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Analysis of the Burgers–Huxley Equation Using the Nondimensionalisation Technique: Universal Solution for Dirichlet and Symmetry Boundary Conditions

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Abstract: The Burgers–Huxley equation is important because it involves the phenomena of accumulation, drag, diffusion, and the generation or decay of species, which are common in various problems in science and engineering, such as heat transmission, the diffusion of atmospheric contaminants, etc. On the other hand, the mathematical technique of nondimensionalisation has proven to be very useful in the appropriate grouping of the variables involved in a physical–chemical phenomenon and in obtaining universal solutions to different complex engineering problems. Therefore, a deep analysis using this technique of the Burgers–Huxley equation and its possible boundary conditions can facilitate a common understanding of these problems through the appropriate grouping of variables and propose common universal solutions. Thus, in this case, the technique is applied to obtain a universal solution for Dirichlet and symmetric boundary conditions. The validation of the methodology is carried out by comparing different cases, where the coefficients or the value of the boundary condition are varied, with the results obtained through a numerical simulation. Furthermore, one of the cases presented presents a boundary condition that changes at a certain time. Finally, after applying the technique, it is studied which phenomenon is predominant, concluding that from a certain value diffusion predominates, with the rest being practically negligible.

Keywords: nondimensionalisation; universal solution; mathematical modelling; numerical simulation; engineering science; ordinary differential equations

MSC: 00A73; 00A69; 00A79

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1. Introduction

Many engineering problems, such as heat transmission, fluid mechanics, contaminant emission, chloride diffusion in concrete, etc., involve the phenomena of diffusion, accumulation, generation, or the decay of species, and drag. In this sense, the Burgers–Huxley equation is a general equation that encompasses all these phenomena [1–28].

Thus, it is necessary, on the one hand, to obtain a universal solution that allows this equation to be easily solved, and on the other, to study the influence of the variables involved. In this way, the nondimensionalisation technique allows us to both obtain universal solutions and study the variables that have been grouped to form monomials [29–31].

The application of the nondimensionalisation technique to engineering problems formulated in ordinary differential equations is well known because it allows one to obtain dimensionless groups by grouping the variables, allowing us to know both the universal solution of the problem and the influence of each variable [31]. This procedure has recently been applied to different engineering problems, such as the diffusion of chlorides in concrete or soil consolidation, and the basis of the methodology has been explained in detail [29,31].

The objective of this study is to apply the nondimensionalisation methodology to the Burgers–Huxley equation to obtain, on the one hand, a universal solution to the problem posed. In this case, the symmetry condition will be applied to one of the boundary conditions, and to the other boundary condition, a constant value of the variable will be applied (Dirichlet condition). On the other hand, another objective is to study the weight of the variables in the problem. For the construction of the universal curves, the Network Simulation Method was used, which has demonstrated its effectiveness in solving this problem, as well as other engineering problems [32–35].

Thus, the Burgers–Huxley equation is an ordinary differential equation that is widely used in physics, biology, economics, etc., and includes terms such as drag, accumulation, generation or decay, and diffusion. This equation has the following form [2,3,32]:

$$\frac{du}{dt} + \alpha u^{\delta}\frac{du}{dx} - \zeta\frac{d^2u}{dx^2} - \beta u\left(1 - u^{\delta}\right)\left(\varepsilon u^{\delta} - \gamma\right) = 0 \; t \geq 0 \quad (1)$$

where u is the variable, such as concentration, temperature, etc.; x is the distance; t is the time; and, finally, α, β, γ, δ, ε, and ζ are coefficients. Thus, the first addend of the equation is associated with accumulation, the second with drag (coefficients α and δ), the third with diffusion phenomena (coefficient ζ), and, finally, the fourth with the generation or decay of species (coefficients β, γ, δ, and ε).

This article is structured as follows: the introduction and the Burgers–Huxley equation are presented in Section 1. In the next section, the procedure for the nondimensionalisation technique is detailed so that it can be applied to the Burgers–Huxley equation in the same section. The results used to validate the proposed methodology after the application of the nondimensionalisation technique are presented in Section 3. Finally, Section 4 presents the conclusions of this study.

2. Nondimensionalisation Technique and Its Application to the Burgers–Huxley Equation

The correct steps for applying the nondimensionalisation technique have been described in the literature, which have recently included those necessary to obtain universal solutions. Furthermore, several articles have focused on defining the behaviour of monomials based on the values obtained, establishing criteria in which some monomials can have little influence on the problem compared with others that govern it. Thus, as a summary, the following steps must be applied [30,31]:

(i) Choice of references

For the correct choice of references, a deep understanding of the problem is necessary because they may appear explicitly in the problem or may be hidden. Furthermore, the values selected for the references are related to each other through a physical interval (temporal or spatial) of the independent variable, limiting the dimensionless variables to the interval of values [0–1]. When the solution to the problem is asymptotic, references close to the limit are taken as the dependent variable, for example, 99% or 90% of the maximum value of the variable. Thus, there was no significant modification in the range [0–1] of the dimensionless variables.

(ii) Dimensionless variables and dimensionless governing equations

The divisions between the dimensional variables and their references are the dimensionless variables, e.g., $x' = \frac{x}{L}$, where x' is the dimensionless variable of the distance, x is the distance variable, and L is the reference, which in this case is the total length of the medium. Thus, these dimensionless variables are introduced into the governing equations of the problem and transform them into dimensionless equations. Each addend of these equations is formed by two factors: one that involves the grouping of boundary conditions, problem parameters, and/or references, and another with dimensionless variables and their changes, which can be assumed to be of the order of unity. Thus, based on this hypothesis, the first factor, known as the coefficient, must also be of the same order of magnitude.

