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Efficiency and Vulnerability in Networks: A Game Theoretical Approach

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Abstract: Defining measures of network efficiency and vulnerability is a pivotal aspect of modern networking paradigms. We approach this issue from a game theoretical perspective, considering networks where actors have social or economic interests modeled through a cooperative game. This allows us to define, for each network, a family of efficiency measures and another of vulnerability measures, parameterized by the game. The proposed measures use the within groups' and the between groups' Myerson values. These values, respectively, measure the portion of the classical Myerson allocation corresponding to the productivity of players and the part related to intermediation costs. Additionally, they indicate the portion of total centrality in social networks attributed to communication or betweenness. In our proposal, the efficiency of a network is the proportion of total productivity (or centrality) that players can retain using the network topology. Intermediation costs (and betweenness centrality) can be seen as a weakness with a negative impact. Therefore, we suggest calculating vulnerability as the proportion of expenses players incur in intermediation payments. We explore the properties of these measures and tailor them to various structures and specific games, also analyzing their asymptotic behavior.

Keywords: game theory; TU-game; efficiency; vulnerability; networks; communication situation; Myerson value

MSC: 91A12; 91A43; 91A80

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Citation: Manuel, C.M.; Ortega, E. Efficiency and Vulnerability in Networks: A Game Theoretical Approach. *Axioms* **2023**, *12*, 1119. https://doi.org/10.3390/axioms12121119

Academic Editors: Zhijiao Du, Sumin Yu and Xuanhua Xu

Received: 7 November 2023
Revised: 5 December 2023
Accepted: 7 December 2023
Published: 13 December 2023

1. Introduction

Networks are the backbone of our interconnected world. They are therefore ubiquitous today. The internet, power grid, infrastructure, transportation, neural, computer and social networks play a crucial role in our daily lives.

Understanding the structural properties of networks is of paramount importance to provide a comprehensive perspective on network dynamics that can help decision-makers and stakeholders across different domains.

This paper focuses on studying two highly relevant properties of social and economic networks: efficiency and vulnerability.

In social networks, efficiency refers to the network's ability to connect individuals, facilitate communication, optimize interactions and enhance social cohesion by improving the flow of information. In economic networks, efficiency pertains to the network's capacity to allocate resources effectively, maximize productivity, reduce transaction costs and minimize waste.

Conversely, vulnerability in a social network indicates the exposure of the network or its members to external or internal risks, such as misinformation, loss of information and harmful interactions. In economic networks, vulnerability refers to the possibility of weaknesses that could disrupt economic activities, generating a negative impact on productivity.

Some authors have emphasized the relevance of the individual power of actors to the group outcomes of the network. Two prominent works in this regard are those of [1,2]. In both (although with different metrics), the importance of each individual in global efficiency is measured through the impact that their removal from the network generates on group activity.

Refs. [3,4] is the seminal work in which the relationship between individuals' centrality in the network and their influence on group processes is assumed. Since then, different sociologists have proposed various measures of centrality, which, broadly speaking, we can classify into two groups: closeness measures [5–7] and betweenness [3,7–10].

However, the relationship between centrality and power generates controversy. Ref. [11] cast doubt on centrality and power being the same in exchange networks. Nevertheless, Ref. [12] introduced a family of measures that incorporates (according to the author) the usual positive relationship between centrality and power as well as exceptional situations in which this relationship does not occur.

Game theory is devoted to analyzing strategic interactions and decision-making processes, and thus it can be effectively used to study dynamics within networks. As mentioned, our objective in this paper is to analyze efficiency and vulnerability in a network, and we will approach this from a game theoretical perspective, which has also proven useful in the calculation of centralities in social networks. While the analysis of efficiency in networks from a game theoretical viewpoint is not a new concept, previous studies on wireline and wireless networks have primarily been conducted from a non-cooperative game perspective. The paper of [13] is a remarkable survey of this type of contribution. The analysis of efficiency and vulnerability in transportation networks has also attracted much interest, but primarily from the perspective of non-cooperative game theory. Refs. [14,15] introduced an efficiency measure for congested networks that [16] generalize to the case in which information about traffic demand is uncertain and modeled through random variables. In our framework, we assume that the nodes of the network simultaneously act as players in a TU-game (a cooperative game with transferable utility), which represents their social or economic interests. Consequently, we propose a family of game theoretical measures of efficiency and vulnerability for a network with the game as a parameter. This implies that efficiency and vulnerability are contingent on the purpose for which individuals use the network. Whether it involves information dissemination, economic productivity or communication, we model this purpose through a TU-game.

As mentioned, the analysis of networks from a game theory perspective has a long tradition. Ref. [17] laid the groundwork with his seminal work, where he proposed to examine the impact of communication restrictions among players in a TU-game using a network (graph). He introduced a new TU-game, known as the graph-restricted game. Given the prominence of the Shapley value [18], Myerson suggested the Shapley value as the solution for graph-restricted games. Following this pioneering work, there is extensive literature on games involving cooperation restricted by a graph. Refs. [19,20] used the Myerson value as a centrality measure for actors in a symmetrical TU-game with communication restricted by a graph. In an attempt to capture both communication and betweenness centrality, they decomposed the Myerson value into the within groups Myerson value (WG-Myerson value, for short) and the between groups Myerson value (BG-Myerson value, for short). These values, when calculated for a general game, respectively, measure the player's productivity in the game and their intermediation opportunities, among others.

The proposed efficiency and vulnerability measures introduced in this paper use this decomposition of the Myerson value. In our approach, the efficiency of a network parameterized by a game is defined as the proportion of the value of the grand coalition that players can retain (this is calculated as the sum of the within groups Myerson values of all players) as a consequence of the network topology. Therefore, more efficient economic networks enable players to retain a greater proportion of productivity, thereby reducing intermediate costs. These intermediate costs are considered a weakness with a negative impact on productivity. In our proposal for the family of vulnerability measures (also parameterized by the game), we calculate the proportion of the value of the grand coalition that players must expend in intermediation payments (the sum of the between groups Myerson values of players). These payments reduce productivity and expose the players to the disruption of economic activities resulting from a lack of communication between different groups of players (coalitions). For social networks, roughly speaking, efficiency is seen as the proportion of total centrality in the network attributed to communication among players. Vulnerability, on the other hand, is the proportion of the total centrality attributed to betweenness. In this context, efficiency is measured in terms of the relational abilities of players, while vulnerability increases when third-party control of communication between pairs of players grows. After defining the measures, we explore their most relevant properties, including their asymptotic behavior for specific games.

In a fixed game, both the efficiency and vulnerability measures we introduce can be considered normalized value functions, following [21,22]. These functions assign an aggregated value to each network, which can then be distributed among its actors. In our case, the value function indeed arises from the sum of Myerson's allocations (WG- or BG-) generated by the purpose (game) with which players use the network.

The remainder of the paper is organized as follows. After some preliminaries on games, graphs and communication situations (graph-games), in Section 3, we introduce the family of efficiency measures and analyze their properties. Section 4 focuses on specifying the values of this family for various games and structures, including an examination of their asymptotic behavior. In Section 5, we present the family of vulnerability measures. The paper concludes with a section summarizing conclusions and final remarks, followed by the references.

2. Preliminaries

2.1. Games

A *cooperative n-person game* of transferable utility (TU-game for short) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and $v : 2^N \to \mathbb{R}$, satisfying $v(\emptyset) = 0$, is the characteristic function that assigns to each coalition $S \subseteq N$ the payoff $v(S)$ that its $s = |S|$ members can obtain by cooperating.

We will denote by G^N the vector space of all TU-games with player set N. A very useful basis of G^N is the unanimity games basis $\{(N, u_S)\}_{\emptyset \neq S \subseteq N}$, with characteristic functions given by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, for each $(N, v) \in G^N$, we can write v as the following linear combination, $v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S$, in which the coefficients, $\Delta_v(S)$, known as *Harsanyi dividends* ([23]), can be calculated as

$$\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T).$$

We will say that $(N, v) \in G^N$ is zero-normalized if $v(\{i\}) = 0$ for all $i \in N$, is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N, S \cap T = \emptyset$, is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, and is *almost-positive* if $\Delta_v(S) \geq 0$ for all $S \subseteq N$. Almost-positive games are superadditive and convex.

Given $(N, v) \in G^N$, a player $i \in N$ is a *null player* in (N, v) if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$ and is a *necessary player* ([24]) if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$.

An *allocation rule* or point solution ψ on G^N assigns to every $(N, v) \in G^N$ a vector $\psi(N, v) \in \mathbb{R}^n$, $\psi_i(N, v)$ representing the outcome of the player i in the game. A very relevant point solution was introduced by [18]. It assigns to every player i in the game (N, v) the following expected value of his marginal contributions:

$$Sh_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!}(v(S \cup \{i\}) - v(S)), \ \ i \in N.$$

2.2. Graphs

A graph or a *network* is a pair (N, γ), where $N = \{1, 2, \ldots, n\}$ is the set of nodes and γ is a set of links included in the complete graph $\gamma_N = \{\{i, j\} \mid i, j \in N, i \neq j\}$. If the *link* $\{i, j\} \in \gamma$, we will say that i and j are directly connected. In another case, i and j could be connected if there exists a path, i.e., a sequence of nodes (intermediaries) $i_1, i_2, \ldots i_k, k > 2$, with $i_1 = i, i_k = j$, such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \ldots, k-1$.

A set $S \subseteq N$, $S \neq \emptyset$, is *connected (internally connected)* in (N, γ) if every pair of nodes in S are directly connected or they can be connected with intermediaries in S. Singletons will be considered connected. *A connected component*, C, of the graph (N, γ) is a maximal connected subset, that is, C is connected and, for all $C' \subseteq N$, if $C \subsetneq C'$, then C' is not connected. A graph (N, γ) induces a partition N/γ of the set N into connected components.

A connected set in (N, γ), S^*, is a *minimal connection set* of $S \subseteq S^*$ if there is no $S' \subsetneq S^*$ with $S \subseteq S'$ and S' connected. $\mathcal{MCS}(S, N, \gamma)$ will denote the family (occasionally empty) of all minimal connection sets of S in (N, γ).

Given (N, γ) and $S \subseteq N$, S/γ will denote the set of all connected components in the restriction of (N, γ) to S, i.e., in $(S, \gamma_{|S})$, where $\gamma_{|S} = \{\{i, j\} \in \gamma \mid i, j \in S\}$.

A *subgraph* of a graph (N, γ) is (N, γ') with $\gamma' \subseteq \gamma$. Given (N, γ) and $l \in \gamma$, $(N, \gamma \setminus \{l\})$ is the subgraph obtained when the link l is removed; (N, γ_i) is the subgraph of (N, γ) of the links incident on i, i.e., $\gamma_i = \{l \in \gamma \mid i \in l\}$ and $(N, \gamma_{-i}) = (N, \gamma \setminus \gamma_i)$.

