axioms

Article

More Effective Criteria for Testing the Asymptotic and Oscillatory Behavior of Solutions of a Class of Third-Order Functional Differential Equations

Fahd Masood [1], Osama Moaaz [2,*], Ghada AlNemer [3] and Hamdy El-Metwally [1]

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; fahdmasoud22@gmail.com (F.M.); helmetwally@mans.edu.eg (H.E.-M.)
[2] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah 51452, Saudi Arabia
[3] Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 105862, Riyadh 11656, Saudi Arabia; gnnemer@pnu.edu.sa
* Correspondence: o_moaaz@mans.edu.eg

Abstract: This paper delves into the investigation of quasi-linear neutral differential equations in the third-order canonical case. In this study, we refine the relationship between the solution and its corresponding function, leading to improved preliminary results. These enhanced results play a crucial role in excluding the existence of positive solutions to the investigated equation. By building upon the improved preliminary results, we introduce novel criteria that shed light on the nature of these solutions. These criteria help to distinguish whether the solutions exhibit oscillatory behavior or tend toward zero. Moreover, we present oscillation criteria for all solutions. To demonstrate the relevance of our results, we present an illustrative example. This example validates the theoretical framework we have developed and offers practical insights into the behavior of solutions for quasi-linear third-order neutral differential equations.

Keywords: oscillatory; nonoscillatory; delay differential equation; third-order; canonical

MSC: 34C10; 34K11

Citation: Masood, F.; Moaaz, O.; AlNemer, G.; El-Metwally, H. More Effective Criteria for Testing the Asymptotic and Oscillatory Behavior of Solutions of a Class of Third-Order Functional Differential Equations. *Axioms* **2023**, *12*, 1112. https://doi.org/10.3390/axioms12121112

Academic Editors: Hatıra Günerhan, Francisco Martínez González, Mohammed K. A. Kaabar and Behzad Djafari-Rouhani

Received: 15 September 2023
Revised: 25 October 2023
Accepted: 30 November 2023
Published: 11 December 2023

1. Introduction

Third-order quasi-linear NDEs, while sounding complex, play a pivotal role in various practical applications, addressing a wide array of real-world problems. These equations emerge in fields such as engineering, physics, and biology, where they are instrumental in modeling dynamic systems exhibiting intricate interactions and time delays. By delving into their solutions and properties, we gain insights into phenomena ranging from electrical circuits with distributed parameters to the behavior of biochemical systems with feedback loops. In this paper, understanding and solving third-order quasi-linear NDEs become invaluable tools for engineers, scientists, and researchers seeking to unravel the mysteries of dynamic systems and optimize their performance in the face of delays and nonlinearities [1–3].

Delay-neutral differential equations are considered one of the most important tools used to describe and represent life models and systems with extreme accuracy. This is due to the nature of the delay-neutral differential equation, which contains both delayed and non-delayed functions. Therefore, many mechanical, physical, chemical, and other science models use delay-neutral differential equations. For example, these equations are used in describing population growth dynamics and in modeling physiological processes with neurotransmission delays, see [4]. For more applications in various sciences, please see [5–7].

In this paper, we study the oscillatory behavior of quasi-linear third-order NDEs. These equations are expressed in the following form:

$$\left(a_2(\ell)\left(\left(a_1(\ell)z'(\ell)\right)'\right)^{\alpha}\right)' + q(\ell)x^{\alpha}(\sigma(\ell)) = 0,\ \ell \geq \ell_0, \tag{1}$$

where $z(\ell) = x(\ell) + \mathrm{p}(\ell)x(\tau(\ell))$. Throughout this study, we make the following assumptions:

(H_1) α is a ratio of two positive odd integers and $\alpha > 1$;
(H_2) $q, \mathrm{p} \in C([\ell_0, \infty))$, $q(\ell) \geq 0$, and $0 \leq \mathrm{p}(\ell) < \mathrm{p}_0 < \infty$;
(H_3) $\tau, \sigma \in C^1([\ell_0, \infty))$, $\tau(\ell) \leq \ell$, $\sigma(\ell) \leq \ell$, $\tau'(\ell) \geq \tau_0 > 0$, $\sigma'(\ell) > 0$, $\left(\sigma^{-1}(\ell)\right)' \geq \sigma_0 > 0$, $\tau \circ \sigma = \sigma \circ \tau$, $\lim_{\ell \to \infty} \tau(\ell) = \infty$, and $\lim_{\ell \to \infty} \sigma(\ell) = \infty$;
(H_4) $a_2 \in C^1([\ell_0, \infty))$, $a_1 \in C^2([\ell_0, \infty))$, $a_1 > 0$, $a_2 > 0$,

$$\int_{\ell_0}^{\infty} \frac{1}{a_1(s)} \mathrm{d}s = \infty, \text{ and } \int_{\ell_0}^{\infty} \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s = \infty. \tag{2}$$

By a solution to (1), we mean a nontrivial function, $x \in C([L_x, \infty), \mathbb{R})$, $L_x \geqslant \ell_0$, which has the property z, $a_1 z'$, $a_2\left(\left(a_1 z'\right)'\right)^{\alpha} \in C^1([L_x, \infty), \mathbb{R})$, and satisfies (1) on $[L_x, \infty)$. We consider only those solutions x of (1) that exist on some half-line $[L_x, \infty)$ and satisfy the condition

$$\sup\{|x(\ell)| : \ell \geqslant L\} > 0, \text{ for all } L \geq L_x.$$

Differential equations (DEs) form a fundamental framework in mathematics, encompassing a variety of applications across science and engineering. Within this field, NDEs hold a special place due to their ability to model systems where the rate of change of a function is affected not only by its past behavior but also by the behavior of the delayed intermediate. This property allows NDEs to capture real-world phenomena that exhibit inherent time lags, making them invaluable tools in various fields, including biology, control theory, economics, and physics, see [8–10].

Oscillation theory, a pivotal facet of differential equation analysis, offers crucial insights into solution behaviors. Oscillatory solutions, reflecting dynamic and periodic phenomena, pervade many natural systems. Hence, investigating oscillation criteria, particularly for third-order NDEs, holds paramount importance in both theoretical and practical contexts. This paper delves into obtaining oscillation criteria for third-order NDEs, aiming to establish more precise conditions governing the occurrence of oscillations in the solutions, see [11–14].

The study of oscillation criteria for higher-order DEs has long captured significant interest within the field, see [15–18]. Notably, the analysis of third-order NDEs has received attention due to its importance in diverse scientific and engineering fields, from control theory to population dynamics. Several preceding studies have contributed valuable insights into the oscillation behavior of such equations. Researchers have proposed varied techniques and methodologies to establish conditions under which solutions of third-order NDEs either oscillate or remain nonoscillatory. These criteria often involve intricate mathematical analyses, including inequalities, integral inequalities, and comparisons with auxiliary functions, see [19–21].

Hanan [22], in 1961, studied third-order differential equations in the linear case, that is, by setting $a_1(\ell) = a_2(\ell) = 1, \alpha = 1$ in (1). She provides one of the most important conditions that cannot be weakened for (1) in the linear case by introducing the condition

$$\liminf_{t \to \infty} t^3 q(t) > \frac{2}{3\sqrt{3}}.$$

Thereafter, many works focused on this type of equation. In 2010, Saker and Džurina [23], extended the study to include the presence of α, i.e., they were interested in studying the oscillatory behavior of the delay differential equation

$$\left(a_2(\ell)\left(x''(\ell)\right)^{\alpha}\right)' + q(\ell)x^{\alpha}(\sigma(\ell)) = 0.$$

They presented sufficient conditions ensuring that every solution of previous equations either oscillates or converges to zero. On the other hand, by using Riccati transformation, Thandapani and Li [24] investigated some asymptotic properties for the neutral differential equation

$$\left(a_2(\ell)\left(z''(\ell)\right)^{\alpha}\right)' + q(\ell)x^{\alpha}(\sigma(\ell)) = 0, \tag{3}$$

with $0 \leq \mathrm{p}(\ell) \leq \mathrm{p}_0 < 0$. They established certain sufficient conditions guaranteeing that every solution of (3) either oscillates or converges to zero.

In 2019, Džurina et al. [25] established necessary conditions for the nonexistence of Kneser solutions in oscillation results for third-order NDEs of the following form

$$\left(a_2(\ell)\left(a_1(\ell)z'(\ell)\right)'\right)' + q(\ell)x(\sigma(\ell)) = 0. \tag{4}$$

By combining their recently acquired results with pre-existing research, they ensured oscillation for all solutions of (4). In the same year, Jadlovská et al. [26] investigated the effective oscillatory criteria associated with third-order delay differential equations, represented by the form

$$\left(a_2(\ell)\left(a_1(\ell)x'(\ell)\right)'\right)' + q(\ell)x(\sigma(\ell)) = 0,$$

with a specific focus on the canonical case, aiming to establish that any nonoscillatory solution converges to zero.

Following a different approach, Chatzarakis et al. [27] introduced improved criteria for oscillatory behavior in third-order NDEs with unbounded neutral coefficients, presented by the form

$$z'''(\ell) + q(\ell)x^{\alpha}(\sigma(\ell)) = 0,$$

where they introduced sharp criteria that demonstrate the nonexistence of Kneser solutions.

On the other hand, higher order equations have been studied using many methods and techniques, see for example [28,29].

This paper aims to establish more stringent and improved criteria that guarantee the oscillation of all solutions of (1) through the use of advanced mathematical tools and techniques. The proposed criteria extend current results and facilitate a deeper understanding of the oscillatory nature of tertiary NDEs, providing more space when modeling.

The rest of this paper is structured as follows. In Section 2, we introduce a set of definitions and lemmas essential for simplifying mathematical operations in our work. Section 3 is dedicated to a series of lemmas that pertain to the asymptotic properties of solutions within the class N_2. These lemmas play a pivotal role in illustrating oscillation results. Section 4 provides results that ensure the asymptotic convergence to zero of any Kneser solution. Moving on to Section 5, we combine the results from the preceding sections to articulate the main results of this paper. Finally, in Section 6, we offer an example that supports and illustrates the validity of our results.

