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A Note on Finite Dimensional Odd Contact Lie Superalgebra in Prime Characteristic

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Abstract: Over a field of characteristic $p > 3$, let $KO(n, n+1; \underline{t})$ denote the odd contact Lie superalgebra. In this paper, the super-biderivations of odd Contact Lie superalgebra $KO(n, n+1; \underline{t})$ are studied. Let T_{KO} be a torus of $KO(n, n+1; \underline{t})$, which is an abelian subalgebra of $KO(n, n+1; \underline{t})$. By applying the weight space decomposition approach of $KO(n, n+1; \underline{t})$ with respect to T_{KO}, we show that all skew-symmetric super-biderivations of $KO(n, n+1; \underline{t})$ are inner super-biderivations.

Keywords: torus; super-biderivations; weight space; odd contact Lie superalgebra

MSC: 17B05; 17B40; 17B50

1. Introduction

Over a field of characteristic $p = 0$, the theory of Lie superalgebras has had noticeable development in recent years [1–5]. For example, one author classified the finite dimensional simple Lie superalgebras and infinite-dimensional simple linearly compact Lie superalgebras [1,2]. Nevertheless, there is an open problem about the complete classification of the finite-dimensional simple modular Lie superalgebras (i.e., Lie superalgebras over a field of prime characteristic) [6]. In the last decade, there has been notable development in the study of modular Lie superalgebras, especially in the structures and representations of simple modular Lie superalgebras of Cartan type. The eight families of finite-dimensional simple modular Lie superalgebras W, S, H, K, HO, KO, SHO, and SKO are discussed in [7–11]. The superderivation algebras, second cohomologies, filtrations, and representations of the eight families of finite-dimensional Cartan-type simple modular Lie superalgebras have also been investigated (see [11–13], for example).

As is well known, the study of derivations is very active because of their importance in Lie algebras and Lie superalgebras. With further research about the theory of derivations, it is therefore natural to begin the investigations of biderivations and commuting maps on Lie algebras [14–20]. The research of biderivations goes back to the investigation of the commuting mapping in the associative ring, which showed that all biderivations on commutative prime rings were inner [21]. In particular, the notations of super-biderivations and skew-symmetric super-biderivations was introduced in [22,23]. The skew-symmetric super-biderivations of any perfect and centerless Lie algebras or Lie superalgebras were proved to be inner in [24]. Meanwhile, applications for and results on biderivations and super-biderivations of simple Lie superalgebras arose in [25]. For example, based on the theory related to super-biderivations, the authors obtained commutative post-Lie superalgebra structures [26]. The skew-symmetric super-biderivations of generalized Witt Lie superalgebra $W(m, n; \underline{t})$ were proved to be inner in [27]. In [28], there were similar results for contact Lie superalgebra $K(m, n; \underline{t})$.

This paper is devoted to studying the super-biderivations of odd contact Lie superalgebra $KO(n, n+1; \underline{t})$. And this essay is structured as follows. In Section 2, we review the basic definitions concerning $KO(n, n+1; \underline{t})$. In Section 3, we get several useful conclusions concerning the skew-symmetric super-biderivations on Lie superalgebras. We use

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the method of the weight space decomposition of $KO(n,n+1;\underline{t})$ with respect to T_{KO} to prove that all skew-symmetric super-biderivation of $KO(n,n+1;\underline{t})$ are inner in Section 4 (Theorem 1). Finally, we summarize the important findings in Section 5.

2. Preliminaries

The fundamental notations concerning the odd contact Lie superalgebras $KO(n,n+1;\underline{t})$ are reviewed in this section [22].

$\mathbb{F}$ denotes an algebraically closed field of characteristic $p>3$, and we all work on field $\mathbb{F}$. Let $Z_2=\{\bar{0},\bar{1}\}$ be the additive group of modular 2. For a vector superspace $Q=Q_{\bar{0}}\oplus Q_{\bar{1}}$, the symbol $\mathrm{d}(x)=\alpha$ means the parity of a homogeneous element $x\in Q_\alpha$, $\alpha\in Z_2$. Let $Q=\oplus_{i\in Z}Q_i$ be a Z-graded vector space. Write $\mathrm{Zd}(x)=i$ for the Z-degree of a Z-homogeneous element $x\in Q_i$, $i\in Z$. Throughout this paper, we should mention that once the symbol $\mathrm{d}(x)$ ($\mathrm{Zd}(x)$) appears, it signifies that x is a Z_2-homogeneous (Z-homogeneous) element.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0$ be the set of non-negative integers. Given $n\in\mathbb{N}, n>2$. For two n-tuple $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathbb{N}_0^n$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)\in\mathbb{N}_0^n$, we write $\binom{\alpha}{\beta}=\prod_{i=1}^{n}\binom{\alpha_i}{\beta_i}$. Over the field $\mathbb{F}$, we call $B(n)$ a divided power algebra with generators $\{x^{(\alpha)}\mid\alpha\in\mathbb{N}_0^n\}$. For $\varepsilon_i=(\delta_{i1},\delta_{i2},\cdots,\delta_{in})\in\mathbb{N}_0^n$, where δ_{ij} is the Kronecker symbol, we abbreviate $x^{(\varepsilon_i)}$ as x_i, $i=1,2,\cdots,n$. We call $\Lambda(n+1)$ the Grassmann superalgebra with generators $x_i, i=n+1,\cdots,2n+1$. Furthermore, we write $\Lambda(n,n+1)$ for the tensor product $B(n)\otimes\Lambda(n+1)$.

For $g\in B(n)$ and $f\in\Lambda(n+1)$, we simply write $g\otimes f$ as gf. The formulas hold for $\Lambda(n,n+1)$ as follows:

$$\begin{aligned}
x^{(\alpha)}x^{(\beta)} &= \binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}, \quad \alpha,\beta\in\mathbb{N}_0^n,\\
x_ix_j &= -x_jx_i, \quad i,j\in\{n+1,\cdots,2n+1\},\\
x^{(\alpha)}x_j &= x_jx^{(\alpha)}, \quad \alpha\in\mathbb{N}_0^n,\ j\in\{n+1,\cdots,2n+1\}.
\end{aligned}$$

For $k=\{1,\cdots,n+1\}$, we set

$$\mathbb{B}_k:=\{\langle i_1,i_2,\cdots,i_k\rangle\mid n+1\le i_1<i_2<\cdots<i_k\le 2n+1\}$$

and $\mathbb{B}:=\cup_{k=0}^{n+1}\mathbb{B}_k, \bar{\mathbb{B}}:=\{u\in\mathbb{B}\mid 2n+1\notin\mathbb{B}\}$, where $\mathbb{B}_0=\emptyset$. For $u=\langle i_1,i_2,\cdots,i_k\rangle\in\mathbb{B}_k$, set $|u|=k$ and

$$\|u\|=\begin{cases}k, & 2n+1\notin\mathbb{B}_k,\\ k+1, & 2n+1\in\mathbb{B}_k,\end{cases}$$

and $x^u=x_{i_1}x_{i_2}\cdots x_{i_k}$. It is obvious that $\{x^{(\alpha)}x^u\mid\alpha\in\mathbb{N}_0^n, u\in\mathbb{B}\}$ is an $\mathbb{F}$-basis of $\Lambda(n,n+1)$.