(N, γ) is a tree if it is connected and it has no cycles. We will denote (N, γ_i^S) the star with center in the node i, and thus $\gamma_i^S = \{\{i, j\}, j \in N, \ j \neq i\}$. (N, γ^C) will denote the chain ordered in the natural way (from 1 to n). (N, γ^W) is the wheel with $\gamma^W = \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$.

2.3. Communication Situations

Refs. [17,25] introduced games with cooperation restricted by a graph (also graph-games or communications situations), (N, v, γ), in which the players in the game (N, v) are restricted to communicate using only the links in the graph (N, γ). $\mathcal{CS}^N$ will denote the set of all communication situations with a set of player-nodes N, and $\mathcal{CS}_0^N$ will denote the subclass of $\mathcal{CS}^N$ of those communication situations in which the game is zero-normalized. Finally $\mathcal{CS}_*^N$ will be the subset of $\mathcal{CS}_0^N$ in which the game (N, v) is non-null and superadditive.

An *allocation rule* ψ on $\mathcal{CS}^N$ is a map $\psi : \mathcal{CS}^N \rightarrow \mathbb{R}^n$, $\psi_i(N, v, \gamma)$ representing the outcome for player i in the communication situation (N, v, γ). An allocation rule ψ defined on $\mathcal{CS}^N$ satisfies *component efficiency* if for all $(N, v, \gamma) \in \mathcal{CS}^N$ and all $C \in N/\gamma$, $\sum_{i \in C} \psi_i(N, v, \gamma) = v(C)$.

Following [17], for $(N, v, \gamma) \in \mathcal{CS}^N$, a new game (N, v^γ) captures the impact in the formation of coalitions (and thus in their value) of the restrictions in the communications imposed by (N, γ). The characteristic function is given by

$$v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \text{ for all } S \subseteq N.$$

Given the relevance of the Shapley value, Myerson defined μ, the allocation rule in $\mathcal{CS}^N$ (nowadays known as the Myerson value), given by $\mu(N, v, \gamma) = Sh(N, v^\gamma)$. Exploring the possibilities of the Myerson value as a centrality measure, Refs. [19,20] highlight the vector character of the Myerson value, decomposing it into two new allocation rules: the within groups Myerson value, μ^W, (WG-Myerson value, for short) and the between groups Myerson value, μ^B, (BG-Myerson value, for short). For each $(N, v, \gamma) \in \mathcal{CS}^N$ and each

$i \in N$, they are defined as $\mu_i^W(N,v,\gamma) = \mu_i(N,v_i,\gamma)$ and $\mu_i^B(N,v,\gamma) = \mu_i(N,v_{-i},\gamma)$, where $v_i = \sum_{S\subseteq N: i\in S} \Delta_v(S)u_S$ and $v_{-i} = v - v_i$.

3. A Family of Efficiency Measures

In the following definition, we introduce a family of game theoretical measures for the efficiency of a network. We will assume that nodes in the network are players in a TU-game, which represents the economic or social interests that motivate the interaction of players. The network represents the feasible communications between players (and also, as a consequence, the existing restrictions that motivate the need for intermediaries when players have no direct communication). The proposed family of efficiency measures assigns to each network and each TU-game a value representing which part of the total productivity of the game, $v(N)$, players can maintain given the geometry of the network.

We will use superadditive games in which cooperation between individuals is not detrimental to them. We will also assume that the games are zero-normalized so that the total absence of communication between players—the impossibility to cooperate or empty graph—corresponds to the minimum (and null) efficiency.

Definition 1. *We define a measure of efficiency, $\mathcal{E}$, for each communication situation $(N,v,\gamma) \in CS_*^N$ as*

$$\mathcal{E}(N,v,\gamma) = \frac{\sum_{i\in N} \mu_i^W(N,v,\gamma)}{v(N)}.$$

Remark 1. *In the special case in which (N,v) is a symmetric game and (N,γ) is a connected graph, the defined measure represents the proportion of $v(N)$ corresponding to the total communication centrality of players introduced by [19].*

To motivate our definition, let us consider the following example, in which we will use several specific games and communication centrality measures also considered in [19].

Example 1. *Let us denote by (N,v_1), (N,v_2) and (N,v_3), respectively, the messages game, the zero-normalized overhead game and the conference game with characteristic functions given by the following:*

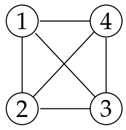
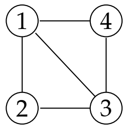
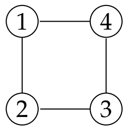
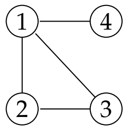
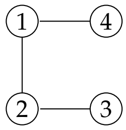
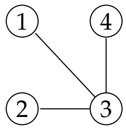
(i) $v_1(S) = s(s-1)$,
(ii) $v_2(S) = s-1$,
(iii) $v_3(S) = 2^s - s - 1$,

for $S \subseteq N = \{1,2,3,4\}$.

The characteristic function of the messages game, v_1, assigns to each coalition the number of messages that can be exchanged among all pairs of players within that coalition. In terms of Harsanyi's dividends, each coalition of two players receives a dividend of two units, while the remaining coalitions receive no dividends. Similarly, the conference game, v_3, assigns to each coalition the number of non-unitary subsets (conferences) that can be formed with its members. In this case, each non-trivial coalition receives a unitary dividend. Finally, in the zero-normalized overhead game, v_2, each player contributes one monetary unit to each coalition it belongs to, and then all coalition members collectively face a unitary cost. The three games are symmetric and convex, and both, the first and the third, are almost-positive. Therefore, in all of them, there is a strong incentive for cooperation.

Suppose (N,γ) is one of the four-node connected graphs. Then, the efficiency (calculated with the proposed measure) of the (eighteen) different communication situations is given in Table 1. For the within groups Myerson values of the nodes–players (communication centrality of the nodes), see Table 2 in [19].

Table 1. $\mathcal{E}(N, v_i, \gamma)$ for $N = \{1,2,3,4\}$, $i = \{1,2,3\}$ and (N, γ) a four-node graph.

(N,γ)	$\mathcal{E}(N, v_1, \gamma)$	$\mathcal{E}(N, v_2, \gamma)$	$\mathcal{E}(N, v_3, \gamma)$
	1	1	1
	$\frac{35}{36}$	$\frac{17}{18}$	$\frac{65}{66}$
	$\frac{34}{36}$	$\frac{8}{9}$	$\frac{32}{33}$
	$\frac{32}{36}$	$\frac{31}{36}$	$\frac{11}{12}$
	$\frac{29}{36}$	$\frac{7}{9}$	$\frac{28}{33}$
	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{39}{44}$

To illustrate the previous calculations, we will reproduce those corresponding to obtaining the efficiency of the star for the messages game, which is $\frac{5}{6}$.

Let us recall that, for a game (N, v) *and player* $i \in N$*, the characteristic function* v_i*, which gives the marginal contributions of player* i *to different coalitions* S*, is obtained as* $\sum_{S \subseteq N: i \in S} \Delta_v(S) u_S$.

For the messages game, (N, v_1)*, with characteristic function* $2(u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}} + u_{\{2,3\}} + u_{\{2,4\}} + u_{\{3,4\}})$ *and for players* $i = 1,2,3,4$*, we have*

$(v_1)_1 = 2(u_{\{1,2\}} + u_{\{1,3\}} + u_{\{1,4\}})$,
$(v_1)_2 = 2(u_{\{1,2\}} + u_{\{2,3\}} + u_{\{2,4\}})$,
$(v_1)_3 = 2(u_{\{1,3\}} + u_{\{2,3\}} + u_{\{3,4\}})$,
$(v_1)_4 = 2(u_{\{1,4\}} + u_{\{2,4\}} + u_{\{3,4\}})$.

Then, if (N, γ_3^S) *is the four-nodes star centered at node* 3*, for player-satellite 1, we have*
$\mu_1^W(N, v_1, \gamma_S) = \mu_1(N, (v_1)_1, \gamma_S) = Sh_1(N, 4u_{\{1,2,3\}} + 2u_{\{1,3\}}) = \frac{4}{3} + 1 = \frac{7}{3}$.
Symmetrically, for the other two satellites,
$\mu_2^W(N, v_1, \gamma_S) = \mu_4^W(N, v_1, \gamma_S) = \frac{7}{3}$*, and for the player-hub,*
$\mu_3^W(N, v_1, \gamma_S) = \mu_3^W(N, (v_1)_3, \gamma_S) = Sh_3(N, 2(u_{\{1,3\}} + u_{\{2,3\}} + u_{\{3,4\}})) = 3$.
Finally, $\mathcal{E}(N, v_1, \gamma_S) = \frac{\frac{7}{3} + \frac{7}{3} + \frac{7}{3} + 3}{12} = \frac{5}{6}$.

In the different communication situations, $\mathcal{E}(N, v_1, \gamma)$ *can be seen as a measure of the efficiency of the network* (N, γ) *in achieving the purpose of communicating pairs of players/nodes. For this purpose, the measure is conceptually very close to that of [1]. Similarly,* $\mathcal{E}(N, v_2, \gamma)$ *and* $\mathcal{E}(N, v_3, \gamma)$*, respectively, measure the efficiency of networks in achieving the goals of facing a unitary cost and making the maximum number of conferences viable.*

As shown by the figures in the table above, efficiency remains between zero and one. It increases when adding edges if the game is convex, it equals 1 *if the graph is complete, it equals* 0 *for the empty graph, and among all trees, it is maximum in the star and minimum in the chain, provided* *that the game is almost-positive, as in cases v_1 and v_3. In this paper, we will prove that this behavior of the metrics is not limited to these specific cases but responds to general properties.*

The following proposition includes some properties related to the range of values of the defined measures. The corresponding proofs can be found in Appendix A.

Proposition 1. *Given $(N, v, \gamma) \in \mathcal{CS}_*^N$, we have the following:*

(i)

$$0 \leq \mathcal{E}(N, v, \gamma) \leq \frac{v^\gamma(N)}{v(N)}.$$

Moreover, if (N, γ) is also a connected graph, then $0 \leq \mathcal{E}(N, v, \gamma) \leq 1$.

(ii) The lower and upper bounds of $\mathcal{E}$ are reached, respectively, for the empty and the complete network. If (N, γ_N) is the complete graph and $(N, \gamma_\emptyset)$ is the empty graph, then

$$\mathcal{E}(N, v, \gamma_N) = 1 \text{ and } \mathcal{E}(N, v, \gamma_\emptyset) = 0,$$

and if the game is not zero-normalized, then $\mathcal{E}(N, v, \gamma) \geq \frac{\sum_{i \in N} v(\{i\})}{v(N)}$.

