; Weihong, Z.; Oitmaa, J. Series expansions for the massive Schwinger model in Hamiltonian lattice theory. *Phys. Rev. D* **1997**, *56*, 55–67. [CrossRef]
73. Carrol, A.; Kogut, J.; Sinclair, D.K.; Susskind, L. Lattice gauge theory calculations in $1+1$ dimensions and the approach to the continuum limit. *Phys. Rev. D* **1976**, *13*, 2270–2277. [CrossRef]
74. Vary, J.P.; Fields, T.J.; Pirner, H.J. Chiral perturbation theory in the Schwinger model. *Phys. Rev. D* **1996**, *53*, 7231–7238. [CrossRef] [PubMed]
75. Adam, C. The Schwinger mass in the massive Schwinger model. *Phys. Lett. B* **1996**, *382*, 383–388. [CrossRef]

76. Striganesh, P.; Hamer, C.J.; Bursill, R.J. A new finite-lattice study of the massive Schwinger model. *Phys. Rev. D* **2000**, *62*, 034508. [CrossRef]
77. Coleman, S. More about the massive Schwinger model. *Ann. Phys.* **1976**, *101*, 239–267. [CrossRef]
78. Hamer, C.J. Lattice model calculations for $SU(2)$ Yang-Mills theory in $1+1$ dimensions. *Nucl. Phys. B* **1977**, *121*, 159–175. [CrossRef]
79. Yukalov, V.I.; Gluzman, S. Self-similar interpolation in high-energy physics. *Phys. Rev. D* **2015**, *91*, 125023. [CrossRef]
80. Banks, T.; Torres, T.J. Two-point Padé approximants and duality. *arXiv* **2013**, arXiv:1307.3689
81. Muthukumar, M.; Nickel, B.G. Perturbation theory for a polymer chain with excluded volume interaction. *J. Chem. Phys.* **1984**, *80*, 5839–5850. [CrossRef]
82. Grosberg, A.Y.; Khokhlov, A.R. *Statistical Physics of Macromolecules*; AIP Press: Woodbury, NY, USA, 1994.
83. Pelissetto, A.; Vicari, E. Critical phenomena and renormalization-group theory. *Phys. Rep.* **2002**, *368*, 549–727. [CrossRef]
84. Muthukumar, M.; Nickel, B.G. Expansion of a polymer chain with excluded volume interaction. *J. Chem. Phys.* **1987**, *86*, 460–476. [CrossRef]
85. Gluzman, S.; Yukalov, V.I. Self-similar continued root approximants. *Phys. Lett. A* **2012**, *377*, 124–128. [CrossRef]
86. Courteille, P.W.; Bagnato, V.S.; Yukalov, V.I. Bose-Einstein condensation of trapped atomic gases. *Laser Phys.* **2001**, *11*, 659–800.
87. Arnold, P.; Moore, G. BEC transition temperature of a dilute homogeneous imperfect Bose gas. *Phys. Rev. Lett.* **2001**, *87*, 120401. [CrossRef]
88. Arnold, P.; Moore, G. Monte Carlo simulation of $O(2)\ \phi^4$ field theory in three dimensions. *Phys. Rev. E* **2001**, *64*, 066113.
89. Nho, K.; Landau, D.P. Bose-Einstein condensation temperature of a homogeneous weakly interacting Bose gas: Path integral Monte Carlo study. *Phys. Rev. A* **2004**, *70*, 053614. [CrossRef]
90. Kastening, B. Shift of BEC temperature of homogenous weakly interacting Bose gas. *Laser Phys.* **2004**, *14*, 586–590.
91. Kastening, B. Bose-Einstein condensation temperature of a homogenous weakly interacting Bose gas in variational perturbation theory through seven loops. *Phys. Rev. A* **2004**, *69*, 043613. [CrossRef]
92. Kastening, B. Nonuniversal critical quantities from variational perturbation theory and their application to the Bose-Einstein condensation temperature shift. *Phys. Rev. A* **2004**, *70*, 043621. [CrossRef]
93. Baker, G.A. Neutron matter model. *Phys. Rev. C* **1999**, *60*, 054311. [CrossRef]
94. Ketterle, W.; Zwierlein, M.W. Making, probing and understanding ultracold Fermi gases. *Riv. Nuovo C.* **2008**, *31*, 247–422.
95. Vidana, I. Low-density neutron matter and the unitary limit. *Front. Phys.* **2021**, *9*, 660662. [CrossRef]
96. Carlson, J.; Gandolfi, S.; Schmidt, K.E.; Zhang, S. Auxiliary-field quantum Monte Carlo method for strongly paired fermions. *Phys. Rev. A* **2011**, *84*, 061602(R). [CrossRef]
97. Ku, M.J.H.; Sommer, A.T.; Cheuk, L.W.; Zwierlein, M.W. Revealing the superfluid lambda transition in the universal thermodynamics of a unitary Fermi gas. *Science* **2012**, *335*, 563–567. [CrossRef]