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Some Connectivity Parameters of Interval-Valued Intuitionistic Fuzzy Graphs with Applications

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Abstract: Connectivity in graphs is useful in describing different types of communication systems like neural networks, computer networks, etc. In the design of any network, it is essential to evaluate the connections based on their strengths. In this manuscript, we comprehensively describe various connectivity parameters related to interval-valued intuitionistic fuzzy graphs (IVIFGs). These are the generalizations of the parameters defined for fuzzy graphs, interval-valued fuzzy graphs, and intuitionistic fuzzy graphs. First, we introduce interval-valued intuitionistic fuzzy bridges (IVIF bridges) and interval-valued intuitionistic fuzzy cut-nodes (IVIF cut-nodes). We discuss the many characteristics of these terms as well as establish the necessary and sufficient conditions for an arc to become an IVIF-bridge and a vertex to be an IVIF-cutnode. Furthermore, we initiate the concepts of interval-valued intuitionistic fuzzy cycles (IVIFCs) and interval-valued intuitionistic fuzzy trees (IVIFTs) and explore few relationships among them. In addition, we introduce the notions of fuzzy blocks and fuzzy block graphs and extend these terms as interval-valued fuzzy blocks (IVF-blocks) and interval-valued intuitionistic fuzzy block graphs (IVIF-block graphs). Finally, we provide the application of interval-valued intuitionistic fuzzy trees (IVIFTs) in a road transport network.

Keywords: IVIF-blocks; IVIF-block graphs; IVIF-bridges; IVIF-cycles; IVIF-trees

MSC: 057C2; 03E72

Citation: Guan, H.; Khan, W.A.; Saleem, S.; Arif, W.; Shafi, J.; Khan, A. Some Connectivity Parameters of Interval-Valued Intuitionistic Fuzzy Graphs with Applications. *Axioms* **2023**, *12*, 1120. https://doi.org/10.3390/axioms12121120

Academic Editors: Yilun Shang, Zeeshan Saleem Mufti and Hsien-Chung Wu

Received: 22 October 2023
Revised: 20 November 2023
Accepted: 29 November 2023
Published: 13 December 2023

1. Introduction

In 1965, Zadeh [1] initiated the concept of a fuzzy sets (FSs) which became an efficient tool to solve the problems containing uncertainties. After this, the theory of FSs became an important area of research for researchers in various fields of natural and social sciences including medical, engineering, artificial intelligence, management sciences, etc. Due to a wide range of applications of FSs, several generalizations of FSs have been introduced. Zadeh himself introduced the generalization of FSs named interval-valued fuzzy sets (IVFSs) [2]. Afterwards, Atanassov [3] proposed the concepts of intuitionistic fuzzy sets (IFSs). Furthermore, Gargov and Atanassov [4] explored the notion of interval-valued intuitionistic fuzzy sets (IVIFSs) with the aim of amalgamating the view and measuring the complex nature of the human mind with more accuracy. The concepts of IVIFS was extensively applied in different fields like decision making [5], medical diagnosis [6], etc.

A decade after Zadeh's landmark paper, Rosenfeld [7] and Yeh et al. [8] introduced fuzzy graphs (FGs). Subsequently, the term M-strong FGs was introduced by Bhutani et al. [9]. Some important terms related to FGs were initiated by Bhattacharya [10]. Peng and Mordeson [11] introduced some important operations on FGs. Different terms of FGs like cycles, co-cycles, and fuzzy line graphs were presented by Mordeson and Nair [12]. Mathew and Sunitha [13] initiated the term node connectivity based on different types of arcs of FGs. The term interval-valued fuzzy graphs (IVFGs) was explored in [14]. Cayley-IVFGs were explored in [15]. Sequentially, the term intuitionistic fuzzy graph (IFGs) was explored in 1994 in [16]. Numerous new operations along with their basic properties on IFGs were introduced in [17]. Similarly, the shortest paths in different networks were examined through IFGs in [18]. The order, size, etc. of IFGs were explored by Nagoor Gani et al. [19]. The extension of IFG, named intuitionistic fuzzy competition graphs [20], intuitionistic fuzzy tolerance graphs [21], etc., have also been discussed in the literature. Similarly, the concepts of connectivity in intuitionistic fuzzy incidence graphs and their applications were explored in [22]. The term internally stable set in IFGs was explained in [23]. Some new terms of IFGs were addressed in [24] The concept of a strong-IVIFG was discussed in [25]. The connectivity status of the IFG with applications in banking was introduced in [26]. Intuitionistic fuzzy trees (IFTs) were introduced by Chountas et al. [27,28]. Various terms related to IF-trees such as the radius, distance, and eccentricity were introduced by Thamizhendhi and Parvathi [29]. The study of IFGs and IFTs were also conducted by many researchers [30,31]. Interval-valued intuitionistic fuzzy graphs (IVIFGs) were studied in [32]. Similarly, many generalizations of IVIFGs were explored in [33–35]. Recently, some other extensions of FGs with applications in various fields have been explored in [36,37]. In the same vein, many new terms related to fuzzy graphs have been explored in [38,39].

The unique structure of IVIFSs is useful to express the membership and non-membership degrees in terms of intervals, which is more helpful in dealing with uncertainties. Also, expressing uncertain information using IVIFSs can effectively avoid the loss of any information. In our study, we introduce various connectivity parameters related to IVIFGs. These are the generalized forms of concepts related to FGs, IVFGs, and IFGs. Moreover, the notions described for IVIFGs combine the qualitative characteristics of FGs, IVFGs, and IFGs.

The novelty and motivations of our work are as follows.

1. We introduce different of strong arcs like $\alpha_{[\phi^-,\phi^+]}$-strong, $\beta_{[\phi^-,\phi^+]}$-strong, $\delta_{[\phi^-,\alpha^+]}$-arc, $\alpha_{[\psi^-,\psi^+]}$-strong, $\beta_{[\psi^-,\psi^+]}$-strong, and $\delta_{[\psi^-,\psi^+]}$-strong. Based on these arcs, we introduce the different types of path and discuss the connectivity of IVIFGs.
2. We extend many notions of IFGs towards IVIFGs such as interval-valued intuitionistic fuzzy bridges (IVIFBs), interval-valued intuitionistic fuzzy cut-notes (IVIFCNs), interval-valued intuitionistic fuzzy trees (IVIFTs), interval-valued intuitionistic fuzzy cycles (IVIFCs), etc.
3. We can hardly find discussions about the fuzzy blocks and fuzzy block graphs. In our work, we first introduce the concepts of interval-valued fuzzy blocks (IVF-blocks) and interval-valued fuzzy block graphs (IVF-block graphs). Then, we extend these notions as interval-valued intuitionistic fuzzy blocks (IVIF-block graphs) and interval-valued intuitionistic fuzzy block graphs (IVIF-block graphs).
4. A traffic control system, computer networking, or any real-life problem involving networking works best in an interval-valued intuitionistic fuzzy environment as compared to the fuzzy, interval-valued fuzzy, and intuitionistic fuzzy environments. As evidence, we provide the application of IVIFTs towards the traffic control system.

The rest of this manuscript is organized as follows: In Section 2, basic useful terminologies are provided. In Section 3, we introduce the notions of interval-valued intuitionistic fuzzy bridges (IVIFBs) and interval-valued intuitionistic fuzzy cut notes (IVIFCNs). Afterwards, the ideas of interval-valued intuitionistic fuzzy trees (IVIFTs) and interval-valued intuitionistic fuzzy cycles (IVIFCs) are discussed in Section 4. In Section 5, first we introduce the notions of interval-valued fuzzy-blocks (IVF-blocks) and interval-valued fuzzy block graphs (IVF-block graphs). Then, we shift these terms towards IVIFGs. In Section 6, we present the application of IVIFTs towards the traffic control system along with numerical calculations. Finally, we conclude our study with future directions.

2. Preliminaries

Definition 1 ([1])**.** *A fuzzy set (FS) is a pair (V, ϕ), where ϕ is a membership function that allocates values to each entity of V from $[0,1]$.*

Definition 2 ([40])**.** *An IVFS is a pair (V, ϕ), where ϕ is a membership function that allocate values in terms of the subintervals of $[0,1]$ to each member of V.*

Definition 3 ([41])**.** *An IFS T on U is of the form*

$$T = \{(u, \phi_T(u), \psi_T(u)) | u \in U\}$$

where both $\phi_T(u)$ and $\psi_T(u)$ are the functions such as $\phi_T : U \to [0,1]$ and $\psi_T : U \to [0,1]$ are the membership values and non-membership values of the given entity $u \in U$, respectively, with $0 \leq \phi_T(u) + \psi_T(u) \leq 1$.

Definition 4 ([4])**.** *An IVIFS T on U can be described as*

$$T = \{(u, \phi_T(u), \psi_T(u)) | u \in U\}$$

where $\phi_T(u) \subset [0,1]$ and $\psi_T(u) \subset [0,1]$ are intervals representing the membership and non-membership values, respectively, with $sup\phi_T(u) + sup\phi_T(u) \leq 1$.

Definition 5 ([7])**.** *A pair $G = (C, D)$ is a fuzzy graph (FG) defined on a crisp graph $G^* = (V, E)$, where $C = \{\phi_C\}$, $D = \{\psi_D\}$, $\phi_C : V \to [0,1]$, and $\psi_D : V \times V \to [0,1]$, satisfying $\psi_D(u, v) \leq min\{\phi_C(u), \phi_C(v)\}$ for all $u, vs. \in V$.*

Definition 6 ([42])**.** *An interval-valued fuzzy graph (IVFG) defined on a crisp graph $G^* = (V, E)$ is $G = (C, D)$, where $C = \{\phi_C\} \subset [0,1]$ and $D = \{\phi_D\} \subset [0,1]$ are the IVFSs on V and $E = V \times V$, respectively.*

Definition 7 ([16])**.** *An intuitionistic fuzzy graph (IFG) defined on a crisp graph $G^* = (V, E)$ is a pair $G = (C, D)$, where $C = (\psi_C, \phi_C)$ is an IFS on V and $D = (\psi_D, \phi_D)$ is an IFS on $E = V \times V$, and for all $u, v \in V$*

$$\psi_D(u, v) \leq min(\psi_C(u), \psi_C(v))$$
$$\phi_D(u, v) \leq max(\phi_C(u), \phi_C(v))$$

Definition 8 ([43])**.** *An IFG $G = (C, D)$ defined on a crisp graph $G^* = (V, E)$ is said to be complete, if for each $(u, v) \in E$,*

$$\psi_D(u, v) = min(\psi_C(u), \psi_C(v))$$

$$\phi_D(u, v) = max(\phi_C(u), \phi_C(v))$$

Definition 9 ([43])**.** *In an IFG $G = (C, D)$, a ϕ-path $P(u, v)$ is a set of distinct vertices $P : u = v_0, v_1, \ldots, v_m = v$ such that $\phi_D(v_{j-1}, v_j) > 0$, and a ψ-path if $\psi_D(v_{j-1}, v_j) > 0$, where $j = 1, 2, \ldots, m$. If $\phi_D(v_{j-1}, v_j) > 0$ and $\psi_D(v_{j-1}, v_j) > 0$, then we call it a path in IFG.*

Definition 10 ([43])**.** *The ϕ-strength of connectedness among the vertices u_i and u_j in an IFG G is $\phi_D^\infty = \sup\{\phi_D^k(u_i, u_j) | k = 1, 2, \ldots, m\}$, and the ψ-strength of connectedness among u_i and u_j is $\psi_D^\infty = \inf\{\psi_D^k(u_i, u_j) | k = 1, 2, \ldots, m\}$. Similarly, $\phi_D^\infty(u_i, u_j) = \sup\{\phi_D(u, u_1) \wedge$*

$\phi_D(u_1, u_2) \wedge ... \wedge \phi_D(u_{k-1}, u) | u, u_1, u_2, \ldots, u_{k-1}\}$, *whenever the nodes* u, v *are joined by* ϕ*-paths having the length* k*. If the nodes* u, v *are joined by* ψ*-paths having the length* k*, then* $\psi_D^{\infty}(u_i, u_j) = \inf\{\psi_D(u, u_1) \vee \psi_D(u_1, u_2) \vee ... \vee \psi_D(u_{k-1}, v)\}$.

Definition 11 ([43])**.** *An IFG* $G = (C, D)$ *described in Definition 7, is said to be*

1. ϕ*-connected, if there is a* ϕ*-path between each pair of the vertices.*
2. ψ*-connected, if there is a* ψ*-path between each pair of vertices.*
3. *a strong connected graph, whenever there exists a path in between each pair of vertices.*

Definition 12 ([44])**.** *An arc* (v_1, v_2) *in an IFG* $G = (C, D)$ *is said to be*

1. α_{ϕ}*-strong if* $\phi_D(v_1, v_2) > \phi_D'^{\infty}(v_1, v_2)$ *and* α_{ψ}*-strong if* $\psi_D(v_1, v_2) < \psi_D'^{\infty}(v_1, v_2)$*.*
2. β_{ϕ}*-strong if* $\phi_D(v_1, v_2) = \phi_D'^{\infty}(v_1, v_2)$ *and* β_{ψ}*-strong if* $\psi_D(v_1, v_2) = \psi_D'^{\infty}(v_1, v_2)$*.*
3. δ_{ϕ}*-arc if* $\phi_D(v_1, v_2) < \phi_D'^{\infty}(v_1, v_2)$ *and* δ_{ψ}*-arc if* $\psi_D(v_1, v_2) > \psi_D'^{\infty}(v_1, v_2)$*.*

We refer the reader to [44] for further discussions on bridges, trees, etc., of IFGs.

Important note: Throughout this article, we use the notations $\phi_C^-(u)$ for $\phi_{LC}(u)$ and $\phi_C^+(u)$ for $\phi_{UC}(u)$, respectively. Similarly, we use $\psi_D^-(uv)$ for $\psi_{LD}(u)$ and $\psi_D^+(uv)$ for $\phi_{UD}(uv)$.

3. IVIF-Bridges and IVIF-Cut Nodes

In this section, we introduce IVIF-bridges and IVIF-cut nodes. We also provide some important results and examples related to them.