The following proposition lists some properties regarding the behavior of these measures in the face of changes in the network, ceteris paribus. Again, the corresponding proofs can be found in Appendix A.

Proposition 2. *Given $(N, v, \gamma) \in \mathcal{CS}_*^N$, the following apply:*

(i) The disconnection of a player who is simultaneously null in the game and terminal in the graph does not affect the efficiency of the network. Then, the network can be lightened by eliminating both null and terminal players, i.e., if $i \in N$ is a null player in (N, v) such that $|\gamma_i| = 1$, we have that $\mathcal{E}(N, v, \gamma) = \mathcal{E}(N, v, \gamma \setminus \gamma_i)$.

(ii) If we isolate an essential player in the game, the efficiency of the network, whatever it is, becomes zero. This property can be seen as a possibility to override the efficiency of potentially dangerous networks (e.g., terrorist networks). Formally, if $i \in N$ is an essential player in (N, v) and isolated in the graph (N, γ), then $\mathcal{E}(N, v, \gamma) = 0$.

(iii) The star whose center is an essential player has maximum efficiency. This justifies that, when building certain networks, a key player controls the entire communication flow. Formally, if $i \in N$ is an essential player in (N, v), which is also the hub of the star (N, γ_i^S), then $\mathcal{E}(N, v, \gamma_i^S) = 1$.

(iv) If the game is convex, the efficiency of a network does not decrease when adding a new link, i.e., if (N, v) is a convex game, then

$$\mathcal{E}(N, v, \gamma) \leq \mathcal{E}(N, v, \gamma \cup \{l\}), \text{ for all } l \notin \gamma.$$

Remark 2. *The analysis of terrorist networks has garnered significant interest in recent times for reasons that are readily apparent. In this context, it is worth highlighting the works of [26–28], wherein the centrality of terrorists (and efficient calculation methods) in the WTC 9/11 attack is examined through the Shapley value calculation in the connectivity game [29] corresponding to the network formed by the terrorists. A connectivity game is dichotomic, assigning a unit value to connected coalitions in the network and zero otherwise. The game is thus created from the network, not given exogenously. It presents an alternative approach to that proposed by [19] but undeniably suggests defining network efficiency based on centralities given by the Shapley value in the connectivity game and comparing that measure with the one defined here by exploring advantages and disadvantages.*

Similarly, other terrorist networks, such as that of Khalid Zercani, have been analyzed using connectivity games but with different allocation rules, such as Owen values and Banzhaf values, as in [30,31].

In *(iv)* of Proposition 2, the convexity condition cannot be weakened, as is seen in the following example.

Example 2. *Consider (N,v) with $N=\{1,2,3\}$ and $v=u_{\{1,2\}}+u_{\{1,3\}}+u_{\{2,3\}}-2u_{\{1,2,3\}}$, which is a superadditive but not convex game. Let $\gamma_1=\{\{1,2\}\}$ and $\gamma_2=\gamma_1\cup\{\{2,3\}\}$, then*

$$\mathcal{E}(N,v,\gamma_1)=\frac{1}{1}=1 \text{ and } \mathcal{E}(N,v,\gamma_2)=\frac{2}{3}.$$

As this example shows, in the case of a non-convex game, it may happen that a non-connected network is more efficient than a connected one.

The following proposition includes some properties relating to the value of the measures for different games. The proofs can be found in Appendix A.

Proposition 3. *$\mathcal{E}$ satisfies the following properties:*

(i) *It is invariant by strategic equivalence of the game. That is to say, the efficiency is not changed if we transform the economic units in which the outcome of coalitions are measured, i.e., given (N,v,γ) and $(N,w,\gamma)\in\mathcal{CS}_*^N$ with (N,v) and (N,w) strategically equivalent, we have $\mathcal{E}(N,v,\gamma)=\mathcal{E}(N,w,\gamma)$.*

(ii) *The measures of communication situations in which the network is not connected can be obtained as a linear combination of the efficiencies corresponding to the respective (component) communication situations, i.e., given $(N,v,\gamma)\in\mathcal{CS}_*^N$, if $N/\gamma=\{C_1,...,C_r\}$, then*

$$\mathcal{E}(N,v,\gamma)=\sum_{k=1}^{r}\frac{v(C_i)}{v(N)}\mathcal{E}(C_k,v_{|C_k},\gamma_{|C_k}).$$

(iii) *It is not linear in the game, ceteris paribus. Nevertheless, the efficiency of a communication situation in which the game is the sum of other games can be calculated as a convex linear combination of the efficiencies in the corresponding games' communication situations, i.e., if $(N,v_1,\gamma),\ldots,(N,v_r,\gamma),(N,\sum_{k=1}^{r}v_k,\gamma)\in\mathcal{CS}_*^N$, then it holds that*

$$\mathcal{E}(N,\sum_{k=1}^{r}v_k,\gamma)=\sum_{k=1}^{r}\frac{v_k(N)}{(\sum_{k=1}^{r}v_k)(N)}\mathcal{E}(N,v_k,\gamma).$$

(iv) *In a fixed network, we can obtain the efficiency in a communication situation in terms of efficiencies in communication situations with unanimity games, i.e., if $(N,v,\gamma)\in CS_*^N$, then*

$$\mathcal{E}(N,v,\gamma)=\frac{1}{v(N)}\sum_{\emptyset\neq S\subseteq N}\Delta_v(S)\mathcal{E}(N,u_S,\gamma).$$

(v) *The efficiency of a connected network to communicate the coalition S is s times the probability of that coalition being connected in the graph, i.e., if $(N,u_S,\gamma)\in CS_*^N$, then*

$$\mathcal{E}(N,u_S,\gamma)=\begin{cases} s\alpha_\gamma(S), & \text{if } \mathcal{MCS}(S,N,\gamma)=\{S_1,...,S_r\}\neq\emptyset \\ 0, & \text{otherwise,}\end{cases}$$

where

$$\alpha_\gamma(S)=\sum_{i=1}^{r}\frac{1}{|S_i|}-\sum_{i<j}\frac{1}{|S_i\cup S_j|}+...(-1)^{r+1}\frac{1}{|\cup_{i=1}^{r}S_i|}.$$

As a consequence of *(iv) and (v)* of the previous proposition, we can easily obtain the expression of $\mathcal{E}(N,v,\gamma)$ in terms of the Harsanyi dividends for some particular structures, such as trees, stars and chains.

Example 3. *For (N,v) a superadditive and zero-normalized game,*

(i) *If (N,γ^T) is a tree, then*

$$\mathcal{E}(N,v,\gamma^T) = \frac{1}{v(N)} \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) \frac{s}{|H(S)|},$$

where $H(S)$ is the connected hull of S in (N,γ^T), i.e., the unique minimal connection set of S in (N,γ^T).

(ii) *If (N,γ^C) is a chain ordered in the natural way, then*

$$\mathcal{E}(N,v,\gamma^C) = \frac{1}{v(N)} \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) \frac{s}{\max_j\{S\} - \min_j\{S\} + 1}.$$

(iii) *If (N,γ_1^S) is a star with center at node 1, then*

$$\mathcal{E}(N,v,\gamma_1^S) = \frac{1}{v(N)} \left[\sum_{\emptyset \neq S \subseteq N, 1 \in S} \Delta_v(S) + \sum_{\emptyset \neq S \subseteq N, 1 \notin S} \Delta_v(S) \frac{s}{s+1} \right].$$

In the following proposition, we prove that if the game is zero-normalized, symmetric and almost positive, then the maximum efficiency of the trees is reached at the star.

Proposition 4. *Let (N,v,γ_1^S) and (N,v,γ^T) be two communication situations in which (N,v) is a zero-normalized, symmetric and almost positive game, (N,γ_1^S) is the star with $1 \in N$ as its hub and (N,γ^T) is a tree. Then,*

$$\mathcal{E}(N,v,\gamma_1^S) \geq \mathcal{E}(N,v,\gamma^T).$$

Proof. We have

$$\mathcal{E}(N,v,\gamma_1^S) = \frac{1}{v(N)} \left[\sum_{S \subseteq N, 1 \in S} \Delta_v(S) + \sum_{S \subseteq N, 1 \notin S} \Delta_v(S) \frac{s}{s+1} \right],$$

and

$$\mathcal{E}(N,v,\gamma^T) = \frac{1}{v(N)} \sum_{S \subseteq N} \Delta_v(S) \frac{s}{|H(S)|}$$

$$= \frac{1}{v(N)} \left[\sum_{\substack{S \subseteq N \\ S \text{ connected}}} \Delta_v(S) + \sum_{\substack{S \subseteq N \\ S \text{ not connected}}} \Delta_v(S) \frac{s}{|H(S)|} \right].$$

As for $S \subseteq N$, S not connected in (N,γ_T), $|H(S)| \geq s+1$, to obtain the result it is sufficient to prove that, for $s = 2,\ldots,n$, the number of connected sets in (N,γ_S) with s elements is greater than or equal to the number of connected sets in (N,γ_T).

Effectively, for $s = 2,\ldots,n$, the number of connected sets in (N,γ_1^S) of size s is $\binom{n-1}{s-1}$, the total number of choosing $s-1$ links from the $n-1$ total links of the star. Clearly, the number of connected sets of size $s = 2,\ldots,n$ in a tree is upper bounded by $\binom{n-1}{s-1}$. This is because the set of s nodes incident on $s-1$ links of a tree is not necessarily connected. $\square$

In the following proposition, we prove that if the game is zero-normalized, symmetric and almost positive, among all the trees, the minimum efficiency is reached in the chain.

Proposition 5. *Let (N, v, γ^T) and (N, v, γ^C) be two communication situations in which (N, v) is a zero-normalized, symmetric and almost positive game, (N, γ^C) is a chain and (N, γ^T) is a tree. Then,*

$$\mathcal{E}(N, v, \gamma^T) \geq \mathcal{E}(N, v, \gamma^C).$$

Let us consider in (N, γ^T) the longest path (chain) and, without loss of generality, let us suppose that γ^T is labeled so that path is $\{1,2\}, \{2,3\}, ..., \{r-1, r\}$.

If $r = n$, then $(N, \gamma^T) = (N, \gamma^C)$, and the result is proven.

Otherwise, we will transform (N, γ^T) in other $(N, \gamma^{T'})$, in which the longest path has a length strictly greater than r and such that $\mathcal{E}(N, v, \gamma^T) \geq \mathcal{E}(N, v, \gamma^{T'})$. It is clear that repeating this process a finite number of times will prove the result.

Let us denote $h = \min_{j \in \{1,...r\}} \{j \mid \deg(j) > 2\}$, the first node from 1 to r having degree greater than 2. As the longest path of (N, γ^T) differs from (N, γ^T), this node must exist.