(iii) Dimensionless groups

The dimensionless groups, or monomials, are the relation between the coefficients mentioned in the previous step, which will be at most as many addends as the dimensionless equation has minus one. Because some groups may be expressed as a combination of other groups with multiplications or divisions, or the same group may appear in more than one equation when the problem is a system of coupled equations, the final number of groups may be reduced. Additionally, the groups can be manipulated such that each unknown appears in a single group.

(iv) The existence of m groups with a different unknown each one (π_u) and n groups without unknowns (π_w)

The solution for each unknown is explicitly expressed as a function of groups that do not contain unknowns. That is, in the form

$$\pi_{u,i} = \Psi_i(\pi_{w,1}, \pi_{w,2}, \ldots, \pi_{w,n}) \text{ where } 1 \leq i \leq m \quad (2)$$

where Ψ_i is an arbitrary function of the n π_w groups. When the groups are of unit order of magnitude, the arbitrary function will also be of this order of magnitude.

(v) Functionals

The Ψ_i functionals presented in the previous step were obtained by adjusting two monomials or dimensionless groups, keeping the rest at a constant value, as will be shown in the resolution of the problem posed in this article.

(vi) Universal solutions

The universal solution is obtained by representing the dimensionless variables defined above, which, as indicated by their own definition, are in the range of values [0–1].

The information provided above is very important because it allows us to both obtain universal solutions to the problem posed and know the influence of the variables on it. Thus, if we apply this methodology to the Burgers–Huxley equation, we can obtain its universal solution and study the influence of its variables.

The study problem must be defined before applying the nondimensionalisation methodology. In this case, we have a variable u that is found in a medium of length L and is subject to the accumulation, drag, generation or decay, and diffusion phenomena. Regarding the boundary conditions, on the left side, a Dirichlet condition is applied with a constant value of u, u_{ext}, and on the right side, there is a Neumann condition to apply with a symmetry condition [32]. Finally, the variable u can present initial values in the medium, u_{ini}, as shown in Figure 1.

Figure 1. Description of the study problem. Geometry, boundary, and initial conditions.

To apply the steps specified above for correct nondimensionalisation to Equation (1), the references must first be defined, and the dimensionless variables must be established (steps i and ii). Thus, dimensionless variables are the division of the variable with its reference.

$$u' = \frac{u - u_{ini}}{u_{ext} - u_{ini}} \; x' = \frac{x}{L} \; t' = \frac{t}{\tau} \tag{3}$$

The reference chosen for the dimensionless variable u' is the difference between the final value it can reach, that is, the value in the boundary condition, u_{ext}, and its initial value, u_{ini}. For the dimensionless variable x', the total length of the medium, L, was chosen. Finally, because the problem has an asymptotic tendency, for the dimensionless time variable t', the time at which 99% of the maximum value of u is reached, that is, 99% of u_{ext}, was chosen. The dimensionless variables were then introduced into the governing equation and the following dimensionless form was obtained:

$$\begin{aligned} &\left[\frac{u_{ext}-u_{ini}}{\tau}\right]\frac{du'}{dt'} + \alpha\left((u_{ext}-u_{ini})u' + u_{ini}\right)^{\delta}\frac{u_{ext}-u_{ini}}{L}\frac{du'}{dx'} - \left[\zeta\frac{u_{ext}-u_{ini}}{L^2}\right]\frac{d^2u'}{dx'^2} - \beta\varepsilon\left((u_{ext}-u_{ini})u' + u_{ini}\right)^{\delta+1} + \\ &\beta\gamma\left((u_{ext}-u_{ini})u' + u_{ini}\right) + \beta\varepsilon\left((u_{ext}-u_{ini})u' + u_{ini}\right)^{2\delta+1} - \beta\gamma\left((u_{ext}-u_{ini})u' + u_{ini}\right)^{\delta+1} = 0. \end{aligned} \tag{4}$$

The dimensionless equation gives us seven coefficients:

$$\frac{u_{ext} - u_{ini}}{\tau},\; \frac{\alpha u_{ext}^{\delta}(u_{ext} - u_{ini})}{L},\; \zeta\frac{u_{ext} - u_{ini}}{L^2},\; \beta\varepsilon u_{ext}^{\delta+1},\; \beta\gamma u_{ext},\; \beta\varepsilon u_{ext}^{2\delta+1},\; \beta\gamma u_{ext}^{\delta+1}.$$

As some coefficients of species generation or decay phenomena are contained in the others, there are finally five coefficients:

$$\frac{\mathrm{u_{ext}} - \mathrm{u_{ini}}}{\tau},\frac{\alpha \mathrm{u}_{\mathrm{ext}}^{\delta}(\mathrm{u_{ext}} - \mathrm{u_{ini}})}{\mathrm{L}},\zeta\frac{\mathrm{u_{ext}} - \mathrm{u_{ini}}}{\mathrm{L}^2},\beta\varepsilon \mathrm{u}_{\mathrm{ext}}^{2\delta+1},\beta\gamma \mathrm{u}_{\mathrm{ext}}^{\delta+1}.$$