Definition 13. *An IVIFG* $G = (C, D)$ *is strong, if for all* $([u_i, u_j], [v_i, v_j]) \in D$, $\phi_D([u_i, u_j], [v_i, v_j]) = \min\{\phi_C[u_i, u_j]\}, \phi_C[v_i, G = (C, D)$ *is strong, if for all* $([u_i, u_j], [v_i, v_j]) \in D$, $\phi_D([u_i, u_j], [v_i, v_j]) = \min(\phi_C[u_i, u_j], \phi_C[v_i, v_j])$ *and* $\psi_D([u_i, u_j], [v_i, v_j]) = \max(\psi_C[u_i, u_j], \phi_C[v_i, v_j])$*. And* $G = (C, D)$ *is complete if for all* $u_i = u_0, u_1, u_2..., u_n = u_j, v_i = v_0, v_1, v_2, ..., v_n = v_j \in C$ $\phi_D([u_i, u_j], [v_i, v_j]) = \min(\phi_C[u_i, u_j], \phi_C[v_i, v_j])$ *and* $\psi_D([u_i, u_j], [v_i, v_j]) = \max(\psi_C[u_i, u_j], \phi_C[v_i, v_j])$*.*

Remark 1. *In an IVIFG, if* $\phi_D([u_i, u_j], [v_i, v_j]) = \psi_D([u_i, u_j], [v_i, v_j]) = 0$ *for any* i *and* j*, then there is no edge among the vertices* $u_i = u_0, u_1, u_2..., u_n = u_j$ *and* $v_i = v_0, v_1, v_2, ..., v_n = v_j$*. However, if* $\phi_D([u_i, u_j], [v_i, v_j]) \neq \psi_D([u_i, u_j], [v_i, v_j]) = 0$*, then there is an edge among the vertices* $u_i = u_0, u_1, u_2..., u_n = u_j$ *and* $v_i = v_0, v_1, v_2, ..., v_n = v_j$*.*

Definition 14. *In an IVIFG* $G = (C, D)$*, a path* P *in* G*, i.e.,* $P : u = u_0, u_1, \ldots, u_m = v$ *is a sequence of distinct vertices. The path* P *is called a* $[\phi^-, \phi^+]$*-path from* u *to* v *if* $\phi_D(u_{j-1}, u_j) > 0$*, and is called a* $[\psi^-, \psi^+]$*-path whenever* $\psi_D(u_{j-1}, u_j) > 0$ *for* $j = 1, 2, \ldots, m$*. We call* P *a path in an IVIFG if it has both* $[\phi^-, \phi^+]$*-path and* $[\psi^-, \psi^+]$*-path. Hence,* P *is a* $(u - v)$ *path of length* m*. If* $u = v$ *and* $m > 3$*, then the path* P *is a* $[\phi^-, \phi^+]$*-cycle and* $[\psi^-, \psi^+]$*-cycle which turns into be a cycle.*

Definition 15. *Let* $G = (C, D)$ *be an IVIFG, then it is called a* $[\phi^-, \phi^+]$*-connected, whenever there is a* $[\phi^-, \phi^+]$*-path among each couple of the nodes in* G*, and is called a* $[\psi^-, \psi^+]$*-connected, if there is a* $[\psi^-, \psi^+]$*-path among each pair of the nodes in* G*. Additionally, an IVIFG* G^* *is strongly connected, if there is a path among every pair of vertices.*

Definition 16. *If* $u_i, u_j \in C \subseteq G$*, then the* $[\phi^-, \phi^+]$*-strength of connectedness among the vertices* u_i, u_j *is* $\phi_D^{\infty} = \sup\{\phi_D^k(u_i, u_j) | k = 1, 2, \ldots, m\}$*, and the* $[\psi^-, \psi^+]$*-strength among* u_i *and* u_j *is* $\psi_D^{\infty} = \inf\{\psi_D^k(u_i, u_j) | k = 1, 2, \ldots, m\}$*. Similarly,* $\phi_D^{\infty}(u_i, u_j)$ *is* $\sup\{\phi_D(u, u_1) \wedge \phi_D(u_1, u_2) \wedge ... \wedge \phi_D(u_{k-1}, u) | u, u_1, u_2, \ldots, u_{k-1}, v \in C\}$*, whenever the nodes* u, v *are joined by* $[\phi^-, \phi^+]$*-paths having the length* k*. If the nodes* u, v *are joined by* $[\psi^-, \psi^+]$*-paths having the length* k*, then* $\psi_D^{\infty}(u_i, u_j)$ *is* $\inf\{\psi_D(u, u_1) \vee \psi_D(u_1, u_2) \vee ... \vee \psi_D(u_{k-1}, v) | u, u_1, u_2, ..., u_{k-1}, v \in C\}$*.*

Remark 2. *If there is a path P* = $[\phi^-, \phi^+]$*-path in an IVIFG* $G = (C, D)$ *from u to v, then the* $[\phi^-, \phi^+]$*-strength of P among the vertices u and v is denoted by* $[\phi^-, \phi^+]_P^\infty(u, v)$*. Also, if there is a path P that is a* $[\psi^-, \psi^+]$*-path in an IVIFG G from u to v, then the* $[\psi^-, \psi^+]$*-strength of P among the vertices u and v is denoted by* $[\psi^-, \psi^+]_P^\infty(u, v)$*. A path P in between a pair of the nodes u and v is a* $[\phi^-, \phi^+]$*-strongest* $(u - v)$ *path and a* $[\psi^-, \psi^+]$*-strongest* $(u - v)$ *path, if the* $[\phi^-, \phi^+]$*-strength is equal to* $[\phi^-, \phi^+]_P^\infty(u, v)$ *and the* $[\psi^-, \psi^+]$*-strength is equal to* $[\psi^-, \psi^+]_P^\infty(u, v)$*, respectively.*

Definition 17. *Let* $G = (C, D)$ *be an IVIFG, then an arc* (r, s) *in G is called* $[\phi^-, \phi^+]$*-strong and* $[\psi^-, \psi^+]$*-strong, if* $[\phi^-, \phi^+]_D(r, s) \geq [\phi^-, \phi^+]'^\infty_D(r, s)$*, and* $[\psi^-, \psi^+]_D(r, s) \leq [\psi^-, \psi^+]'^\infty_D(r, s)$*. And, an arc* (r, s) *in G is called strong if it is either* $[\phi^-, \phi^+]$*-strong or* $[\psi^-, \psi^+]$*-strong.*

Definition 18. *Let* $G = (C, D)$ *be an IVIFG. Then, an arc* (r, s) *in G is* $\alpha_{[\phi^-,\phi^+]}$*-strong,* $\beta_{[\phi^-,\phi^+]}$*-strong, and* $\delta_{[\phi^-,\phi^+]}$*-arc satisfies*

$$[\phi^-, \phi^+]_D(r, s) > [\phi^-, \phi^+]'^\infty_D(r, s)$$

$$[\phi^-, \phi^+]_D(r, s) = [\phi^-, \phi^+]'^\infty_D(r, s)$$

$$[\phi^-, \phi^+]_D(r, s) < [\phi^-, \phi^+]'^\infty_D(r, s)$$

and is said to be $\alpha_{[\psi^-,\psi^+]}$*-strong,* $\beta_{[\psi^-,\psi^+]}$*-strong and* $\delta_{[\psi^-,\psi^+]}$*-arc if*

$$[\psi^-, \psi^+]_D(r, s) < [\psi^-, \psi^+]'^\infty_D(r, s)$$

$$[\psi^-, \psi^+]_D(r, s) = [\psi^-, \psi^+]'^\infty_D(r, s)$$

$$[\psi^-, \psi^+]_D(r, s) > [\psi^-, \psi^+]'^\infty_D(r, s)$$

Example 1. *In an IVIFG* $G = (C, D)$ *shown in Figure 1, the arcs* (w_1, w_2) *and* (w_1, w_3) *are* $\alpha_{[\phi^-,\phi^+]}$*-strong arcs and* $\alpha_{[\psi^-,\psi^+]}$*-strong arcs, which implies that the arcs* (w_1, w_2) *and* (w_1, w_3) *are strong arcs. However, the arc* (w_3, w_4) *is neither an* $\alpha_{[\phi^-,\phi^+]}$*-strong arc, nor* $\beta_{[\phi^-,\phi^+]}$*-strong arc, and nor* $\delta_{[\phi^-,\phi^+]}$*-arc, but it is a* $\delta_{[\psi^-,\psi^+]}$*-arc. Similarly, the arc* (w_2, w_4) *is neither an* $\alpha_{[\phi^-,\phi^+]}$*-strong arc, nor* $\beta_{[\phi^-,\phi^+]}$*-strong arc, and nor* $\delta_{[\phi^-,\phi^+]}$*-arc, but it is an* $\alpha_{[\psi^-,\psi^+]}$*-strong arc. But, the arc* (w_2, w_3) *is* $\delta_{[\phi^-,\phi^+]}$*-arc and* $\delta_{[\psi^-,\psi^+]}$*-arc.*

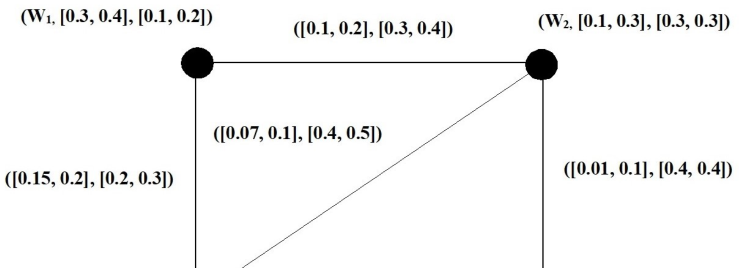

Figure 1. Interval-valued intuitionistic fuzzy graph.

Definition 19. *Let* $P : u = u_0, u_1, \ldots, u_m = v$ *be a* $[\phi^-, \phi^+]$*-path from u to v in an IVIFG* $G = (C, D)$*. The path P is called a* $[\phi^-, \phi^+]$*-strong (*$\alpha_{[\phi^-,\phi^+]}$*-strong), if the arcs* (u_{j-1}, u_j)*, where* $j = 1, 2, ..., m$ *are* $[\phi^-, \phi^+]$*-strong (*$\alpha_{[\phi^-,\phi^+]}$*-strong). Again, P is a* $[\psi^-, \psi^+]$*-path, then the path P is called* $[\psi^-, \psi^+]$*-strong (*$\alpha_{[\psi^-,\psi^+]}$*-strong), if the arcs* (u_{j-1}, u_j)*, where* $j = 1, 2, ..., m$ *are* $[\psi^-, \psi^+]$*-strong (*$\alpha_{[\psi^-,\psi^+]}$*-strong).*

A path P in G is termed as strong (α-strong), and is either $[\phi^-, \phi^+]$*-strong or* $[\psi^-, \psi^+]$*-strong (*$\alpha_{[\phi^-,\phi^+]}$*-strong or* $\alpha_{[\psi^-,\psi^+]}$*-strong).*

Example 2. *In an IVIFG $G = (C, D)$ given in Figure 2a, $C = \{u_1, u_2, u_3\}$ and $D = \{(u_1, u_2), (u_2, u_3), (u_1, u_3)\}$. The arc (u_1, u_2) is in G is $\alpha_{[\phi^-,\phi^+]}$-strong arc but the arc (u_1, u_2) is neither $\alpha_{[\psi^-,\psi^+]}$-strong arc, nor $\beta_{[\psi^-,\psi^+]}$-strong arc, and nor $\delta_{[\psi^-,\psi^+]}$-arcl the arc (u_1, u_3) is $\alpha_{[\phi^-,\phi^+]}$-arc, but the arc (u_1, u_3) is neither $\alpha_{[\psi^-,\psi^+]}$-strong arc, nor $\beta_{[\psi^-,\psi^+]}$-strong arc, and nor $\delta_{[\psi^-,\psi^+]}$-arc; and the arc (u_2, u_3) is $\delta_{[\phi^-,\phi^+]}$-strong but the arc (u_2, u_3) is neither $\alpha_{[\psi^-,\psi^+]}$-strong arc, nor $\beta_{[\phi^-,\phi^+]}$-strong arc, and nor $\delta_{[\psi^-,\psi^+]}$-arc. Similarly, in an IVIFG shown in Figure 2b, the arc (v_1, v_3) is the only $\alpha_{[\psi^-,\psi^+]}$-strong arc.*

In Figure 2a, the path $P_1 : u_1, u_2$ is a $\alpha_{[\phi^-,\phi^+]}$-strong path while the path $(P_3)^* : v_1, v_3$ in Figure 2b is a $\alpha_{[\psi^-,\psi^+]}$-strong path. Thus, both the paths P_1 and P_3^* are α-strong paths.

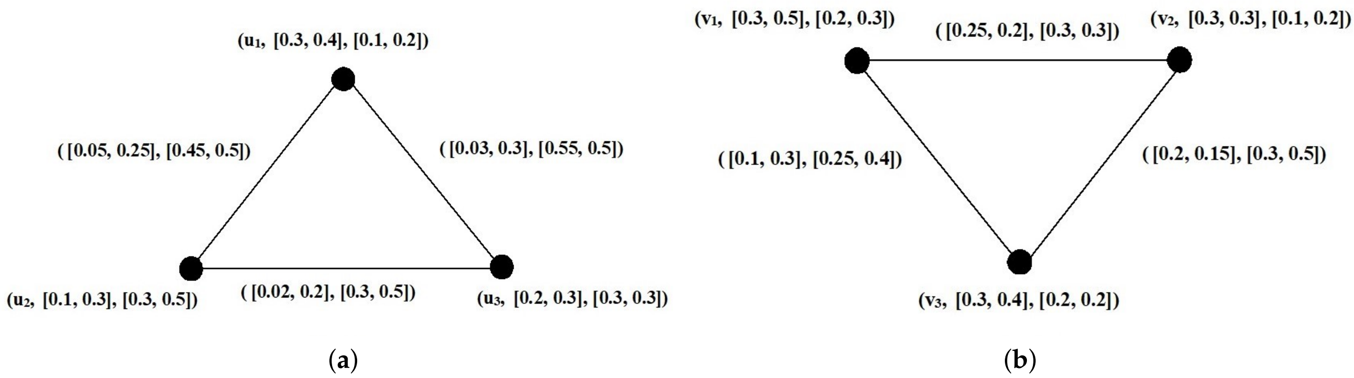

Figure 2. Interval-valued intuitionistic fuzzy graphs (**a**,**b**).