Let k be a node directly connected with h, $k \neq h-1$ and $k \neq h+1$, and let us define $A = \{i \in N \mid \text{the path joining } i \text{ with } h \text{ contains } k\}$.

Consider the new tree $(N, \gamma^{T'})$ defined as $\gamma^{T'} = (\gamma^T \setminus \{\{h,k\}\}) \cup \{\{1,k\}\}$.

To prove that $\mathcal{E}(N, v, \gamma^T) \geq \mathcal{E}(N, v, \gamma^{T'})$, let $\emptyset \neq S \subseteq N$. Then, we can distinguish three different situations:

(a) $h \notin H_{\gamma^T}(S)$.
Then, $H_{\gamma^T}(S) = H_{\gamma^{T'}}(S)$, and for these sets,

$$\frac{s}{|H_{\gamma^T}(S)|} = \frac{s}{|H_{\gamma^{T'}}(S)|}.$$

(b) $h \in H_{\gamma^T}(S) \cap H_{\gamma^{T'}}(S)$.
Then, clearly $H_{\gamma^T}(S) \subseteq H_{\gamma^{T'}}(S)$, and thus,

$$\frac{s}{|H_{\gamma^T}(S)|} \geq \frac{s}{|H_{\gamma^{T'}}(S)|}.$$

(c) $h \in H_{\gamma^T}(S)$ and $h \notin H_{\gamma^{T'}}(S)$.
In this case, $h \notin S$. Let us denote $B_S = S \cap \{1, ..., h\}$. Then,

$$|H_{\gamma^T}(S)| - |H_{\gamma^{T'}}(S)| = h - \min_{i \in B_S}\{i\} + 1 - \max_{i \in B_S}\{i\}.$$

If $h - \min_{i \in B_S}\{i\} + 1 - \max_{i \in B_S}\{i\} \leq 0$, then the result is proven.
Otherwise, there is exactly one coalition $S' \subseteq N$, such that
$|S| = |S'|$ and $|H_{\gamma^T}(S')| - |H_{\gamma^{T'}}(S')| = |H_{\gamma^{T'}}(S)| - |H_{\gamma^T}(S)|$.
S' is defined as $S' = (S \setminus B_S) \cup \{h - j + 1, j \in B_S\}$. If $B'_S = S' \cap \{1, ..., h\}$, then

$$\begin{aligned} |H_{\gamma^T}(S')| - |H_{\gamma^{T'}}(S')| &= h - \min_{i \in B'_S}\{i\} + 1 - \max_{i \in B'_S}\{i\} \\ &= h - (h - \max_{i \in B_S}\{i\} + 1) + 1 - (h - \min_{i \in B_S}\{i\} + 1) \\ &= \min_{i \in B_S}\{i\} - 1 + \max_{i \in B_S}\{i\} - h = |H_{\gamma^{T'}}(S)| - |H_{\gamma^T}(S)|, \end{aligned}$$

which concludes the proof.

Remark 3. *If the game is not almost-positive, even if it is superadditive or even convex, the efficiency of a chain can be higher than the efficiency of the star, as can be seen in Example 1 for the overhead game.*

4. Efficiency and Its Asymptotic Behavior in Some Specific Cases

In this section, we calculate the efficiency of the star, the chain and the wheel for a general superadditive symmetric and zero-normalized game. We particularize the obtained results for the three games considered in *Example 1*. Moreover, we analyze the asymptotic behavior of these efficiencies as the number of players/nodes increases.

4.1. The Star

Consider the communication situation (N, v, γ_1^S) in which (N, v) is a superadditive, symmetric and zero-normalized game and (N, γ_1^S) is a star with node 1 at the hub.

In this case,

$$\mu_1^W(N, v, \gamma_1^S) = \sum_{S \subseteq N \setminus \{1\}} \frac{(n-s-1)!s!}{n!} [v(C_1^{S\cup\{1\}, \gamma_1^S}) - v(C_1^{S\cup\{1\}, \gamma_1^S} \setminus \{1\})]$$

$$= \sum_{S \subseteq N \setminus \{1\}} \frac{(n-s-1)!s!}{n!} [v(S \cup \{1\}) - v(S)]$$

$$= \sum_{s=0}^{n-1} \frac{(n-s-1)!s!}{n!} \binom{n-1}{s} [v(s+1) - v(s)] = \sum_{s=0}^{n-1} \frac{1}{n} [v(s+1) - v(s)],$$

where the symmetry of the game is used in the third equality.

The within groups Myerson value for node 2 (and thus for any other node different from the node 1) is

$$\mu_2^W(N, v, \gamma_1^S) = \sum_{S \subseteq N \setminus \{2\}} \frac{(n-s-1)!s!}{n!} [v(C_2^{S\cup\{2\}, \gamma_1^S}) - v(C_2^{S\cup\{2\}, \gamma_1^S} \setminus \{2\})]$$

$$= \sum_{S \subseteq N \setminus \{2\}, 1 \in S} \frac{(n-s-1)!s!}{n!} [v(S \cup \{2\}) - v(S)]$$

$$= \sum_{s=1}^{n-1} \frac{(n-s-1)!s!}{n!} \binom{n-2}{s-1} [v(s+1) - v(s)] = \sum_{s=1}^{n-1} \frac{s}{n(n-1)} [v(s+1) - v(s)].$$

The second equality holds as if the player at the hub does not belong to a coalition, the connected components of that coalition are singletons and worthless (as the game is zero-normalized). The third equality is due to the symmetry of the game.

Then,

$$\mathcal{E}(N, v, \gamma_1^S) = \frac{1}{v(n)} \sum_{i=1}^{n} \mu_i^W(N, v, \gamma_1^S)$$

$$= \frac{1}{v(n)} \left[\sum_{s=0}^{n-1} \frac{1}{n} [v(s+1) - v(s)] + (n-1) \sum_{s=1}^{n-1} \frac{s}{n(n-1)} [v(s+1) - v(s)] \right]$$

$$= \frac{1}{nv(n)} \sum_{s=1}^{n-1} (s+1)[v(s+1) - v(s)] = 1 - \sum_{s=2}^{n-1} \frac{v(s)}{nv(n)}.$$

Particularizing to game (N, v_1), with characteristic function $v_1(S) = s(s-1)$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_1, \gamma_1^S) = 1 - \frac{1}{n^2(n-1)} \sum_{s=2}^{n-1} (s^2 - s) = \frac{2(n+1)}{3n},$$

and thus

$$\lim_{n \to \infty} \mathcal{E}(N, v_1, \gamma_1^S) = \frac{2}{3}.$$

This means that when the number of players increases, the proportion of messages that can be sent without intermediaries approaches $\frac{2}{3}$.

If the game is (N, v_2), which has a characteristic function equal to $v_2(S) = s - 1$, for all $s \geq 1$, we have

$$\mathcal{E}(N, v_2, \gamma_1^S) = 1 - \frac{1 + 2 + 3 + ... + (n-2)}{n(n-1)} = \frac{n+2}{2n},$$

which converges to $\frac{1}{2}$, and thus, in the limit, only half of the total value can be retained by the grand coalition.

Finally, for the game (N, v_3), with $v_3(S) = 2^s - s - 1$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_3, \gamma_1^S) = 1 - \sum_{2}^{n-1} \frac{(2^s - s - 1)}{n(2^n - n - 1)},$$

and, after some manipulations,

$$\mathcal{E}(N, v_3, \gamma_1^S) = \frac{(n-1)2^{n+1} - n^2 - n + 2}{2n(s^n - n - 1)},$$

which converges to 1 when n tends to infinity.

In this case, the star is "efficient" in the sense that the total worth of the grand coalition can be retained by players as their number increases.

As it can be seen, the asymptotic behavior of the efficiency also depends on the game considered.

4.2. The Chain

Consider the communication situation (N, v, γ^C) in which (N, v) is a superadditive, symmetric and zero-normalized game and (N, γ^C) is a chain of n nodes, which we assume to be numbered consecutively starting with 1 at one end.

In this case, we have

$$\mathcal{E}(N, v, \gamma^C) = \frac{1}{v(n)} \left[\sum_{s=2}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} [v(s) - v(s-1)] + [v(n) - v(n-1)] \right]. \quad (1)$$

The proof of this result is in Appendix B.

In the following, we particularize the previous expression to the three games considered above. Also, we analyze their asymptotic behavior.

For game (N, v_1), with characteristic function $v_1(S) = s(s-1)$ for all $s \geq 1$, we have

$$\begin{aligned}
\mathcal{E}(N, v_1, \gamma^C) &= \frac{1}{n(n-1)} \left[\sum_{s=2}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} 2(s-1) + 2(n-1) \right] \\
&= \frac{1}{n(n-1)} \left[4(n+1) \sum_{s=2}^{n-1} \frac{s-1}{(s+2)(s+1)} + 2(n-1) \right] \\
&= \frac{4(n+1)}{n(n-1)} \sum_{s=2}^{n-1} \frac{1}{(s+1)} + \frac{6-2n}{n(n-1)}.
\end{aligned}$$

If $H_n = \sum_{i=1}^{n} \frac{1}{i}$, we have $H_n = \int_0^1 \frac{1-x^n}{1-x} dx \approx ln(n) + K$, with K a constant between 0 and 1, and thus,

$$\begin{aligned}
&\lim_{n \to \infty} \mathcal{E}(N, v_1, \gamma^C) \\
&= \lim_{n \to \infty} \frac{4(n+1)}{n(n-1)} \left[ln(n) - \frac{3}{2} + K \right] + \lim_{n \to \infty} \frac{6-2n}{n(n-1)} = 0.
\end{aligned}$$

For game (N, v_2), with characteristic function $v_2(S) = s - 1$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_2, \gamma^C) = \frac{1}{n-1}\left[\sum_{s=2}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} + 1\right]$$

$$= \frac{2(n+1)}{n-1}\sum_{s=2}^{n-1} \frac{1}{(s+2)(s+1)} + \frac{1}{n-1}$$

$$= \frac{2(n+1)}{(n-1)}\left(\frac{1}{3} - \frac{1}{(n+1)}\right) + \frac{1}{(n-1)} = \frac{2n-1}{3(n-1)},$$

and thus, $\lim_{n\to\infty} \mathcal{E}(N, v_2, \gamma^C) = \frac{2}{3}$.

For game (N, v_3), with characteristic function $v_3(S) = 2^s - s - 1$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_3, \gamma^C) = \frac{2(n+1)}{2^n - n - 1}\sum_{s=2}^{n-1} \frac{2^{s-1} - 1}{(s+2)(s+1)} + \frac{1}{2^n - n - 1}(2^{n-1} - 1),$$

and thus, $\lim_{n\to\infty} \mathcal{E}(N, v_3, \gamma^C) = \frac{1}{2}$.