These give rise to four dimensionless monomials:

$$\pi_1 = \frac{\zeta\tau}{L^2},\; \pi_2 = \frac{\zeta}{\alpha L u_{ext}^{\delta}},\; \pi_3 = \frac{\beta\gamma L u_{ext}}{\alpha(u_{ext} - u_{ini})},\; \pi_4 = \frac{\varepsilon}{\gamma}u_{ext}^{\delta}.$$

The first monomial, π_1, is typical of problems involving diffusion phenomena, relating time to the distance squared and the diffusion coefficient, in this case, coefficient ζ. This monomial is found in the literature on heat transmission problems, known as the Fourier number [9], chloride diffusion [29], etc. The second monomial, π_2, indicates the relationship between diffusion and drag phenomena. The third, π_3, relates to the drag phenomena and the generation or decay of species. Finally, π_4 relates the coefficients of generation and decay of the species. It should be noted that when β is zero, the monomial π_3 is zero, and the fourth monomial π_4 has no influence, since there would be no generation or decay of the species.

Applying the π theorem, $\pi_1 = \Psi(\pi_2, \pi_3, \pi_4)$, the same characteristic time can be expressed from each of the equations in the form

$$\tau = \frac{L^2}{\zeta}\Psi\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}, \frac{\beta\gamma L u_{ext}}{\alpha(u_{ext} - u_{ini})}, \frac{\varepsilon}{\gamma}u_{ext}^{\delta}\right). \tag{5}$$

where Ψ is an unknown function of the arguments.

If we apply step (v) of the nondimensionalisation technique, the value of the functional can be obtained. In this case, because we depend on three monomials, we adjust the monomials π_1 and π_2, keeping the rest, π_3 and π_4, with constant values. To perform the adjustment, the problem was simulated using the Network Simulation Method to determine the time necessary for the concentration at the right-hand end, $x = L$, to be 99% of the u_{ext} value, as defined above. Thus, Equations (6)–(10) show the fits for different values of π_3 and π_4.

$$\pi_3 = 0 \ \forall \ \pi_4 \ \pi_1 = 1.974 + 0.8186\pi_2{}^{-1.105} \ \mathrm{R}^2 = 0.9987 \ \tau = \frac{L^2}{\zeta}\left(1.974 + 0.8186\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}\right)^{-1.105}\right) \tag{6}$$

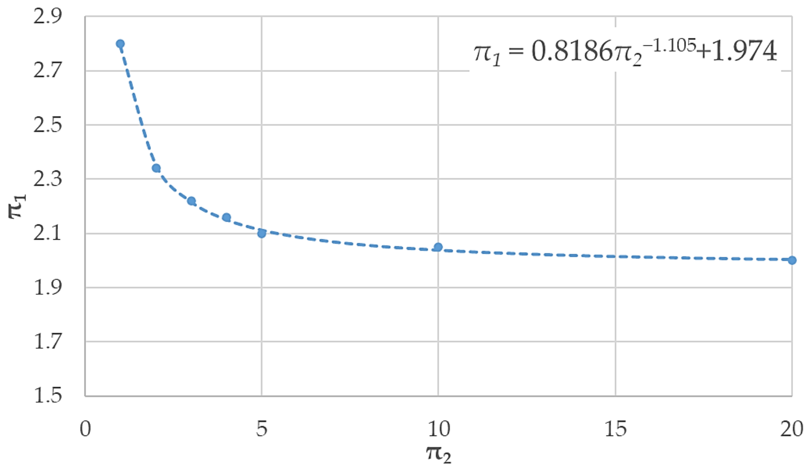
$$\pi_3 = 1 \ \pi_4 = 1 \ \pi_1 = 1.974 + 0.6781\pi_2{}^{-1.13} \ \mathrm{R}^2 = 0.9978 \ \tau = \frac{L^2}{\zeta}\left(1.974 + 0.6781\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}\right)^{-1.13}\right) \tag{7}$$

$$\pi_3 = 1 \ \pi_4 = 2 \ \pi_1 = 2.284 + 9708\pi_2{}^{-13.33} \ \mathrm{R}^2 = 1.0000 \ \tau = \frac{L^2}{\zeta}\left(2.284 + 9708\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}\right)^{-13.33}\right) \tag{8}$$

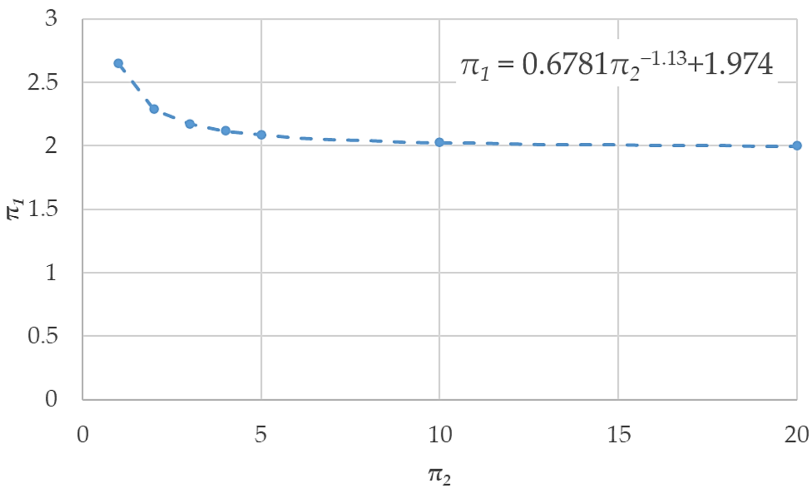
$$\pi_3 = 2 \ \pi_4 = 1 \ \pi_1 = 1.978 + 0.562\pi_2{}^{-1.102} \ \mathrm{R}^2 = 0.9990 \ \tau = \frac{L^2}{\zeta}\left(1.978 + 0.562\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}\right)^{-1.102}\right) \tag{9}$$