Obviously, $\Lambda(n,n+1)$ is an associative superalgebra with a Z-gradation:

$$\Lambda(n,n+1)=\Lambda(n,n+1)_{\bar{0}}\oplus\Lambda(n,n+1)_{\bar{1}},$$

where $\Lambda(n,n+1)_{\bar{0}}=B(n)\otimes\Lambda(n+1)_{\bar{0}}, \Lambda(n,n+1)_{\bar{1}}=B(n)\otimes\Lambda(n+1)_{\bar{1}}$.

Let $I_0:=\{1,2,\cdots,n\}, I_1:=\{n+1,n+2,\cdots,2n+1\}$ and $I:=I_0\cup I_1$. Put $J_1:=I_1\setminus\{2n+1\}$.

Let $D_1,D_2,\cdots,D_{2n+1}$ be the linear transformations of $\Lambda(n,n+1)$ such that

$$D_i(x^{(\alpha)}x^u)=\begin{cases}x^{(\alpha-\varepsilon_i)}x^u, & i\in I_0,\\ x^{(\alpha)}\cdot\partial x^u/\partial x_i, & i\in I_1.\end{cases}$$

Then it is easy to see that $D_1, D_2, \cdots, D_{2n+1}$ are derivations of the superalgebra $\Lambda(n, n+1)$, and $\mathrm{d}(D_i)$=$\tau(i)$, where

$$\tau(i) = \begin{cases} \bar{0}, & i \in I_0, \\ \bar{1}, & i \in I_1. \end{cases}$$

Let

$$W(n, n+1) := \{ \sum_{i=1}^{2n+1} a_i D_i \mid a_i \in \Lambda(n, n+1), i \in I \}.$$

Then $W(n, n+1)$ is an infinite-dimension Lie superalgebra that is contained in $\mathrm{Der}(\Lambda(n, n+1))$ and the following formula holds:

$$[aD_i, bD_j] = aD_i(b)D_j - (-1)^{\mathrm{d}(aD_i)\mathrm{d}(bD_j)} bD_j(a)D_i,$$

where a, $b \in \Lambda(n, n+1)$, i, $j \in I$.

Over the algebraically closed field $\mathbb{F}$ of characteristic $p > 3$, we choose two n-tuples of positive integers $\underline{t} = (t_1, t_2, \cdots, t_n) \in \mathbb{N}_0^n$ and $\pi = (\pi_1, \pi_2, \cdots, \pi_n) \in \mathbb{N}_0^n$, where $\pi_i = p^{t_i} - 1$ for all $i \in I_0$.

Let

$$\Lambda(n, n+1, \underline{t}) := \mathrm{span}_{\mathbb{F}}\{ x^{(\alpha)} x^u \mid \alpha \in A(n, \underline{t}), u \in \mathbb{B} \},$$

where $A(n, \underline{t}) = \{ \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n \mid 0 \leq \alpha_i \leq \pi_i, i \in I_0 \}$.

Set

$$W(n, n+1; \underline{t}) := \{ \sum_{i=1}^{2n+1} a_i D_i \mid a_i \in \Lambda(n, n+1; \underline{t}),\ i \in I \}.$$

Then $W(n, n+1; \underline{t})$ is a finite-dimensional simple Lie superalgebra. Note that $W(n, n+1; \underline{t})$ possesses a Z-graded structure:

$$W(n, n+1; \underline{t}) = \oplus_{i=-1}^{\xi-1} W(n, n+1; \underline{t})_i,$$

by letting $W(n, n+1; \underline{t})_i := \{ x^{(\alpha)} x^u D_j \mid |\alpha| + |u| = i+1,\ j \in I \}$ and $\xi := \sum_{i=1}^{n} \pi_i + n + 1$.

Put

$$i' = \begin{cases} i+n, & i \in I_0, \\ i-n, & i \in J_1. \end{cases}$$

We define the linear operator $T_K : \Lambda(n, n+1; \underline{t}) \to W(n, n+1; \underline{t})$ as follows:

$$T_K(a) := \sum_{l=1}^{2n} ((-1)^{\tau(l')\mathrm{d}(a)} (D_{l'}(a)) + (-1)^{\mathrm{d}(a)} (D_{2n+1}(a) x_l)) D_l + (\sum_{l=1}^{2n} x_l D_l(a) - 2a) D_{2n+1}.$$

Put

$$KO(n, n+1; \underline{t}) := \mathrm{span}_{\mathbb{F}}\{ T_K(a) \mid a \in \Lambda(n, n+1; \underline{t}) \}.$$

For a, $b \in \Lambda(n, n+1; \underline{t})$, the formula holds:

$$[T_K(a), T_K(b)] = T_K(\langle a, b \rangle), \tag{1}$$

where $\langle a, b \rangle := T_K(a)(b) - (-1)^{\mathrm{d}(a)} 2(D_{2n+1}(a)(b))$ is the Lie bracket in $\Lambda(n, n+1; \underline{t})$.

Then it is easy to show that $KO(n, n+1; \underline{t})$ is a simple Lie superalgebra. And we call $KO(n, n+1; \underline{t})$ the odd contact Lie superalgebra. Moreover, the principal Z-graded is listed below:

$$KO(n, n+1; \underline{t}) = \oplus_{i=-2}^{\xi-2} KO(n, n+1; \underline{t})_i,$$

where $KO(n, n+1; \underline{t})_i = \{T_K(x^{(\alpha)} x^u) \mid |\alpha| + \|u\| = i+2\}$. In particular,

$$KO(n, n+1; \underline{t})_{-2} = \mathbb{F}T_K(1).$$

3. The Notions of Super-Biderivation

The properties of super-biderivations on centerless super-Virasoro algebras were introduced in [22]. Our aim in this section is to introduce a more-general definition concerning super-biderivations of Lie superalgebras. In order to prove the main conclusions, we need some preparations.