4.3. The Wheel

Consider the communication situation (N, v, γ^W) in which (N, v) is a superadditive, symmetric and zero-normalized game and (N, γ^W) is a wheel of n nodes, which we assume to be numbered consecutively. We have

$$\sum_{i=1}^{n} \mu_i^W(N, v, \gamma^W)$$

$$= \sum_{s=1}^{n-2} ns \sum_{k=0}^{n-s-2} \binom{n-s-2}{k} \frac{(n-s-k)!(s+k-1)!}{n!}[v(s) - v(s-1)]$$

$$+ \frac{n(n-1)1!(n-2)!}{n!}[v(n-1) - v(n-2)] + \frac{n0!(n-1)!}{n!}[v(n) - v(n-1)]$$

$$= \sum_{s=1}^{n-2} 2n \frac{s!}{(s+2)!} \sum_{k=0}^{n-s-2} \frac{\binom{n-s-k}{n-s-k-2}\binom{s+k-1}{k}}{\binom{n}{n-s-2}}[v(s) - v(s-1)]$$

$$+ [v(n-1) - v(n-2)] + [v(n) - v(n-1)]$$

$$= \sum_{s=1}^{n-2} \frac{2n}{(s+1)(s+2)}[v(s) - v(s-1)] + [v(n) - v(n-2)],$$

and thus,

$$\mathcal{E}(N, v, \gamma^W) = \frac{1}{v(n)}\left[\sum_{s=2}^{n-2} \frac{2n}{(s+1)(s+2)}[v(s) - v(s-1)] + [v(n) - v(n-2)]\right].$$

In the following, we calculate expressions for the efficiency of the wheel in the three games considered, and its asymptotic behavior as the number of players increases.

For game (N, v_1), with characteristic function $v_1(S) = s(s-1)$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_1, \gamma^W) = \frac{1}{v(n)}\left[\sum_{s=2}^{n-2} \frac{2n}{(s+1)(s+2)}[v(s) - v(s-1)] + [v(n) - v(n-2)]\right]$$

$$= \frac{2n}{n(n-1)}\left[\sum_{s=2}^{n-2}\frac{s(s-1)-(s-1)(s-2)}{(s+1)(s+2)}\right] + \frac{1}{n(n-1)}[n(n-1)-(n-2)(n-3)]$$

$$= \frac{4}{(n-1)}\left[\sum_{s=2}^{n-2}\frac{s-1}{(s+1)(s+2)}\right] + [1 - \frac{(n-2)(n-3)}{n(n-1)}]$$

$$= \frac{4}{(n-1)}\left[\sum_{s=2}^{n-2}\frac{-2}{s+1} + \frac{3}{s+2}\right] + [1 - \frac{(n-2)(n-3)}{n(n-1)}]$$

$$= \frac{4}{(n-1)}\left[\sum_{s=4}^{n-1}\frac{1}{s} + \frac{3}{n} - \frac{2}{3}\right] + [1 - \frac{(n-2)(n-3)}{n(n-1)}],$$

and thus, $\lim_{n\to\infty} \mathcal{E}(N, v_1, \gamma^W) = 0$.

For game (N, v_2), with characteristic function $v_2(S) = s - 1$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_2, \gamma^W) = \frac{1}{v(n)}\left[\sum_{s=2}^{n-2}\frac{2n}{(s+1)(s+2)}[v(s)-v(s-1)] + [v(n)-v(n-2)]\right]$$

$$= \frac{2n}{(n-1)}\left[\sum_{s=2}^{n-2}\frac{1}{(s+1)(s+2)}\right] + \frac{2}{(n-1)} = \frac{2n}{(n-1)}\left[\sum_{s=2}^{n-2}\frac{1}{s+1} - \frac{1}{s+2}\right] + \frac{2}{(n-1)}$$

$$= \frac{2n}{(n-1)}(\frac{1}{3} - \frac{1}{n}) + \frac{2}{(n-1)} = \frac{2n(n-3)}{3n(n-1)} + \frac{2}{(n-1)},$$

and thus, $\lim_{n\to\infty} \mathcal{E}(N, v_2, \gamma^W) = \frac{2}{3}$.

For game (N, v_3), with characteristic function $v_3(S) = 2^s - s - 1$ for all $s \geq 1$, we have

$$\mathcal{E}(N, v_3, \gamma^W)\frac{1}{v(n)}\left[\sum_{s=2}^{n-2}\frac{2n}{(s+1)(s+2)}[v(s)-v(s-1)] + [v(n)-v(n-2)]\right]$$

$$= \frac{2n}{(2^n-n-1)}\left[\sum_{s=2}^{n-2}\frac{1}{(s+1)(s+2)}(2^{(s-1)}+1)\right] + \frac{3.2^{(n-2)}-2}{(2^n-n-1)},$$

and $\lim_{n\to\infty} \mathcal{E}(N, v_3, \gamma_W) = \frac{3}{4}$.

In the following Table 2, we summarize the asymptotic behavior of the efficiency of chains, stars and wheels for the tree games considered. It can be seen that the introduced measure is sensitive to the geometry of the network and the considered game.

Table 2. Asymptotic behavior of the efficiency of chains, stars and wheels.

$n \longmapsto \infty$	$\mathcal{E}(N, v_1, \gamma)$	$\mathcal{E}(N, v_2, \gamma)$	$\mathcal{E}(N, v_3, \gamma)$
$\gamma = \gamma^C$	0	$\frac{2}{3}$	$\frac{1}{2}$
$\gamma = \gamma_1^S$	$\frac{2}{3}$	$\frac{1}{2}$	1
$\gamma = \gamma^W$	0	$\frac{2}{3}$	$\frac{3}{4}$

5. A Family of Vulnerability Measures

In this section, we introduce a game theoretic family of measures for vulnerability of a network. We will assume the same setting of communication situations used for efficiency measures. In our approach, the vulnerability in (N, v, γ) can have two different origins.

On one hand it can be due to lack of connection between components of the graph. This causes a portion of the value of the grand coalition, $v(N)$, to be lost, and then players can only obtain $v^{\gamma}(N) = \sum_{C \in N/\gamma} v(C)$. Then, we define the vulnerability due to disconnection in (N, v, γ) as

$$\mathcal{V}_d(N, v, \gamma) = \frac{v(N) - v^{\gamma}(N)}{v(N)},$$

i.e., as the proportion of the total productivity in the game that is lost due to lack of connection in the graph.

On the other hand, by looking inside the components, the lack of links generates costs of intermediation. The proportion of the productivity lost because of these intermediation costs can be seen as another type of vulnerability of the network, the betweenness vulnerability $\mathcal{V}_b$. In our definition, it is given by

$$\mathcal{V}_b(N, v, \gamma) = \frac{\sum_{i \in N} \mu_i^B(N, v, \gamma)}{v(N)}.$$

We will assume that these two types of vulnerability are additive, and thus we propose the following definition.

Definition 2. *Given a communication situation $(N, v, \gamma) \in \mathcal{CS}_*^N$, we define a measure of vulnerability of the network (N, γ) for the game (N, v) as*

$$\mathcal{V}(N, v, \gamma) = \frac{v(N) - v^{\gamma}(N)}{v(N)} + \frac{\sum_{i \in N} \mu_i^B(N, v, \gamma)}{v(N)} = \mathcal{V}_d(N, v, \gamma) + \mathcal{V}_b(N, v, \gamma).$$

The following proposition includes some properties related to the range of values of the defined measures. The corresponding proofs can be found in Appendix C.

Proposition 6. *Given $(N, v, \gamma) \in \mathcal{CS}_*^N$, we have the following:*

(i) $0 \leq \mathcal{V}(N, v, \gamma) \leq 1$.
(ii) $\mathcal{V}(N, v, \gamma) + \mathcal{E}(N, v, \gamma) = 1$.
(iii) *For connected graphs (ceteris paribus), the disconnection vulnerability vanishes, i.e., if (N, γ) is a connected graph, then*
$$\mathcal{V}(N, v, \gamma) = \mathcal{V}_b(N, v, \gamma).$$
(iv) *In any communication situation in which the network is complete, vulnerabilities due to lack of connection and intermediation are both null, i.e., for all $(N, v, \gamma_N) \in \mathcal{CS}_*^N$, where (N, γ_N) is the complete graph, we have*
$$\mathcal{V}_d(N, v, \gamma_N) = \mathcal{V}_b(N, v, \gamma_N) = 0.$$
(v) *In the case of an empty graph, all vulnerability is due to disconnection, i.e., for all $(N, v, \gamma_{\emptyset}) \in \mathcal{CS}_*^N$, where $(N, \gamma_{\emptyset})$ is the empty graph, we have*
$$\mathcal{V}(N, v, \gamma_{\emptyset}) = \mathcal{V}_d(N, v, \gamma_{\emptyset}) = 1.$$

The following proposition lists some properties regarding the behavior of these measures in the face of changes in the network, ceteris paribus. Again, the corresponding proofs can be found in Appendix C.

Proposition 7. *Given $(N, v, \gamma) \in \mathcal{CS}_*^N$.*

(i) *If $i_0 \in N$ is a terminal node in the graph and additionally it is a null player in the game, the disconnection of i_0 neither $\mathcal{V}_b(N, v, \gamma)$ nor $\mathcal{V}_d(N, v, \gamma)$ changes, i.e., if $i_0 \in N$ is a null player in (N, v) such that $|\gamma_{i_0}| = 1$, it holds that $\mathcal{V}_b(N, v, \gamma) = \mathcal{V}_b(N, v, \gamma \setminus \gamma_{i_0})$ and $\mathcal{V}_d(N, v, \gamma) = \mathcal{V}_d(N, v, \gamma \setminus \gamma_{i_0})$.*

(ii) *If all the links incident in the node of an essential player in the game are broken, the disconnection vulnerability becomes* 1 *and the betweenness vulnerability vanishes, i.e., if* $i_0 \in N$ *is an essential player in* (N, v)*, we have that* $\mathcal{V}_b(N, v, \gamma \setminus \gamma_{i_0}) = 0$ *and* $\mathcal{V}_d(N, v, \gamma \setminus \gamma_{i_0}) = 1$.