2. Preliminary Results

In this section, we present a set of definitions and assumptions that are needed in this paper to simplify the mathematical calculations. For the sake of brevity, we define

$$\mathrm{p}_0(\ell) := (1 - \mathrm{p}(\sigma(\ell)))^{\alpha},$$

$$\phi(\ell) := \min\{q(\ell), q(\tau(\ell))\},$$

$$L_0 z = z,\ L_1 z = a_1 z',\ L_2 z = a_2\Big(\big(a_1 z'\big)'\Big)^{\alpha}, L_3 z = \Big(a_2\Big(\big(a_1 z'\big)'\Big)^{\alpha}\Big)',$$

$$\pi_1(\ell) := \int_{\ell_0}^{\ell} \frac{1}{a_1(s)}\mathrm{d}s,\ \pi_2(\ell) := \int_{\ell_0}^{\ell} \frac{1}{a_2^{1/\alpha}(s)}\mathrm{d}s,\ \pi_{12}(\ell) := \int_{\ell_0}^{\ell} \frac{\pi_2(s)}{a_1(s)}\mathrm{d}s,$$

$$\pi_1(\varsigma, \varrho) := \int_{\varrho}^{\varsigma} \frac{1}{a_1(s)}\mathrm{d}s,\ \pi_2(\varsigma, \varrho) := \int_{\varrho}^{\varsigma} \frac{1}{a_2^{1/\alpha}(s)}\mathrm{d}s,\ \pi_{12}(\varsigma, \varrho) := \int_{\varrho}^{\varsigma} \frac{\pi_2(s)}{a_1(s)}\mathrm{d}s,$$

$$F^{[0]}(\ell) := F(\ell) \text{ and } F^{[j]}(\ell) := F\Big(F^{[j-1]}(\ell)\Big), \text{ for } j = 1, 2, \ldots, n,$$

$$\mathrm{p}_1(\ell; n) := \sum_{k=0}^{n}\left(\prod_{i=0}^{2k}\mathrm{p}\Big(\tau^{[i]}(\ell)\Big)\right)\left[\frac{1}{\mathrm{p}(\tau^{[2k]}(\ell))} - 1\right]\frac{\pi_{12}(\tau^{[2k]}(\ell))}{\pi_{12}(\ell)},$$

$$\widehat{\mathrm{p}}_1(\ell, n) := \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\mathrm{p}(\tau^{[-i]}(\ell))}\right)\left[\frac{\pi_{12}\Big(\tau^{[-2k+1]}(\ell)\Big)}{\pi_{12}(\tau^{[-2k]}(\ell))} - \frac{1}{\mathrm{p}(\tau^{[-2k]}(\ell))}\right],$$

$$\mathrm{B}(\ell, n) := \begin{cases} \max\{\mathrm{p}_0(\ell), \mathrm{p}_1(\ell; n)\} & \text{for } p_0 < 1, \\ \widehat{\mathrm{p}}_1(\ell; n) & \text{for } p_0 > R_{12}(\ell, \ell_1)/R_{12}(\tau(\ell), \ell_1), \end{cases}$$

$$\lambda_* := \liminf_{\ell \to \infty}\frac{\pi_{12}(\ell)}{\pi_{12}(\sigma(\ell))},$$

$$\beta_* := \liminf_{\ell \to \infty}\frac{1}{\alpha}a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\sigma(\ell))\pi_2(\ell)q(\ell)\mathrm{B}^{\alpha}(\sigma(\ell), n),$$

and

$$k_* := \liminf_{\ell \to \infty}\frac{\pi_2^{\beta_*}(\ell)}{\pi_{12}(\ell)}\int_{\ell_0}^{\ell}\frac{\pi_2^{1-\beta_*}(s)}{a_1(s)}\mathrm{d}s, \text{ for } \beta_* \in (0, 1).$$

Remark 1. *For our purposes, we must define the following conditions*

$$\frac{\pi_{12}(\ell)}{\pi_{12}(\sigma(\ell))} \geq \lambda, \textit{ where } \lambda \in (1, \lambda_*), \tag{5}$$

$$\frac{1}{\alpha}a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\sigma(\ell))\pi_2(\ell)q(\ell)\mathrm{B}^{\alpha}(\sigma(\ell), n) \geq \beta, \textit{ where } \beta \in (0, \beta_*), \tag{6}$$

and

$$\frac{\pi_2^{\beta}(\ell)}{\pi_{12}(\ell)}\int_{\ell_0}^{\ell}\frac{\pi_2^{1-\beta}(s)}{a_1(s)}\mathrm{d}s \geq k, \textit{ where } k \in [1, \infty). \tag{7}$$

Lemma 1 ([30])**.** *Assume that A and B are real numbers, $A > 0$, then,*

$$BU - AU^{(\alpha+1)/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}. \tag{8}$$

Lemma 2 ([31])**.** *Assume that $x_1,\ x_2 \in [0, \infty)$. Then,*

$$(x_1 + x_2)^{\alpha} \leq \mu(x_1^{\alpha} + x_2^{\alpha})$$

and

$$\mu = \begin{cases} 1 & \textit{for } 0 < \alpha \leq 1; \\ 2^{\alpha-1} & \textit{for } \alpha > 1. \end{cases}$$

Lemma 3 ([32])**.** *Let $y \in C^n([\ell_0,\infty),(0,\infty))$, $y^{(i)}(\ell) > 0$ for $i = 1,2,\ldots,n$, and $y^{(n+1)}(\ell) \leq 0$, eventually. Then, eventually,*

$$\frac{y(\ell)}{y'(\ell)} \geq \frac{\epsilon}{n}\ell,$$

for every $\epsilon \in (0,1)$.

Lemma 4 ([33])**.** *Suppose that x is a solution to (1) that is positive eventually. In such a case, z satisfies one of the following cases*

$$\begin{array}{lll} \mathrm{N}_1 & : & z > 0,\ L_1 z < 0,\ L_2 z > 0, \text{ and } L_3 z \leq 0, \\ \mathrm{N}_2 & : & z > 0,\ L_1 z > 0, \text{ and } L_2 z > 0, \end{array}$$

for ℓ large enough. The symbol Ω_i (Category Ω_i) represents the set of all solutions that are positive eventually and where the corresponding function fulfills condition (N_i) for $i = 1,2$. The solutions within the category Ω_1 are referred to as Kneser solutions.

Lemma 5 ([34])**.** *Assume that x is an eventually positive solution of (1). If $\mathrm{p}_0 < 1$, then, eventually*

$$x(\ell) > \sum_{k=0}^{n}\left(\prod_{i=0}^{2k}\mathrm{p}\left(\tau^{[i]}(\ell)\right)\right)\left[\frac{z\left(\tau^{[2k]}(\ell)\right)}{\mathrm{p}\left(\tau^{[2k]}(\ell)\right)} - z\left(\tau^{[2k+1]}(\ell)\right)\right],$$

for any integer $n \geq 0$.

Lemma 6. *Assume that x is an eventually positive solution of (1). If $\mathrm{p}_0 > 1$, then,*

$$x(\ell) > \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\mathrm{p}\left(\tau^{[-k]}(\ell)\right)}\right)\left[z\left(\tau^{[-2k+1]}(\ell)\right) - \frac{1}{\mathrm{p}\left(\tau^{[-2k]}(\ell)\right)}z\left(\tau^{[-2k]}(\ell)\right)\right].$$

Proof. From

$$z(\ell) = x(\ell) + \mathrm{p}(\ell)x(\tau(\ell)),$$

we deduce that

$$\begin{aligned} x(\ell) &= \frac{1}{\mathrm{p}(\tau^{-1}(\ell))}\left[z\left(\tau^{-1}(\ell)\right) - x\left(\tau^{-1}(\ell)\right)\right] \\ &= \frac{1}{\mathrm{p}\left(\tau^{[-1]}(\ell)\right)}z\left(\tau^{[-1]}(\ell)\right) \\ &\quad -\frac{1}{\mathrm{p}\left(\tau^{[-1]}(\ell)\right)}\frac{1}{\mathrm{p}\left(\tau^{[-2]}(\ell)\right)}\left[z\left(\tau^{[-2]}(\ell)\right) - x\left(\tau^{[-2]}(\ell)\right)\right] \\ &= \frac{1}{\mathrm{p}\left(\tau^{[-1]}(\ell)\right)}z\left(\tau^{[-1]}(\ell)\right) - \prod_{i=1}^{2}\frac{1}{\mathrm{p}\left(\tau^{[-i]}(\ell)\right)}z\left(\tau^{[-2]}(\ell)\right) \\ &\quad +\prod_{i=1}^{3}\frac{1}{\mathrm{p}\left(\tau^{[-i]}(\ell)\right)}\left[z\left(\tau^{[-3]}(\ell)\right) - x\left(\tau^{[-3]}(\ell)\right)\right]. \end{aligned}$$

By repeating the same technique a number of times, we obtain

$$x(\ell) > \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\mathrm{p}\left(\tau^{[i]}(\ell)\right)}\right)\left[z\left(\tau^{[-2k+1]}(\ell)\right) - \frac{1}{\mathrm{p}\left(\tau^{[-2k]}(\ell)\right)}z\left(\tau^{[-2k]}(\ell)\right)\right].$$

Therefore, we have successfully demonstrated the proof. □

3. Nonexistence of N_2-Type Solutions

In this section, we introduce several lemmas that pertain to the asymptotic properties of solutions within the class N_2. These lemmas will play a pivotal role in demonstrating our primary results regarding oscillations.

Lemma 7. *Suppose that $\beta_* > 0$ and $x \in \Omega_2$. Then for ℓ sufficiently large*
$(A_{1,1})$ $\lim_{\ell\to\infty} L_2z(\ell) = \lim_{\ell\to\infty} L_1z(\ell)/\pi_2(\ell) = \lim_{\ell\to\infty} z(\ell)/\pi_{12}(\ell) = 0$;
$(A_{1,2})$ *L_1z/π_2 is decreasing and $L_1z \geq \pi_2(L_2z)^{1/\alpha}$;*
$(A_{1,3})$ *z/π_{12} is decreasing and $x > (\pi_{12}/\pi_2)L_1z$.*

Proof. Let $x \in \Omega_2$ and choose $\ell_1 \geq \ell_0$, such that $x(\tau(\ell)) > 0$ and β satisfies (6) for $\ell \geq \ell_1$.
$(A_{1,1})$: Since L_2z is a positive decreasing function, obviously

$$\lim_{\ell\to\infty} L_2z = l \geq 0.$$

If $l > 0$, then $L_2z \geq l > 0$, and so for any $\varepsilon \in (0,1)$, we have

$$z(\ell) \geq l^{1/\alpha}\int_{\ell_1}^{\ell}\frac{1}{a_1(u)}\int_{\ell_1}^{u}\frac{1}{a_2^{1/\alpha}(s)}\mathrm{d}s\mathrm{d}u \geq \widetilde{l}\pi_{12}(\ell),\ \widetilde{l} = \varepsilon l^{1/\alpha}. \tag{9}$$

Since

$$z(\ell) = x(\ell) + \mathrm{p}(\ell)x(\tau(\ell)),$$

then $z(\ell) \geq x(\ell)$ and

$$\begin{aligned} x(\ell) &= z(\ell) - \mathrm{p}(\ell)x(\tau(\ell)) \\ &\geq z(\ell) - \mathrm{p}(\ell)z(\tau(\ell)). \end{aligned}$$

Since $z' > 0$, then

$$x(\ell) \geq (1 - \mathrm{p}(\ell))z(\ell).$$

Using this in (1), we obtain

$$\begin{aligned} -L_3z(\ell) &= q(\ell)x^{\alpha}(\sigma(\ell)) \\ &\geq q(\ell)(1 - \mathrm{p}(\sigma(\ell)))^{\alpha}z(\sigma(\ell))^{\alpha}. \end{aligned}$$

From (9), we find

$$-L_3z(\ell) \geq -\widetilde{l}^{\alpha}q(\ell)\mathrm{B}(\ell)\pi_{12}^{\alpha}(\sigma(\ell)).$$