G denotes a Lie algebra over an arbitrary field. A linear mapping $D : G \to G$ is called a derivation if the following axioms are satisfied:

$$D([x, y]) = [D(x), y] + [x, D(y)],$$

for all $x, y \in G$. And we say a bilinear map $\psi : G \times G \longrightarrow G$ is a biderivation if the following axioms are satisfied:

$$\psi(x, [y, z]) = [\psi(x, y), z] + [y, \psi(x, z)],$$
$$\psi([x, y], z) = [\psi(x, z), y] + [x, \psi(y, z)],$$

for all $x, y, z \in G$. Meanwhile, we say a biderivation ψ is a skew-symmetric biderivation if it satisfies

$$\psi(x, y) = -\psi(y, x)$$

for all $x, y \in G$. Specially, a bilinear map $\psi_\lambda : G \times G \longrightarrow G$ is an inner biderivation if it satisfies $\psi_\lambda(x, y) = \lambda[x, y]$ for all $x, y \in G$ (see [22]).

L denotes a Lie superalgebra. Recall that a linear map $D : L \times L \to L$ is a superderivation of L if the following axiom is satisfied:

$$D([x, y]) = [D(x), y] + (-1)^{\mathrm{d}(D)\mathrm{d}(x)}[x, D(y)],$$

for all x, $y \in L$. Meanwhile, we write $\mathrm{Der}_{\bar{0}}(L)$ (resp. $\mathrm{Der}_{\bar{1}}(L)$) for the set of all superderivations of Z_2-degree $\bar{0}$ (resp. $\bar{1}$) of L.

A Z_2-homogeneous bilinear map φ with Z_2-degree γ of L is a bilinear map such that $\varphi(L_\alpha, L_\beta) \subset L_{\alpha+\beta+\gamma}$ for any $\alpha, \beta \in Z_2$. Specially, we say φ fits these criteria even if $\mathrm{d}(\varphi) = \gamma = \bar{0}$.

Definition 1. *A bilinear mapping $\varphi : L \times L \to L$ is a super-biderivation of L if the following axioms are satisfied:*

$$\varphi([x, y], z) = [x, \varphi(y, z)] + (-1)^{(\mathrm{d}(z)+\mathrm{d}(\varphi))\mathrm{d}(y)}[\varphi(x, z), y], \quad (2)$$
$$\varphi(x, [y, z]) = [\varphi(x, y), z] + (-1)^{(\mathrm{d}(x)+\mathrm{d}(\varphi))\mathrm{d}(y)}[y, \varphi(x, z)], \quad (3)$$

for all Z_2-homogeneous elements $x, y, z \in L$.

And we say a biderivation φ is a skew-symmetric biderivation if it satisfies:

$$\varphi(x, y) = -(-1)^{\mathrm{d}(x)\mathrm{d}(y)+(\mathrm{d}(x)+\mathrm{d}(y))\mathrm{d}(\varphi)}\varphi(y, x)$$

for all $x, y \in L$.

Denote by $\mathrm{BDer}_{\gamma}(L)$ the set of all skew-symmetric super-biderivations of Z_2-degree γ. It is obvious that

$$\mathrm{BDer}(L) = \mathrm{BDer}_{\bar{0}}(L) \oplus \mathrm{BDer}_{\bar{1}}(L).$$

Lemma 1. *Let $\varphi_\lambda : L \times L \to L$ be a bilinear map with $\lambda \in \mathbb{F}$. Then φ_λ is a skew-symmetric super-biderivation on L if it satisfies $\varphi_\lambda(x,y) = \lambda[x,y]$ for all x, $y \in L$. This class of super-biderivations is called inner.*

Proof. Obviously, it is easy to obtain that φ_λ is an even bilinear map, i.e., $\mathrm{d}(\varphi_\lambda) = 0$. By the skew-symmetry of Lie superalgebras , we have

$$\varphi_\lambda(x,y) = -(-1)^{\mathrm{d}(x)\mathrm{d}(y)+\mathrm{d}(y)\mathrm{d}(\varphi_\lambda)+\mathrm{d}(x)\mathrm{d}(\varphi_\lambda)}\varphi_\lambda(y,x)$$

for any $x, y \in L$.

Due to the definition of graded Jacobi identity $[[x,y],z] = [x,[y,z]] + (-1)^{\mathrm{d}(z)\mathrm{d}(y)}[[x,z],y]$, we have that

$$\varphi_\lambda([x,y],z) = [x,\varphi_\lambda(y,z)] + (-1)^{(\mathrm{d}(z)+\mathrm{d}(\varphi_\lambda))\mathrm{d}(y)}[\varphi_\lambda(x,z),y]$$

for any $x, y, z \in L$.

Similarly, it follows that

$$\varphi_\lambda(x,[y,z]) = [\varphi_\lambda(x,y),z] + (-1)^{(\mathrm{d}(x)+\mathrm{d}(\varphi_\lambda))\mathrm{d}(y)}[y,\varphi_\lambda(x,z)]$$

for any $x, y, z \in L$. $\square$

Lemma 2. *Let φ be a skew-symmetric super-biderivation on L. Then for any $x, y, u, v \in L$, we have*

$$[\varphi(x,y),[u,v]] = (-1)^{(\mathrm{d}(y)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,y],\varphi(u,v)].$$

Proof. Due to Definition 1, there are two different ways to compute $\varphi([x,u],[y,v])$.

From Equation (2), we have

$$\begin{aligned}
\varphi([x,u],[y,v]) &= [x,\varphi(u,[y,v])] + (-1)^{(\mathrm{d}(y)+\mathrm{d}(v)+\mathrm{d}(\varphi))\mathrm{d}(u)}[\varphi(x,[y,v]),u] \\
&= [x,[\varphi(u,y),v]] + (-1)^{(\mathrm{d}(u)+\mathrm{d}(\varphi))\mathrm{d}(y)}[x,[y,\varphi(u,v)]] \\
&\quad +(-1)^{(\mathrm{d}(y)+\mathrm{d}(v)+\mathrm{d}(\varphi))\mathrm{d}(u)}[[\varphi(x,y),v],u] \\
&\quad +(-1)^{(\mathrm{d}(y)+\mathrm{d}(v)+\mathrm{d}(\varphi))\mathrm{d}(u)+(\mathrm{d}(x)+\mathrm{d}(\varphi))\mathrm{d}(y)}[[y,\varphi(x,v)],u].
\end{aligned}$$