(iii) $\mathcal{V}_d(N, v, \gamma)$ *reduces when adding links to a graph, ceteris paribus, i.e., given* (N, v, γ) *and* $(N, v, \gamma') \in \mathcal{CS}_*^N$ *with* $\gamma \subseteq \gamma'$*, we have* $\mathcal{V}_d(N, v, \gamma) \geq \mathcal{V}_d(N, v, \gamma')$*. This property is not satisfied for the other component of the vulnerability, the betweenness vulnerability, as the intermediation can increase. This is shown in the example following this proposition.*

Example 4. *Consider* (N, v, γ) *and* $(N, v, \gamma') \in \mathcal{CS}_*^N$ *with* $v = u_{\{1,2\}} + u_{\{1,3\}} - u_{\{1,2,3\}}$*,* $\gamma = \{\{1,2\}\}$ *and* $\gamma' = \{\{1,2\}, \{2,3\}\}$*. Then, it is easy to verify that* $\mathcal{V}_d(N, v, \gamma) = 0$ *and* $\mathcal{V}_d(N, v, \gamma') = \frac{1}{3}$.

The following proposition includes other properties relating to the efficiency measures corresponding to communication situations with different games. The proofs can be found in Appendix C.

Proposition 8. $\mathcal{V}$ *satisfies the following properties:*

(i) *The strategic equivalence of games preserves the vulnerability, i.e., given* (N, v, γ) *and* $(N, w, \gamma) \in \mathcal{CS}_*^N$*,* (N, v) *and* (N, w) *being strategically equivalent, we have that* $\mathcal{V}_d(N, v, \gamma) = \mathcal{V}_d(N, w, \gamma)$ *and* $\mathcal{V}_b(N, v, \gamma) = \mathcal{V}_b(N, w, \gamma)$.

(ii) *The betweenness vulnerability of an unconnected graph can be written in terms of the betweenness vulnerability of the graph restricted to different components, i.e., suppose* $(N, v, \gamma) \in \mathcal{CS}_*^N$ *and* $N/\gamma = \{C_1, C_2, \ldots, C_r\}$*, then*

$$\mathcal{V}_b(N, v, \gamma) = \sum_{j=1}^{r} \frac{v(C_j)}{v(N)} \mathcal{V}_b(N, v_{|C_j}, \gamma_{|C_j}).$$

(iii) *The betweenness vulnerability can be written as a linear combination of betweenness vulnerabilities in communication situations corresponding to unanimity games (with a fixed graph), i.e., given* $(N, v, \gamma) \in \mathcal{CS}_*^N$*, we have*

$$\mathcal{V}_b(N, v, \gamma) = \sum_{\emptyset \neq S \subseteq N} \frac{\Delta_v(S)}{v(N)} \mathcal{V}_b(N, u_S, \gamma).$$

6. Conclusions and Final Remarks

In this paper, we have introduced a family of efficiency measures and a family of vulnerability measures for a network from a game theory perspective. We have assumed that actors can use the network for different purposes or with different social or economic interests modeled by means of a cooperative game or transferable utility. In this way, both families are parameterized by the game. With a fixed game, both the efficiency and vulnerability measures we have introduced can be considered normalized value functions, as in [21,22].

In our proposal, given the game, the efficiency of a network is the proportion of the value of the grand coalition corresponding to the within group Myerson values. And thus, it can be seen as the proportion of the productivity of the grand coalition that the network allows players to retain. But also, for social networks, it is related to the proportion of total centrality due to communication. The other ratio, corresponding to expenses or centrality through intermediation, can be seen as a measure of network vulnerability.

The proposed measures satisfy properties that can be considered appealing when measuring efficiency or vulnerability in a network. The calculation of efficiency and vulnerability for different networks and specific purposes has provided us with values and asymptotic behaviors that seem consistent with the (somewhat vague) idea of what both should measure.

Given the increasingly extensive literature on the use of game theory to measure centrality in networks, we believe that some of the ideas presented here may be useful in defining other theoretical measures of efficiency and vulnerability arising from different centralities. Examples include the Shapley value, Owen value and Banzhaf value for connectivity games ([29]), which have been widely used to analyze terrorist networks. See, for example, [9,26–28,30].

As the Myerson value and its decomposition are Shapley values, the results in [27,32] are useful to calculate the measures introduced here.

Open and relevant problems that can be considered are the analysis of the relation between these measures and others existing (not necessarily game-theoretic) in the literature and the asymptotic behavior of the measures in other different structures and games than those studied here.

Author Contributions: Conceptualization, C.M.M. and E.O.; methodology, C.M.M. and E.O.; software, C.M.M. and E.O.; validation, C.M.M. and E.O.; formal analysis, C.M.M. and E.O.; investigation, C.M.M. and E.O.; resources, C.M.M. and E.O.; data curation, C.M.M. and E.O.; writing—original draft preparation, C.M.M. and E.O.; writing—review and editing, C.M.M. and E.O.; visualization, C.M.M. and E.O.; supervision, C.M.M. and E.O.; project administration, C.M.M. and E.O.; funding acquisition, C.M.M. and E.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been partially supported by the "Plan Nacional de I+D+i" of the Spanish Government under the project PID2020-116884GB-I00.

Data Availability Statement: Data sharing is not applicable to this article.

Acknowledgments: We would like to thank two anonymous referees and the editor for all their suggestions and comments that has helped us to deeply improve this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Proof of Proposition 1.

(i) Given that (N, v) is a non-null, superadditive and zero-normalized game, we have the following:

(a) $v(N) > 0$.

(b) $\sum_{i\in N} \mu_i^W(N, v, \gamma)$

$$= \sum_{i\in N} \sum_{S\subseteq N\setminus\{i\}} \frac{(n-s-1)!s!}{n!}[v(C_i^{S\cup\{i\},\gamma}) - v(C_i^{S\cup\{i\},\gamma} \setminus \{i\})] \geq 0.$$

(c) $\sum_{i\in N} \mu_i^B(N, v, \gamma)$

$$= \sum_{i\in N} \sum_{S\subseteq N\setminus\{i\}} \frac{(n-s-1)!s!}{n!}[v(C_i^{S\cup\{i\},\gamma} \setminus \{i\}) - \sum_{C\in(C_i^{S\cup\{i\},\gamma}\setminus\{i\})_{|\gamma}} v(C)] \geq 0,$$

and thus, using $a)$ and $b)$, we obtain $\mathcal{E}(N, v, \gamma) \geq 0$. On the other hand, as

$$\sum_{i\in N} \mu_i^W(N, v, \gamma) + \sum_{i\in N} \mu_i^B(N, v, \gamma) = \sum_{i\in N} \mu_i(N, v, \gamma) = v^\gamma(N),$$

we have $\mathcal{E}(N, v, \gamma) \leq \frac{v^\gamma(N)}{v(N)}$.

The second part is straightforward because, if a graph is connected, then $v^\gamma(N) = v(N)$.

(ii) To prove that $\mathcal{E}(N, v, \gamma_N) = 1$, we have

$$\sum_{i\in N} \mu_i^W(N, v, \gamma)$$

$$= \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(C_i^{S \cup \{i\}, \gamma_N}) - v(C_i^{S \cup \{i\}, \gamma_N} \setminus \{i\})]$$

$$= \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(S \cup \{i\}) - v(S)]$$

$$= \sum_{i \in N} Sh_i(N, v) = v(N),$$

the first equality holding by the definition of the within groups Myerson value; the second one, because of the connected component of $S \cup \{i\}$ in (N, γ_N), for all $S \subseteq N \setminus \{i\}$, $i \in N$, is $S \cup \{i\}$; the third one because of the definition of the Shapley value; and the last one using the efficiency of the Shapley value.
To prove that $\mathcal{E}(N, v, \gamma_\emptyset) = 0$, we have

$$\sum_{i \in N} \mu_i^W(N, v, \gamma)$$

$$= \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(C_i^{S \cup \{i\}, \gamma_\emptyset}) - v(C_i^{S \cup \{i\}, \gamma_\emptyset} \setminus \{i\})]$$

$$= \sum_{i \in N} \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(\{i\}) - v(\emptyset)] = \sum_{i \in N} v(\{i\}) = 0,$$

as, in this case, the connected component of $S \cup \{i\}$ in $(N, \gamma_\emptyset)$ for all $S \subseteq N \setminus \{i\}$, $i \in N$, is $\{i\}$, the sum of the weights of the marginal contributions in the Shapley formula is equal to one, and the game is zero-normalized.
□

Proof of Proposition 2.

(i) If i is a null player in (N, v), then

$$\mu_i^W(N, v, \gamma) = \mu_i^W(N, v, \gamma \setminus \gamma_i) = 0.$$

On the other hand, for all $j \in N, j \neq i$,

$$\mu_j^W(N, v, \gamma) = \sum_{S \subseteq N \setminus \{j\}} \frac{(n-s-1)!s!}{n!} [v(C_j^{S \cup \{j\}, \gamma}) - v(C_j^{S \cup \{j\}, \gamma} \setminus \{j\})].$$

If $i \notin S$, then $C_j^{S \cup \{j\}, \gamma} = C_j^{S \cup \{j\}, \gamma \setminus \{l\}}$, and if $i \in S$, then either $C_j^{S \cup \{j\}, \gamma} = C_j^{S \cup \{j\}, \gamma \setminus \{l\}}$ or $C_j^{S \cup \{j\}, \gamma} = C_j^{S \cup \{j\}, \gamma \setminus \{l\}} \cup \{i\}$. As i is a null player in (N, v), in any case,

$$v(C_j^{S \cup \{j\}, \gamma}) = v(C_j^{S \cup \{j\}, \gamma \setminus \{l\}})$$

and

$$v(C_j^{S \cup \{j\}, \gamma} \setminus \{j\}) = v(C_j^{S \cup \{j\}, \gamma \setminus \{l\}} \setminus \{j\}).$$

Then, for all $k \in N$, $\mu_k^W(N, v, \gamma) = \mu_k^W(N, v, \gamma \setminus \{l\})$ and the result is proved.

(ii) For all $j \in N, j \neq i$,

$$\mu_j^W(N, v, \gamma) = \sum_{S \subseteq N \setminus \{j\}} \frac{(n-s-1)!s!}{n!} [v(C_j^{S \cup \{j\}, \gamma}) - v(C_j^{S \cup \{j\}, \gamma} \setminus \{j\})] = 0,$$

as i does not belong to the connected component of j (and thus $i \notin C_j^{S \cup \{j\}, \gamma} \setminus \{j\}$) because it is isolated.

On the other hand,

$$\mu_i^W(N,v,\gamma) = \sum_{S\subseteq N\setminus\{i\}} \frac{(n-s-1)!s!}{n!}[v(C_i^{S\cup\{i\},\gamma}) - v(C_i^{S\cup\{i\},\gamma}\setminus\{i\})]$$

$$= \sum_{S\subseteq N\setminus\{i\}} \frac{(n-s-1)!s!}{n!}[v(\{i\}) - v(\emptyset)] = v(\{i\}) = 0,$$

as, for all $S \subseteq N \setminus \{i\}$, $C_i^{S\cup\{i\},\gamma} = \{i\}$ and (N,v) is zero-normalized.