Integrating from ℓ_1 to ℓ, we have

$$\begin{aligned} L_2z(\ell_1) &\geq \widetilde{l}^{\alpha}\int_{\ell_1}^{\ell} q(s)\mathrm{B}(s)\pi_{12}^{\alpha}(\sigma(s))\mathrm{d}s \\ &\geq \alpha\beta\widetilde{l}^{\alpha}\int_{\ell_1}^{\ell}\frac{1}{a_2^{1/\alpha}(s)\pi_2(s)}\mathrm{d}s \\ &= \alpha\beta\widetilde{l}^{\alpha}\ln\frac{\pi_2(\ell)}{\pi_2(\ell_1)} \to \infty \text{ as } \ell \to \infty, \end{aligned}$$

which is a contradiction. Hence, $l = 0$. Applying l'Hôpital's rule, we see that $(A_{1,1})$ holds.
$(A_{1,2})$: Using the fact that L_2z is positive and decreasing, we see that

$$
\begin{aligned}
L_1 z(\ell) &= L_1 z(\ell_1) + \int_{\ell_1}^{\ell} (L_1 z(s))' \mathrm{d}s \\
&\geq L_1 z(\ell_1) + \int_{\ell_1}^{\ell} \frac{L_2^{1/\alpha} z(s)}{a_2^{1/\alpha}(s)} \mathrm{d}s \\
&\geq L_1 z(\ell_1) + L_2^{1/\alpha} z(\ell) \int_{\ell_1}^{\ell} \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s \\
&= L_1 z(\ell_1) + L_2^{1/\alpha} z(\ell) \int_{\ell_1}^{\ell} \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s - L_2^{1/\alpha} z(\ell) \int_{\ell_0}^{\ell_1} \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s.
\end{aligned}
$$

In view of $(A_{1,1})$, there is a $\ell_2 > \ell_1$, such that

$$L_1 z(\ell_1) - L_2^{1/\alpha} z(\ell) \int_{\ell_0}^{\ell_1} \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s > 0, \ \ell \geq \ell_2.$$

Thus

$$L_1 z(\ell) > \pi_2(\ell) L_2^{1/\alpha} z(\ell),$$

and consequently

$$\left(\frac{L_1 z}{\pi_2}\right)'(\ell) = \frac{\pi_2(\ell) L_2^{1/\alpha} z(\ell) - L_1 z(\ell)}{a_2^{1/\alpha}(\ell)\pi_2^2(\ell)} < 0.$$

$(A_{1,3})$: Since $L_1 z/\pi_2$ is a decreasing function tending to zero, then

$$
\begin{aligned}
z(\ell) &= z(\ell_2) + \int_{\ell_2}^{\ell} \frac{L_1 z(s)}{\pi_2(s)} \frac{\pi_2(s)}{a_1(s)} \mathrm{d}s \\
&\geq z(\ell_2) + \frac{L_1 z(\ell)}{\pi_2(\ell)} \int_{\ell_2}^{\ell} \frac{\pi_2(s)}{a_1(s)} \mathrm{d}s \\
&\geq z(\ell_2) + \frac{L_1 z(\ell)}{\pi_2(\ell)} \pi_{12}(\ell) + \frac{L_1 z(\ell)}{\pi_2(\ell)} \int_{\ell_0}^{\ell_2} \frac{\pi_2(s)}{a_1(s)} \mathrm{d}s \\
&> \frac{L_1 z(\ell)}{\pi_2(\ell)} \pi_{12}(\ell).
\end{aligned}
$$

Therefore

$$\left(\frac{z}{\pi_{12}}\right)'(\ell) = \frac{L_1 z(\ell)\pi_{12}(\ell) - z(\ell)\pi_2(\ell)}{a_1(\ell)\pi_{12}^2(\ell)} < 0.$$

□

Lemma 8. *Assume that $x \in \Omega_2$. Then*

$$x(\ell) > \mathrm{B}(\ell, n) z(\ell) \tag{10}$$

and

$$\left(a_2(\ell)\left(\left(a_1(\ell) z'(\ell)\right)'\right)^{\alpha}\right)' \leq -q(\ell)\mathrm{B}^{\alpha}(\sigma(\ell), n) z^{\alpha}(\sigma(\ell)). \tag{11}$$

Proof. If $\mathrm{p}_0 < 1$, then, due to the fact that $z(\ell)$ is increasing and $\tau^{[2k]}(\ell) \geq \tau^{[2k+1]}(\ell)$, we have

$$z(\tau^{[2k]}(\ell)) \geq z(\tau^{[2k+1]}(\ell)),$$

which, along with Lemma 5, implies that

$$
\begin{aligned}
x(\ell) &> \sum_{k=0}^{n} \left(\prod_{i=0}^{2k} \mathrm{p}\left(\tau^{[i]}(\ell)\right)\right) \left[\frac{z(\tau^{[2k]}(\ell))}{\mathrm{p}(\tau^{[2k]}(\ell))} - z(\tau^{[2k+1]}(\ell))\right] \\
&\geq \sum_{k=0}^{n} \left(\prod_{i=0}^{2k} \mathrm{p}\left(\tau^{[i]}(\ell)\right)\right) \left[\frac{1}{\mathrm{p}(\tau^{[2k]}(\ell))} - 1\right] z(\tau^{[2k]}(\ell)).
\end{aligned} \tag{12}
$$

Moreover, as z/π_{12} is decreasing and $\tau^{[2k]}(\ell) \leq \ell$, we have

$$\frac{z(\tau^{[2k]}(\ell))}{\pi_{12}(\tau^{[2k]}\ell)} \geq \frac{z(\ell)}{\pi_{12}(\ell)}$$

and

$$z(\tau^{[2k]}(\ell)) \geq \frac{\pi_{12}(\tau^{[2k]}(\ell))}{\pi_{12}(\ell)} z(\ell).$$

Thus, using the above inequality and substituting in (12), we obtain

$$\begin{aligned} x(\ell) &> \sum_{k=0}^{n}\left(\prod_{i=0}^{2k} \mathrm{p}\left(\tau^{[i]}(\ell)\right)\right)\left[\frac{1}{\mathrm{p}(\tau^{[2k]}(\ell))} - 1\right]\frac{\pi_{12}(\tau^{[2k]}(\ell))}{\pi_{12}(\ell)} z(\ell) \\ &= \mathrm{p}_1(\ell; n) z(\ell). \end{aligned} \tag{13}$$

On the other hand, if $\mathrm{p}_0 > 1$, then z/π_{12} is decreasing and $\tau^{[-2k]}(\ell) \geq \tau^{[-2k+1]}(\ell)$, implying that

$$\frac{z\left(\tau^{[-2k+1]}(\ell)\right)}{\pi_{12}(\tau^{[-2k+1]}(\ell))} \geq \frac{z\left(\tau^{[-2k]}(\ell)\right)}{\pi_{12}(\tau^{[-2k]}(\ell))}$$

and

$$z\left(\tau^{[-2k+1]}(\ell)\right) \geq \frac{\pi_{12}\left(\tau^{[-2k+1]}(\ell)\right)}{\pi_{12}(\tau^{[-2k]}(\ell))} z\left(\tau^{[-2k]}(\ell)\right).$$

Using Lemma 6, we can conclude that

$$x(\ell) > \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\mathrm{p}(\tau^{[-i]}(\ell))}\right)\left[\frac{\pi_{12}\left(\tau^{[-2k+1]}(\ell)\right)}{\pi_{12}(\tau^{[-2k]}(\ell))} - \frac{1}{\mathrm{p}(\tau^{[-2k]}(\ell))}\right] z\left(\tau^{[-2k]}(\ell)\right).$$

As $z(\ell)$ is increasing and $\tau^{[-2k]}(\ell) \geq \ell$, we have

$$\begin{aligned} x(\ell) &> \sum_{k=1}^{n}\left(\prod_{i=1}^{2k-1}\frac{1}{\mathrm{p}(\tau^{[-i]}(\ell))}\right)\left[\frac{\pi_{12}\left(\tau^{[-2k+1]}(\ell)\right)}{\pi_{12}(\tau^{[-2k]}(\ell))} - \frac{1}{\mathrm{p}(\tau^{[-2k]}(\ell))}\right] z(\ell) \\ &= \widehat{\mathrm{p}}_1(\ell, n) z(\ell). \end{aligned} \tag{14}$$

From (1), we have

$$L_3 z(\ell) = -q(\ell) x^{\alpha}(\sigma(\ell)).$$

Using (13) and (14), we obtain

$$L_3 z(\ell) \leq -q(\ell) \mathrm{B}^{\alpha}(\sigma(\ell), n) z^{\alpha}(\sigma(\ell)).$$

Hence, we have successfully demonstrated the proof of the lemma. □

The following lemma gives some additional properties of solutions belonging to the class N_2.

Lemma 9. *Assume that $\beta_* > 0$ and $x \in \Omega_2$. Then for $\beta \in (0, \beta_*)$ and ℓ sufficiently large*
$(A_{2,1})$ $L_1 z/\pi_2^{1-\beta_*}$ *is decreasing and* $(1-\beta_*)L_1 z > \pi_2 (L_2 z)^{1/\alpha}$;
$(A_{2,2})$ $\lim_{\ell\to\infty} L_1 z(\ell)/\pi_2^{1-\beta_*}(\ell) = 0$;
$(A_{2,3})$ $z/\pi_{12}^{1/k}$ *is decreasing and* $z > k(\pi_{12}/\pi_2) L_1 z$.