$$\pi_3 = 2 \ \pi_4 = 2 \ \pi_1 = 2.290 + 4674\pi_2{}^{-12.32} \ \mathrm{R}^2 = 1.0000 \ \tau = \frac{L^2}{\zeta}\left(2.290 + 4674\left(\frac{\zeta}{\alpha L u_{ext}^{\delta}}\right)^{-12.32}\right) \tag{10}$$

As can be seen, in Equations (6)–(10), the R^2 fits are very close to unity. It should be noted that for Equation (6) β takes zero and, therefore, the monomial π_4 has no influence. Finally, Figure 2 shows the fit of Equations (6)–(10).



(a)



(b)

Figure 2. *Cont.*

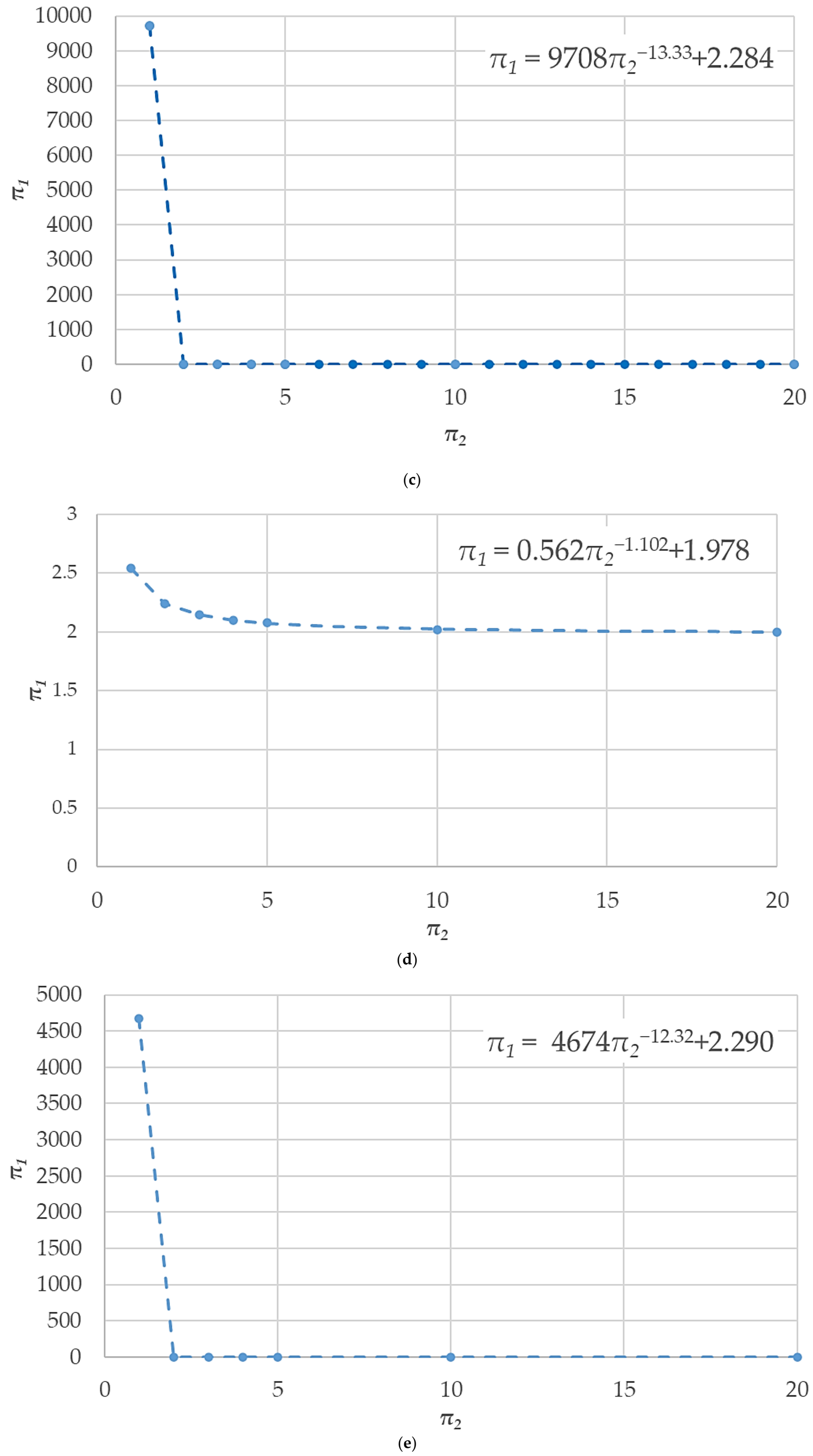


Figure 2. Representation of π_1 versus π_2 (**a**) with $\pi_3 = 0$ and independent of π_4, (**b**) with $\pi_3 = 1$ and $\pi_4 = 1$, (**c**) with $\pi_3 = 1$ and $\pi_4 = 2$, (**d**) with $\pi_3 = 2$ and $\pi_4 = 1$, and (**e**) with $\pi_3 = 2$ and $\pi_4 = 2$.