According to Equation (3), one gets

$$\begin{aligned}
\varphi([x,u],[y,v]) &= [\varphi([x,u],y),v] + (-1)^{(\mathrm{d}(x)+\mathrm{d}(u)+\mathrm{d}(\varphi))\mathrm{d}(y)}[y,\varphi([x,u],v)] \\
&= [[x,\varphi(u,y)],v] + (-1)^{(\mathrm{d}(y)+\mathrm{d}(\varphi))\mathrm{d}(u)}[[\varphi(x,y),u],v] \\
&\quad +(-1)^{(\mathrm{d}(x)+\mathrm{d}(u)+\mathrm{d}(\varphi))\mathrm{d}(y)}[y,[x,\varphi(u,v)]] \\
&\quad +(-1)^{(\mathrm{d}(x)+\mathrm{d}(u)+\mathrm{d}(\varphi))\mathrm{d}(y)+(\mathrm{d}(v)+\mathrm{d}(\varphi))\mathrm{d}(u)}[y,[\varphi(x,v),u]].
\end{aligned}$$

Comparing the two sides of the above two equations, we have that

$$\begin{aligned}
&[\varphi(x,y),[u,v]] - (-1)^{(\mathrm{d}(y)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,y],\varphi(u,v)] \\
= \ &(-1)^{\mathrm{d}(v)\mathrm{d}(u)+\mathrm{d}(y)\mathrm{d}(v)+\mathrm{d}(y)\mathrm{d}(u)}([\varphi(x,v),[u,y]] - (-1)^{(\mathrm{d}(v)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,v],\varphi(u,y)]).
\end{aligned}$$

And we set

$$f(x,y;u,v) = [\varphi(x,y),[u,v]] - (-1)^{(\mathrm{d}(y)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,y],\varphi(u,v)],$$

$$f(x,v;u,y) = [\varphi(x,v),[u,y]] - (-1)^{(\mathrm{d}(v)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,v],\varphi(u,y)].$$

According to the above equation, it can be easily seen that

$$f(x,y;u,v) = (-1)^{\mathrm{d}(y)\mathrm{d}(u)+\mathrm{d}(v)\mathrm{d}(u)+\mathrm{d}(y)\mathrm{d}(v)}f(x,v;u,y).$$

On the one hand, one goes

$$\begin{aligned} f(x,y;u,v) &= -(-1)^{\mathrm{d}(u)\mathrm{d}(v)}f(x,y;v,u) \\ &= -(-1)^{\mathrm{d}(u)\mathrm{d}(v)}(-1)^{\mathrm{d}(u)\mathrm{d}(v)+\mathrm{d}(y)\mathrm{d}(v)+\mathrm{d}(u)\mathrm{d}(y)}f(x,u;v,y) \\ &= (-1)^{\mathrm{d}(u)\mathrm{d}(y)}f(x,u;y,v). \end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned} f(x,y;u,v) &= (-1)^{\mathrm{d}(u)\mathrm{d}(v)+\mathrm{d}(y)\mathrm{d}(v)+\mathrm{d}(u)\mathrm{d}(y)}f(x,v;u,y) \\ &= -(-1)^{\mathrm{d}(u)\mathrm{d}(y)}(-1)^{\mathrm{d}(u)\mathrm{d}(v)+\mathrm{d}(y)\mathrm{d}(v)+\mathrm{d}(u)\mathrm{d}(y)}f(x,v;y,u) \\ &= -(-1)^{\mathrm{d}(u)\mathrm{d}(y)}f(x,u;y,v). \end{aligned}$$

Hence, we get

$$f(x,y;u,v) = -f(x,y;u,v).$$

Due to $\mathrm{char}(\mathbb{F}) \neq 2$, we have that $f(x,y;u,v) = 0$. Furthermore, we obtain

$$[\varphi(x,y),[u,v]] = (-1)^{(\mathrm{d}(y)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,y],\varphi(u,v)].$$

□

Lemma 3. *If* $\mathrm{d}(x) + \mathrm{d}(y) = \bar{0}$ *for any* $x,\ y \in L$*, then we have*

$$[\varphi(x,y),[x,y]] = 0.$$

Proof. By Lemma 2, it is easily seen that

$$[\varphi(x,y),[x,y]] = (-1)^{(\mathrm{d}(y)+\mathrm{d}(x))\mathrm{d}(\varphi)}[[x,y],\varphi(x,y)].$$

Since $\mathrm{d}(x) + \mathrm{d}(y) = \bar{0}$, we have

$$\begin{aligned} [\varphi(x,y),[x,y]] &= [[x,y],\varphi(x,y)] \\ &= -[\varphi(x,y),[x,y]]. \end{aligned}$$

Thus, we obtain $[\varphi(x,y),[x,y]] = 0$. □

Lemma 4. *Let* $C_L([L,L])$ *be the centralizer of* $[L,L]$*. If* $[x,y] = 0$*, then we have* $\varphi(x,y) \in C_L([L,L])$.

Proof. If $[x,y] = 0$, for any $u,v \in L$, we obtain

$$[\varphi(x,y),[u,v]] = -(-1)^{(\mathrm{d}(y)+\mathrm{d}(u))\mathrm{d}(\varphi)}[[x,y],\varphi[u,v]] = 0.$$

So we get $\varphi(x,y) \in C_L([L,L])$. □

4. Skew-Symmetric Super-Biderivations of $KO(n,n+1;\underline{t})$

In this section, we prove that all skew-symmetric super-biderivations of $KO(n,n+1;\underline{t})$ are inner. For simplicity, we write KO for $KO(n,n+1;\underline{t})$. In order to prove the main theory, we need some preparations.

Set $T_{KO} = \mathrm{span}_{\mathbb{F}}\{T_K(x_i x_{i'}) \mid i \in I_0\}$. It is easy to see that T_{KO} is an abelian subalgebra of KO. From Equation (1), for all $T_K(x^{(\alpha)}x^u) \in KO$, we have

$$[T_K(x_i x_{i'}), T_K(x^{(\alpha)}x^u)] = (\delta_{(i' \in u)} - \alpha_i)T_K(x^{(\alpha)}x^u), \tag{4}$$

where

$$\delta_{(\mathrm{P})} = \begin{cases} 1, & \text{P is ture,} \\ 0, & \text{P is false.} \end{cases}$$

For fixed $\alpha \in \mathbb{N}^n$ and $u \in \mathbb{B}$, we define a linear function $(\alpha + \langle u \rangle) : T_{KO} \to \mathbb{F}$ by means of

$$(\alpha + \langle u \rangle)T_K(x_i x_{i'}) = \delta_{(i' \in u)} - \alpha_i.$$

Therefore, KO has a weight-space decomposition with respect to T_{KO}:

$$KO = \bigoplus_{(\alpha+\langle u \rangle)} KO_{(\alpha+\langle u \rangle)},$$

where

$$\begin{aligned} KO_{(\alpha+\langle u \rangle)} &= \mathrm{span}_{\mathbb{F}}\{T_K(x^{(\beta)}x^v) \in KO \mid [T_K(x_i x_{i'}), T_K(x^{(\beta)}x^v)] \\ &= (\delta_{(i' \in u)} - \alpha_i)T_K(x^{(\beta)}x^v), \forall\, T_K(x_i x_{i'}) \in T_{KO}, i \in I_0\}. \end{aligned}$$

Not specifically, ϕ denotes a Z_2-homogeneous skew-symmetric super-biderivation on KO in the proof below.