(iii) Under the hypothesis

$$\mu_i^B(N,v,\gamma_i^S) = \sum_{S\subseteq N\setminus\{i\}} \frac{(n-s-1)!s!}{n!}[v(C_i^{S\cup\{i\},\gamma_i^S}\setminus\{i\}) - \sum_{C\in\left(C_i^{S\cup\{i\},\gamma_i^S}\setminus\{i\}\right)/\gamma_i^S} v(C)],$$

which is equal to zero because i is an essential player and therefore the value of any coalition to which he does not belong is null.

On the other hand, for $j \neq i$,

$$\mu_j^B(N,v,\gamma) = \sum_{S\subseteq N\setminus\{j\}} \frac{(n-s-1)!s!}{n!}[v(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}) - \sum_{C\in\left(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}\right)/\gamma_i^S} v(C)].$$

If $i \notin S \subseteq N \setminus \{j\}$, then $C_j^{S\cup\{j\},\gamma_i^S} = \{j\}$, and thus $C_j^{S\cup\{j\},\gamma_i^S} \setminus \{j\} = \emptyset$. Then, in such a case,

$$v(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}) - \sum_{C\in\left(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}\right)/\gamma_i^S} v(C)] = v(\emptyset) - v(\emptyset) = 0.$$

If, on the other hand, $i \in S \subseteq N \setminus \{j\}$, then $C_j^{S\cup\{j\},\gamma_i^S}$ is a connected set, and thus

$$v(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}) - \sum_{C\in\left(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}\right)/\gamma_i^S} v(C)]$$

$$= v(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}) - v(C_j^{S\cup\{j\},\gamma_i^S}\setminus\{j\}) = 0.$$

Finally, for all $j \in N$,
$\mu_j(N,v,\gamma_i^S) = \mu_j^W(N,v,\gamma_i^S) + \mu_j^B(N,v,\gamma_i^S) = \mu_j^W(N,v,\gamma_i^S) + 0$, and then

$$\mathcal{E}(N,v,\gamma_i^S) = \frac{\sum_{j\in N} \mu_j^W(N,v,\gamma_i^S)}{v(N)} = \frac{\sum_{j\in N} \mu_j(N,v,\gamma_i^S)}{v(N)} = \frac{v(N)}{v(N)} = 1,$$

given that, as it is known, the Myerson value satisfies component efficiency and the star is a connected graph.

(iv) As proven in Proposition 3.3. (pag. 590) of [20], for communication situations in which the game is convex, we have that

$$\mu_i^W(N,v,\gamma) \leq \mu_i^W(N,v,\gamma\cup\{l\}), \text{ for all } i \in N,$$

and thus the result.

□

Proof of Proposition 3.

(i) If (N, v) and (N, w) are strategically equivalent and zero-normalized, then for all $S \subseteq N$, $w(S) = \alpha v(S)$, $\alpha \in \mathbb{R}^+$. As μ^W is clearly linear in the game, we have

$$\mathcal{E}(N, w, \gamma) = \frac{\sum_{i \in N} \mu_i^W(N, w, \gamma)}{w(N)} = \frac{\sum_{i \in N} \mu_i^W(N, \alpha v, \gamma)}{\alpha v(N)}$$

$$= \frac{\alpha \sum_{i \in N} \mu_i^W(N, v, \gamma)}{\alpha v(N)} = \mathcal{E}(N, v, \gamma).$$

(ii) We have

$$\mathcal{E}(N, v, \gamma) = \frac{1}{v(N)} \sum_{i \in N} \mu_i^W(N, v, \gamma) = \frac{1}{v(N)} \sum_{k=1}^{r} \sum_{i \in C_k} \mu_i^W(N, v, \gamma)$$

$$\mu_i^W(N, v_{|C_k}, \gamma_{|C_k}) = \frac{1}{v(N)} \sum_{k=1}^{r} v(C_k) \sum_{i \in C_k} \frac{\mu_i^W(C_k, v_{|C_k}, \gamma_{|C_k})}{v(C_k)}$$

$$= \sum_{k=1}^{r} \frac{v(C_i)}{v(N)} \mathcal{E}(C_k, v_{|C_k}, \gamma_{|C_k}),$$

where the third equality holds as the within groups Myerson value of a player only depends on the component to which it belongs. The other equalities are straightforward.

(iii)

$$\mathcal{E}(N, \sum_{k=1}^{r} v_k, \gamma) = \frac{1}{(\sum_{k=1}^{r} v_k)(N)} \sum_{i \in N} \mu_i^W(N, \sum_{k=1}^{r} v_k, \gamma)$$

$$= \frac{1}{(\sum_{k=1}^{r} v_k)(N)} \sum_{i \in N} \sum_{k=1}^{r} \mu_i^W(N, v_k, \gamma) = \frac{1}{(\sum_{k=1}^{r} v_k)(N)} \sum_{k=1}^{r} \sum_{i \in N} \mu_i^W(N, v_k, \gamma)$$

$$= \sum_{k=1}^{r} \frac{v_k(N)}{(\sum_{k=1}^{r} v_k)(N)} \mathcal{E}(N, v_k, \gamma).$$

The first and third equalities are by the definition of the measure and the second one is by the additivity of μ^W.

(iv) The result is straightforwardly obtained particularizing (iii) of this proposition to communication situations $\{(N, \Delta_v(S) u_S, \gamma)\}_{\emptyset \neq S \subseteq N}$ and (N, v, γ) and using the linearity of the Myerson value.

(v) We have that

$$\mathcal{E}(N, u_S, \gamma) = 0 \text{ if } \mathcal{MCS}(S, N, \gamma) = \emptyset$$

and

$$\mathcal{E}(N, u_S, \gamma) = \sum_{i \in N} \mu_i^W(N, u_S, \gamma)$$

$$= s[\sum_{i=1}^{r} \frac{1}{|S_i|} - \sum_{i<j} \frac{1}{|S_i \cup S_j|} + ...(-1)^{r+1} \frac{1}{|\cup_{i=1}^{r} S_i|}] = s\alpha_\gamma(S),$$

if $\mathcal{MCS}(S, N, \gamma) = \{S_1, ..., S_r\}$, where the second equality is obtained in [20].

□

Appendix B

Proof of Equation (1). Consider the communication situation (N, v, γ^C) in which (N, v) is a superadditive, symmetric and zero-normalized game and (N, γ^C) is a chain of n nodes, which we assume to be numbered consecutively starting with 1 at one end.

In this case, we have that

$$\sum_{i=1}^{n} \mu_i^W(N, v, \gamma^C) =$$

$$\sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} \sum_{l=0}^{r-1} \sum_{k=0}^{n-r-s-1} s \binom{r-1}{l} \binom{n-r-s-1}{k} \frac{(n-s-l-k)!(s+l+k-1)!}{n!}$$
$$\times [v(s) - v(s-1)]$$

$$+ 2 \sum_{s=1}^{n-2} \sum_{k=0}^{n-s-1} s \binom{n-s-1}{k} \frac{(n-s-k)!(s+k-1)!}{n!} [v(s) - v(s-1)]$$

$$+ 2(n-1) \frac{1!(n-2)!}{n!} [v(n-1) - v(n-2)] + 2n \frac{0!(n-1)!}{n!} [v(n) - v(n-1)].$$

The first term corresponds to the efficiency generated by the n players from coalitions that have a component of size s, $s = 1, ..., n-2$, having these coalitions of other components than variable size. Moreover, the component of size s possibly has other components "before" and "after" in the chain, labeled from 1 to n. The second term corresponds to coalitions in which the component of size s is on the left or on the right of the chain. In the last two terms, it is assumed, respectively, that the component has size $n-1$ or n.

We have

$$\sum_{l=0}^{r-1} \sum_{k=0}^{n-r-s-1} s \binom{r-1}{l} \binom{n-r-s-1}{k} \frac{(n-s-l-k)!(s+l+k-1)!}{n!}$$
$$\times [v(s) - v(s-1)]$$

$$= s \sum_{l=0}^{r-1} \frac{(r+l-1)!(s+l-1)!}{(r+s+1)!} \frac{(r-1)!}{l!(r-l-1)!}$$
$$\times \sum_{k=0}^{n-r-s-1} \frac{\binom{n-s-l-k}{n-r-s-k-1}\binom{s+l+k-1}{k}}{\binom{n}{n-r-s-1}} [v(s) - v(s-1)]$$
$$= s[v(s) - v(s-1)] \sum_{l=0}^{r-1} \frac{(r+l-1)!(s+l-1)!}{(r+s+1)!} \frac{(r-1)!}{l!(r-l-1)!}$$
$$= 2 \frac{s!}{(s+2)!} [v(s) - v(s-1)] \sum_{l=0}^{r-1} \frac{\binom{s+l-1}{l}\binom{r-l+1}{r-l-1}}{\binom{r+s+1}{r-1}}$$
$$= 2 \frac{s!}{(s+2)!} [v(s) - v(s-1)].$$

And thus,

$$\sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} \sum_{l=0}^{r-1} \sum_{k=0}^{n-r-s-1} s \binom{r-1}{l} \binom{n-r-s-1}{k} \frac{(n-s-l-k)!(s+l+k-1)!}{n!}$$
$$\times [v(s) - v(s-1)]$$

$$= \sum_{s=1}^{n-2} \sum_{r=1}^{n-s-1} 2 \frac{s!}{(s+2)!} [v(s) - v(s-1)] = \sum_{s=1}^{n-2} \frac{2(n-s-1)}{(s+2)(s+1)} [v(s) - v(s-1)].$$

On the other hand, for the second term,

$$2 \sum_{s=1}^{n-2} \sum_{k=0}^{n-s-1} s \binom{n-s-1}{k} \frac{(n-s-k)!(s+k-1)!}{n!} [v(s) - v(s-1)]$$
$$= 2 \sum_{s=1}^{n-2} \frac{s!}{(s+1)!} \sum_{k=0}^{n-s-1} \frac{\binom{n-s-k}{n-s-k-1}\binom{s+k-1}{k}}{\binom{n}{n-s-1}} [v(s) - v(s-1)]$$
$$= 2 \sum_{s=1}^{n-2} \frac{1}{s+1} [v(s) - v(s-1)].$$

Finally,

$$\sum_{i=1}^{n} \mu_i^W(N, v, \gamma_C)$$
$$= \sum_{s=1}^{n-2} \frac{2(n-s-1)}{(s+2)(s+1)} [v(s) - v(s-1)] + \sum_{s=1}^{n-2} \frac{2}{(s+1)} [v(s) - v(s-1)]$$
$$+ \frac{2}{n} [v(n-1) - v(n-2)] + [v(n) - v(n-1)]$$
$$= \sum_{s=1}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} [v(s) - v(s-1)] + [v(n) - v(n-1)]$$
$$= \sum_{s=2}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} [v(s) - v(s-1)] + [v(n) - v(n-1)],$$

the last equality holding as for $s = 1$, and (N, v) a zero-normalized game, we have that $v(s) - v(s-1) = v(1) - v(0) = 0$.