If the expressions obtained and Figure 2 are analysed, it can be seen that for very low values of π_2, very high values of π_1 are required, showing a tendency towards infinity. This indicates that the diffusion phenomenon is practically negligible, and with the drag phenomenon, it would be very difficult to reach a value of 99% of u_{ext}. This trend increased with increasing π_4, as shown in Figure 2. Owing to this tendency, the values provided by the fits for π_2 values lower than unity may have a higher error. On the other hand, for all the cases studied, from when π_2 takes a value between two and five, π_1 tends to be a value close to two, becoming practically independent of π_2, π_3, and π_4; therefore, the diffusion phenomenon governs the problem, as shown in Figure 2.

Finally, if step (vi) is applied by simulating the problem using the Network Simulation Method [32], and presenting u' versus x' for different values of t', the universal solution to the problem is obtained, as shown in Figure 3.

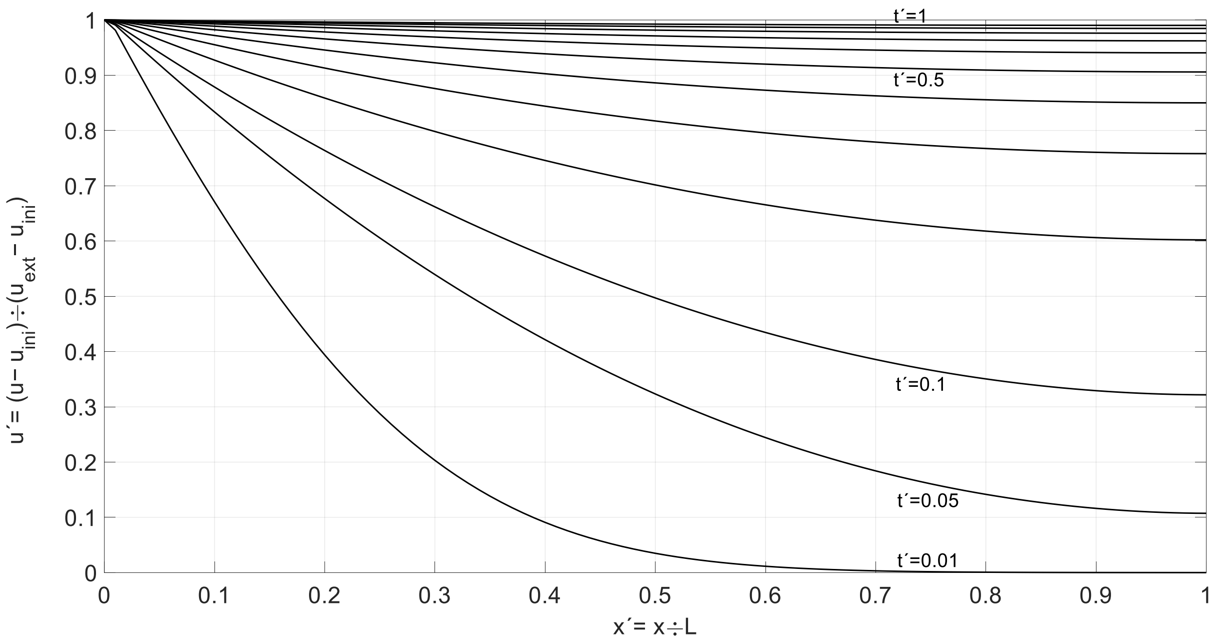


Figure 3. Universal curve for Burgers–Huxley equation with Dirichlet and symmetry (Neumann) boundary conditions.

The methodology for using the universal solution to obtain the value of u at a given position and time is as follows.

1. The values of position x and time t at which the u value is to be determined are known. In addition, the value of u at the boundary condition, u_{ext}, the initial value of u at position x, u_{ini}, and the length of the medium, L, are known. In addition, the coefficients α, β, γ, δ, ε, and ζ are known.
2. The monomials π_2, π_3, and π_4 are calculated.
3. The value of τ is determined using Equations (6)–(10). If the values of π_3 and π_4 are not those given in Equations (6)–(10), one can interpolate.
4. Calculate t' and x' with $t' = \frac{t}{\tau}$ and $x\prime = \frac{x}{L}$
5. The value of the curve t' is taken for position x' in Figure 3, and the value of u is obtained from the expression given for u', $u = u'(u_{ext} - u_{ini}) + u_{ini}$. If the value of t' lies between two curves, it is necessary to interpolate.

Similarly, the methodology can be used to obtain the time at which a given value of u is reached at a given position, the position at which a value of u is reached for a given time, etc.