Proposition 1. *If an IVIFG $G = (C, D)$ is $[\phi^-, \phi^+]$-connected, then there must exist a $[\phi^-, \phi^+]$-strong path among every couple of vertices of a graph G.*

Proof. It is straightforward. □

Proposition 2. *Let $G = (C, D)$ be a $[\psi^-, \psi^+]$-connected IVIFG. Then, there is a $[\psi^-, \psi^+]$-strong path in the set of vertices of graph G.*

Proof. Let G be an $[\psi^-, \psi^+]$-connected IVIFG. Then, there must exist a $[\psi^-, \psi^+]$-path between each couple of the vertices u, v. However, if the arc (u, v) between u, v is not a $[\psi^-, \psi^+]$-strong arc, then $[\psi^-, \psi^+]_D(u, v) > [\psi^-, \psi^+]'^{\infty}_D(u, v)$. Thus, there exists a $P = [\psi^-, \psi^+]$-path from the vertex u to v in which the $[\psi^-, \psi^+]$-strength of P is less than $[\psi^-, \psi^+]_D(u, v)$. Here, if a few arcs of the path P are not $[\psi^-, \psi^+]$-strong, then we have the same argument. Lastly, we must have a $[\psi^-, \psi^+]$-path from u to v which is $[\psi^-, \psi^+]$-strong. □

Remark 3. *The converse of Proposition 2 is not true, in general.*

Proposition 3. *If a $[\phi^-, \phi^+]$-path from the vertex u to v in an IVIFG $G = (C, D)$ is $\alpha_{[\phi^-,\phi^+]}$-strong, then this path P in G is a $[\phi^-, \phi^+]$-strongest $(u - v)$ path.*

Proof. Let $G = (C, D)$ be an IVIFG. Consider a path $P : u = u_0, u_1, u_2, ..., u_m = v$ in G which is an $\alpha_{[\phi^-,\phi^+]}$-strong and further assume that it is not a $[\phi^-, \phi^+]$-strongest $(u - v)$ path in G. Consider a $[\phi^-, \phi^+]$-strongest $(u - v)$ path in G which is expressible as $P' : u = u'_0, u'_1, u'_2, ..., u'_m = v$. Hence, for $j = 1, 2, 3, ..., m$, we have $[\phi^-, \phi^+]_D(u'_{j-1}, u'_j) > u^{\infty}_P(u, v)$. Moreover, both the paths P and P' make a cycle, denoted by C^*. In a path P, $[\phi^-, \phi^+]$-arc of C^* is the weakest arc. Assuming an arc (r, s) be the weakest $[\phi^-, \phi^+]$-arc in the path P. Let $(r - s)$ be the path in a cycle C^* not including $(r - s)$, which is represented by P''. Consequently, we have $[\phi^-, \phi^+]_D(r, s) \leq [\phi^-, \phi^+]''^{\infty}_P(r, s) \leq [\phi^-, \phi^+]_D(r, s)$, which shows that (r,s) is not identified to be an $\alpha_{[\phi^-,\phi^+]}$-strong arc, which disproves of the supposition. Hence, the path P in G is a $[\phi^-,\phi^+]$-strongest $(u-v)$ path. $\square$

Remark 4. *The converse of Proposition 3 is not true, i.e., if there exists a $[\phi^-,\phi^+]$-strongest $(u-v)$ path in G, then it is not necessary to be an $\alpha_{[\phi^-,\phi^+]}$-strong path.*

Proposition 4. *In an IVIFG $G=(C,D)$, if a $[\psi^-,\psi^+]$-path P from the vertex u to v is $\alpha_{[\psi^-,\psi^+]}$-strong, then the path P is a $[\psi^-,\psi^+]$-strongest $(u-v)$ path in G.*

Proof. Let $G=(C,D)$ be an IVIFG. Let $P: u=u_0,u_1,u_2,...,u_m=v$ in G be $\alpha_{[\psi^-,\psi^+]}$-strong, and assume that P in G is not a $[\psi^-,\psi^+]$-strongest $(u-v)$ path. In an IVIFG, a $[\psi^-,\psi^+]$-strongest $(u-v)$ path in G, denoted by $P^{'}: u=u^{'}_0,u^{'}_1,u^{'}_2,...,u^{'}_m=v$. Thus, for $j=1,2,3,...,m$, $[\psi^-,\psi^+]_D(u^{'}_{j-1},u^{'}_j)<v^{\infty}_P(u,v)$. Furthermore, a cycle C^* is formed by both the paths P and $P^{'}$. The weakest arc of C^* in the path P is $[\psi^-,\psi^+]$-arc. Let an arc (r,s) in the path P as the weakest $[\psi^-,\psi^+]$-arc. In cycle C^*, let $P^{''}$ be the $(r-s)$ path excluding $(r-s)$. Consequently, for an arc (r,s) not to be an $\alpha_{[\psi^-,\psi^+]}$-strong arc, we must have $[\psi^-,\psi^+]_D(r,s)\geq[\psi^-,\psi^+]^{''\infty}_P(r,s)\geq[\psi^-,\psi^+]_D(r,s)$. Which disproves our assumption. Hence, a path P in G is satisfied to be a $[\psi^-,\psi^+]$-strongest $(u-v)$ path. $\square$

Remark 5. *The converse of Proposition 4 does not need to be true, i.e., it is not mandatory to the $[\psi^-,\psi^+]$-strongest $(u-v)$ path in G to be an $\alpha_{[\psi^-,\psi^+]}$-strong path.*

Definition 20. *Let $G=(C,D)$ be an IVIFG. An arc (u,v), i.e., $([u_1,u_2],[v_1,v_2])$ in G is said to be an interval-valued intuitionistic fuzzy $[\phi^-,\phi^+]$-bridge (IVIF $[\phi^-,\phi^+]$-bridge), whenever an arc $([u_1,u_2],[v_1,v_2])$ in between some couple of the nodes is removed, which reduces the $[\phi^-,\phi^+]$-strength of connectedness among some couple of the nodes. Similarly, if the nodes $r,s\in C$, that is, $r=([r_1,r_2],[r_3,r_4])$ and $s=([s_1,s_2],[s_3,s_4])$ is such that there is an edge $([u_1,u_2],[v_1,v_2])$ in every $[\phi^-,\phi^+]$-strongest $(r-s)$ path. An arc (u,v), i.e., $([u_1,u_2],[v_1,v_2])$ in an IVIFG G is an interval-valued intuitionistic fuzzy $[\psi^-,\psi^+]$-bridge (IVIF $[\psi^-,\psi^+]$-bridge), and whenever this arc $([u_1,u_2],[v_1,v_2])$ is removed, it extends the $[\psi^-,\psi^+]$ connectedness strength among a few couples of the nodes. Similarly, the nodes $r,s\in C$, i.e., $r=([r_1,r_2],[r_3,r_4])$ and $s([s_1,s_2],[s_3,s_4])$, such that, in every $[\psi^-,\psi^+]$-strongest $(r-s)$ path, the arc $([u_1,u_2],[v_1,v_2])$ exists. An arc $([u_1,u_2],[v_1,v_2])$ in an IVIFG G is an interval-valued intuitionistic fuzzy bridge (IVIFB), which is either an IVIF $[\phi^-,\phi^+]$-bridge or an IVIF $[\psi^-,\psi^+]$-bridge.*

Definition 21. *Let $G=(C,D)$ be an IVIFG. When A node $u\in C$, where $u=([u_1,u_2],[u_3,u_4])$ in G is an interval-valued intuitionistic fuzzy $[\phi^-,\phi^+]$-cut node (IVIF $[\phi^-,\phi^+]$-cut node), and this u is removed, it minimizes the $[\phi^-,\phi^+]$ connectedness strength among a few other couples of nodes. Likewise, if the nodes $r,s\in C$ exist where $r=([r_1,r_2],[r_3,r_4])$ and $s=([s_1,s_2],[s_3,s_4])$ in such a way that a node u is involved in every $[\phi^-,\phi^+]$-strongest $(r-s)$ path. A node $u\in C$ where $u=([u_1,u_2],[u_3,u_4])$ in G is an interval-valued intuitionistic fuzzy $[\psi^-,\psi^+]$-cut node (IVIF $[\psi^-,\psi^+]$-cut node), whenever this node u is removed, the $[\psi^-,\psi^+]$ connectedness strength among a few other couples of nodes increases. Similarly, there exist $r,s\in C$ where $r=([r_1,r_2],[r_3,r_4])$ and $s=([s_1,s_2],[s_3,s_4])$, so that this u is involved in every $[\psi^-,\psi^+]$-strongest $(r-s)$path. A node $u\in C$, where $u=([u_1,u_2],[u_3,u_4])$ in an IVIFG G is called an interval-valued intuitionistic fuzzy cut-node (IVIFCN) if it is either an IVIF $[\phi^-,\phi^+]$-cut node or an IVIF $[\psi^-,\psi^+]$-cut node.*

Example 3. *Consider an IVIFG $G=(C,D)$ given in Figure 3, where $C=\{u,w,v,x\}$ and $D=\{(u,v),(v,w),(u,w),(w,x),(v,x)(u,x),(v,w)\}$. Here, the arc (u,v) is an $\delta_{[\phi^-,\phi^+]}$-arc and $\delta_{[\psi^-,\psi^+]}$-arc, (u,w) and (v,x) are $\delta_{[\phi^-,\phi^+]}$-arcs and $\alpha_{[\psi^-,\psi^+]}$-arcs, the arcs (u,x) and (v,w) are $\alpha_{[\phi^-,\phi^+]}$-strong and $\alpha_{[\psi^-,\psi^+]}$-strong arcs, i.e., (u,x) and (v,w) are strong arcs, while the arc (w,x) is an $\alpha_{[\phi^-,\phi^+]}$-strong and $\delta_{[\psi^-,\psi^+]}$-arc. Consequently, all the arcs in G are strong. Further to this, both the arcs (u,v) and (v,x) are neither an IVIF $[\phi^-,\phi^+]$-bridges, nor an IVIF $[\psi^-,\psi^+]$-*

bridges. Hence, the arcs (u,v) and (v,x) do not form IVIFBs. The arcs (u,x) and (v,w) are both IVIF $[\phi^-,\phi^+]$-bridges and IVIF $[\psi^-,\psi^+]$-bridges, i.e., both are IVIFBs. The arc (w,x) is an IVIF $[\phi^-,\phi^+]$-bridge for some couple of vertices as well as an IVIF $[\psi^-,\psi^+]$-bridge for some couple of vertices, which implies the IVIFB. Hence, all the arcs except (u,v) and (v,x) are forming IVIFBs. The vertex u is $[\psi^-,\psi^+]$-cut vertex for the pair of vertices v and x, and the vertex x is a $[\phi^-,\phi^+]$-cut vertex for the pair of vertices u and v, and the vertex w is both $[\phi^-,\phi^+]$, and the $[\psi^-,\psi^+]$-cut vertices are for the vertices u and v. Hence, all the vertices u, x, and w are IVIFCNs, except v.

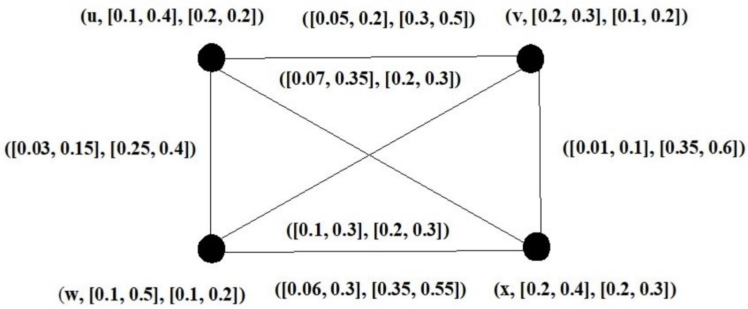

Figure 3. Interval-valued intuitionistic fuzzy graph.

Theorem 1. *The following statements are equivalent.*

1. *An arc (u,v) in G is an IVIF-bridge.*
2. *An arc (u,v) in G is not the weakest arc of any cycle.*

Theorem 2. *Let (u,v) be an IVIF-bridge of an IVIFG $G=(C,D)$. Then, $\phi_D^{\infty}(u,v) = \phi_D(u,v)$ and $\psi_D^{\infty}(u,v) = \psi_D(u,v)$.*

Proof. Assume an arc (u,v) is an IVIF-bridge and $\phi_D^{\infty}(u,v)$ exceeds over $\phi_D(u,v)$; however, $\psi_D^{\infty}(u,v)$ falls behind from $\psi_D(u,v)$. So, there is a strongest $(u-v)$ path that has a strength that is greater than $\phi_D(u,v)$ and less than $\psi_D(u,v)$. Also, every arc of $(u-v)$ has a strongest path that is has a strength greater than $\phi_D(u,v)$ and less than $\psi_D(u,v)$. Along with the arc (u,v), the path $(u-v)$ makes an IVIF-cycle, which considers (u,v) to be the weakest arc. This contradicts the assumption, i.e., that (u,v) is an IVIF-bridge. □

Theorem 3. *Let $G=(C,D)$ be an IVIFG and (u,v) be an arc in G. Then, we have the following.*

(i) *An arc (u,v) in G is an IVIF-$[\phi^-,\phi^+]$-bridge if and only if $[\phi^-,\phi^+]_D(u,v) > [\phi^-,\phi^+]'^{\infty}{}_D(u,v)$.*

(ii) *An arc (u,v) in G is an IVIF-$[\psi^-,\psi^+]$-bridge if and only if $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]'^{\infty}{}_D(u,v)$.*

(iii) *An arc (u,v) in G is called an IVIFB if and only if either $[\phi^-,\phi^+]_D(u,v) > [\phi^-,\phi^+]'^{\infty}{}_D(u,v)$ or $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]'^{\infty}{}_D(u,v)$.*

Proof. (i) Let an arc (u,v) in G be an IVIF $[\phi^-,\phi^+]$-bridge. Then, the nodes $r,s \in C$ exist in such a way that for all (u,v) in G, there is an arc which is a $[\phi^-,\phi^+]$-strongest (r,s) P-path. Now, let P' be a $[\phi^-,\phi^+]$-path from the node r to s, where the arc (u,v) does not include any of the $[\phi^-,\phi^+]$-paths from the node r to s, and the $[\phi^-,\phi^+]$-strength of it is at its maximum, where the arc (u,v) is not present.