Proof. Let $x \in \Omega_2$ and choose $\ell_1 \geq \ell_0$, such that $z(\sigma(\ell)) > 0$ and parts $(A_{1,1})$–$(A_{1,3})$ in Lemma 7 hold for $\ell \geq \ell_1 \geq \ell_0$ and choose fixed but arbitrarily large $\beta \in (\beta_*/(1+\beta_*), \beta_*)$

and $k \leq k_*$ satisfying (6) and (7), respectively, for $\ell \geq \ell_1$.
Since

$$\frac{\beta}{1-\beta} > \beta_*,$$

there exist constants $c_1 \in (0,1)$ and $c_2 > 0$, such that

$$\frac{c_1\beta}{1-\beta} > \beta_* + c_2. \tag{15}$$

$(A_{2,1})$: Define

$$w(\ell) = L_1 z(\ell) - \pi_2(\ell)(L_2 z(\ell))^{1/\alpha}, \tag{16}$$

which is clearly positive by $(A_{1,2})$. Differentiating w and using (11) and (6), we see that

$$\begin{aligned} w'(\ell) &= \left(L_1 z(\ell) - (L_2 z(\ell))^{1/\alpha}\pi_2(\ell)\right)' \\ &= -\frac{1}{\alpha}\pi_2(\ell)(L_2 z(\ell))^{1/\alpha-1} L_3 z(\ell) \\ &\geq \frac{1}{\alpha} q(\ell)\pi_2(\ell)\mathrm{B}^{\alpha}(\sigma(\ell), n) z^{\alpha}(\sigma(\ell))(L_2 z(\ell))^{1/\alpha-1} \\ &\geq \beta\frac{z^{\alpha}(\sigma(\ell))}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\sigma(\ell))}(L_2 z(\ell))^{1/\alpha-1}. \end{aligned} \tag{17}$$

By virtue of $(A_{1,3})$, we have

$$w'(\ell) \geq \beta\frac{z^{\alpha}(\ell)}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\ell)}(L_2 z(\ell))^{1/\alpha-1}. \tag{18}$$

Considering $(A_{1,2})$ and $(A_{1,3})$, we obtain the following inequality:

$$\frac{z(\ell)}{\pi_{12}(\ell)} > \frac{L_1 z(\ell)}{\pi_2(\ell)} > (L_2 z(\ell))^{1/\alpha}.$$

Since $\alpha > 1$, then

$$\left(\frac{z(\ell)}{\pi_{12}(\ell)}\right)^{1-\alpha} < \left(\frac{L_1 z(\ell)}{\pi_2(\ell)}\right)^{1-\alpha} < (L_2 z(\ell))^{(1-\alpha)/\alpha}. \tag{19}$$

Substituting the previous inequality in (18), we obtain

$$w'(\ell) \geq \beta\frac{z^{\alpha}(\ell)}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\ell)}\left(\frac{z(\ell)}{\pi_{12}(\ell)}\right)^{1-\alpha} = \beta\frac{z(\ell)}{a_2^{1/\alpha}(\ell)\pi_{12}(\ell)} \geq \beta\frac{L_1 z(\ell)}{a_2^{1/\alpha}(\ell)\pi_2(\ell)}.$$

Integrating from ℓ_2 to ℓ and using the fact that $L_1 z/\pi_2$ is decreasing and tends to zero asymptotically, we have

$$\begin{aligned} w(\ell) &\geq w(\ell_2) + \beta\int_{\ell_2}^{\ell}\frac{L_1 z(s)}{a_2^{1/\alpha}(s)\pi_2(s)}\mathrm{d}s \geq w(\ell_2) + \beta\frac{L_1 z(\ell)}{\pi_2(\ell)}\int_{\ell_2}^{\ell}\frac{1}{a_2^{1/\alpha}(s)}\mathrm{d}s \\ &= z(\ell_2) + \beta\frac{L_1 x(\ell)}{\pi_2(\ell)}\pi_2(\ell) - \beta\frac{L_1 x(\ell)}{\pi_2(\ell)}\int_{\ell_0}^{\ell_2}\frac{1}{a_2^{1/\alpha}(s)}\mathrm{d}s > \beta L_1 x(\ell). \end{aligned} \tag{20}$$

Then

$$(1-\beta)L_1 z(\ell) > \pi_2(\ell)(L_2 z(\ell))^{1/\alpha}$$

and

$$\left(\frac{L_1 z(\ell)}{\pi_2^{1-\beta}(\ell)}\right)' = \frac{(L_2 z(\ell))^{1/\alpha}\pi_2(\ell) - (1-\beta)L_1 z(\ell)}{a_2^{1/\alpha}(\ell)\pi_2^{2-\beta}(\ell)} < 0. \tag{21}$$

It can be deduced straightforwardly from (21) and the observation that $L_1 z$ is increasing that $\beta < 1$. Using this in (20) and taking (15) into account, we find that

$$\begin{aligned} w(\ell) &\geq w(\ell_3) + \beta \int_{\ell_3}^{\ell} \frac{L_1 z(s)}{a_2^{1/\alpha}(s)\pi_2(s)} \mathrm{d}s \\ &\geq w(\ell_3) + \beta \frac{L_1 z(\ell)}{\pi_2^{1-\beta}(\ell)} \int_{\ell_3}^{\ell} \frac{1}{a_2^{1/\alpha}(s)\pi_2^{\beta}(s)} \mathrm{d}s \\ &\geq \frac{\beta}{1-\beta} \frac{L_1 z(\ell)}{\pi_2^{1-\beta}(\ell)} \left(\pi_2^{1-\beta}(\ell) - \pi_2^{1-\beta}(\ell_3)\right) \\ &\geq \frac{c_1 \beta}{1-\beta} L_1 z(\ell) \\ &\geq (\beta_* + c_2) L_1 z(\ell), \end{aligned}$$

which implies

$$(1-\beta_*) L_1 z(\ell) > (1 - \beta_* - c_2) L_1 z(\ell) > (L_2 z(\ell))^{1/\alpha} \pi_2(\ell)$$

and

$$\left(\frac{L_1 z(\ell)}{\pi_2^{1-\beta_*-c_2}(\ell)}\right)' < 0, \tag{22}$$

the conclusion then immediately follows.

$(A_{2,2})$: Obviously, (22) also implies that $L_1 z / \pi_2^{1-\beta_*} \to 0$ as $\ell \to \infty$, since otherwise

$$\frac{L_1 z(\ell)}{\pi_2^{1-\beta_*-c_2}(\ell)} = \frac{L_1 z(\ell)}{\pi_2^{1-\beta_*}(\ell)} \pi_2^{c_2}(\ell) \to \infty \text{ as } \ell \to \infty, \tag{23}$$

which is a contradiction.

$(A_{2,3})$: By utilizing $(A_{2,1})$ and $(A_{2,2})$, as well as $L_1 z / \pi_2^{1-\beta_*}$ as a decreasing function tending towards zero, we can derive:

$$\begin{aligned} z(\ell) &= z(\ell_4) + \int_{\ell_4}^{\ell} \frac{L_1 z(s)}{\pi_2^{1-\beta_*}(s)} \frac{\pi_2^{1-\beta_*}(s)}{a_1(s)} \mathrm{d}s \\ &\geq z(\ell_4) + \frac{L_1 z(\ell)}{\pi_2^{1-\beta_*}(\ell)} \int_{\ell_4}^{\ell} \frac{\pi_2^{1-\beta_*}(s)}{a_1(s)} \mathrm{d}s \\ &= z(\ell_4) + \frac{L_1 z(\ell)}{\pi_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_*}(s)}{a_1(s)} \mathrm{d}s - \frac{L_1 z(\ell)}{\pi_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell_4} \frac{\pi_2^{1-\beta_*}(s)}{a_1(s)} \mathrm{d}s \\ &> \frac{L_1 z(\ell)}{\pi_2^{1-\beta_*}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_*}(s)}{a_1(s)} \mathrm{d}s \\ &\geq k \frac{\pi_{12}(\ell)}{\pi_2(\ell)} L_1 z(\ell). \end{aligned}$$

Therefore

$$\left(\frac{z(\ell)}{\pi_{12}^{1/k}(\ell)}\right)' = \frac{k\pi_{12}(\ell) L_1 z(\ell) - \pi_2(\ell) z(\ell)}{k a_1(\ell) \pi_2^{1/k+1}(\ell)} < 0.$$

As a result, we have successfully concluded the proof of the Lemma. □

Corollary 1. *If $\beta_* \geq 1$ then $\Omega_2 = \varnothing$.*

Proof. This can be deduced from the inequality:

$$(1-\beta_*)L_1z(\ell) > (L_2z(\ell))^{1/\alpha}\pi_2(\ell),$$

taking into account the positivity of L_2z. □

Corollary 2. *If $\beta_* > 0$ and $\lambda_* = \infty$, then $\Omega_2 = \varnothing$.*

Proof. Let $x \in \Omega_2$, and choose $\ell_1 \geq \ell_0$, such that $z(\sigma(\ell)) > 0$ and parts $(A_{2,1})$–$(A_{2,3})$ in Lemma 7 hold for $\ell \geq \ell_1 \geq \ell_0$ and choose fixed but arbitrarily large $\lambda \leq \lambda_*$, $\beta \leq \beta_*$, and $k \leq k_*$ satisfying (5), (6), and (7), respectively, for $\ell \geq \ell_1$. Using (17), and the decreasing of $z/\pi_{12}^{1/k}$, we have

$$\begin{aligned} w'(\ell) &\geq \beta \frac{z^\alpha(\sigma(\ell))}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha/k}(\sigma(\ell))\pi_{12}^{\alpha(1-1/k)}(\sigma(\ell))}(L_2z(\ell))^{1/\alpha-1} \\ &\geq \beta \frac{z^\alpha(\ell)}{\pi_{12}^{\alpha/k}(\ell)} \frac{1}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha(1-1/k)}(\sigma(\ell))}(L_2z(\ell))^{1/\alpha-1}. \end{aligned}$$

Using $(A_{2,3})$, (19), and (5), we obtain

$$\begin{aligned} w'(\ell) &\geq \beta \frac{z^\alpha(\ell)}{\pi_{12}^{\alpha/k}(\ell)} \frac{1}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha(1-1/k)}(\sigma(\ell))}\left(\frac{z(\ell)}{\pi_{12}(\ell)}\right)^{1-\alpha} \\ &= \beta \frac{\pi_{12}^{\alpha(1-1/k)}(\ell)}{a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha(1-1/k)}(\sigma(\ell))} \frac{z(\ell)}{\pi_{12}(\ell)} \\ &\geq \beta \frac{\lambda^{\alpha(1-1/k)}}{a_2^{1/\alpha}(\ell)} \frac{z(\ell)}{\pi_{12}(\ell)} \geq \beta k \lambda^{\alpha(1-1/k)} \frac{L_1z(\ell)}{a_2^{1/\alpha}(\ell)\pi_2(\ell)}. \end{aligned}$$

Integrating the last inequality from ℓ_2 to ℓ and using that L_1z/π_2 is a decreasing function tending to zero, we obtain

$$w(\ell) \geq k\beta\lambda^{\alpha(1-1/k)}L_1z(\ell). \tag{24}$$

Thus

$$\left(1-k\beta\lambda^{\alpha(1-1/k)}\right)L_1z(\ell) \geq (L_2z(\ell))^{1/\alpha}\pi_2(\ell).$$

As λ can assume arbitrarily large values, we can choose λ such that $\lambda > (1/k\beta)^{k/\alpha(k-1)}$, thereby leading to a contradiction with the positivity L_2z. This concludes the proof of Corollary 2. □

Corollary 3. *Assume that $\beta_* > 0$ and $k_* = \infty$. Then, $\Omega_2 = \varnothing$.*

Proof. The proof follows with the same steps from Corollary 2, and the fact that k can be arbitrarily large, we omit it. □

Definition 1. *For our purposes, let us define the following sequence $\{\beta_n\}_{n=0}^\infty$, assuming it exists:*

$$\beta_0 = \beta_*, \text{ where } \beta_* \in (0,1),$$

$$\beta_n = \frac{\beta_0 k_{n-1}\lambda_*^{\alpha(1-1/k_{n-1})}}{1-\beta_{n-1}}, \text{ where } \lambda_* \in [1,\infty), \tag{25}$$

and k_n satisfies the condition:

$$k_n = \liminf_{\ell\to\infty} \frac{\pi_2^{\beta_n}(\ell)}{\pi_{12}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_n}(s)}{a_1(s)} \mathrm{d}s,\ n \in \mathbb{N}_0. \tag{26}$$

Remark 2. *Clearly, β_{n+1} exists if $\beta_i < 1$ and $k_i \in [1, \infty)$ for $i = 0, 1, \ldots, n$. In this scenario, we can derive the following inequality:*

$$\frac{\beta_1}{\beta_0} = \frac{k_0 \lambda^{\alpha(1-1/k_0)}}{1-\beta_0} > 1$$

and

$$k_1 \geq k_0.$$

We can easily establish, through the use of mathematical induction on n, the following inequality

$$\frac{\beta_{n+1}}{\beta_n} \geq l_n > 1, \tag{27}$$

where

$$l_0 := \frac{k_0 \lambda_*^{\alpha(1-1/k_{n-1})}}{1-\beta_0},$$

$$l_n := \frac{k_n \lambda_*^{\alpha(1/k_{n-1}-1/k_n)}(1-\beta_{n-1})}{k_{n-1}(1-\beta_n)},\ n \in \mathbb{N}, \tag{28}$$

with

$$k_n \geq k_{n-1}.$$

Next, we will demonstrate how iterative improvements can be made to the results presented in Lemma 9.