3. Result and Validation

In the following, two types of studies are carried out to validate the methodology. In the first study, cases 1 to 3 (Table 1), we analyse different cases where the parameters of the problem change, and therefore the values of the monomials change. The results obtained using the universal solution were compared with those obtained by simulation using the Network Simulation Method [32]. In case 1, it is a problem where the variable

u is raised to one, since δ takes unit value. Moreover, the variable does not have initial values. In this case, all phenomena are present, that is, accumulation, drag, diffusion, and the generation and decay of species. If we compare the results obtained, following the methodology specified in the previous section, for the value of u at the 0.8 m position at a time of 0.458 s with those obtained by the simulation (Figure 4), we can see that the values obtained with both the universal solution and simulation are very similar. In the second case, where all the phenomena described above are involved again, the variable u is squared (δ = 2), affecting the phenomena of drag, generation, and the decay of species, and it also has initial values. Because these initial values influence the monomial π_3, it has a value of 0.667. Therefore, to obtain the value of τ, it is necessary to interpolate between Equations (5) and (6). Once again, the results obtained by the simulation (Figure 5) and those obtained with the universal solution have practically the same value; the difference is due to the carryover of errors in the application of the methodology. Finally, in case 3, where the phenomena of species generation and decay are not present, the variable u is raised to 0.5 (δ = 0.5) for the drag phenomena. Furthermore, the variable does not have initial values. In this case, it is necessary to interpolate between the curves of the universal solution, Figure 3, because t' takes a value of 0.45. If we compare the results with those obtained through the simulation (Figure 6), again, they are practically the same.

Table 1. Comparison between simulated values and those calculated with the universal curve.

Case	α	β	γ	δ	ε	ζ	L (m)	u_{ext}	u_{ini}	x (m)	t (s)
1	0.5	0.5	0.5	1	0.5	6	2	2	0	0.8	0.458
2	2	1	1	2	0.25	16	1	2	0.5	0.3	0.0865
3	1	0	1	0.5	0	4	0.25	4	0	0.1	0.0145
								Universal Solution		**Simulation**	
Case	π_3	π_4	π_2	π_1	τ	t'	x'	u		u	
1	1	2	3	2.288	1.525	0.3	0.4	1.688		1.698	
2	0.667	1	2	2.307	0.144	0.6	0.3	1.927		1.943	
3	0	0	8	2.056	0.0321	0.45	0.4	3.673		3.667	

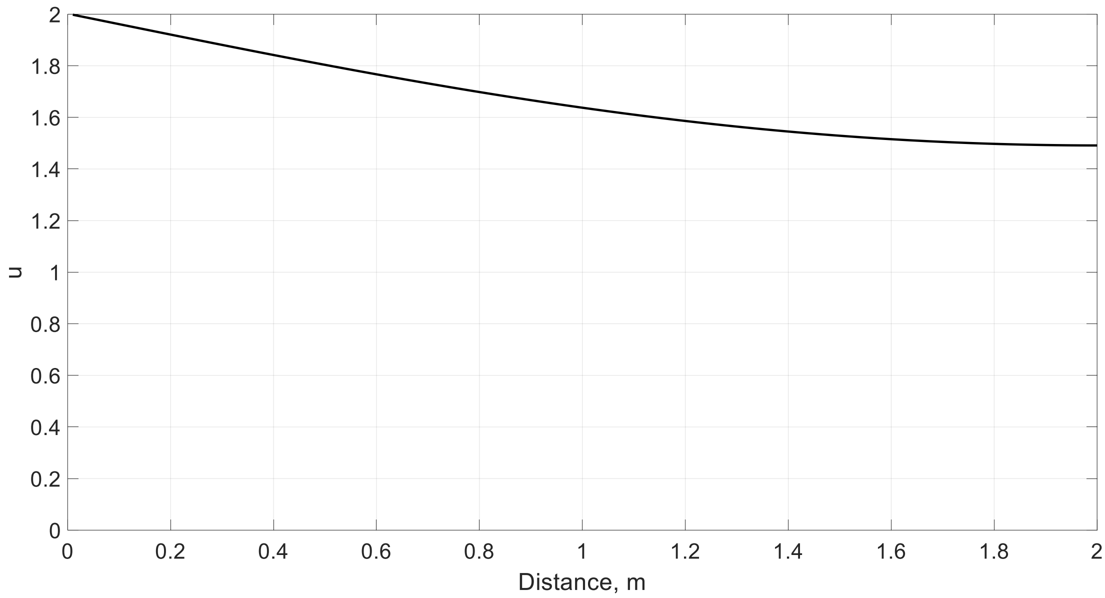


Figure 4. Simulation of case 1 for a time of 0.458 s.