So, both the paths P and P' make a cycle C^*, and there is an another $[\phi^-,\phi^+]$-path as $C^*-(u,v)$ which is a P''. We assert that there exists a $[\phi^-,\phi^+]$-strongest path as P'' in between nodes u and v. Consider the path P' in between the nodes u and v as a $[\phi^-,\phi^+]$-strongest path, and the $[\phi^-,\phi^+]$-strength between r and s does not reduce after deleting the arc (u,v). This proves that our supposition is wrong. Therefore, $[\phi^-,\phi^+]_P''^{\infty}(u,v) = [\phi^-,\phi^+]_D'^{\infty}(u,v)$. Furthermore, the cycle C^* of the weakest $[\phi^-,\phi^+]$-arc is held on P'; thus, $[\phi^-,\phi^+]_D(u,v) > [\phi^-,\phi^+]_P''^{\infty}(u,v)$ refers to the fact that $[\psi^-,\psi^+]_D(u,v) > [\phi^-,\phi^+]_D'^{\infty}(u,v)$.

On the other hand, if we assume that $[\phi^-,\phi^+]_D(u,v) > [\phi^-,\phi^+]'^{\infty}_D(u,v)$, then the $[\phi^-,\phi^+]$-strength of connectedness in between the nodes u and v is reduced by deleting an arc (u,v). Hence, the arc (u,v) in G is IVIF $[\phi^-,\phi^+]$-strong.

(ii) Let an arc (u,v) in G be an IVIF $[\psi^-,\psi^+]$-bridge. Then, the nodes $r,s \in C$ exist in such a way that, for all (u,v) in G, there is an arc which is a $[\psi^-,\psi^+]$-strongest (r,s) P-path. Now, let the path P' be a $[\psi^-,\psi^+]$-path from the node r to s in which the arc (u,v) is not included, and among all of the $[\psi^-,\psi^+]$-paths from node r to s, the $[\psi^-,\psi^+]$-strength is the minimum, where the arc (u,v) is not present.

So, both paths P and P' generate a cycle C^*, and there is an another $[\psi^-,\psi^+]$-path as $C^* - (u,v)$, i.e., P''. We claim that there exists a $[\psi^-,\psi^+]$-strongest path P'' in between the nodes u and v. Assuming that the path P' in between the nodes u and v is a $[\psi^-,\psi^+]$-strongest path, then the $[\psi^-,\psi^+]$-strength between r and s does not increase after deleting the arc (u,v). This proves that our supposition is wrong. Therefore, $[\psi^-,\psi^+]''^{\infty}_P(u,v) = [\psi^-,\psi^+]'^{\infty}_D(u,v)$. Furthermore, cycle C^* of the weakest $[\psi^-,\psi^+]$-arc is relying on P'. Thus, $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]''^{\infty}_P(u,v)$ implies that $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]'^{\infty}_D(u,v)$. On the other hand, if we assume that $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]'^{\infty}_D(u,v)$, then the $[\psi^-,\psi^+]$ connectedness strength among the nodes u and v increases by removing an arc (u,v). Hence, the arc (u,v) in G is an IVIF $[\psi^-,\psi^+]$-strength.

(iii) From parts (i) and (ii), this is evident. □

Corollary 1. *Let $G = (C,D)$ be an IVIFG and (u,v) be an arc in G. Then:*

(i) *An arc (u,v) is defined as an IVIF $[\phi^-,\phi^+]$-bridge if and only if this arc (u,v) is an $\alpha_{[\phi^-,\phi^+]}$-strong arc.*

(ii) *An arc (u,v) is defined as an IVIF $[\psi^-,\psi^+]$-bridge if and only if this arc (u,v) is an $\alpha_{[\psi^-,\psi^+]}$-strong arc.*

(iii) *An arc (u,v) in G is called an IVIFB if and only if either it is an $\alpha_{[\phi^-,\phi^+]}$-strong arc or $\alpha_{[\psi^-,\psi^+]}$-strong arc.*

Corollary 2. *In an IVIFG $G = (C,D)$, every IVIFB is a strong arc.*

Remark 6. *The converse of Corollary 2 is not valid and it means there is a strong arc (r,s) in IVIFG $G = (C,D)$, which does not need to be an IVIFB.*

Proposition 5. *Let (u,v) be an arc in an IVIFG $G = (V,E)$. Then,*

(i) *If the arc (u,v) in G is $[\phi^-,\phi^+]$-strong, then $[\phi^-,\phi^+]_D(u,v) = [\phi^-,\phi^+]^{\infty}_D(u,v)$.*

(ii) *If the arc (u,v) in G is $[\psi^-,\psi^+]$-strong, then $[\psi^-,\psi^+]_D(u,v) = [\psi^-,\psi^+]^{\infty}_D(u,v)$.*

(iii) *If the arc (u,v) in G is strong, then either $\phi^+]_D(u,v) = [\phi^-,\phi^+]^{\infty}_D(u,v)$ or $\psi^+]_D(u,v) = [\psi^-,\psi^+]^{\infty}_D(u,v)$.*

Proof. (i) In an IVIFG $G = (C,D)$, we evidently have $[\phi^-,\phi^+]_D(u,v) = [\phi^-,\phi^+]^{\infty}_D(u,v)$.
(ii) In an IVIFG $G = (C,D)$, we have $[\psi^-,\psi^+]_D(u,v) \geq [\psi^-,\psi^+]'^{\infty}_D(u,v)$. Let an arc (u,v) in G be an $[\psi^-,\psi^+]$-strong arc, then $[\psi^-,\psi^+]_D(u,v) \leq [\psi^-,\psi^+]'^{\infty}_D(u,v)$. If $[\psi^-,\psi^+]_D(u,v) = [\psi^-,\psi^+]'^{\infty}_D(u,v)$, then we have $[\psi^-,\psi^+]_D(u,v) = [\psi^-,\psi^+]^{\infty}_D(u,v)$. Moreover, if $[\psi^-,\psi^+]_D(u,v) < [\psi^-,\psi^+]'^{\infty}_D(u,v)$, then $[\psi^-,\psi^+]_D(u,v) = [\psi^-,\psi^+]^{\infty}_D(u,v)$.

(iii) By parts (i) and (ii), it follows directly. □

Proposition 6. *Let $G = (C,D)$ be an IVIFG. Then, an arc (r,s) in G is an IVIF $[\phi^-,\phi^+]$-bridge if and only if (r,s) in G is not present as a weakest $[\phi^-,\phi^+]$-arc in all the cycles in G.*

Proof. Consider the arc (r,s) of a cycle C^* as the weakest $[\phi^-,\phi^+]$-arc of C^* in G and the path P from the vertex r to s is denoted by $C^* - (r,s)$. So, $[\phi^-,\phi^+]_D(u,v) \leq [\phi^-,\phi^+]^{\infty}_P(u,v)$. Anyway, if we have $[\phi^-,\phi^+]'^{\infty}_D(u,v) \geq [\phi^-,\phi^+]^{\infty}_P(u,v)$, this further implies that $[\phi^-,\phi^+]_D(u,v) \leq [\phi^-,\phi^+]'^{\infty}_D(u,v)$. Finally, the arc (r,s) is not proven to be an $\alpha[\phi^-,\phi^+]$-strong arc. So, it is not defined to be an IVIF $[\phi^-, \phi^+]$-bridge by the Corollary 1. Conversely, this assumes that an arc (r,s) is not an IVIF $[\phi^-, \phi^+]$-bridge. By Corollary 1, it is not $\alpha[\phi^-, \phi^+]$-strong. Hence, $[\phi^-, \phi^+]_D(u,v) \leq [\phi^-, \phi^+]'^{\infty}_D(u,v)$. Consider a path P from the vertex r to s in $G-(r,s)$ satisfying $[\phi^-, \phi^+]'^{\infty}_D(u,v) = [\phi^-, \phi^+]^{\infty}_P(u,v)$. As a result, $[\phi^-, \phi^+]_D(u,v) \leq [\phi^-, \phi^+]^{\infty}_P(u,v)$. A cycle C^* is formed by adding the path denoted by P together with the arc (r,s). It is obvious that (r,s) is the weakest $[\phi^-, \phi^+]$-arc in cycle C^*, which disproves our supposition. □

Proposition 7. *Let $G = (C, D)$ be an IVIFG. An arc (r,s) in G is an IVIF $[\psi^-, \psi^+]$-bridge if and only if it is not present as a weakest $[\psi^-, \psi^+]$-arc in all the cycles of G.*

Proof. Consider an arc (r,s) of a cycle C^* as the weakest $[\psi^-, \psi^+]$-arc of C^* in G and the path P from the node r to s is $C^* - (r,s)$. So, $[\psi^-, \psi^+]_D(u,v) \geq [\psi^-, \psi^+]^{\infty}_P(u,v)$. However, if we have $[\psi^-, \psi^+]'^{\infty}_D(u,v) \leq [\psi^-, \psi^+]^{\infty}_P(u,v)$, this implies that $[\psi^-, \psi^+]_D(u,v) \geq [\psi^-, \psi^+]'^{\infty}_D(u,v)$. Consequently, the arc (r,s) is not proven to be an $\alpha[\psi^-, \psi^+]$-strong arc. So, it is not an IVIF $[\psi^-, \psi^+]$-bridge by Corollary 1.

Conversely, assume that an arc (r,s) is not an IVIF $[\psi^-, \psi^+]$-bridge. By Corollary 1, it is not $\alpha[\psi^-, \psi^+]$-strong. Hence, $[\psi^-, \psi^+]_D(u,v) \geq [\psi^-, \psi^+]'^{\infty}_D(u,v)$. Consider a path P from the vertex r to s in $G-(r,s)$ satisfying $[\psi^-, \psi^+]'^{\infty}_D(u,v) = [\psi^-, \psi^+]^{\infty}_P(u,v)$. As a result, $[\psi^-, \psi^+]_D(u,v) \geq [\psi^-, \psi^+]^{\infty}_P(u,v)$. A cycle C^* is formed by adding the path P together with arc (r,s). It is obvious that (r,s) is the weakest $[\psi^-, \psi^+]$-arc in cycle C^*, which disproves our supposition. □

Proposition 8. *In an IVIFG G, the arc (u,v) is an IVIF-bridge if and only if there does not exist any cycle in which (u,v) is the weakest bridge.*

Proposition 9. *Let $G = (C, D)$ be an $IVIFG$. Then, any vertex is an IVIF cut-node if and only if it is the common node between two $IVIF - bridges$.*

Proof. Let $G = (C, D)$ be an IVIFG with edges (u,v) and (r,s) and assume that the vertex v is an IVIF cut-node. The node v as an IVIF cut-node decrease the strength of connectedness when it is removed. Assume that at least one of the edges incident to v, namely (u,v) and (r,s), is not an IVIF-bridge. Hence, it is possible that such an edge in G is the weakest edge. Thus, by removing vertex v, the strength of connectedness is not decreased. Therefore, both of the given edges (u,v) and (r,s) are essentially IVIF-bridges.

On the other hand, assume that, in between two IVIF-bridges (u,v) and (r,s), v is the common node. To prove that v is the IVIF cut-node of the two IVIF-bridges (u,v) and (r,s), we need to show that removing either of these bridges reduces the strength of connectedness. Evidently, if two incident edges are IVIF-bridges, then their removal will decrease the strength of connectedness of the graph. Therefore, it follows that, if v is the IVIF cut-node of the bridges (u,v) and (r,s), then removing either of these bridges will reduce the strength of connectedness. Hence, the converse is also true. □

Theorem 4. *Let $G = (C, D)$ be an IVIFG that has a cycle C^*. Then, any node is an IVIF cut-node of G if and only if it is a common node in between two $IVIF - bridges$.*

Proof. Let t be an $IVIF$-cut node in G. So, there are two distinct vertices u and v such that either u or v is not t. In this way, the node t lies on each $IVIFstrongest$ $u - v$ path. Since G^* is a cycle, there is only one strongest path from u to v that includes node v, and by Remark 2, all the arcs on this path are $IVIF - bridges$. Hence, in between two $IVIF - bridges$, t is a common node.

On the other hand, if two $IVIF - bridges$ (u,t) and (t,v) exist that share a common node at t, then according to Proposition 8, neither of the arcs (u,t) and (t,v) can be the weakest in the graph G. Furthermore, if a path exists between nodes u and v in G that does not include the arcs (u,t) and (t,v), then its strength is guaranteed to be less than or equal to the minimum of $\phi_D(u,t)$ and $\phi_D(t,v)$ and greater than the maximum of $\psi_D(u,t)$ and $\psi_D(t,v)$. Therefore, it can be concluded that the path $u-v$ is the strongest among all possible paths connecting nodes u and v in G and $\phi_D^{\infty}(u,v) = \phi_D(u,t) \wedge \phi_D(t,v)$ and $\psi_D^{\infty}(u,v) = \psi_D(u,t) \wedge \psi_D(t,v)$. Hence, t is an $IVIF$-cut node. □

Theorem 5. *Let C^* be a cycle in an IVIFG G. If there is at most one α-strong arc in G, then G does not have an $IVIF$-cut node.*

Proof. The proof is obvious. □

Remark 7. *In general, the converse of Theorem 5 is not true.*

Theorem 6. *In an IVIFG G, if there is only one path in between any pair of vertices v_1, v_2, then that path constitutes a strong $p_1 - p_2$ path.*

4. IVIF-Trees and IVIF-Cycles

In this section, we introduce and discuss the terms IVIF-trees and IVIF-cycles along with examples and the necessary results.