Lemma 10. *Suppose that $\delta_* > 0$ and $x \in \Omega_2$. Then, for any $n \in \mathbb{N}_0$ and ℓ sufficiently large*
$(A_{n,1})$ *$L_1 z / \pi_2^{1-\beta_n}$ is decreasing and $(1-\beta_n) L_1 z > (L_2 z)^{1/\alpha} \pi_2$;*
$(A_{n,2})$ *$\lim_{\ell \to \infty} L_1 z(\ell) / \pi_n^{1-\beta_n}(\ell) = 0$;*
$(A_{n,3})$ *$z / \pi_{12}^{1/\varepsilon_n k_n}$ is decreasing and $z > \varepsilon_n k_n (\pi_{12}/\pi_2) L_1 z$ for any $\varepsilon_n \in (0,1)$.*

Proof. Let $x \in \Omega_2$ with $z(\sigma(\ell)) > 0$ and parts $(A_{1,1})$–$(A_{1,3})$ in Lemma 7 hold for $\ell \geq \ell_1 \geq \ell_0$ and choose fixed but arbitrarily large $\beta \leq \beta_*$, and $k \leq k_*$ satisfying (6) and (7), respectively, for $\ell \geq \ell_1$. We will proceed by induction on n. For $n = 0$, the conclusion follows from Lemma 9 with $\varepsilon_0 = k/k_*$. Next, assume that $(A_{n,1})$–$(A_{n,3})$ hold for $n \geq 1$ for $\ell \geq \ell_n \geq \ell_1$. We need to show that they each hold for $n+1$.
$(A_{n+1,1})$: Using $(A_{n,3})$ in (17), we obtain

$$\begin{aligned} w'(\ell) &\geq \beta \frac{z^\alpha(\sigma(\ell))}{a_2^{1/\alpha}(\ell) \pi_{12}^{\alpha/\varepsilon_n k_n}(\sigma(\ell)) \pi_{12}^{\alpha(1-1/\varepsilon_n k_n)}(\sigma(\ell))} (L_2 z(\ell))^{1/\alpha - 1} \\ &\geq \beta \frac{z^\alpha(\ell)}{a_2^{1/\alpha}(\ell) \pi_{12}^{\alpha/\varepsilon_n k_n}(\ell) \pi_{12}^{\alpha(1-1/\varepsilon_n k_n)}(\sigma(\ell))} \left(\frac{z(\ell)}{\pi_{12}(\ell)} \right)^{1-\alpha} \\ &= \beta \frac{\pi_{12}^{\alpha(1-1/\varepsilon_n k_n)}(\ell)}{\pi_{12}^{\alpha(1-1/\varepsilon_n k_n)}(\sigma(\ell))} \frac{z(\ell)}{a_2^{1/\alpha}(\ell) \pi_{12}(\ell)} \\ &\geq \varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)} \frac{L_1 z(\ell)}{a_2^{1/\alpha}(\ell) \pi_2(\ell)}. \end{aligned}$$

By integrating the aforementioned inequality from ℓ_n to ℓ and employing $(A_{n,1})$ and $(A_{n,2})$, we obtain

$$\begin{aligned}
w(\ell) &\geq w(\ell_n) + \varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)} \int_{\ell_n}^{\ell} \frac{L_1 z(s)}{a_2^{1/\alpha}(s)\pi_2(s)} \mathrm{d}s \\
&\geq w(\ell_n) + \varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)} \frac{L_1 z(\ell)}{\pi_2^{1-\beta_n}(\ell)} \int_{\ell_n}^{\ell} \frac{1}{a_2^{1/\alpha}(s)\pi_2^{\beta_n}(s)} \mathrm{d}s \\
&\geq w(\ell_n) + \frac{\varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)}}{1-\beta_n} \frac{L_1 z(\ell)}{\pi_2^{1-\beta_n}(\ell)} \left[\pi_2^{1-\beta_n}(\ell) - \pi_2^{1-\beta_n}(\ell_n) \right] \\
&> \frac{\varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)}}{1-\beta_n} L_1 z(\ell) = \eta \beta_{n+1} L_1 z(\ell),
\end{aligned} \tag{29}$$

where

$$\eta = \frac{\beta}{\beta_*} \varepsilon_n \frac{\lambda^{\alpha(1-1/\varepsilon_n k_n)}}{\lambda_*^{\alpha(1-1/k_n)}} \in (0,1),$$

and $\eta \to 1$ where $(\lambda, \varepsilon_n, \beta) \to (\lambda_*, 1, \beta_*)$. Choose η, such that

$$\eta > \frac{1}{1-\beta_n+\beta_{n+1}} = \frac{1}{1+\beta_n(l_n-1)}, \tag{30}$$

where l_n satisfies (27). Then

$$\frac{\eta\beta_{n+1}}{1-\eta\beta_{n+1}} > \frac{\beta_{n+1}}{(1+\beta_n(l_n-1))\left(1 - \frac{l_n\beta_n}{1+\beta_n(l_n-1)}\right)} = \frac{\beta_{n+1}}{1-\beta_n},$$

and there exist two constants, $c_1 \in (0,1)$ and $c_2 > 0$, such that

$$c_1 \frac{\eta(1-\beta_n)\beta_{n+1}}{1-\eta\beta_{n+1}} > \beta_{n+1} + c_2.$$

According to the definition (16) of w, we deduce that

$$(1-\eta\beta_{n+1}) L_1 z(\ell) = (L_2 z(\ell))^{1/\alpha} \pi_2(\ell)$$

and

$$\left(\frac{L_1 z(\ell)}{\pi_2^{1-\eta\beta_{n+1}}(\ell)} \right)' < 0.$$

Using the above monotonicity in (29), we see that

$$\begin{aligned}
w(\ell) &\geq w(\ell_n) + \varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)} \int_{\ell_n}^{\ell} \frac{L_1 z(s)}{a_2^{1/\alpha}(s)\pi_2(s)} \mathrm{d}s \\
&\geq \frac{\varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)}}{1-\eta\beta_{n+1}} \frac{L_1 z(\ell)}{\pi_2^{1-\eta\beta_{n+1}}(\ell)} \left(\pi_2^{1-\eta\beta_{n+1}}(\ell) - \pi_2^{1-\eta\beta_{n+1}}(\ell_n) \right) \\
&\geq \frac{c_1 \varepsilon_n k_n \beta \lambda^{\alpha(1-1/\varepsilon_n k_n)}}{1-\eta\beta_{n+1}} L_1 z(\ell) \\
&= c_1 \eta \beta_{n+1} \frac{1-\beta_n}{1-\eta\beta_{n+1}} L_1 z(\ell) \\
&> (\beta_{n+1} + c_2) L_1 z(\ell).
\end{aligned}$$

Then

$$(1-\beta_{n+1}-c_2) L_1 x(\ell) > (L_2 x(\ell))^{1/\alpha} \pi_2(\ell), \tag{31}$$

and

$$\left(\frac{L_1 x(\ell)}{\pi_2^{1-\beta_{n+1}-c_2}(\ell)} \right)' < 0, \tag{32}$$

which leads to the conclusion.
$(A_{n+1,2})$: Obviously, (32) also implies that $L_1z/\pi_2^{1-\beta_{n+1}} \to 0$ as $\ell \to \infty$, since otherwise

$$\frac{L_1z(\ell)}{\pi_2^{1-\beta_{n+1}-c_2}(\ell)} = \frac{L_1(z\ell)}{\pi_2^{1-\beta_{n+1}}(\ell)}\pi_2^{c_2}(\ell) \to \infty \text{ as } \ell \to \infty, \tag{33}$$

which is a contradiction.
$(A_{n+1,3})$: By utilizing that $(A_{n+1,1})$ and $(A_{n+1,2})$, as well as $L_1z/\pi_2^{1-\beta_{n+1}}$ as a decreasing function tending towards zero, we can derive:

$$\begin{aligned}
z(\ell) &= z(\ell''_n) + \int_{\ell''_n}^{\ell} \frac{L_1z(s)}{\pi_2^{1-\beta_{n+1}}(s)} \frac{\pi_2^{1-\beta_{n+1}}(s)}{a_1(s)} \mathrm{d}s \\
&\geq z(\ell''_n) + \frac{L_1z(\ell)}{\pi_2^{1-\beta_{n+1}}(\ell)} \int_{\ell''_n}^{\ell} \frac{\pi_2^{1-\beta_{n+1}}(s)}{a_1(s)} \mathrm{d}s \\
&= z(\ell''_n) + \frac{L_1z(\ell)}{\pi_2^{1-\beta_{n+1}}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_{n+1}}(s)}{a_1(s)} \mathrm{d}s - \frac{L_1z(\ell)}{\pi_2^{1-\beta_{n+1}}(\ell)} \int_{\ell_0}^{\ell''_n} \frac{\pi_2^{1-\beta_{n+1}}(s)}{a_1(s)} \mathrm{d}s \\
&> \frac{L_1z(\ell)}{\pi_2^{1-\beta_{n+1}}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_{n+1}}(s)}{a_1(s)} \mathrm{d}s \\
&\geq \varepsilon_{n+1}k_{n+1}\frac{\pi_{12}(\ell)}{\pi_2(\ell)}L_1z(\ell),
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{z(\ell)}{\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}}(\ell)}\right)' &= \frac{\varepsilon_{n+1}k_{n+1}\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}}(\ell)L_1z(\ell) - \pi_{12}^{1/\varepsilon_{n+1}k_{n+1}-1}(\ell)\pi_2(\ell)z(\ell)}{\varepsilon_{n+1}k_{n+1}a_1(\ell)\pi_{12}^{2/\varepsilon_{n+1}k_{n+1}}(\ell)} \\
&= \frac{\varepsilon_{n+1}k_{n+1}\pi_{12}(\ell)L_1z(\ell) - \pi_2(\ell)z(\ell)}{\varepsilon_{n+1}k_{n+1}a_1(\ell)\pi_{12}^{1/\varepsilon_{n+1}k_{n+1}+1}(\ell)} < 0,
\end{aligned}$$

for any $\varepsilon_n \in (0,1)$. The proof of this Lemma is complete. □

Corollary 4. *Assume that $\beta_i < 1$, $i = 0,1,2,\dots,n-1$, and $\beta_n \geq 1$. Then, $\Omega_2 = \varnothing$.*

Proof. This follows directly from

$$(1-\beta_n)L_1z(\ell) > (L_2z(\ell))^{1/\alpha}\pi_2(\ell),$$

and the fact that L_2 is positive. □

In view of the previous corollary and (27), the sequence $\{\beta_n\}$ given by (25) is increasing and bounded from above, i.e, there exists a limit

$$\lim_{n\to\infty} \beta_n = \beta_j \in (0,1),$$

satisfying the equation

$$\beta_j = \frac{\beta_* k_j \lambda_*^{\alpha(1-1/k_j)}}{1-\beta_j}, \tag{34}$$

where

$$k_j = \liminf_{\ell\to\infty} \frac{\pi_2^{\beta_j}(\ell)}{\pi_{12}(\ell)} \int_{\ell_0}^{\ell} \frac{\pi_2^{1-\beta_j}(s)}{a_1(s)} \mathrm{d}s.$$

Then, the next important resulting in the nonexistence of N_2-type solutions are direct.