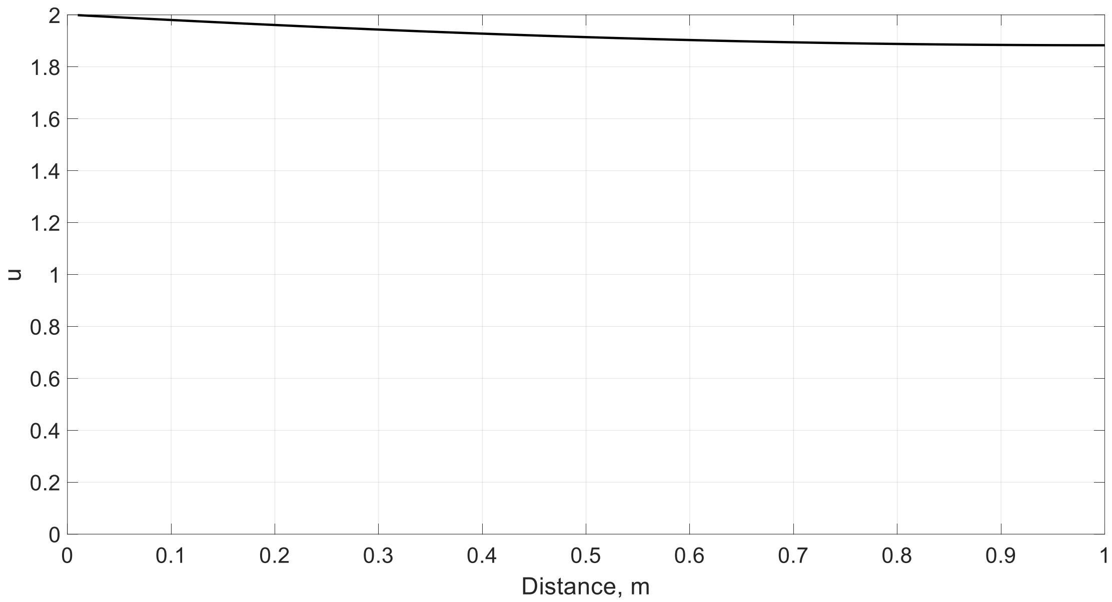


Figure 5. Simulation of case 2 for a time of 0.0865 s.

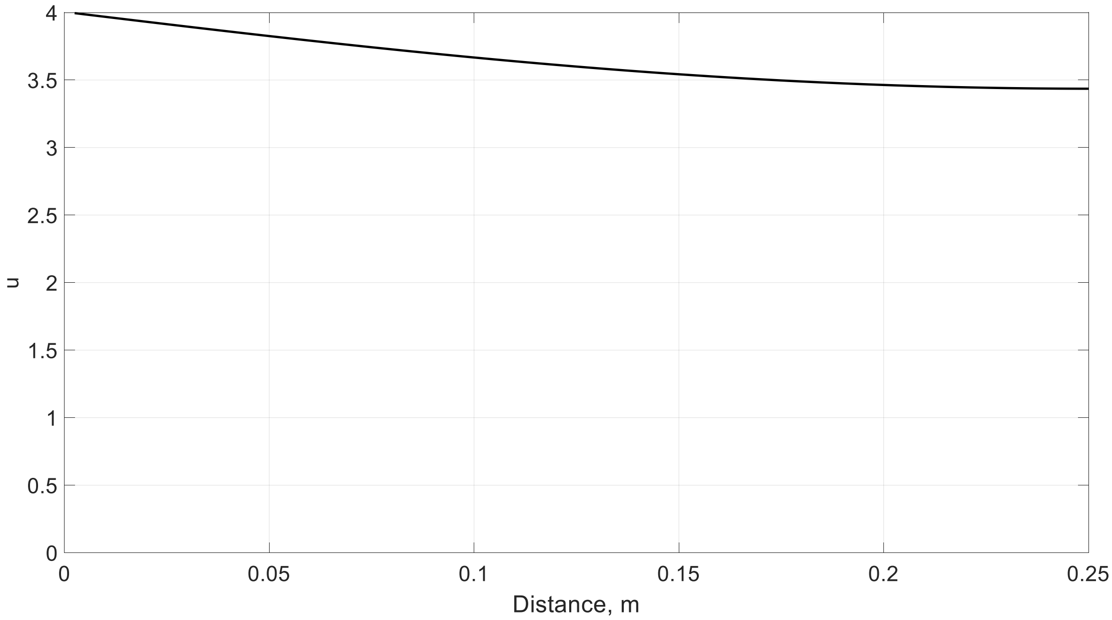


Figure 6. Simulation of case 3 for a time of 0.0145 s.

In the second study, the proposed methodology was applied to a case where the boundary condition changed value (case 4). Thus, for a time between 0 and 0.0269 s, the value of u_{ext} is unity, and then doubles for a time between 0.0269 and 0.3212 s, as shown in Table 2. To solve this case, it must be divided into two parts, cases 4a and 4b, by applying the indicated procedure twice. For case 4a, a time of 0.0269 s was used, and there were no initial values for the study variable. Once case 4a was solved, the procedure was applied, taking as initial values for case 4b the results obtained in case 4a. On the other hand, the time used for case 4b was 0.2943 s, the difference between 0.3212 s and 0.0269 s. For both cases 4a and 4b, three positions were taken, and their results were compared with those obtained by the simulation (Figure 7). As can be seen, for case 4a there are greater differences than in case 4b with respect to the values obtained by the simulation because the methodology is more sensitive to small times due to the possible accumulated errors when applying the procedure. However, for case 4b, the results were practically the same.

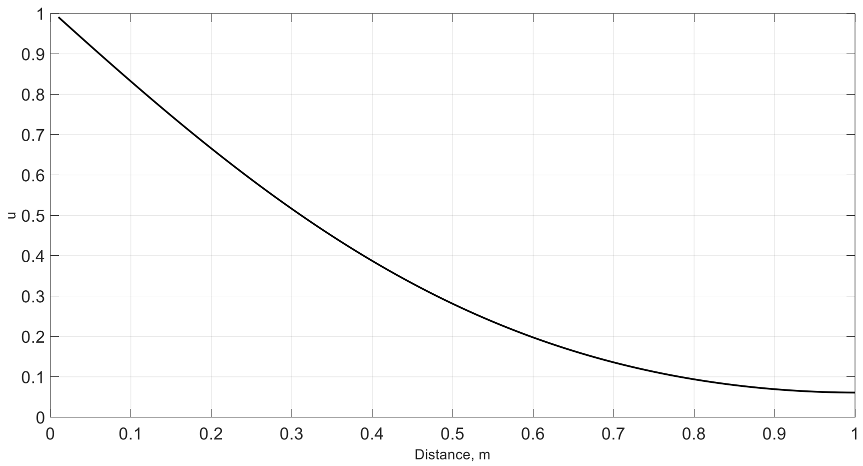
On the other hand, if we were in a case where the boundary condition will change more over time, we would have to apply the methodology explained in case 4 as many times as these changes occur.