Definition 22. *In an IVIFG $G = (C, D)$, a $[\phi^-, \phi^+]$-connected graph G is an interval-valued intuitionistic fuzzy $[\phi^-, \phi^+]$-tree (IVIF $[\phi^-, \phi^+]$-tree) if it contains an interval-valued intuitionistic fuzzy spanning subgraph (IVIFSSG) F and is itself a $[\phi^-, \phi^+]$-tree such that for all arcs (r,s) not in F, we have $[\phi^-, \phi^+]_D(r,s) < [\phi^-, \phi^+]_F^{\infty}(r,s)$. In addition, an IVIFSSG denoted by F is referred to as a spanning $[\phi^-, \phi^+]$-tree of G.*

Definition 23. *In an IVIFG $G = (C, D)$, a $[\psi^-, \psi^+]$-connected graph G is an interval-valued intuitionistic fuzzy $[\psi^-, \psi^+]$-tree, also represented as (IVIF $[\psi^-, \psi^+]$-tree), whenever an interval-valued intuitionistic fuzzy spanning subgraph IVIFSSG exists there, which is denoted by F', and itself is a $[\psi^-, \psi^+]$-tree such that, for every arc (r,s) not in F', it ensures that $[\psi^-, \psi^+]_D(r,s) > [\psi^-, \psi^+]_{F'}^{\infty}(r,s)$. Furthermore, an IVIFSSG F' is a spanning $[\psi^-, \psi^+]$-tree of an IVIFG G.*

Definition 24. *Let $G = (C, D)$ be an IVIFG which is strongly connected graph. Then, G is an interval-valued intuitionistic fuzzy tree (IVIFT) whenever an IVIFSSG denoted by F'' is itself a tree such that, for every arc (r,s) not in F'', we have $[\phi^-, \phi^+]_D(r,s) < [\phi^-, \phi^+]_F^{\infty}(r,s)$ as well as satisfying $[\psi^-, \psi^+]_D(r,s) > [\psi^-, \psi^+]_{F'}^{\infty}(r,s)$. In addition, an IVIFSSG F'' is a spanning tree of G.*

Proposition 10. *If an IVIFG $G = (C, D)$ is an IVIFT, then G must be an IVIF $[\phi^-, \phi^+]$-tree, and IVIF $[\psi^-, \psi^+]$-tree.*

Remark 8. *The converse of Proposition 10 is not true in general as depicted in Example 4.*

Example 4. *In Figure 4, consider $G = (C, D)$, where $C = \{u_1, u_2, u_3, u_4, u_5\}$ and $D = \{(u_1,u_2), (u_2,u_3), (u_3,u_4), (u_4,u_5), (u_1,u_5), (u_2,u_5), (u_3,u_5)\}$, G is an IVIF $[\phi^-, \phi^+]$-tree and IVIF $[\psi^-, \psi^+]$-tree; however, G is not an IVIFT because a spanning tree F'' does not exist which is equal to both F and F'.*

Theorem 7. *Let (u,v) be an arc in an IVIF $[\phi^-, \phi^+]$-tree. Then, an IVIFG G is an $\alpha_{[\phi^-, \phi^+]}$-strong if and only if the arc (u,v) is present in $[\phi^-, \phi^+]$-spanning tree F of G.*

Proof. It is straightforward. □

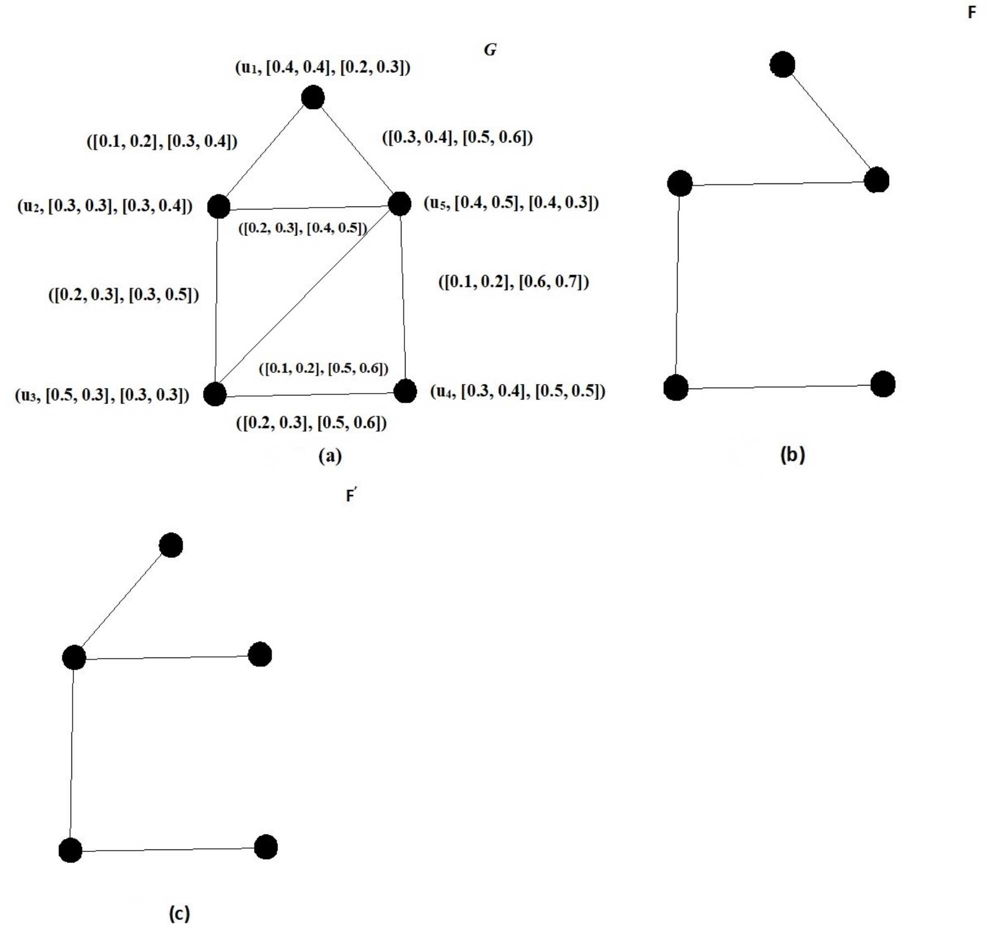

Figure 4. Interval-valued intuitionistic fuzzy $[\phi^-, \phi^+]$-tree and $[\psi^-, \psi^+]$-tree G.

By Theorem 7, F is an IVIF $[\phi^-, \phi^+]$-tree which includes all the $\alpha_{[\phi^-,\phi^+]}$-strong arcs.

Corollary 3. *If an IVIFG $G = (C, D)$ is an IVIF $[\phi^-, \phi^+]$-tree, then an IVIFSSG F is a unique spanning $[\phi^-, \phi^+]$-tree.*

Theorem 8. *If there is an arc (u, v) in the IVIF $[\psi^-, \psi^+]$-tree, then an IVIFG G is $\alpha_{[\psi^-,\psi^+]}$-strong if and only if the arc (u, v) is lying in the IVIFG G of the spanning $[\psi^-, \psi^+]$-tree of F'.*

Proof. Let (u, v) be an arc in G which is an $\alpha_{[\psi^-,\psi^+]}$-strong arc, so by Definition 22, we have $[\psi^-, \psi^+]_D(u, v) < [\psi^-, \psi^+]^{\infty}_{G-(u,v)}(u, v)$. Whenever (u, v) is not a part of an IVIFSSG F', then consequently $[\psi^-, \psi^+]_D(u, v) > [\psi^-, \psi^+]^{\infty}_{F'}(u, v)$. Furthermore, the $[\psi^-, \psi^+]$-tree that is an IVIFSSG F' is an IVISFS of $G - (u, v)$. Therefore, this condition is satisfied $[\psi^-, \psi^+]^{\infty}_{F'}(u, v) \geq [\psi^-, \psi^+]^{\infty}_{G-(u,v)}(u, v)$. We observe that $[\psi^-, \psi^+]_D(u, v) > [\psi^-, \psi^+]^{\infty}_{G-(u,v)}(u, v)$, which disproves our supposition. Thus, the arc (u, v) is present in an IVIVSSG F'. Alternatively, assume an arc (u, v) be a part of F'. In an IVIFG G, if the arc (u, v) is not a $\alpha_{[}\psi^-, \psi^+]$-strong, so we have $[\psi^-, \psi^+]_D(u, v) \geq [\psi^-, \psi^+]^{\infty}_{G-(u,v)}(u, v)$. A cycle C^* is considered as a $[\psi^-, \psi^+]$-cycle that includes the arc (u, v). So, in the cycle C^*, there is an arc (r, s), not be a part of an IVIFSSG F'. Then, it follows $[\psi^-, \psi^+]_D(r, s) > [\psi^-, \psi^+]^{\infty}_{F'}(r, s)$. We obtain the $[\psi^-, \psi^+]$-path $P = C^* - (r, s)$ from the node r to s in F' of G, and consequently $[\psi^-, \psi^+]^{\infty}_P(r, s) = [\psi^-, \psi^+]^{\infty}_{F'}(r, s)$, since an IVIFSSG F' is proven to be a $[\psi^-, \psi^+]$-tree. Moreover, $[\psi^-, \psi^+]^{\infty}_P(r, s) \geq [\psi^-, \psi^+]_D(u, v)$ implies that

$[\psi^-,\psi^+]^{\infty}_{F'}(r,s) \geq [\psi^-,\psi^+]_D(u,v)$ implies that $[\psi^-,\psi^+]_D(r,s) > [\psi^-,\psi^+]_D(u,v)$. Hence, the arc (u,v) in each cycle C^* of an IVIFG G is not proven to be the weakest $[\psi^-,\psi^+]$-arc. Thus, this arc (u,v) in G defined an IVIF $[\psi^-,\psi^+]$-bridge by Proposition 7. Hence, the arc (u,v) in G is an $\alpha_[\psi^-,\psi^+]$-strong. □

Corollary 4. *If an IVIFG G is an IVIF $[\psi^-,\psi^+]$-tree, then an IVIFSSG F' of G contains a distinct spanning $[\psi^-,\psi^+]$-tree.*

Proposition 11. *A unique spanning tree F'' exists in an IVIFT $G = (V,E)$, satisfying $F = F' = F''$.*

Proof. Let G be an IVIFT. Then, a unique spanning tree F'' exists such that, for every arc (r,s) not included in F'', we have

$$[\phi^-,\phi^+]_D(r,s) < [\phi^-,\phi^+]^{\infty}_{F''}(r,s)$$

and also we have

$$[\psi^-,\psi^+]_D(r,s) > [\psi^-,\psi^+]^{\infty}_{F''}(r,s)$$

Then, a unique spanning $[\phi^-,\phi^+]$-tree F exists with $F'' = F$, and similarly, a unique spanning $[\psi^-,\psi^+]$-tree F' exists with $F'' = F'$. As a result, we obtain a distinct spanning tree F'' with $F = F' = F''$. □

Corollary 5. *An IVIFG $G = (C,D)$ is an IVIFT if and only if it is an IVIF $[\phi^-,\phi^+]$-tree and an IVIF $[\psi^-,\psi^+]$-tree together with the condition that $F = F' = F''$.*

Proof. Let an IVIFG $G = (C,D)$ be an IVIFT; then, by Proposition 10, G must be an IVIF $[\phi^-,\phi^+]$-tree and IVIF $[\psi^-,\psi^+]$-tree, and by Proposition 11, we have $F'' = F' = F$. Conversely, let there exist a spanning $[\phi^-,\phi^+]$-tree and a $[\psi^-,\psi^+]$-tree, represented by F and F' with $F = F'$ with $F'' = F' = F$. Then, for an arc (u,v) not in IVIFSSG F'' equal to F implies $[\phi^-,\phi^+]_D(u,v) < [\phi^-,\phi^+]^{\infty}_{F''}(u,v)$ and $[\psi^-,\psi^+]_D(u,v) > [\psi^-,\psi^+]^{\infty}_{F''}(u,v)$. Thus, an IVIFG G is an IVIFT together with the spanning tree F''. □

Example 5. *In Figure 5, consider $G = (C,D)$, where $C = \{v_1,v_2,v_3,v_4,v_5\}$ and $D = \{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_5),(v_1,v_5), and (v_2,v_5)\}$. IVIFG given in Figure 5 does not satisfy the property of the IVIF $[\phi^-,\phi^+]$-tree because it contains $\beta_{[\phi^-,\phi^+]}$-strong arcs; however, it is an IVIF $[\psi^-,\psi^+]$-tree as it does not contain any $\beta_{[\psi^-,\psi^+]}$-strong arcs. Thus, $G = (C,D)$ is not an IVIFT.*
The arcs (v_2,v_5) and (v_2,v_3) in Figure 5 are $\beta_{[\phi^-,\phi^+]}$-strong and $\delta_{[\psi^-,\psi^+]}$-strong, the arcs (v_1,v_2) and (v_1,v_5) in the graph given in Figure 5 are $\beta_{[\phi^-,\phi^+]}$-strong and $\alpha_{[\psi^-,\psi^+]}$-strong. Lastly, the arcs (v_3,v_4) and (v_4,v_5) in G are $\alpha_{[\phi^-,\phi^+]}$-strong and $\alpha_{[\psi^-,\psi^+]}$-strong, i.e., both of these arcs are strong.

Corollary 6. *In an IVIFT $G = (C,D)$, arc (u,v) is an $\alpha_{[\phi^-,\phi^+]}$-strong if and only if $\alpha_{[\psi^-,\psi^+]}$-strong in G.*

Proof. Let (u,v) be an $\alpha_{[\phi^-,\phi^+]}$-strong arc in an IVIFG G. Then, the arc (u,v) is present in the IVIFSSG F, and by Theorem 7, G is an IVIFT. Therefore, F'' is equal to F' and is also equal to F, i.e., $F = F' = F''$. It follows that the arc (u,v) is present in the IVIFSSG F'. Hence, the arc (u,v) in G is $\alpha_{[\psi^-,\psi^+]}$-strong by Theorem 8. The converse is the same as the above. □

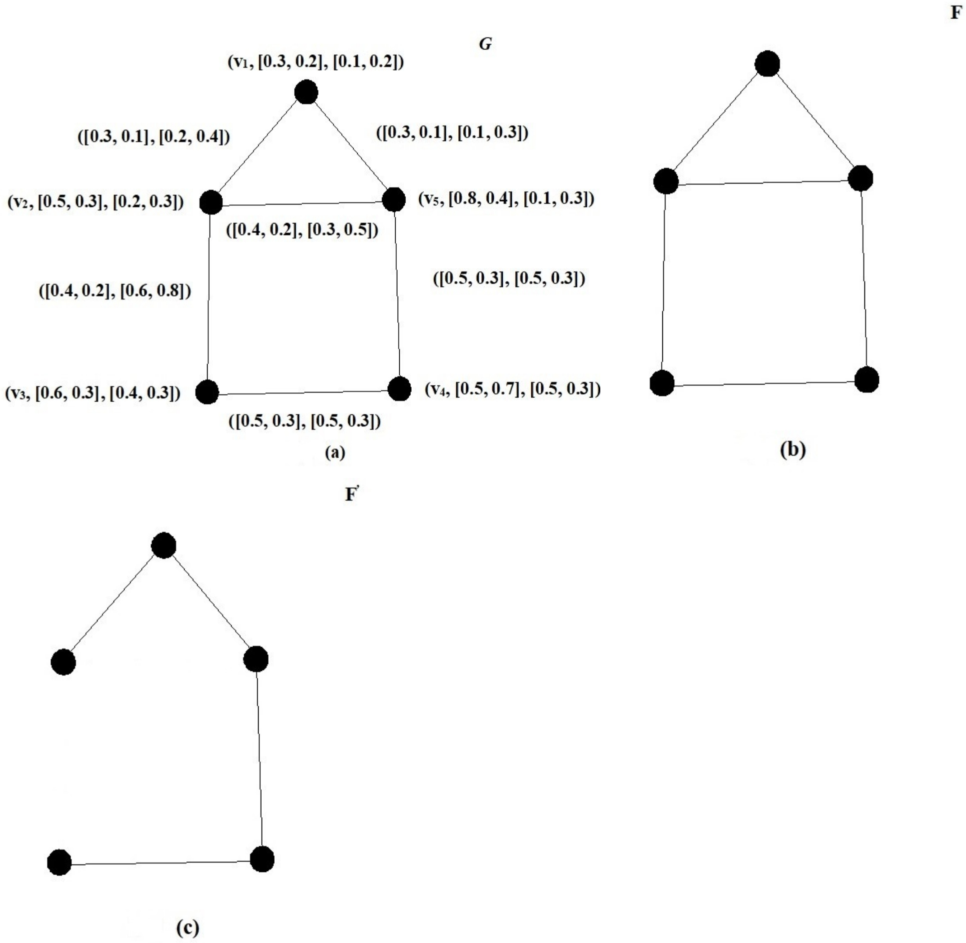

Figure 5. Interval-valued intuitionistic fuzzy $[\psi^-, \psi^+]$-tree $\boldsymbol{G}$.