Lemma 5. *If $[x, y] = 0$ for $x, y \in KO$, we have $\phi(x, y) = 0$.*

Proof. By applying Lemma 4, we obtain that $\phi(x, y) \in C(KO)$. As KO is a simple Lie superalgebra, $\phi(x, y) = 0$. □

Lemma 6. *For $T_K(x_i x_{i'})$ and $T_K(x^{(\alpha)}x^u)$, we have*

$$\phi(T_K(x_i x_{i'}), T_K(x^{(\alpha)}x^u)) \in KO_{(\alpha+\langle u \rangle)}.$$

Proof. By applying Lemma 5, it is obvious that $\phi(T_K(x_i x_{i'}), T_K(x_j x_{j'})) = 0$ for any $i, j \in I_0$ from $[T_K(x_i x_{i'}), T_K(x_j x_{j'})] = 0$. Note that $\mathrm{d}(T_K(x_l x_{l'})) = \bar{0}$ for all $i \in I_0$. For $T_K(x^{(\alpha)}x^u) \in KO$, one gets

$$\begin{aligned} & (-1)^{(\mathrm{d}(\phi)+\mathrm{d}(T_K(x_i x_{i'})))\mathrm{d}(T_K(x_l x_{l'}))}[T_K(x_l x_{l'}), \phi(T_K(x_i x_{i'}), T_K(x^{(\alpha)}x^u))] \\ =\; & \phi(T_K(x_i x_{i'}), [T_K(x_l x_{l'}), T_K(x^{(\alpha)}x^u)]) - [\phi(T_K(x_i x_{i'}), T_K(x_l x_{l'})), T_K(x^{(\alpha)}x^u)] \\ =\; & (\delta_{(l' \in u)} - \alpha_l)\phi(T_K(x_i x_{i'}), T_K(x^{(\alpha)}x^u)). \end{aligned}$$

□

Lemma 7. *Let $i, j \in I_0, i \neq j$. Then the statements below hold:*

$$(i)\; KO_{(\varepsilon_i)} = KO_{(\varepsilon_i + \langle 2n+1 \rangle)} = \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} \mathbb{F}T_K\Big(\big(\prod_{l \in I_0 \setminus \{i\}, h' \in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)}\big)x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u\Big); \tag{5}$$

$$(ii)\; KO_{(\langle i' \rangle)} = KO_{(\langle i' \rangle + \langle 2n+1 \rangle)} = \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} \mathbb{F}T_K\Big(\big(\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)}\big)x_{i'}x^u\Big); \tag{6}$$

$$(iii)\; KO_{(\varepsilon_i + \langle j' \rangle)} = \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} \mathbb{F}\Big(T_K\big(\big(\prod_{l \in I_0 \setminus \{i\}, h' \in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)}\big)x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x_{j'}x^u\big), \tag{7}$$

where $\alpha_l^{\bar{q}}$ *denotes some integer, and* $\alpha_l^{\bar{q}} \equiv q \pmod p$.

Proof. (i) We may choose a fixed element $i \in I_0$. By applying (1), we can directly obtain that

$$[T_K(x_l x_{l'}), T_K(x^{(\varepsilon_i)})] = -\delta_{li} T_K(x^{(\varepsilon_i)}),$$

for any $l \in I_0$. From Equation (4), we get

$$\delta_{(l' \in u)} - \alpha_l = -\delta_{li},$$

for any $l \in I_0$. If $l \in I_0 \backslash \{i\}$, then $\delta_{(l' \in u)} - \alpha_l \equiv 0 \pmod p$. If $l = i$, it is easy to see that $\delta_{(i' \in u)} - \alpha_i \equiv -1 \pmod p$. Then we obtain the desired result.

(ii) We also choose a fixed element $i' \in J_1$. A straightforward computation proves that

$$[T_K(x_l x_{l'}), T_K(x_{i'})] = \delta_{l'i'} T_K(x_{i'}),$$

for any $l \in I_0$. Equation (4) then yields

$$\delta_{(l' \in u)} - \alpha_l = \delta_{l'i'},$$

for any $l \in I_0$. If $l \in I_0 \backslash \{i\}$, it is easily seen that $\delta_{(l' \in u)} - \alpha_l \equiv 0 \pmod p$. If $l = i$, we have that $\delta_{(i' \in u)} - \alpha_i \equiv 1 \pmod p$. Then the assertion follows.

(iii) The proof is similar to (i) and (ii). $\square$

Lemma 8. *For any* $i \in I \backslash \{2n+1\}$, $\lambda_i \in \mathbb{F}$, *we have*

$$\phi(T_K(x_i x_{i'}), T_K(x_i)) = \lambda_i [T_K(x_i x_{i'}), T_K(x_i)],$$

where λ_i *depends on* i.

Proof. (i) For $i' \in J_1$, according to Equality (6), one may assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_{i'})) = \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i') T_K((\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u),$$

where $c(\alpha, u, i') \in \mathbb{F}$. By Lemma 5, we have that

$$\begin{aligned} 0 &= (-1)^{(\mathrm{d}(\phi) + \mathrm{d}(T_K(x_i x_{i'})))\mathrm{d}(T_K(1))} (\phi(T_K(x_i x_{i'}), [T_K(1), T_K(x_{i'})]) \\ &\quad - [\phi(T_K(x_i x_{i'}), T_K(1)), T_K(x_{i'})]) \\ &= [T_K(1), \phi(T_K(x_i x_{i'}), T_K(x_{i'}))] \\ &= [T_K(1), \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i') T_K((\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u)] \\ &= \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i') T_K(\langle 1, (\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u \rangle). \end{aligned}$$

So we can conclude that $c(\alpha, u, i') = 0$ if $2n+1 \in u$. Then, we may suppose that

$$\phi(T_K(x_i x_{i'}), T_K(x_{i'})) = \sum_{0 \le \alpha \le \pi, u \in \bar{\mathbb{B}}} c(\alpha, u, i') T_K((\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u).$$