Lemma 11. *Assume that $\lambda_* < \infty$ and (34) does not possess a root on $(0,1)$. Then, $\Omega_2 = \varnothing$.*

Corollary 5. *Assume that $\lambda_* < \infty$. If*

$$\beta_* > \max\left\{ \frac{\beta_j(1-\beta_j)\lambda_*^{\alpha(1/k_j - 1)}}{k_j} : 0 < \beta_j < 1 \right\}. \tag{35}$$

Then, $\Omega_2 = \varnothing$.

Lemma 12. *Assume that (2) holds. Furthermore, assume that there exists $\rho \in C^1([\ell_0,\infty),(0,\infty))$, such that*

$$\limsup_{\ell\to\infty} \int_{\ell_0}^{\ell} \left(\rho(s)q(s)\mathrm{B}^\alpha(\sigma(s),n)\left(\frac{\sigma(s)}{s}\right)^{2\alpha/\epsilon} - \frac{a_1^\alpha(s)(\rho'(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}\pi_2^\alpha(s)\rho^\alpha(s)} \right) \mathrm{d}s = \infty, \tag{36}$$

where $(\rho'(\ell))_+ = \max\{0,\rho'(\ell)\}$. Then, $\Omega_2 = \varnothing$.

Proof. Assume the contrary, that $x \in \Omega_2$. Now define

$$\mathrm{w}(\ell) = \rho(\ell)\frac{L_2 z(\ell)}{z^\alpha(\ell)},\ \ell \geq \ell_1, \tag{37}$$

then, $\mathrm{w}(\ell) > 0$ and

$$\begin{aligned}
\mathrm{w}'(\ell) &= \rho'(\ell)\frac{L_2 z(\ell)}{z^\alpha(\ell)} + \rho(\ell)\frac{L_3 z(\ell)}{z^\alpha(\ell)} - \alpha\rho(\ell)\frac{L_2 z(\ell)}{z^\alpha(\ell)}\frac{z'(\ell)}{z(\ell)} \\
&= \rho'(\ell)\frac{L_2 z(\ell)}{z^\alpha(\ell)} + \rho(\ell)\frac{L_3 z(\ell)}{z^\alpha(\ell)} - \alpha\rho(\ell)\frac{L_2 z(\ell)}{z^\alpha(\ell)}\frac{1}{a_1(\ell)}\frac{L_1 z(\ell)}{z(\ell)} \\
&\leq -\rho(\ell)q(\ell)\mathrm{B}^\alpha(\sigma(\ell),n)\frac{z^\alpha(\sigma(\ell))}{z^\alpha(\ell)} + \frac{\rho'(\ell)}{\rho(\ell)}\mathrm{w}(\ell) - \alpha\mathrm{w}(\ell)\frac{1}{a_1(\ell)}\frac{L_1 z(\ell)}{z(\ell)}.
\end{aligned}$$

Then, in view of (11) and $(A_{1,2})$-part of Lemma 7, we have

$$\begin{aligned}
w'(\ell) &\leq -\rho(\ell)q(\ell)\mathrm{B}^\alpha(\sigma(\ell),n)\frac{z^\alpha(\sigma(\ell))}{z^\alpha(\ell)} + \frac{\rho'(\ell)}{\rho(\ell)}\mathrm{w}(\ell) - \alpha\frac{\pi_2(\ell)}{a_1(\ell)}\mathrm{w}(\ell)\frac{(L_2 z)^{1/\alpha}}{z(\ell)} \\
&= -\rho(\ell)q(\ell)\mathrm{B}^\alpha(\sigma(\ell),n)\frac{z^\alpha(\sigma(\ell))}{z^\alpha(\ell)} + \frac{\rho'(\ell)}{\rho(\ell)}\mathrm{w}(\ell) - \frac{\alpha\pi_2(\ell)}{a_1(\ell)\rho^{1/\alpha}(\ell)}\mathrm{w}^{1+1/\alpha}(\ell).
\end{aligned}$$

Since $z > 0$, $L_1 z > 0$, and $L_2 z > 0$, then from Lemma 3 we obtain

$$\frac{z(\ell)}{z'(\ell)} \geq \frac{\varepsilon}{2}\ell.$$

By integrating the preceding inequality over the interval from $\tau(\ell)$ to ℓ, we obtain

$$\frac{z(\sigma(\ell))}{z(\ell)} \geq \left(\frac{\sigma(\ell)}{\ell}\right)^{2/\epsilon},$$

which implies that

$$\mathrm{w}'(\ell) \leq -\rho(\ell)q(\ell)\mathrm{B}^\alpha(\sigma(\ell),n)\left(\frac{\sigma(\ell)}{\ell}\right)^{2\alpha/\epsilon} + \frac{(\rho'(\ell))_+}{\rho(\ell)}\mathrm{w}(\ell) - \frac{\alpha\pi_2(\ell)}{a_1(\ell)\rho^{1/\alpha}(\ell)}\mathrm{w}^{1+1/\alpha}(\ell). \tag{38}$$

Setting

$$B = \frac{(\rho'(\ell))_+}{\rho(\ell)} \text{ and } A = \frac{\alpha\pi_2(\ell)}{a_1(\ell)\rho^{1/\alpha}(\ell)},$$

and using Lemma 1, we see that

$$\frac{(\rho'(\ell))_+}{\rho(\ell)}\mathrm{w}(\ell) - \frac{\alpha\pi_2(\ell)}{a_1(\ell)\rho^{1/\alpha}(\ell)}\mathrm{w}^{1+1/\alpha}(\ell) \le \frac{a_1^\alpha(\ell)(\rho'(\ell))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}\pi_2^\alpha(\ell)\rho^\alpha(\ell)}. \tag{39}$$

Thus, from (38) and (39), we obtain

$$\mathrm{w}'(\ell) \le -\left(\rho(\ell)q(\ell)\mathrm{B}^\alpha(\sigma(\ell),n)\left(\frac{\sigma(\ell)}{\ell}\right)^{2\alpha/\epsilon} - \frac{a_1^\alpha(\ell)(\rho'(\ell))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}\pi_2^\alpha(\ell)\rho^\alpha(\ell)}\right). \tag{40}$$

Integrating (40) from ℓ_1 to ℓ, we obtain

$$\int_{\ell_1}^{\ell}\left(\rho(s)q(s)\mathrm{B}^\alpha(\sigma(s),n)\left(\frac{\sigma(s)}{s}\right)^{2/\epsilon} - \frac{a_1^\alpha(s)(\rho'(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}\pi_2^\alpha(s)\rho^\alpha(s)}\right)\mathrm{d}s \le w(\ell_1),$$

for all large ℓ. This is a contradiction to (36). □

4. Convergence to Zero of Kneser Solutions

In this section, we establish certain conditions that guarantee the absence of Kneser solutions satisfying (N_1) within Category Ω_1.

Theorem 1. *If there exists a function $\zeta \in C([\ell_0,\infty),(0,\infty))$ satisfying $\sigma(\ell) < \zeta(\ell)$ and $\tau^{-1}(\zeta(\ell)) < \ell$, such that the differential equation*

$$\omega'(\ell) + \frac{1}{\mu}\frac{\tau_0}{\tau_0 + \mathrm{p}_0^\alpha}\phi(\ell)\pi_{12}^\alpha(\zeta(\ell),\sigma(\ell))\omega\left(\tau^{-1}(\zeta(\ell))\right) \le 0, \tag{41}$$

is oscillatory, then $\Omega_1 = \varnothing$.

Proof. Let $x \in \Omega_1$, say $x(\ell) > 0$ and $x(\sigma(\ell)) > 0$ for $\ell \ge \ell_1 \ge \ell_0$. This implies that

$$z > 0,\ L_1 z < 0,\ L_2 z > 0,\ \text{and } L_3 z \le 0. \tag{42}$$

From (1), we see that

$$\begin{aligned} 0 \quad &\ge \quad \frac{\mathrm{p}_0^\alpha}{\tau'(\ell)}\left(a_2(\tau(\ell))\left(\left(a_1(\tau(\ell))z'(\tau(\ell))\right)'\right)^\alpha\right)' + \mathrm{p}_0^\alpha q(\tau(\ell))x^\alpha(\sigma(\tau(\ell)))\\ &\ge \quad \frac{\mathrm{p}_0^\alpha}{\tau_0}L_3 z(\tau(\ell)) + \mathrm{p}_0^\alpha q(\tau(\ell))x^\alpha(\sigma(\tau(\ell)))\\ &= \quad \frac{\mathrm{p}_0^\alpha}{\tau_0}L_3 z(\tau(\ell)) + \mathrm{p}_0^\alpha q(\tau(\ell))x^\alpha(\tau(\sigma(\ell))). \end{aligned} \tag{43}$$

Combining (1) and (43), we obtain

$$\begin{aligned} 0 &\ge L_3 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0}L_3 z(\tau(\ell)) + q(\ell)x^\alpha(\sigma(\ell)) + \mathrm{p}_0^\alpha q(\tau(\ell))x^\alpha(\tau(\sigma(\ell)))\\ &\ge L_3 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0}L_3 z(\tau(\ell)) + \phi(\ell)(x^\alpha(\sigma(\ell)) + \mathrm{p}_0^\alpha x^\alpha(\tau(\sigma(\ell)))). \end{aligned}$$

Using Lemma (2), we obtain

$$0 \ge L_3 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0}L_3 z(\tau(\ell)) + \frac{1}{\mu}\phi(\ell)(x(\sigma(\ell)) + \mathrm{p}_0 x(\tau(\sigma(\ell))))^\alpha. \tag{44}$$

From the definition of z, we have

$$z(\sigma(\ell)) = x(\sigma(\ell)) + \mathrm{p}(\sigma(\ell))x(\tau(\sigma(\ell))) \leq x(\sigma(\ell)) + \mathrm{p}_0 x(\tau(\sigma(\ell))).$$