Proposition 12. *Let $G = (C, D)$ be an IVIFG. Then:*

(i) *If $G = (C, D)$ is an IVIF $[\phi^-, \phi^+]$-tree and (u, v) is not in F, then $[\phi^-, \phi^+]_F(u, v) = [\phi^-, \phi^+]'^{\infty}_D(u, v)$.*

(ii) *If $G = (C, D)$ is an IVIF $[\psi^-, \psi^+]$-tree and (u, v) is not in F', then $[\psi^-, \psi^+]_{F'}(u, v) = [\psi^-, \psi^+]'^{\infty}_D(u, v)$.*

(iii) *If an IVIFG $G = (C, D)$ is an IVIFT and (u, v) is not in F'', then $[\phi^-, \phi^+]_{F''}(u, v) = [\phi^-, \phi^+]'^{\infty}_D(u, v)$ and $[\psi^-, \psi^+]_{F''}(u, v) = [\psi^-, \psi^+]'^{\infty}_D(u, v)$.*

Proof. (i) Consider a path P in F as a $[\phi^-, \phi^+]$-path from node u to v. All the arcs present in P are $\alpha_{[\phi^-, \phi^+]}$-strong by Theorem 7. Consequently, P is $\alpha_{[\phi^-, \phi^+]}$-strong. Hence, by Proposition 3, P is a $[\phi^-, \phi^+]$-strongest $(u - v)$ path. This implies that $[\phi^-, \phi^+]_F(u, v) = [\phi^-, \phi^+]'^{\infty}_D(u, v)$.

(ii) Consider a path P in F' as a $[\psi^-, \psi^+]$-path from node u to v. All the arcs present in P are $\alpha_{[\psi^-, \psi^+]}$-strong by the Theorem 8. As a result, P is $\alpha_{[\psi^-, \psi^+]}$-strong. Hence, by Proposition 4, P is satisfied as a $[\psi^-, \psi^+]$-strongest $(u - v)$ path. which implies that $[\psi^-, \psi^+]_{F'}(u, v) = [\psi^-, \psi^+]'^{\infty}_D(u, v)$.

(iii) The third part of Proposition follows directly from the parts ((i) and (ii)), i.e., $[\phi^-, \phi^+]_{F''}(u, v) = [\phi^-, \phi^+]'^{\infty}_D(u, v)$ as well as $[\psi^-, \psi^+]_{F''}(u, v) = [\psi^-, \psi^+]'^{\infty}_D(u, v)$. □

Example 6. *In Figure 6, $G = (C, D)$, where $C = \{w_1, w_2, w_3, w_4, w_5\}$ and $D = \{(w_1, w_2), (w_2, w_3), (w_3, w_4), (w_4, w_5), (w_1, w_5), (w_2, w_5), (w_3, w_5)\}$. An IVIFG $G = (C, D)$ is an IVIF $[\phi^-, \phi^+]$*

-tree as well as an IVIF $[\phi^-, \phi^+]$ -tree, and we also have the equality that is $F = F'$. Consequently, G is an IVIFT.

The arcs (w_2, w_5), (w_2, w_3), and (w_3, w_4) in G are $\alpha_{[\phi^-,\phi^+]}$-strong and $\alpha_{[\psi^-,\psi^+]}$-strong. However, the arcs (w_1, w_2), (w_3, w_5) and (w_4, w_5) in G are $\delta_{[\phi^-,\phi^+]}$-arcs and $\delta_{[\psi^-,\psi^+]}$-arcs.

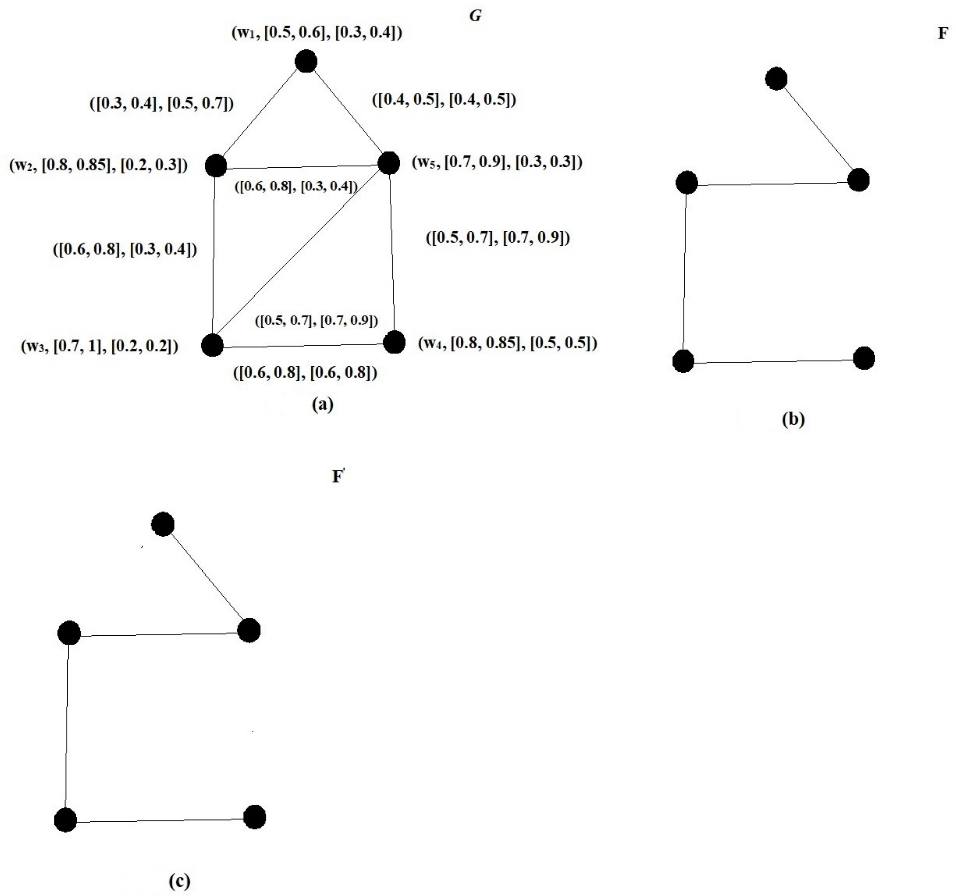

Figure 6. Interval-valued intuitionistic fuzzy graph G.

Finally, we analyze several types of IVIFCs and present a few relationships among IVIFTs and IVIFCs.

Definition 25. *Let G be a $[\phi^-, \phi^+]$-cycle. Then, we call G an interval-valued intuitionistic fuzzy $[\phi^-, \phi^+]$-cycle (IVIF-$[\phi^-, \phi^+]$-cycle) whenever G includes more than one weakest $[\phi^-, \phi^+]$-arcs. Similarly, an IVIFG G is a $[\psi^-, \psi^+]$-cycle, then we call an IVIFG G an interval-valued intuitionistic fuzzy $[\psi^-, \psi^+]$-cycle (IVIF $[\psi^-, \psi^+]$-cycle), whenever G has more than one weakest $[\psi^-, \psi^+]$-arc. Finally, G is termed an IVIFC, and either it is an IVIF-$[\phi^-, \phi^+]$-cycle or an IVIF- $[\psi^-, \psi^+]$-cycle.*

Proposition 13. *Let $G = (C, D)$ be an IVIFG. Then, we have the following:*

(i) *If $G = (C, D)$ is an IVIF-$[\phi^-, \phi^+]$-cycle, then there is no $\delta_{[\phi^-,\phi^+]}$-arc in G.*

(ii) *If $G = (C, D)$ is an IVIF $[\psi^-, \psi^+]$-cycle, then there is no $\delta_{[\psi^-,\psi^+]}$-arc in G.*

(iii) *If an IVIFG $G = (C, D)$ is an IVIFC, then there are no $\delta_{[\phi^-,\phi^+]}$-arcs or $\delta_{[\psi^-,\psi^+]}$-arc in G.*

Proof. (i) If (u, v) is an $\delta_{[\phi^-,\phi^+]}$-arc lying in G, then (u, v) is a single weakest $[\phi^-, \phi^+]$-arc, a contradiction to Definition 25.

(ii) If (u, v) in G is a $\delta_{[\psi^-,\psi^+]}$-arc, then (u, v) in G turns into a single weakest $[\psi^-, \psi^+]$-arc, a contradiction to Definition 25.

(iii) The third part of the proposition directly follows from the parts $((i)$ and $(ii))$. $\square$

Theorem 9. *A path P in an IVIFG G is an IVIF-$[\phi^-,\phi^+]$-cycle if and only if it is not an IVIF-$[\phi^-,\phi^+]$-tree.*

Proof. Let C be an IVIF-$[\phi^-,\phi^+]$-cycle. Then, by Proposition 13, no $\delta_{[\phi^-,\phi^+]}$-arcs exist in G. Consider C be an IVIF $[\phi^-,\phi^+]$-tree, then a distinct spanning $[\phi^-,\phi^+]$-tree represented by IVIFSSG F must exist there. If the arc (u,v) is not present in the unique spanning tree F, then $[\phi^-,\phi^+]_D(u,v) < [\phi^-,\phi^+]'^{\infty}_F(u,v)$ and by Proposition 13, we have $[\phi^-,\phi^+]'^{\infty}_F(u,v) = [\phi^-,\phi^+]'^{\infty}_D(u,v)$. Hence, $[\phi^-,\phi^+]_D(u,v) < [\phi^-,\phi^+]'^{\infty}_D(u,v)$. Thus, G is not an IVIF $[\phi^-,\phi^+]$-cycle. Conversely, do not let C be an IVIF-$[\phi^-,\phi^+]$-tree. Then, arc (r,s) is a random distinct $(r-s)$ path such as $P = G-(r,s)$ exists there in G in such a way that we have $[\phi^-,\phi^+]_D(r,s) \geq [\phi^-,\phi^+]^{\infty}_P(r,s)$. This means that there exists no unique weakest $[\phi^-,\phi^+]$-arc. As a result, C is proven to be an IVIF-$[\phi^-,\phi^+]$-cycle. □

Theorem 10. *Let G be a $[\psi^-,\psi^+]$-cycle. Then, G is an IVIF-$[\psi^-,\psi^+]$-cycle if and only if G is not an IVIF-$[\psi^-,\psi^+]$-tree.*

Proof. Let G be an IVIF-$[\psi^-,\psi^+]$-cycle, then by Proposition 13, no $\delta_{[\psi^-,\psi^+]}$-arcs exists in G. Let G be an IVIF-$[\psi^-,\psi^+]$-tree, then a distinct spanning $[\psi^-,\psi^+]$-tree represented by IVIFSSG F' exists there. If the arc (u,v) is not present in the unique spanning tree F', then $[\psi^-,\psi^+]_D(u,v) > [\psi^-,\psi^+]'^{\infty}_{F'}(u,v)$ and by Proposition 12 $[\psi^-,\psi^+]^{\infty}_{F'}(u,v) = [\psi^-,\psi^+]'^{\infty}_D(u,v)$. This implies that $[\psi^-,\psi^+]_D(u,v) > [\psi^-,\psi^+]'^{\infty}_D(u,v)$. Thus, G is not an IVIF-$[\psi^-,\psi^+]$-cycle. Conversely, let G be an IVIF-$[\psi^-,\psi^+]$-tree. Therefore, arc (r,s) in an IVIFG G, a distinct $(r-s)$ path such as $P = G-(r,s)$ exists in G in such a way that $[\psi^-,\psi^+]_D(r,s) \leq [\psi^-,\psi^+]_P(r,s)$. It implies that in an IVIFG G, no unique weakest $[\psi^-,\psi^+]$-arc exists. Thus, G is proven to be an IVIF-$[\psi^-,\psi^+]$-cycle. □

Corollary 7. *If an IVIFG G is an IVIFC, then it is not an IVIFT.*

Proof. Let G be an IVIFC. Then, either G is an IVIF-$[\phi^-,\phi^+]$-cycle or IVIF-$[\psi^-,\psi^+]$-cycle. Assume that G is an IVIF-$[\phi^-,\phi^+]$-cycle; then, by Theorem 9, G, which is an IVIFC, is not an IVIF-$[\phi^-,\phi^+]$-tree. Consequently, G is also not to be an IVIFT. Consider G as an IVIF-$[\psi^-,\psi^+]$-cycle; then, by Theorem 10, G is not an IVIF-$[\psi^-,\psi^+]$-tree, which implies that G is not an IVIFT. □

Remark 9. *The converse of Corollary 7 is not valid.*

We can observe the converse of Corollary 7 in the next example.