Putting $k \in I \backslash \{i, i'\}$, we obtain

$$\begin{aligned} 0 &= (-1)^{(\mathrm{d}(\phi)+\mathrm{d}(T_K(x_i x_{i'})))\mathrm{d}(T_K(x_k))}(\phi(T_K(x_i x_{i'}), [T_K(x_k), T_K(x_{i'})]) \\ &\quad -[\phi(T_K(x_i x_{i'}), T_K(x_k)), T_K(x_{i'})]) \\ &= [T_K(x_k), \phi(T_K(x_i x_{i'}), T_K(x_{i'}))] \\ &= [T_K(x_k), \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i') T_K((\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u)] \\ &= \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i') T_K(\langle x_k, (\prod_{l \in I_0, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x_{i'} x^u \rangle), \end{aligned}$$

where $c(\alpha, u, i') \in \mathbb{F}$. Hence, $c(\alpha, u, i') = 0$ if $k' \in u$ or $\alpha_{k'}^{\bar{0}} > 0$ by calculating the above equation. Therefore, we assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_{i'})) = \sum_{0 \le \alpha \le \pi} c(\alpha, i') T_K(x^{(\alpha_i^{\bar{0}} \varepsilon_i)} x_{i'}).$$

Since $\mathrm{d}(T_K(x_i x_{i'})) + \mathrm{d}(T_K(x_{i'})) = \bar{0}$ for any $i \in I_0$ and Lemma 2, we have that

$$\begin{aligned} 0 &= [\phi(T_K(x_i x_{i'}), T_K(x_{i'})), [T_K(x_i x_{i'}), T_K(x_{i'})]] \\ &= [[T_K(x_i x_{i'}), T_K(x_{i'})], \phi(T_K(x_i x_{i'}), T_K(x_{i'}))] \\ &= [T_K(x_{i'}), \sum_{0 \le \alpha \le \pi} c(\alpha, i') T_K(x^{(\alpha_i^{\bar{0}} \varepsilon_i)} x_{i'})] \\ &= \sum_{0 \le \alpha \le \pi} c(\alpha, i') T_K(\langle x_{i'}, x^{(\alpha_i^{\bar{0}} \varepsilon_i)} x_{i'} \rangle). \end{aligned}$$

Based on computing the above equation, we deduce that $c(\alpha, i') = 0$ if $\alpha_i^{\bar{0}} > 0$. Then, we suppose that

$$\phi(T_K(x_i x_{i'}), T_K(x_{i'})) = c(i') T_K(x_{i'}).$$

Put $\lambda_{i'} := -c(i')$. By the discussions above, for any $i \in I_0$, one gets

$$\phi(T_K(x_i x_{i'}), T_K(x_{i'})) = \lambda_{i'} [T_K(x_i x_{i'}), T_K(x_{i'})],$$

where $\lambda_{i'}$ is dependent on i'.

(*ii*) According to Equality (5), for $i \in I_0$, we may assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_i)) = \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i) T_K((\prod_{l \in I_0 \backslash \{i\}, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}} \varepsilon_i)} x^u),$$

where $c(\alpha, u, i) \in \mathbb{F}$. By Lemma 5, it is easily seen that

$$\begin{aligned} 0 &= (-1)^{(\mathrm{d}(\phi)+\mathrm{d}(T_K(x_i x_{i'})))\mathrm{d}(T_K(1))}(\phi(T_K(x_i x_{i'}), [T_K(1), T_K(x_i)]) \\ &\quad -[\phi(T_K(x_i x_{i'}), T_K(1)), T_K(x_i)]) \\ &= [T_K(1), \phi(T_K(x_i x_{i'}), T_K(x_i))] \\ &= [T_K(1), \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i) T_K((\prod_{l \in I_0 \backslash \{i\}, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}} \varepsilon_i)} x^u)] \\ &= \sum_{0 \le \alpha \le \pi, u \in \mathbb{B}} c(\alpha, u, i) T_K(\langle 1, (\prod_{l \in I_0 \backslash \{i\}, h' \in u} x^{(\alpha_l^{\bar{0}} \varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}} \varepsilon_i)} x^u \rangle). \end{aligned}$$

A simple calculation shows that $c(\alpha,u,i)=0$ if $2n+1\in u$. Then, we may assume that

$$\phi(T_K(x_ix_{i'}),T_K(x_i))=\sum_{0\leq\alpha\leq\pi,u\in\bar{\mathbb{B}}}c(\alpha,u,i)T_K((\prod_{l\in I_0,h'\in u}x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u).$$

Setting $k\in I\backslash\{i,i'\}$, one gets

$$\begin{aligned}
0 &= (-1)^{(\mathrm{d}(\phi)+\mathrm{d}(T_K(x_ix_{i'})))\mathrm{d}(T_K(x_k))}(\phi(T_K(x_ix_{i'}),[T_K(x_k),T_K(x_i)]) \\
&\quad -[\phi(T_K(x_ix_{i'}),T_K(x_k)),T_K(x_i)]) \\
&= [T_K(x_k),\phi(T_K(x_ix_{i'}),T_K(x_i))] \\
&= [T_K(x_k),\sum_{0\leq\alpha\leq\pi,u\in\bar{\mathbb{B}}}c(\alpha,u,i)T_K((\prod_{l\in I_0,h'\in u}x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u)] \\
&= \sum_{0\leq\alpha\leq\pi,u\in\bar{\mathbb{B}}}c(\alpha,u,i)T_K(\langle x_k,(\prod_{l\in I_0,h'\in u}x^{(\alpha_l^{\bar{0}}\varepsilon_l)}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u\rangle).
\end{aligned}$$

By calculating the above equation, we have $c(\alpha,u,i)=0$ if $\alpha_{k'}^{\bar{0}}>0$ or $k'\in u$. Then, we can suppose that

$$\phi(T_K(x_ix_{i'}),T_K(x_i))=\sum_{0\leq\alpha\leq\pi,u\in\{i'\}}c(\alpha,u,i)T_K((\prod_{h'\in u}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u).$$