By using the latter inequality in (44), we find

$$0 \geq L_3 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0} L_3 z(\tau(\ell)) + \frac{1}{\mu}\phi(\ell) z^\alpha(\sigma(\ell)).$$

That is

$$\left(L_2 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0} L_2 z(\tau(\ell))\right)' + \frac{1}{\mu}\phi(\ell) z^\alpha(\sigma(\ell)) \leq 0. \tag{45}$$

However, it can be deduced from the monotonicity of $L_2 z(\ell)$ that

$$\begin{aligned} -L_1 z(\varrho) &\geq L_1 z(\varsigma) - L_1 z(\varrho) = \int_\varrho^\varsigma (L_1 z(s))' \mathrm{d}s = \int_\varrho^\varsigma \frac{L_2^{1/\alpha} z(s)}{a_2^{1/\alpha}(s)} \mathrm{d}s \\ &\geq L_2^{1/\alpha} z(\varsigma) \int_\varrho^\varsigma \frac{1}{a_2^{1/\alpha}(s)} \mathrm{d}s = L_2^{1/\alpha} z(\varsigma) \pi_2(\varsigma, \varrho). \end{aligned} \tag{46}$$

Integrating (46) from ϱ to ς, and using (42), we obtain

$$z(\varrho) \geq L_2^{1/\alpha} z(\varsigma) \pi_{12}(\varsigma, \varrho). \tag{47}$$

Thus, we have

$$z(\sigma(\ell)) \geq L_2^{1/\alpha} z(\zeta(\ell)) \pi_{12}(\zeta(\ell), \sigma(\ell)),$$

which, by virtue of (45), yields that

$$\left(L_2 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0} L_2 z(\tau(\ell))\right)' + \frac{1}{\mu}\phi(\ell) \pi_{12}^\alpha(\zeta(\ell), \sigma(\ell)) L_2 z(\zeta(\ell)) \leq 0. \tag{48}$$

Now, set

$$\omega(\ell) = L_2 z(\ell) + \frac{\mathrm{p}_0^\alpha}{\tau_0} L_2 z(\tau(\ell)) > 0.$$

From the fact that $L_2 z(\ell)$ is non-increasing, we have

$$\omega(\ell) \leq L_2 z(\tau(\ell)) \left(1 + \frac{\mathrm{p}_0^\alpha}{\tau_0}\right),$$

or equivalently,

$$L_2 z(\zeta(\ell)) \geq \frac{\tau_0}{\tau_0 + \mathrm{p}_0^\alpha} \omega\left(\tau^{-1}(\zeta(\ell))\right). \tag{49}$$

Using (49) in (48), we show that ω is a positive solution of the differential inequality

$$\omega'(\ell) + \frac{1}{\mu} \frac{\tau_0}{\tau_0 + \mathrm{p}_0^\alpha} \phi(\ell) \pi_{12}^\alpha(\zeta(\ell), \sigma(\ell)) \omega\left(\tau^{-1}(\zeta(\ell))\right) \leq 0.$$

Considering ([35], Theorem 1), we can deduce that (41) also possesses a positive solution, which contradicts our previous assertion. Thus, we can conclude that the proof is now fully established. □

Corollary 6. *If there exists a function $\zeta \in C([\ell_0, \infty), (0, \infty))$ satisfying $\sigma(\ell) < \zeta(\ell)$ and $\tau^{-1}(\zeta(\ell)) < \ell$, such that*

$$\liminf_{\ell \to \infty} \int_{\tau^{-1}(\zeta(\ell))}^{\ell} \phi(s) \pi_{12}^\alpha(\zeta(s), \sigma(s)) \mathrm{d}s > \frac{\mu(\tau_0 + \mathrm{p}_0^\alpha)}{\tau_0 \mathrm{e}}, \tag{50}$$

then, $\Omega_1 = \varnothing$.

Theorem 2. *If there exists a function* $\delta \in C([\ell_0,\infty), (0,\infty))$ *satisfying* $\delta(\ell) < \ell$, *and* $\sigma(\ell) < \tau(\delta(\ell))$, *such that*

$$\limsup_{\ell\to\infty} \pi_{12}^{\alpha}(\tau(\delta(\ell)),\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s > \frac{\mu(\tau_0 + \mathrm{p}_0^{\alpha})}{\tau_0}, \tag{51}$$

then, $\Omega_1 = \varnothing$.

Proof. Using the same method as demonstrated in the proof of Theorem 1, we obtain the following inequality:

$$0 \geq \left(L_2 z(\ell) + \frac{\mathrm{p}_0^{\alpha}}{\tau_0} L_2 z(\tau(\ell)) \right)' + \frac{1}{\mu}\phi(\ell) z^{\alpha}(\sigma(\ell)).$$

By integrating the previous inequality from $\delta(\ell)$ to ℓ, and considering the fact that z is a decreasing function, we derive:

$$\begin{aligned} L_2 z(\delta(\ell)) + \frac{\mathrm{p}_0^{\alpha}}{\tau_0} L_2 z(\tau(\delta(\ell))) \quad &\geq \quad L_2 z(\ell) + \frac{\mathrm{p}_0^{\alpha}}{\tau_0} L_2 z(\tau(\ell)) + \frac{1}{\mu} z^{\alpha}(\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s \\ &\geq \quad \frac{1}{\mu} z^{\alpha}(\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s. \end{aligned}$$

Since $\tau(\delta(\ell)) < \tau(\ell)$, and $L_2 z(\ell)$ is non-increasing, we have

$$L_2 z(\tau(\delta(\ell))) \left(1 + \frac{\mathrm{p}_0^{\alpha}}{\tau_0} \right) \geq \frac{1}{\mu} z^{\alpha}(\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s. \tag{52}$$

By using (47) with $\varsigma = \tau(\delta(\ell))$ and $\varrho = \sigma(\ell)$ in (52), we obtain

$$L_2 z(\tau(\delta(\ell))) \left(1 + \frac{\mathrm{p}_0^{\alpha}}{\tau_0} \right) \geq \frac{1}{\mu} L_2 z(\tau(\delta(\ell))) \pi_{12}^{\alpha}(\tau(\delta(\ell)),\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s.$$

That is

$$\frac{\tau_0 + \mathrm{p}_0^{\alpha}}{\tau_0} \geq \frac{1}{\mu} \pi_{12}^{\alpha}(\tau(\delta(\ell)),\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s.$$

Next, we calculate the lim sup for both sides of the preceding inequality, which leads to a contradiction with (51). This concludes the proof. □

5. Oscillation Theorems

In this section, we are prepared to present the main results of this paper. By combining the results from the preceding two sections, we can readily derive the following theorems without providing proof.

Theorem 3. *Assume that* $\beta_* \geq 1$, *and either (50) or (51) holds. Then, (1) is oscillatory.*

Theorem 4. *Assume that* $\beta_* > 0$, $\lambda_* = \infty$, *and either (50) or (51) holds. Then, (1) is oscillatory.*

Theorem 5. *Assume that* $\beta_i < 1$, $i = 0,1,2,\ldots,n-1$, *and* $\beta_n \geq 1$ *and either (50) or (51) holds. Then, (1) is oscillatory.*

Theorem 6. *Assume that* $\lambda_* < \infty$, *(35), and either (50) or (51) holds. Then, (1) is oscillatory.*

Theorem 7. *Assume that (36) and either (50) or (51) holds. Then, (1) is oscillatory.*

In the following, we provide an example that supports and illustrates our results.

Example 1. *Consider*

$$\left(\left((x(\ell)+\mathrm{p}_0x(\tau_0\ell))''\right)^5\right)'+\frac{q_0}{\ell^{11}}x^5(\sigma_0\ell)=0, \tag{53}$$

where $0\leq \mathrm{p}_0<1$, *and* $\tau_0,\sigma_0\in(0,1)$. *Clearly,*

$$a_1(\ell)=1,\ a_2(\ell)=1,\ \pi_1(\ell)\sim\ell,\ \pi_2(\ell)\sim\ell,\ \pi_{12}(\ell)\sim\ell^2/2.$$

We can calculate:

$$\lambda_*=\liminf_{\ell\to\infty}\frac{\pi_{12}(\ell)}{\pi_{12}(\sigma(\ell))}=\frac{1}{\sigma_0^2},$$

$$\mathrm{p}_1(\ell;n)=[1-\mathrm{p}_0]\sum_{k=0}^{n}\mathrm{p}_0^{2k}\tau_0^{4k},$$

$$\widehat{\mathrm{p}}_1(\ell,n)=\left[\mathrm{p}_0\tau_0^2-1\right]\sum_{k=1}^{n}\frac{1}{\mathrm{p}_0^{2k}},$$

and

$$\mathrm{B}(\ell,n)=\mathrm{B}_0=\begin{cases}p_1(\ell;n) & \text{for } p_0<1,\\ \widehat{\mathrm{p}}_1(\ell;n) & \text{for } p_0>1/\tau_0^2.\end{cases}$$

Then

$$\begin{aligned}\beta_* &= \liminf_{\ell\to\infty}\frac{1}{\alpha}a_2^{1/\alpha}(\ell)\pi_{12}^{\alpha}(\sigma(\ell))\pi_2(\ell)q(\ell)\mathrm{B}_0^{\alpha}(\sigma(\ell),n)\\ &= \liminf_{\ell\to\infty}\frac{1}{5}\frac{\sigma_0^{10}\ell^{10}}{2^5}\ell\frac{q_0}{\ell^{11}}\mathrm{B}_0^5=\frac{1}{160}\sigma_0^{10}q_0\mathrm{B}_0^5.\end{aligned}$$

For $\beta_*\geq 1$, *we have*

$$q_0>\frac{160}{\sigma_0^{10}\mathrm{B}_0^5}. \tag{54}$$

Now, for $\rho(\ell)=\ell^{\nu}$, *where* $\nu\geq 10$, *condition (36) leads to*

$$\begin{aligned}&\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left(\rho(s)q(s)\mathrm{B}^{\alpha}(\sigma(s),n)\left(\frac{\sigma(s)}{s}\right)^{2\alpha/\epsilon}-\frac{a_1^{\alpha}(s)(\rho'(s))_+^{\alpha+1}}{(\alpha+1)^{\alpha+1}\pi_2^{\alpha}(s)\rho^{\alpha}(s)}\right)\mathrm{d}s\\ =&\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left(s^{\nu}\frac{q_0}{s^{11}}\mathrm{B}_0^5\sigma_0^{10/\epsilon}-\frac{1}{6^6}\frac{\nu^6s^{6\nu-6}}{s^5s^{5\nu}}\right)\mathrm{d}s\\ =&\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left(q_0\mathrm{B}_0^5\sigma_0^{10/\epsilon}-\frac{\nu^6}{6^6}\right)s^{\nu-11}\mathrm{d}s=\infty,\end{aligned}$$