Example 7. *Consider an IVIFG shown in Figure 7, where $C = \{x_1,x_2,x_3,x_4,x_5\}$ and $D = \{(x_1,x_2),(x_2,x_3),(x_3,x_4),(x_4,x_5), and (x_1,x_5)\}$. Then, the arcs (x_1,x_5), (x_2,x_3) and (x_3,x_4) are $\alpha_{[\phi^-,\phi^+]}$-strong and $\alpha_{[\psi^-,\psi^+]}$-strong; the arc (x_1,x_2) is $\delta_{[\phi^-,\phi^+]}$-arc and $\alpha_{[\psi^-,\psi^+]}$-strong; and the arc (x_4,x_5) is $\alpha_{[\phi^-,\phi^+]}$-strong and $\delta_{[\psi^-,\psi^+]}$-arc. Hence, a graph G is an IVIF-$[\phi^-,\phi^+]$-tree and IVIF- $[\psi^-,\psi^+]$-tree, but this IVIFG G is not an IVIFT as $F \neq F'$. Moreover, it is not an IVIFC as there is no weakest $[\phi^-,\phi^+]$-arc or weakest $[\psi^-,\psi^+]$-arc.*

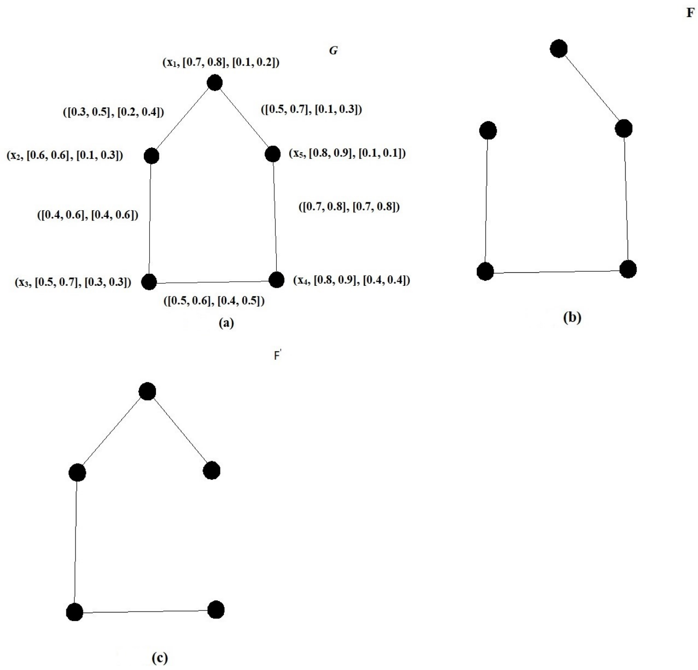

Figure 7. Interval-valued intuitionistic fuzzy $[\phi^-, \phi^+]$-tree and $[\psi^-, \psi^+]$-tree G.

5. IVIF-Blocks and $IVIF$-Block Graphs

The structure of IVIFG G can be manipulated in different ways in terms of IVIF-intersection graphs. An $IVIF$-block graph is an important structure based on the IVIF-intersection graph. We can construct an IVIF-intersection graph by considering an IVIF-block in an IVIFG G as a family of sets and generate an IVIF-intersection graph on this collection by considering each set as a vertex. Before introducing the $IVIF$-block graph, first we propose the notion of fuzzy block graphs (F-block graphs) which is missing in the existing literature about fuzzy graphs.

Definition 26. *A fuzzy block graph is the fuzzy intersection graph in which the vertex set consists of fuzzy blocks.*

Example 8. *Consider an FG shown in Figure 8.*

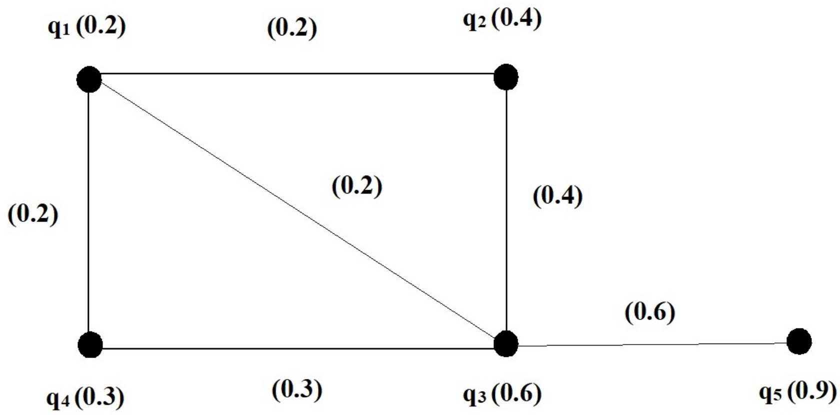

Figure 8. A fuzzy graph G.

It is easy to verify that an FG has three fuzzy blocks, namely B_1, B_2, and B_3 given in Figure 9.

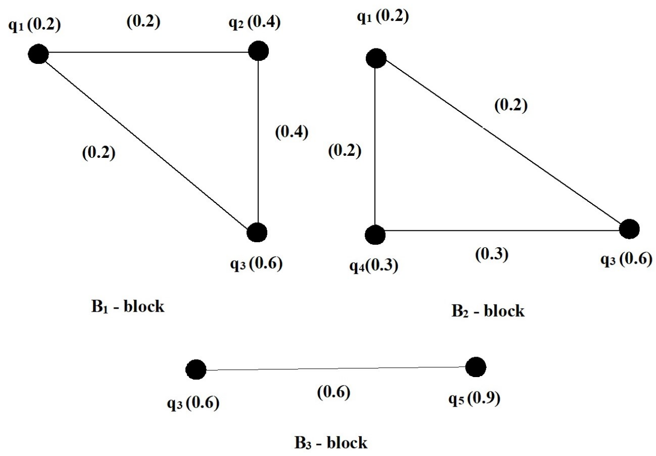

Figure 9. Fuzzy blocks in an FG G.

We have a fuzzy intersection graph $P(G) = (B, Q)$, for all B_i, $B_j \in B$ and $B_iB_j \in Z$. In $P(G) = (B, Q)$, we have

$$B = \{B_1, B_2, B_3\},$$

and

$$Z = \{B_1B_2, B_2B_3, B_1B_3\},$$

where $B_1 = \{q_1, q_2, q_3\}$, $B_2 = \{q_1, q_3, q_4\}$, $B_3 = \{q_3, q_5\}$.

For a fuzzy block graph, we allocate the values under the given conditions:

(1) $\phi_B(B_i) = \phi_C(q_i)$

(2) $\phi_Q(B_iB_j) = \phi_D(p_ip_j)$

Thus, the values are: $\phi_B(B_1) = \phi_C(q_1) = 0.2$, $\phi_B(B_2) = \phi_C(q_2) = 0.4$, $\phi_B(B_3) = \phi_C(q_3) = 0.6$ and $\phi_Q(B_1B_2) = \phi_D(q_1q_2) = 0.2$, $\phi_Q(B_2B_3) = \phi_D(q_2q_3) = 0.4$, $\phi_Q(B_1B_3) = \phi_D(q_1q_3) = 0.2$

Using the above values, we have a new graph, a fuzzy block graph shown in Figure 10.

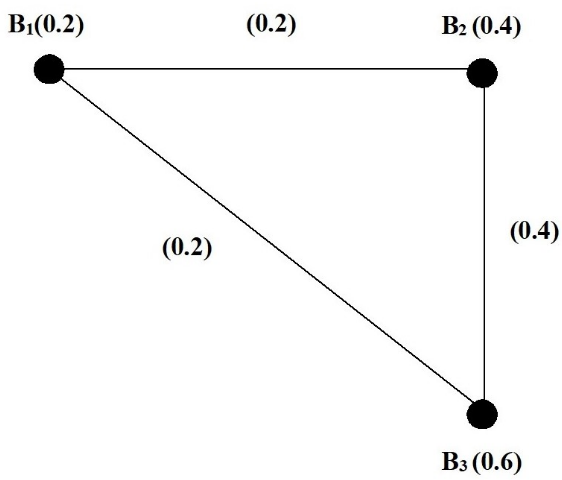

Figure 10. A fuzzy block graph $G^\star$ of G.

Now, we begin our discussion about the term $IVIF$-block graphs.

Definition 27. *In an IVIFG $G = (C, D)$, a maximally connected IVIF subgraph is said to be a block in an $IVIFG$ if it is induced by a subset of vertices that does not contain any IVIF-cut vertex. If a graph G is the $IVIF$-block, then G is itself an $IVIF$-block in G.*

Remark 10. *An IVIFG is an $IVIF$-block if there does not exist any IVIF-cut nodes.*

Theorem 11. *A connected IVIFG is a block if and only if, for every two vertices which are not joined by $IVIF$-bridges, these are connected by two non-intersecting and distinct strongest $IVIF$-path.*

Definition 28. *An $IVIPF$-block graph is the $IVIF$-intersection graph of an IVIFG in which the vertex set consists of $IVIF$-blocks.*

Example 9. *Consider an $IVIFG$ G shown in Figure 11. It is easy to observe that the vertices q_2 and q_4 are IVIF-cut nodes. Hence, G itself is not an IVIF-block. However, with the help of IVIF-cut nodes in an IVIFG G, one can analyze the $IVIF-$ blocks in an IVIFG G.*

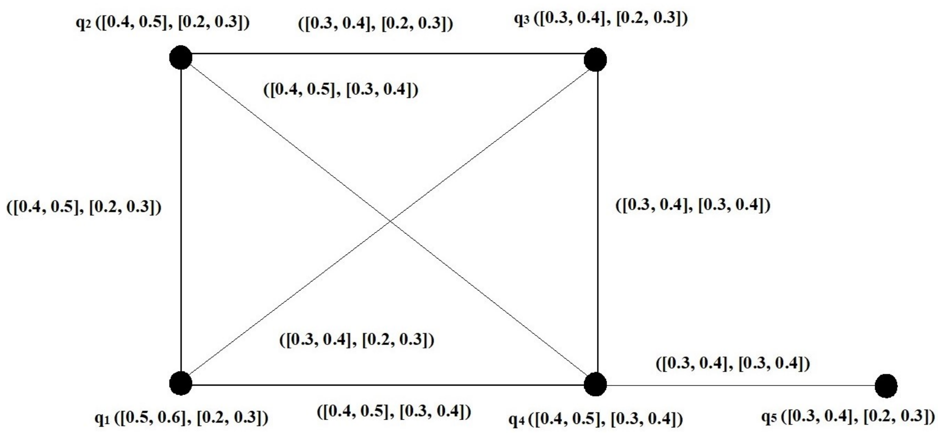

Figure 11. An IVIFG *G*.

Here, $IVIFB_1$, $IVIFB_2$, $IVIFB_3$, $IVIFB_4$, and $IVIFB_5$ shown in Figure 12 are the $IVIF$-blocks in an $IVIFG$ G.

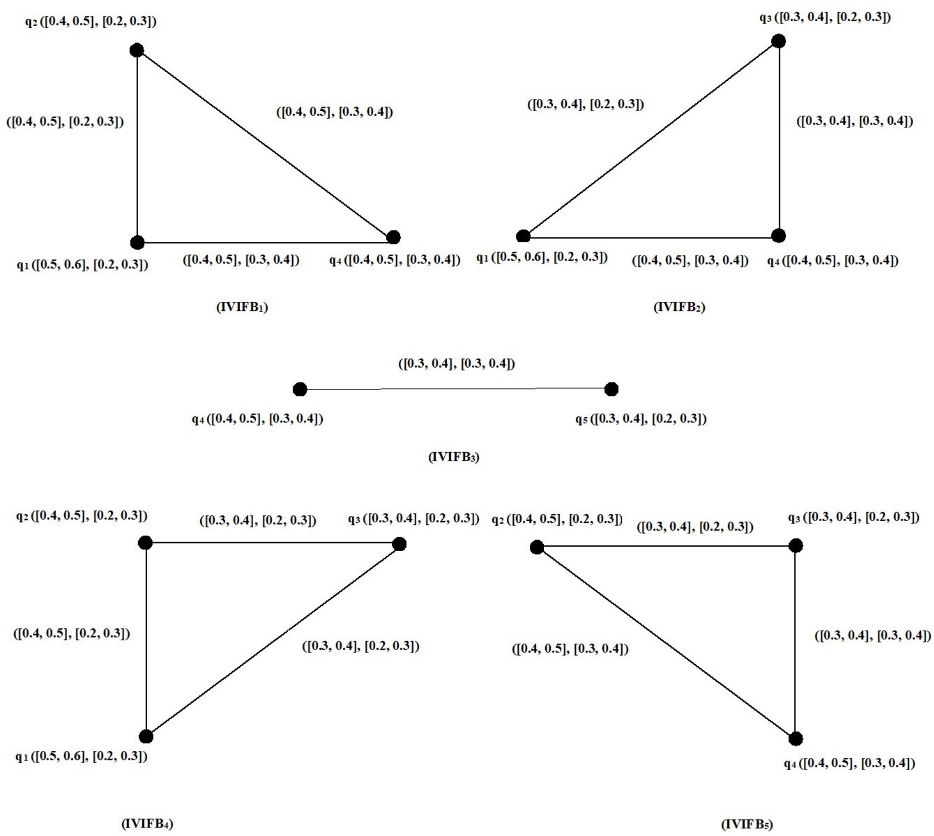

Figure 12. IVIFBs of IVIFG *G*.

Next, we produce an $IVIF$-block graph $G^\star$ of IVIFG G.

Since we have five IVIF-blocks of an IVIFG shown in Figure 12, we can consider an IVIF-intersection graph, $P(G) = (B, Q)$, for all B_i, $B_j \in B$, and $B_iB_j \in Q$. We can compute the values using the conditions of IVIF-intersection graphs given below:

(1) $[\phi^-, \phi^+]_B(B_i) = [\phi^-, \phi^+]_C(q_i)$ *and* $[\psi^-, \psi^+]_B(B_i) = [\psi^-, \psi^+]_C(q_i)$.

(2) $[\phi^-, \phi^+]_Q(B_iB_j) = [\phi^-, \phi^+]_D(q_iq_j)$ *and* $[\psi^-, \psi^+]_Q(B_iB_j) = [\psi^-, \psi^+]_D(q_iq_j)$.

From Figures 11 and 12, we observe that

$$B = \{B_1, B_2, B_3, B_4, B_5\},$$

and

$$Z = \{B_1B_2, B_1B_3, B_1B_4, B_2B_3, B_2B_4, B_3B_4, B_4B_5\},$$

where $B_1 = \{q_1, q_2, q_4\}$, $B_2 = \{q_1, q_3, q_4\}$, $B_3 = \{q_4, q_5\}$, $B_4 = \{q_1, q_2, q_3\}$, *and* $B_5 = \{q_2, q_3, q_4\}$.