By Lemma 2, we have

$$\begin{aligned}
\lambda_{i'}T_K(1) &= [\phi(T_K(x_ix_{i'}),T_K(x_{i'})),[T_K(x_ix_{i'}),T_K(x_i)]] \\
&= [[T_K(x_ix_{i'}),T_H(x_{i'})],\phi(T_K(x_ix_{i'},T_H(x_i))] \\
&= [T_K(x_{i'}),\sum_{0\leq\alpha\leq\pi,u\in\{i'\}}c(\alpha,u,i)T_K((\prod_{h'\in u}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u)] \\
&= \sum_{0\leq\alpha\leq\pi,u\in\{i'\}}c(\alpha,u,i)T_K(\langle x_{i'},(\prod_{h'\in u}x^{(\varepsilon_h)})x^{(\alpha_i^{\bar{1}}\varepsilon_i)}x^u\rangle).
\end{aligned}$$

Based on computing the above equation, we obtain that $c(\alpha,u,i)=0$ if $i'\in u$ or ${\alpha_i}^{\bar{1}}>1$. Then, we assume that

$$\phi(T_K(x_ix_{i'}),T_K(x_i))=c(i)T_K(x_i).$$

Put $\lambda_i:=-c(i)$. By the discussions above, for any $i\in I_0$, we conclude that

$$\phi(T_K(x_ix_{i'}),T_K(x_i))=\lambda_i[T_K(x_ix_{i'}),T_K(x_i)],$$

where λ_i depends on i. And our assertion is affirmed. □

Lemma 9. *All Z_2-homogeneous skew-symmetric super-biderivations of KO are even.*

Proof. Due to Lemma 8, all Z_2-homogeneous skew-symmetric super-biderivations of KO are even mapping. Since $\phi(T_K(x_ix_{i'}),T_K(x_i))$ and $[T_K(x_ix_{i'}),T_K(x_i)]$ have the same Z_2-degree, the Z_2-degree of ϕ is even. □

Lemma 10. *For $T_K(x_ix_{j'})$, $i,j\in I_0, i\neq j$, we have*

$$\phi(T_K(x_ix_{i'}),T_K(x_ix_{j'}))=\lambda_{i'}[T_K(x_ix_{i'}),T_K(x_ix_{j'})],$$

where $\lambda_{i'}\in\mathbb{F}$.

Proof. By virtue of Equality (7), we may assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_i x_{j'})) = \sum_{0\le\alpha\le\pi, u\in\mathbb{B}} c(\alpha, u, i, j') T_K((\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u).$$

It is easily seen that

$$\begin{aligned}
0 &= \phi(T_K(x_i x_{i'}), [T_K(1), T_K(x_i x_{j'})]) - [\phi(T_K(x_i x_{i'}), T_K(1)), T_K(x_i x_{j'})] \\
&= [T_K(1), \phi(T_K(x_i x_{i'}), T_K(x_i x_{j'}))] \\
&= [T_K(1), \sum_{0\le\alpha\le\pi, u\in\mathbb{B}} c(\alpha, u, i, j') T_K((\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u)] \\
&= \sum_{0\le\alpha\le\pi, u\in\mathbb{B}} c(\alpha, u, i, j') T_K(\langle 1, (\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u\rangle).
\end{aligned}$$

By a direct computation, we have that $c(\alpha, u, i, j') = 0$ if $2n+1 \in u$. Then, we may assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_i x_{j'})) = \sum_{0\le\alpha\le\pi, u\in\bar{\mathbb{B}}} c(\alpha, u, i, j') T_K((\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u).$$

By Lemma 5, for $k \in I\backslash\{i, j, i', j'\}$, one gets

$$\begin{aligned}
0 &= \phi(T_K(x_i x_{i'}), [T_K(x_k), T_K(x_i x_{j'})]) - [\phi(T_K(x_i x_{i'}), T_K(x_k)), T_K(x_i x_{j'})] \\
&= [T_K(x_k), \phi(T_K(x_i x_{i'}), T_K(x_i x_{j'}))] \\
&= [T_K(x_k), \sum_{0\le\alpha\le\pi, u\in\bar{\mathbb{B}}} c(\alpha, u, i, j') T_K((\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u)] \\
&= \sum_{0\le\alpha\le\pi, u\in\bar{\mathbb{B}}} c(\alpha, u, i, j') T_K(\langle x_k, (\prod_{l\in I_0\backslash\{i\}, h'\in u} x^{(\alpha_l^{\bar{0}}\varepsilon_l)} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x_{j'} x^u\rangle).
\end{aligned}$$

Hence, $c(\alpha, u, i, j) = 0$ if $\alpha_{k'}^{\bar{0}} > 0$ or $k' \in u$ by calculating the equation above. Then, we assume that

$$\phi(T_K(x_i x_{i'}), T_K(x_i x_{j'})) = \sum_{0\le\alpha\le\pi, u\in\{i'\}} c(\alpha, i, j') T_K((\prod_{h'\in u} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x^{(\alpha_j^{\bar{0}}\varepsilon_j)} x_{j'} x^u).$$

By Lemmas 2 and 10, we have

$$\begin{aligned}
\lambda_{i'} T_K(x_{j'}) &= [\lambda_{i'}[T_K(x_i x_{i'}), T_K(x_{i'})], [T_K(x_i x_{i'}), T_K(x_i x_{j'})]] \\
&= [\phi(T_K(x_i x_{i'}), T_K(x_{i'})), [T_K(x_i x_{i'}), T_K(x_i x_{j'})]] \\
&= [[T_K(x_i x_{i'}), T_K(x_{i'})], \phi(T_K(x_i x_{i'}), T_K(x_i x_{j'}))] \\
&= [T_K(x_{i'}), \sum_{0\le\alpha\le\pi, u\in\{i'\}} c(\alpha, i, j') T_K((\prod_{h'\in u} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x^{(\alpha_j^{\bar{0}}\varepsilon_j)} x_{j'} x^u)] \\
&= \sum_{0\le\alpha\le\pi, u\in\{i'\}} c(\alpha, i, j') T_K(\langle x_{i'}, (\prod_{h'\in u} x^{(\varepsilon_h)}) x^{(\alpha_i^{\bar{1}}\varepsilon_i)} x^{(\alpha_j^{\bar{0}}\varepsilon_j)} x_{j'} x^u\rangle).
\end{aligned}$$

Based on computing the above equation, we have that $c(\alpha, i, j) = 0$ if $\alpha_i^{\bar{1}} > 1$, $\alpha_j^{\bar{0}} > 0$ or $i' \in u$. So we suppose that

$$\phi(T_K(x_i x_{i'}), T_K(x_i x_{j'})) = -c(i, j') T_K(x_i x_{j'}).$$

By Lemmas 2 and 8, we have

$$\begin{aligned}
\lambda_{i'} T_K(x_{j'}) &= [\lambda_{i'}[T_K(x_i x_{i'}), T_K(x_{i'})], [T_K(x_i x_{i'}), T_H(x_i x_{j'})]] \\
&= [\phi(T_K(x_i x_{i'}), T_K(x_{i'})), [T_K(x_i x_{i'}), T_K(x_i x_{j'})]] \\
&= [[T_K(x_i x_{i'}), T_K(x_{i'})], \phi(T_K(x_i x_{i'}), T_K(x_i x_{j'}))] \\
&= c(i, j') T_K(\langle x_{i'}, x_i x_{j'} \rangle).
\end{aligned}$$