And thus,

$$\mathcal{E}(N, v, \gamma^C) = \frac{1}{v(n)} \left[\sum_{s=2}^{n-1} \frac{2(n+1)}{(s+2)(s+1)} [v(s) - v(s-1)] + [v(n) - v(n-1)] \right].$$

□

Appendix C

Proof of Proposition 6.

(i) As $v^\gamma(N) = \sum_{i \in N} \mu_i^W(N, v, \gamma) + \sum_{i \in N} \mu_i^B(N, v, \gamma)$, substituting in the definition of $\mathcal{V}(N, v, \gamma)$,

$$\mathcal{V}(N, v, \gamma) = \frac{v(N) - v^\gamma(N)}{v(N)} + \frac{\sum_{i \in N} \mu_i^B(N, v, \gamma)}{v(N)}$$
$$= \frac{v(N) - \sum_{i \in N} \mu_i^W(N, v, \gamma)}{v(N)} = 1 - \mathcal{E}(N, v, \gamma),$$

and, given that $0 \le \mathcal{E}(N, v, \gamma) \le 1$, the result is proven.

(ii) If $(N, v, \gamma) \in \mathcal{CS}_*^N$ and (N, γ) is a connected graph, we have that $v^\gamma(N) = v(N)$ and thus $\mathcal{V}_d(N, v, \gamma) = 0$.

(iii) For $(N, v, \gamma_N) \in \mathcal{CS}^N_*$ we have that $v^{\gamma_N}(N) = v(N)$, and thus $\mathcal{V}_d(N, v, \gamma_N) = 0$. Then,

$$\mathcal{V}(N, v, \gamma_N) = \mathcal{V}_d(N, v, \gamma_N) + \mathcal{V}_b(N, v, \gamma_N) = \mathcal{V}_b(N, v, \gamma_N).$$

On the other hand, using *i)* and that the efficiency in communication situations with the complete graph is 1,

$$\mathcal{V}(N, v, \gamma_N) = \mathcal{V}_b(N, v, \gamma_N) = 1 - \mathcal{E}(N, v, \gamma_N) = 1 - 1 = 0.$$

(iv) For all $(N, v, \gamma_\emptyset) \in \mathcal{CS}^N_*$, we have that $v^{\gamma_\emptyset}(N) = \sum_{i \in N} v(\{i\} = 0$ (since the game is zero-normalized) and $\mu^B(N, v, \gamma_\emptyset)$ is the null vector. Therefore, $\mathcal{V}(N, v, \gamma_\emptyset) = \mathcal{V}_d(N, v, \gamma_\emptyset) = 1$.

□

Proof of Proposition 7.

(i) If $i_0 \in N$ is such that $|\gamma_{i_0}| = 1$ and it is also a null player, we will prove that $\mu^B(N, v, \gamma) = \mu^B(N, v, \gamma \setminus \gamma_{i_0})$, which implies that $\mathcal{V}_b(N, v, \gamma) = \mathcal{V}_b(N, v, \gamma \setminus \gamma_{i_0})$. For i_0,

$$\mu^B_{i_0}(N, v, \gamma) = \sum_{S \subseteq N \setminus \{i_0\}} \frac{(n-s-1)!s!}{n!} [v(C^{S \cup \{i_0\}, \gamma}_{i_0} \setminus \{i_0\}) - \sum_{C \in (C^{S \cup \{i_0\}, \gamma}_{i_0} \setminus \{i_0\})_{|\gamma}} v(C)]$$

$$= \sum_{S \subseteq N \setminus \{i_0\}} \frac{(n-s-1)!s!}{n!} [v(C^{S \cup \{i_0\}, \gamma}_{i_0} \setminus \{i_0\}) - v(C^{S \cup \{i_0\}, \gamma}_{i_0} \setminus \{i_0\})] = 0.$$

On the other hand,

$$\mu^B_{i_0}(N, v, \gamma \setminus \gamma_{i_0}) = \sum_{S \subseteq N \setminus \{i_0\}} \frac{(n-s-1)!s!}{n!} [v(C^{S \cup \{i_0\}, \gamma \setminus \gamma_{i_0}}_{i_0} \setminus \{i_0\})$$

$$- \sum_{C \in (C^{S \cup \{i_0\}, \gamma \setminus \gamma_{i_0}}_{i_0} \setminus \{i_0\})_{|\gamma \setminus \gamma_{i_0}}} v(C)]$$

$$= \sum_{S \subseteq N \setminus \{i_0\}} \frac{(n-s-1)!s!}{n!} [v(\emptyset) - v(\emptyset)] = 0.$$

For $i \neq i_0$,

$$\mu^B_i(N, v, \gamma) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(C^{S \cup \{i\}, \gamma}_i \setminus \{i\}) - \sum_{C \in (C^{S \cup \{i\}, \gamma}_i \setminus \{i\})_{|\gamma}} v(C)]$$

$$= \sum_{S \subseteq N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} [v(C^{S \cup \{i\}, \gamma \setminus \gamma_{i_0}}_i \setminus \{i\}) - \sum_{C \in (C^{S \cup \{i\}, \gamma \setminus \gamma_{i_0}}_i \setminus \{i\})_{|\gamma \setminus \gamma_{i_0}}} v(C)],$$

the equality holding as for all $S \subseteq N \setminus \{i\}$, we have that

$$C^{S \cup \{i\}, \gamma}_i = C^{S \cup \{i\}, \gamma \setminus \gamma_{i_0}}_i$$

or

$$C^{S \cup \{i\}, \gamma}_i = C^{S \cup \{i\}, \gamma \setminus \gamma_{i_0}}_i \cup \{i_0\},$$

and thus,

$$v(C^{S \cup \{i\}, \gamma}_i) = v(C^{S \cup \{i\}, \gamma \setminus \gamma_{i_0}}_i),$$

because i_0 is a null player. Similarly for each $C \in (C_i^{S\cup\{i\},\gamma} \setminus \{i\})_{|\gamma}$.
Then,

$$\sum_{i\in N} \mu_i^B(N,v,\gamma) = \sum_{i\in N} \mu_i^B(N,v,\gamma \setminus \gamma_{i_0}),$$

and thus,

$$\mathcal{V}_b(N,v,\gamma) = \mathcal{V}_b(N,v,\gamma \setminus \gamma_{i_0}).$$

On the other hand,

$$v^{\gamma}(N) - v^{\gamma\setminus\gamma_{i_0}}(N) = v(C_{i_0}^{N,\gamma}) - v(C_{i_0}^{N,\gamma} \setminus \{i_0\}) \setminus v(\{i_0\}) = 0,$$

the first equality holding as the unique connected component that changes is the one to which i_0 belongs, and the second equality holding because i_0 is a null player in the zero-normalized game (N,v). Therefore, $\mathcal{V}_d(N,v,\gamma) = \mathcal{V}_d(N,v,\gamma \setminus \gamma_{i_0})$.

(ii) If $i_0 \in N$ is an essential player in (N,v) and isolated in $(N,\gamma \setminus \gamma_{i_0})$, we have that

$$v^{\gamma\setminus\gamma_{i_0}}(N) = \sum_{C\in N/(\gamma\setminus\gamma_{i_0})} v(C) = 0,$$

the second equality holding as all the components to which i_0 does not belong have null value and the game is zero-normalized.

(iii) Suppose (N,v,γ) and $(N,v,\gamma') \in \mathcal{CS}_*^N$ with $\gamma \subseteq \gamma'$, then

$$v^{\gamma'}(N) = \sum_{C'\in N/\gamma'} v(C') \geq \sum_{C\in N/\gamma} v(C) = v^{\gamma}(N),$$

the inequality holding as possibly several (disjoint) connected components in (N,γ) become a unique connected component in (N,γ'), and the game is superadditive. As a consequence,

$$\mathcal{V}_d(N,v,\gamma) = 1 - \frac{v^{\gamma}(N)}{v(N)} \geq 1 - \frac{v^{\gamma'}(N)}{v(N)} = \mathcal{V}_d(N,v,\gamma'),$$

and the result is proven.

□

Proof of Proposition 8.

(i) If (N,v) and (N,w) are strategically equivalent, there exists $\alpha \in \mathbb{R}^+$, such that $w = \alpha v$ and thus $w(N) = \alpha v(N)$, $w^{\gamma}(N) = \alpha v^{\gamma}(N)$, and $\mu^B(N,w,\gamma) = \alpha\mu^B((N,v,\gamma)$ because the between groups Myerson value is linear in the game (ceteris paribus). Then, we have the result.

(ii) Under the given hypothesis,

$$\mathcal{V}_b(N,v,\gamma) = \frac{\sum_{i=1}^{n} \mu_i^B(N,v,\gamma)}{v(N)} = \frac{\sum_{j=1}^{r}\sum_{i\in C_j} \mu_i^B(N,v,\gamma_{|C_j})}{v(N)}$$

$$= \frac{\sum_{j=1}^{r}\sum_{i\in C_j} \mu_i^B(N,v_{|C_j},\gamma_{|C_j})}{v(N)} = \sum_{j=1}^{r} \frac{v(C_j)}{v(N)} \frac{\sum_{i\in C_j} \mu_i^B(N,v_{|C_j},\gamma_{|C_j})}{v(C_j)}$$

$$= \sum_{j=1}^{r} \frac{v(C_j)}{v(N)} \frac{\sum_{i\in C_j} \mu_i^B(N,v_{|C_j},\gamma_{|C_j})}{v_{|C_j}(C_j)} = \sum_{j=1}^{r} \frac{v(C_j)}{v(N)} \mathcal{V}_b(N,v_{|C_j},\gamma_{|C_j}),$$

where the third equality holds as the between groups Myerson value of a player only depends on the component to which it belongs. The other equalities are straightforward.

(iii) Using the linearity of the between groups Myerson value, we have

$$\mathcal{V}_b(N,v,\gamma) = \frac{\sum_{i=1}^{n} \mu_i^B(N,v,\gamma)}{v(N)} = \frac{\sum_{i\in N} \sum_{\emptyset\neq S\subseteq N} \Delta_v(S)\mu_i^B(N,u_S,\gamma)}{v(N)}$$

$$= \sum_{\emptyset\neq S\subseteq N} \frac{\Delta_v(S)}{v(N)} \sum_{i\in N} \mu_i^B(N,u_S,\gamma) = \sum_{\emptyset\neq S\subseteq N} \frac{\Delta_v(S)}{v(N)} \mathcal{V}_b(N,u_S,\gamma).$$

□

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