Following the definition, we have

$[\phi^-, \phi^+]_B(B_1) = [\phi^-, \phi^+]_C(q_1) = [0.5, 0.6]$,
$[\phi^-, \phi^+]_B(B_2) = [\phi^-, \phi^+]_C(q_2) = [0.4, 0.5]$,
$[\phi^-, \phi^+]_B(B_3) = [\phi^-, \phi^+]_C(q_3) = [0.3, 0.4]$,
$[\phi^-, \phi^+]_B(B_4) = [\phi^-, \phi^+]_C(q_4) = [0.4, 0.5]$,
$[\phi^-, \phi^+]_B(B_5) = [\phi^-, \phi^+]_C(q_5) = [0.3, 0.4]$;
$[\psi^-, \psi^+]_B(B_1) = [\psi^-, \psi^+]_C(q_1) = [0.2, 0.3]$,
$[\psi^-, \psi^+]_B(B_2) = [\psi^-, \psi^+]_C(q_2) = [0.2, 0.3]$,
$[\psi^-, \psi^+]_B(B_3) = [\psi^-, \psi^+]_C(q_3) = [0.2, 0.3]$,
$[\psi^-, \psi^+]_B(B_4) = [\psi^-, \psi^+]_C(q_4) = [0.3, 0.4]$,
$[\psi^-, \psi^+]_B(B_5) = [\psi^-, \psi^+]_C(q_5) = [0.2, 0.3]$;
$[\phi^-, \phi^+]_Q(B_1B_2) = [\phi^-, \phi^+]_D(q_1q_2) = [0.4, 0.5]$,
$[\phi^-, \phi^+]_Q(B_1B_3) = [\phi^-, \phi^+]_D(q_1q_3) = [0.3, 0.4]$,
$[\phi^-, \phi^+]_Q(B_1B_4) = [\phi^-, \phi^+]_D(q_1q_4) = [0.4, o.5]$,
$[\phi^-, \phi^+]_Q(B_2B_3) = [\phi^-, \phi^+]_D(q_2q_3) = [0.3, 0.4]$,
$[\phi^-, \phi^+]_Q(B_2B_4) = [\phi^-, \phi^+]_D(q_2q_4) = [0.4, 0.5]$,
$[\phi^-, \phi^+]_Q(B_3B_4) = [\phi^-, \phi^+]_D(q_3q_4) = [0.3, 0.4]$,
$[\phi^-, \phi^+]_Q(B_4B_5) = [\phi^-, \phi^+]_D(q_4q_5) = [0.3, 0.4]$;

$[\psi^-, \psi^+]_Q(B_1B_2) = [\psi^-, \psi^+]_D(q_1q_2) = [0.2, 0.3]$, $[\psi^-, \psi^+]_Q(B_1B_3) = [\psi^-, \psi^+]_D(q_1q_3) = [0.2, 0.3]$, $[\psi^-, \psi^+]_Q(B_1B_4) = [\psi^-, \psi^+]_D(q_1q_4) = [0.2, 0.3]$, $[\psi^-, \psi^+]_Q(B_2B_3) = [\psi^-, \psi^+]_D(q_2q_3) = [0.2, 0.3]$, $[\psi^-, \psi^+]_Q(B_2B_4) = [\psi^-, \psi^+]_D(q_2q_4) = [0.3, 0.4]$, $[\psi^-, \psi^+]_Q(B_3B_4) = [\psi^-, \psi^+]_D(q_3q_4) = [0.3, 0.4]$, $[\psi^-, \psi^+]_Q(B_4B_5) = [\psi^-, \psi^+]_D(q_4q_5) = [0.3, 0.4]$.

Hence, the resulting graph shown in Figure 13 is the corresponding $IVIF$-block graph $G^\star$ of an $IVIFG$ G.

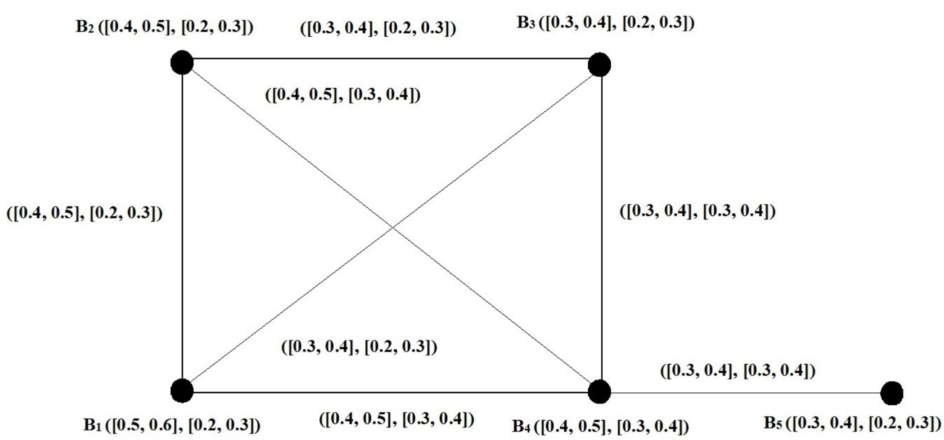

Figure 13. $IVIF$-block graph $G^\star$ of an IVIFG G.

Theorem 12. *In an $IVIFG$ G, $G^\star = (C^\star, D^\star)$ is an $IVIF$-block graph if and only if every $IVIF - block$ in $G^\star$ is a complete $IVIFG$.*

Proof. Let $G^\star$ be an $IVIF$-block graph and ${G_k}^\star$ be an $IVIF - block$ in $G^\star$. Let ${G_k}^\star$ be an incomplete IVIFG, then we have two nodes q_1, q_2 in ${G_k}^\star$ with

1. $[\phi^-, \phi^+]_{D^\star}(q_1, q_2) < [\phi^-, \phi^+]_{C^\star} q_1 \wedge [\phi^-, \phi^+]_{C^\star} q_2$
2. $[\psi^-, \psi^+]_{D^\star}(q_1, q_2) > [\psi^-, \psi^+]_{C^\star} q_1 \vee [\psi^-, \psi^+]_{C^\star} q_2$

Thus, ${G_k}^\star$ is not a strong $IVIFG$ and neither q_1 nor q_2 are $IVIF$-cut-nodes in an IVIFG G. This contradicts the maximality of ${G_k}^\star$.

Conversely, let $G^\star$ be an IVIFG in which every IVIF-block ${G_k}^\star$ is complete. Construct $B(G^\star)$ using the conditions of the $IVIF$-intersection graph, and then generate a new $IVIFG$ by adding every vertex ${G_k}^\star$ of $B(G^\star)$ having a number of end lines which is the same as the counting of the nodes of the block ${G_k}^\star$ which are not the IVIF-cut nodes of G. Thus, it is very clear that $B(G^\star)$ is isomorphic to $G^\star$. □

6. Application

Assuming a traffic blockage on a road where we want to control the flow of traffic. By the use of an IVIF-tree, we can select the proper settings for a traffic light based on the traffic conditions. The input variables of this problem are given in the following:

NS stands for 'North Side': All the vehicles standing at the northern side of the lane;

SS stands for 'South Side': All the vehicles standing at the southern side of the lane;

ES stands for 'East Side': All the vehicles standing at the eastern side of the lane;

WS stands for 'West Side': All the vehicles standing at the western side of the lane;

Utilizing the aforementioned input variables like north, south, east, and west; the following structure will be helpful to produce an IVIF-tree:

- Whenever NS is L for low, THEN move to the vertex c;
- Whenever NS is M for medium, THEN move to the vertex i;
- Whenever NS is H for high, THEN move to the vertex b;
- Whenever SS is L for low, THEN move to the vertex e;
- Whenever SS is M for medium, THEN move to the vertex j;
- Whenever SS is H for high, THEN move to the vertex d;
- Whenever ES is L for low, THEN move to the vertex k;
- Whenever ES is M for medium, THEN move to the vertex g;
- Whenever ES is H for high, THEN move to the vertex l;
- Whenever WS is L for low, THEN move to the vertex f;
- Whenever WS is M for medium, THEN move to the vertex m;
- Whenever WS is H for high, THEN move to the vertex h.

Vertexb: IF the aggregate vehicle count at the SS is specified as L, THEN for the northward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the SS is specified as M, THEN for the northward road, the output is shown by a yellow light, ELSE, for the northward road, the output is shown by a red light;

Vertexc: IF the aggregate vehicle count at the SS is specified as H, THEN for the southward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the SS is specified as M, THEN for the eastward road, the output is shown by a yellow light, ELSE, for the eastward road, the output is shown by a red light;

Vertexd: IF the aggregate vehicle count at the WS is specified as L, THEN for the eastward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the WS is specified as M, THEN for the eastward road, the output is shown by a yellow light, ELSE, for the eastward road, the output is shown by a red light;

Vertexe: IF the aggregate vehicle count at the WS is specified as H, THEN for the eastward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the WS is specified as M, THEN for the southward road, the output is shown by a yellow light, ELSE, for the southward road, the output is shown by a red light;

$Vertex f$: IF the aggregate vehicle count at the NS is specified as L, THEN for the eastward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the NS is specified as M, THEN for the eastward road, the output is shown by a yellow light, ELSE, for the eastward road, the output is shown by a red light;

$Vertex g$: IF the aggregate vehicle count at the NS is specified as H, THEN for the westward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the NS is specified as M, THEN for the northward road, the output is shown by a yellow light, ELSE, for the northward road, the output is shown by a red light;

$Vertex h$: IF the aggregate vehicle count at the ES is specified as L, THEN for the northward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the ES is specified as M, THEN for the northward road, the output is shown by a yellow light, ELSE, for the northward road, the output is shown by a red light;

$Vertex i$: IF the aggregate vehicle count at the ES is specified as H, THEN for the southward road, the output is shown by a green light, ELSE, IF the aggregate vehicle count at the ES is specified as M, THEN for the westward road, the output is shown by a yellow light, ELSE, for the westward road, the output is shown by a red light;

$Vertex j$: IF the aggregate vehicle count at the SS is specified as L, THEN for the northward road, the output is shown by a yellow light, ELSE, IF the aggregate vehicle count at the SS is specified as M, THEN for the northward road, the output is shown by a red light, ELSE, for the northward road, the output is shown by a red light;

$Vertex k$: IF the aggregate vehicle count at the WS is Low, THEN for the eastward road, the output is shown by a yellow light, ELSE, IF the aggregate vehicle count at the WS is specified as M, THEN for the eastward road, the output is shown by a red light, ELSE, for the eastward road, the output is shown by a red light;

$Vertex l$: IF the aggregate vehicle count at the NS is specified as L, THEN for the eastward road, the output is shown by a yellow light, ELSE, IF the aggregate vehicle count at the NS is specified as M, THEN for the eastward road, the output is shown by a red light, ELSE, for the eastward road, the output is shown by a red light;

$Vertex m$: IF the aggregate vehicle count at the ES is specified as L, THEN for the northward road, the output is shown by a yellow light, ELSE, IF the aggregate vehicle count at the ES is specified as M, THEN for the northward road, the output is shown by a red light, ELSE, for the northward road, the output is shown by a red light.

We would further need to calculate the an element's membership degree represented by ϕ and non-membership degree represented by ψ in an IVIFT for the "Medium" condition, based on the peculiar traffic situations.

Suppose that, for all the aforementioned input variables, like NS, SS, ES, and WS, the degree of membership for the "medium" condition is [0.5, 0.6]

NS: $\phi_{NS} high = [0.6, 0.7]$, $\psi_{NS} high = [0.2, 0.3]$;
NS: $\phi_{NS} medium = [0.5, 0.6]$, $\psi_{NS} medium = [0.5, 0.6]$;
NS: $\phi_{NS} low = [0.2, 0.3]$, $\psi_{NS} low = [0.2, 0.3]$;
SS: $\phi_{SS} high = [0.8, 1]$, $\psi_{SS} high = [0.1, 0.2]$;
SS: $\phi_{SS} medium = [0.5, 0.6]$, $\psi_{SS} medium = [0.5, 0.6]$;
SS: $\phi_{SS} low = [0.1, 0.2]$, $\psi_{SS} low = [0.1, 0.2]$;
ES: $\phi_{ES} high = [0.4, 0.5]$, $\psi_{ES} high = [0.3, 0.4]$;
ES: $\phi_{ES} medium = [0.5, 0.6]$, $\psi_{ES} medium = [0.5, 0.6]$;
ES: $\phi_{ES} low = [0.3, 0.4]$, $\psi_{ES} low = [0.3, 0.4]$;
WS: $\phi_{WS} high = [0.2, 0.3]$, $\psi_{WS} high = [0.4, 0.5]$;
WS: $\phi_{WS} medium = [0.5, 0.6]$, $\psi_{WS} medium = [0.5, 0.6]$;
WS: $\phi_{WS} low = [0.4, 0.5]$, $\psi_{WS} low = [0.4, 0.5]$.

By utilizing the membership and non-membership values as mentioned above, we can apply the IVIF-tree to determine the correct traffic light setting for any sequence of low, medium, and high traffic situations.

7. Conclusions

In this research work, we developed the theory of connectivity of IVIFGs and provided application towards road map designs. In this context, we initiated the concepts of IVIF-bridges, IVIF-cut vertices, IVIF-trees, IVIF-cycles. Moreover, we have also introduced the concepts of IVIF-blocks and IVIF-block graphs. Throughout, we provided suitable examples and counter examples to furnish our results. We have also provided several inter-relationships among the newly established terms. Many characterizations of IVIFGs have also been provided. Finally, we provided the application of IVIFTs towards road map designing. One can shift all the terms introduced in our study towards interval-valued picture fuzzy graphs, bipolar picture fuzzy graphs, etc.

Author Contributions: Conceptualization, H.G., W.A.K., S.S. and W.A.; methodology, J.S., A.K., W.A., W.A.K. and S.S.; validation, H.G., W.A.K. and J.S.; formal analysis, S.S. and A.K.; investigation, A.K. and W.A.K.; writing—original draft preparation, H.G., S.S. and W.A.K.; writing—review and editing, J.S. and A.K.; supervision, H.G. and W.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data is contained within the article.

Conflicts of Interest: The authors declare no conflict of interest.

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