Thus, we conclude that

$$\phi(T_K(x_i x_{i'}), T_K(x_i)) = \lambda_{i'}[T_K(x_i x_{i'}), T_K(x_i x_{j'})].$$

□

Remark 1. *For $i, j \in I_0, i \neq j$, we have $\lambda_1 = \cdots = \lambda_n = \cdots = \lambda_{2n}$. Due to Lemmas 8 and 10, one gets*

$$\begin{aligned}
0 &= [\phi(T_K(x_j x_{j'}), T_K(x_j)), [T_K(x_i x_{i'}), T_K(x_i x_{j'})]] \\
&\quad -[[T_K(x_j x_{j'}), T_K(x_j)], \phi(T_K(x_i x_{i'}), T_K(x_i x_{j'}))] \\
&= (\lambda_j - \lambda_{i'}) T_K(x_i).
\end{aligned}$$

Thus, we deduce that $\lambda_{i'} = \lambda_j$ for $i, j \in I_0, i \neq j$. Set $\lambda := \lambda_1 = \cdots = \lambda_n = \cdots = \lambda_{2n}$. By a direct computation, we can conclude that

$$\begin{aligned}
\phi(T_K(x_i x_{i'}), T_K(x_i)) &= \lambda[T_K(x_i x_{i'}), T_K(x_i)], \\
\phi(T_K(x_i x_{i'}), T_K(x_{i'})) &= \lambda[T_K(x_i x_{i'}), T_K(x_{i'})], \\
\phi(T_K(x_i x_{i'}), T_K(x_i x_{j'})) &= \lambda[T_K(x_i x_{i'}), T_K(x_i x_{j'})],
\end{aligned}$$

where λ is dependent on neither i nor j.

Lemma 11. *For any $T_K(x^{(\varepsilon_i)} x_{2n+1}), i \in I_0$, we have that*

$$\phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})) = \lambda[T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})],$$

Proof. By Lemmas 2 and 5, and Remark 1, for $i \in I_0$, we have

$$\begin{aligned}
& [\phi(T_K(x_k x_{k'}), T_K(x_{k'})), [T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})]] \\
& -[[T_K(x_k x_{k'}), T_K(x_{k'})], \phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1}))] \\
= \; & [\lambda[T_K(x_k x_{k'}), T_K(x_{k'})], [T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})]] \\
& -[[T_K(x_k x_{k'}), T_K(x_{k'})], \phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1}))] \\
= \; & [T_K(x_{k'}), \phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})) - \lambda[T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})]] \\
= \; & 0.
\end{aligned}$$

Since $C_{KO}(KO_{-1}) = KO_{-2} = \mathbb{F} T_K(1)$, one gets

$$\phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})) = \lambda[T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)}) x_{2n+1}] + b T_K(1),$$

where $b \in \mathbb{F}$. By virtue of Lemma 7, it follows that $KO_{-2} \cap KO_{(\varepsilon_i + \langle 2n+1 \rangle)} = 0$. So we obtain $b = 0$. Furthermore, we conclude that

$$\phi(T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})) = \lambda[T_K(x_i x_{i'}), T_K(x^{(\varepsilon_i)} x_{2n+1})].$$

□

Theorem 1. *Let KO be the odd contact Lie superalgebra over an algebraically closed field of characteristic $p > 3$. Then, we have*

$$\mathrm{BDer}(KO) = \mathrm{IBDer}(KO).$$

Proof. By Lemma 2 and Remark 1, it is follows that

$$\begin{aligned}
0 &= [\phi(T_K(x_i x_{i'}), T_K(x_i)), [T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)]] \\
&\quad -[[T_K(x_i x_{i'}), T_K(x_i)], \phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t))] \\
&= [[T_K(x_i x_{i'}), T_K(x_i)], \lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)]] \\
&\quad -[[T_K(x_i x_{i'}), T_K(x_i)], \phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t))] \\
&= [T_K(x_i), \lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)] - \phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t))]
\end{aligned}$$

for any $T_K(x^{(\iota)}x^s),\ T_K(x^{(\kappa)}x^t) \in KO$.

Because of $C_{KO}(KO_{-1}) = KO_{-2} = \mathbb{F}T_K(1)$, we have

$$\phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)) = \lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)] + bT_K(1).$$

Due to Lemmas 2 and 11, it is easily seen that

$$\begin{aligned}
0 &= [\phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)), [T_K(x_i x_{i'}), T_K(x_i x_{2n+1})]] \\
&\quad -[[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)], \phi(T_K(x_i x_{i'}), T_K(x_i x_{2n+1}))] \\
&= [\lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)] + bT_K(1), [T_K(x_i x_{i'}), T_K(x_i x_{2n+1})]] \\
&\quad -[[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)], \lambda[T_K(x_i x_{i'}), T_K(x_i x_{2n+1})]] \\
&= [\lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)] + bT_K(1) \\
&\quad -\lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)], [T_K(x_i x_{i'}), T_K(x_i x_{2n+1})]] \\
&= [bT_K(1), T_K(x_i x_{2n+1})] \\
&= (-1)^{\tau(i)} 2bT_K(x_i).
\end{aligned}$$

Since $T_K(x_i) \neq 0$, we obtain $b = 0$. Furthermore, we conclude that

$$\phi(T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)) = \lambda[T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)]$$

for any $T_K(x^{(\iota)}x^s), T_K(x^{(\kappa)}x^t)) \in KO$. Therefore, we prove that ϕ is an inner super-biderivation. □

5. Conclusions

In this section, we summarize the important findings.

Firstly, Definition 1 and Lemmas 2 and 3 are a more-general definition and properties for skew-symmetric super-biderivations. Meanwhile, they are very helpful tools to prove all skew-symmetric super-biderivations of KO are inner super-biderivations.

Thereafter, we obtain the weight space decomposition with respect to T_{KO}. Lemmas 7–9 and Remark 1 show that λ is dependent on neither i nor j. Thus, we obtain Lemma 11.

Lastly, we prove that all skew-symmetric super-biderivations of $KO(n, n+1; \underline{t})$ are inner super-biderivations (Theorem 1) by the results above.

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