Table 2. Comparison between simulated values and those calculated with the universal curve for a case with changing boundary conditions.

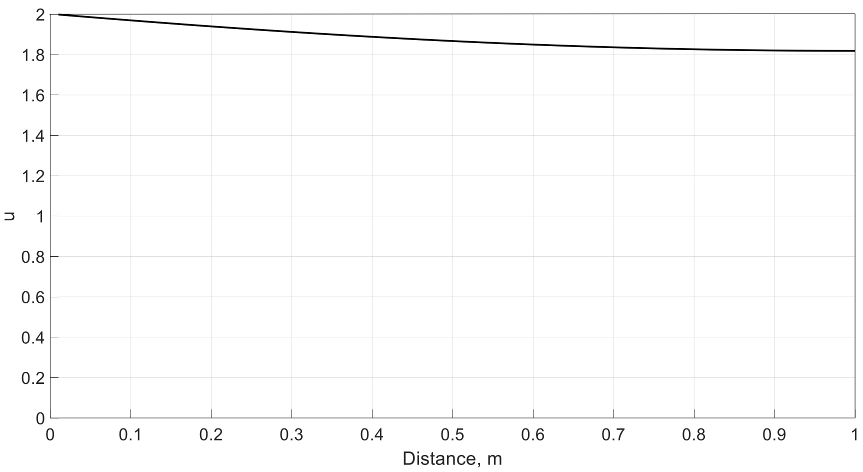
Case	α	β	γ	δ	ε	ζ	**L (m)**	u_{ext}	**t (s)**	**Time Range (s)**
4a	1	0	1	1	0	4	1	1	0.0269	$0 \le t \le 0.0269$
4b	1	0	1	1	0	4	1	2	0.2943	$0.0269 \le t \le 0.3212$

									Universal Solution	**Simulation**
Case	x	u_{ini}	π_3	π_4	π_2	π_1	τ	t'	x'	u	u
4a	0.3	0	0	0	4	2.151	0.5378	0.05	0.3	0.54	0.52
	0.6	0							0.6	0.24	0.20
	0.9	0							0.9	0.11	0.07

										Universal Solution	**Simulation**
Case	x	u_{ini}	π_3	π_4	π_2	π_1	τ	t'	x'	u	u
4b	0.3	0.54	0	0	2	2.355	0.5888	0.5	0.3	1.92	1.91
	0.6	0.24							0.6	1.85	1.85
	0.9	0.11							0.9	1.82	1.82



(**a**)



(**b**)

Figure 7. Simulation of case 4. (**a**) Case 4a: t = 0.0269 s and u_{ext} = 1, and (**b**) case 4b: t = 0.3212 s and u_{ext} = 2.

Finally, the proposed methodology allows us to obtain, by means of a relatively fast procedure, the same results as those obtained by means of a numerical simulation, which means a considerable saving of time, since, in some cases, depending on the discretisation of the problem used for the simulation and the time at which the solution is to be obtained, the calculation time can be large. On the other hand, the nondimensionalisation technique can be compared with other techniques used to obtain analytical solutions of linear and nonlinear differential equations such as the differential transformation method (DTM) [36]. This method is based on obtaining the analytical solution through the Taylor series expansion. One of the main differences between both methods is that when applying the DTM, the information provided by the nondimensionalisation technique is lost, since by grouping the variables into monomials with physical meaning, it is possible to determine the importance of each of the variables or to know which phenomenon predominates over the rest.

4. Conclusions

In this paper, the nondimensionalisation technique was applied to the problem of the Burgers–Huxley equation with Dirichlet and Neumann boundary conditions to obtain a universal solution and study the behaviour of the variables of the problem.

First, after applying the nondimensionalisation technique, it can be observed that for low values of the monomial π_2, the relationship between the diffusion and drag phenomena, very high values of π_1 are needed, the relationship between the time to reach a certain value in the medium and the diffusion phenomena. Thus, in this case, as the drag phenomenon predominates over the diffusion phenomenon, it becomes difficult to reach the required value in the medium because very large time values are required. This behaviour occurs for all the studied cases of π_3 and π_4, the relationship between the drag phenomenon and the generation or decay of species, and the relationship between the generation and decay coefficients of species, respectively. On the other hand, also for all cases of π_3 and π_4, from when π_2 takes a value between two and five, π_1 tends to take a value close to two; thus, from then on, the diffusion phenomenon predominates over the rest, which are practically negligible.

Regarding the validation of the proposed procedure for a universal solution to the proposed problem, several cases have been studied by comparing the results of this procedure with those obtained through simulation, observing that they are practically the same, and the difference may be due to errors in the application of the procedure. On the other hand, the methodology has been applied to a problem where the value of the boundary changes at a certain time, observing that the results are very similar.

Finally, as a strength of the work presented, it is worth highlighting that a simple methodology is presented that allows for universal solutions to various problems in science and engineering to be obtained. As a weakness, it is highlighted that for low values of the monomial π_2 the error in the proposed equations increases, because in these cases, the monomial π_1 tends to reach infinity.

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