Which is satisfied when

$$q_0>\frac{\nu^6}{6^6\mathrm{B}_0^5\sigma_0^{10/\epsilon}}. \tag{55}$$

Condition (50) leads to:

$$\begin{aligned}\liminf_{\ell\to\infty}\int_{\tau^{-1}(\zeta(\ell))}^{\ell}\phi(s)\pi_{12}^{\alpha}(\zeta(s),\sigma(s))\mathrm{d}s &= \liminf_{\ell\to\infty}\int_{\tau_0^{-1}\zeta_0\ell}^{\ell}\frac{q_0}{s^{11}}\frac{\left(\zeta_0^2-\sigma_0^2\right)^5}{2^5}s^{10}\mathrm{d}s\\ &= \frac{1}{32}q_0\left(\zeta_0^2-\sigma_0^2\right)^5\ln\frac{\tau_0}{\zeta_0}.\end{aligned}$$

which is satisfied when:

$$q_0 > \frac{32\mu\left(\tau_0 + \mathrm{p}_0^{\alpha}\right)}{\tau_0 \mathrm{e}\left(\zeta_0^2 - \sigma_0^2\right)^5 \ln\frac{\tau_0}{\zeta_0}} \tag{56}$$

Condition (51) leads to:

$$\begin{aligned}
\limsup_{\ell\to\infty} \pi_{12}^{\alpha}(\tau(\delta(\ell)),\sigma(\ell)) \int_{\delta(\ell)}^{\ell} \phi(s)\mathrm{d}s &= \limsup_{\ell\to\infty} \pi_{12}^{5}(\tau_0\delta_0\ell,\sigma_0\ell) \int_{\delta_0\ell}^{\ell} \frac{q_0}{s^{11}}\mathrm{d}s \\
&= \limsup_{\ell\to\infty} \frac{\left(\tau_0^2\delta_0^2 - \sigma_0^2\right)^5\left(1-\delta_0^{10}\right)\ell^{10}}{320\delta_0^{10}} \frac{q_0}{\ell^{10}} \\
&= \frac{\left(\tau_0^2\delta_0^2 - \sigma_0^2\right)^5\left(1-\delta_0^{10}\right)}{320\delta_0^{10}} q_0,
\end{aligned}$$

which is satisfied when:

$$q_0 > \frac{320\left(\tau_0 + \mathrm{p}_0^{5}\right)\delta_0^{10}}{\left(\tau_0^2\delta_0^2 - \sigma_0^2\right)^5\left(1-\delta_0^{10}\right)}. \tag{57}$$

Now, by applying conditions (54)–(57), *we can show that Theorems (3) and (7) exhibit oscillatory behavior. This can be confirmed by assigning particular values to (53).*

Example 2. *Consider*

$$\left(x(\ell) + 0.5x(0.9\ell)\right)''' + \frac{q_0}{\ell^3} x(0.5\ell) = 0. \tag{58}$$

Clearly,

$$\lambda_* = 4,$$

$$\mathrm{p}_1(\ell;10) = (1-0.5)\sum_{k=0}^{10} (0.5)^{2k}(0.9)^{4k} = 0.5981,$$

and

$$\mathrm{B}(\ell,10) = \mathrm{B}_0 = \mathrm{p}_1(\ell;10) = 0.5981.$$

Then

$$\beta_* = \liminf_{\ell\to\infty} \frac{(0.5)^2\ell^2}{2} \ell \frac{q_0}{\ell^3}(0.598\,1) = 0.07476 q_0.$$

For $\beta_* \geq 1$, we have

$$q_0 > 13.376.$$

Conditions (36) and (51) are satisfied when

$$q_0 > 26.751,\ \rho(\ell) = \ell^2,\ \epsilon = 0.5$$

and

$$q_0 > 22.274,\ \delta_0 = 0.7, \tag{59}$$

respectively. Thus, from Theorems 3 and 7, we conclude that (58) is oscillatory.

6. Conclusions

This paper has studied the oscillatory behavior of a quasi-linear NDE of the third order. Through our research efforts, we have significantly enhanced the understanding of the relationship between the solution, x, and the corresponding function, z. This improvement has led to the derivation of improved preliminary results, which play a crucial role in excluding positive solutions for the studied equation. Building upon these refined preliminary results, we have developed novel criteria for determining the nature of the solutions, whether they exhibit oscillatory behavior or tend towards zero. These criteria contribute to a deeper comprehension of the dynamic behavior of the systems described by these equations. In the future, an intriguing avenue for research involves broadening the scope of this study to encompass NDEs of higher orders.

Author Contributions: Conceptualization, F.M., O.M., G.A. and H.E.-M.; methodology, F.M., O.M., G.A. and H.E.-M.; investigation, F.M., O.M., G.A. and H.E.-M.; writing—original draft, F.M. and G.A.; writing—review and editing, O.M. and H.E.-M. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R45), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2023R45), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jayaraman, G.; Padmanabhan, N.; Mehrotra, R. Entry flow into a circular tube of slowly varying cross-section. *Fluid Dyn. Res.* **1986**, *1*, 131. [CrossRef]
2. Vreeke, S.A.; Sandquist, G.M. Phase space analysis of reactor kinetics. *Nucl. Sci. Eng.* **1970**, *42*, 295–305. [CrossRef]
3. Gregus, M. *Third Order Linear Differential Equations*; Springer Science & Business Media: Berlin, Germany, 2012; Volume 22.
4. Villagomez, A.N.; Muoz, F.M.; Peterson, R.L.; Colbert, A.M.; Gladstone, M.; MacDonald, B. Collaboration Neurodevelopmental Delay Working Group. Neurodevelopmental delay: Case definition and guidelines for data collection, analysis, and presentation of immunization safety data. *Vaccine* **2019**, *37*, 7623. [CrossRef]
5. Liu, M.; Dassios, I.; Tzounas, G.; Milano, F. Stability analysis of power systems with inclusion of realistic-modeling WAMS delays. *IEEE Trans. Power Syst.* **2018**, *34*, 627–636. [CrossRef]
6. Milano, F.; Dassios, I. Small-signal stability analysis for non-index 1 Hessenberg form systems of delay differential-algebraic equations. *IEEE Trans. Circuits Syst. Regul. Pap.* **2016**, *63*, 1521–1530. [CrossRef]
7. Agarwal, R.P.; Berezansky, L.; Braverman, E.; Domoshnitsky, A. *Nonoscillation Theory of Functional Differential Equations with Applications*; Springer Science and Business Media: Berlin, Germany, 2012.
8. Norkin, S.B. *Second Order Differential Equations with Retarded Argument*; Nauk: Moscow, Russia, 1965.
9. Braun, M. *Qualitative Theory of Differential Equations: Differential Equations and Their Applications*; Springer: New York, NY, USA, 1993.
10. Hale, J.K. *Functional Differential Equation*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1971.
11. Agarwal, R.P.; Bohner, M.; Li, W.T. *Nonoscillation and Oscillation: Theory for Functional Differential Equations*; Marcel Dekker, Inc.: New York, NY, USA, 2004.
12. Bainov, D.D.; Mishev, D.P. *Oscillation Theory for Neutral Differential Equations with Delay*; CRC Press: Boca Raton, FL, USA, 1991.
13. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations*; Oxford University Press: New York, NY, USA, 1991.
14. Moaaz, O.; Masood, F.; Cesarano, C.; Alsallami, S.A.M.; Khalil, E.M.; Bouazizi, M.L. Neutral Differential Equations of Second-Order: Iterative Monotonic Properties. *Mathematics* **2022**, *10*, 1356. [CrossRef]
15. Baculíková, B.; Džurina, J. Oscillation theorems for higher order neutral differential equations. *Appl. Math. Comput.* **2012**, *219*, 3769–3778. [CrossRef]
16. Jadlovská, I.; Džurina, J.; Graef, J.R.; Grace, S.R. Sharp oscillation theorem for fourth-order linear delay differential equations. *J. Inequal. Appl.* **2022**, *2022*, 122. [CrossRef]
17. Masood, F.; Moaaz, O.; Santra, S.S.; Fernandez-Gamiz, U.; El-Metwally, H.A. Oscillation theorems for fourth-order quasi-linear delay differential equations. *AIMS Math.* **2023**, *8*, 16291–16307. [CrossRef]
18. Alnafisah, Y.; Masood, F.; Muhib, A.; Moaaz, O. Improved Oscillation Theorems for Even-Order Quasi-Linear Neutral Differential Equations. *Symmetry* **2023**, *15*, 1128. [CrossRef]
19. Baculikova, B.; Rani, B.; Selvarangam, S.; Thandapani, E. Properties of Kneser's solution for half-linear third order neutral differential equations. *Acta Math. Hung.* **2017**, *152*, 525–533. [CrossRef]
20. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I. Oscillation criteria for third-order delay differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 330. [CrossRef]
21. Chatzarakis, G.E.; Grace, S.R. Third-order nonlinear differential equations with nonlinear neutral terms. *Funct. Differ. Equ.* **2020,** *27*, 3–13. [CrossRef]
22. Hanan, M. Oscillation criteria for third-order linear differential equations. *Pac. J. Math.* **1961**, *11*, 919–944. [CrossRef]
23. Saker, S.H.; Džurina, J. On the oscillation of certain class of third-order nonlinear delay differential equations. *Math. Bohem.* **2010**, *135*, 225–237. [CrossRef]

24. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math.* **2011**, *47*, 181–199.
25. Džurina, J.; Grace, S.R.; Jadlovská, I. On nonexistence of Kneser solutions of third-order neutral delay differential equations. *Appl. Math. Lett.* **2019,** *88*, 193–200. [CrossRef]
26. Jadlovská, I.; Chatzarakis, G.E.; Džurina, J.; Grace, S.R. On sharp oscillation criteria for general third-order delay differential equations. *Mathematics* **2021**, *9*, 1675. [CrossRef]
27. Chatzarakis, G.E.; Grace, S.R.; Jadlovská, I.; Li, T.; Tunç, E. Oscillation criteria for third-order Emden–Fowler differential equations with unbounded neutral coefficients. *Complexity* **2019**, *2019*, 5691758. [CrossRef]
28. Karpuz, B.; Ocalan, O.; Ozturk, S. Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations. *Glasg. Math. J.* **2010**, *52*, 107–114. [CrossRef]
29. Xing, G.; Li, T.; Zhang, C. Oscillation of higher-order quasi-linear neutral differential equations. *Adv. Differ. Equ.* **2011**, *2011*, 45. [CrossRef]
30. Zhang, C.; Agarwal, R.; Bohner, M.; Li, T. New results for oscillatory behavior of even-order half-linear delay differential equations. *Appl. Math. Lett*. **2013**, *26*, 179–183. [CrossRef]
31. Hilderbrandt, T.H. *Introduction to the Theory of Integration*; Academic Press: New York, NY, USA, 1963.
32. Kiguradze, I.T.; Chanturiya, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1993.
33. Kusano, T.; Naito, M. Comparison theorems for functional differential equations with deviating arguments. *J. Math. Soc. Jpn.* **1981,** *33*, 509–532. [CrossRef]
34. Moaaz, O.; Cesarano, C.; Almarri, B. An Improved Relationship between the Solution and Its Corresponding Function in Fourth-Order Neutral Differential Equations and Its Applications. *Mathematics* **2023**, *11*, 1708. [CrossRef]
35. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **1981**, *36*, 168–178. [CrossRef]