axioms

Article

Entropy and Multi-Fractal Analysis in Complex Fractal Systems Using Graph Theory

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Abstract: In 1997, Sierpinski graphs, $S(n,k)$, were obtained by Klavzar and Milutinovic. The graph $S(1,k)$ represents the complete graph K_k and $S(n,3)$ is known as the graph of the Tower of Hanoi. Through generalizing the notion of a Sierpinski graph, a graph named a generalized Sierpinski graph, denoted by $Sie(\Lambda,t)$, already exists in the literature. For every graph, numerous polynomials are being studied, such as chromatic polynomials, matching polynomials, independence polynomials, and the M-polynomial. For every polynomial there is an underlying geometrical object which extracts everything that is hidden in a polynomial of a common framework. Now, we describe the steps by which we complete our task. In the first step, we generate an M-polynomial for a generalized Sierpinski graph $Sie(\Lambda,t)$. In the second step, we extract some degree-based indices of a generalized Sierpinski graph $Sie(\Lambda,t)$ using the M-polynomial generated in step 1. In step 3, we generate the entropy of a generalized Sierpinski graph $Sie(\Lambda,t)$ by using the Randić index.

Keywords: M-polynomial; Shannon's entropy; graph entropy; Randić index; fractals; generalized Sierpinski graphs

MSC: 05C07; 05C09

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1. Introduction

M-Polynomials and Fractals

Polynomials are also connected to some other fields of science such as graph theory, networking, artificial Intelligence, machine learning, and neural networks. Polynomials related to graph theory play a vital role. In graph theory, the Hosoya polynomial is used [1] to find the distance-based topological indices. Several polynomials can be extracted from Hosoya polynomials. Cach [2] introduced the importance of Hosoya and hyper-wiener polynomials. Chou et al. [3] found the Zhang–Zhang polynomials of benzenoid chemical structures. Matching polynomials were introduced in [4,5]. Clar covered polynomials [3,6,7] and Schultz polynomials [8] have been discussed in the literature.

M-polynomials were introduced by Deutsch and Klavzar in 2015. Polynomials are very much famous in chemical graph theory and particularly in the field of mathematical chemistry. Topological indices are the sub-parts of mathematical chemistry. A topological index is the graph invariant that unveils the hidden properties of the chemical structures. Three types of topological indices are used in mathematical chemistry such as degree-based, distance-based, and spectrum-based indices. Degree-based topological indices are entirely based on the valency of the atom in the relevant chemical structure. The best property of an M-polynomial is its extraction of different degree-based topological indices from a graph.

Different degree-based topological indices can be demonstrated as particular derivatives or integrals related to the M-polynomial. The benefit of M-polynomials is that, if you know the M-polynomial then you do not need to find degree-based topological indices one by one because the M-polynomial can extract multiple degree-based topological indices at once. Due to this feature of polynomials, the laborious work of finding the indices is now not necessary.

Fractals are shapes that repeat again and again in the same pattern. They occur naturally in nature. For instance, you can see the mountains, the leaves of the trees, the motion of waves, and many other shapes. There are so many shapes that are irregular in nature but fractals are not. Fractals are self-similar shapes that repeat infinitely. They simply paste copies of the same shape at different scales [9,10].

The Sierpinski graph was introduced by Klavazar and Milutinovic [11] in 1997. For further study of these graphs see reference [12]. This graph is named after the polish mathematician Waclaw Sierpinski. We denote the Sierpinski graph with $S(K;n)$, where $S(K;1)$ is simply a complete graph and, after that, $S(K;3)$ is called the Hanoi tower of the Sierpinski graph. This is a kind of fractal. Their introduction was first propelled by topological indices in which $S(K;1)$ is isomorphic to a complete graph on the K vertices; $S(K;n+1)$ is constructed by $S(K;n)$ by adding just one edge between each copy-pair. The connectivity of the Sierpinski sieve is obtained from the Sierpinski graph of order n. The Sierpinski graph $S(K;3)$ is the tower of Hanoi, while the Sierpinski gasket graphs are naturally defined by the finite number of iterations. We are motivated by the work of Klavazar and Milutinovic [11] and we have generalized the graph using this concept for degree-based topological indices and M-Polynomials.

2. Preliminaries

This definition has been taken from [11,12]. Let $\Lambda = (V,E)$ be a graph of order n and vertex set $V(\Lambda)$. We denote by $V^t(\Lambda)$ the set of words of size t on alphabet $V(\Lambda)$. The letters of a word p of length t are denoted by $p_1p_2p_3\dots p_t$. The concatenation of two words p and q is denoted by pq. We can define the t-th generalized Sierpinski graph denoted by $Sie(\Lambda,t)$ having a vertex set $V^t(\Lambda)$ and the edge set $E(Sie(\Lambda,t))$ is of the form pq, which is an edge $\Leftrightarrow, \exists\, i \in \{1,2,3,\dots,t\}$. Then, we have the following:

1. $p_b = q_b$, if $b < a$;
2. $p_a \neq q_a$ and $pq \in E$;
3. $p_b = q_a$ and $p_a = q_b$ if $b > a$.

Observe that, if $pq \in E(Sie(\Lambda,t))$, then $rs \in E(\Lambda)$ and a word w such that $p = wrss\dots s$ and $q = wsrr\dots r$. We can generate the graph $Sie(\Lambda,t)$ as follows.

Note that $S(\Lambda,1) = \Lambda$ and $t \geq 2$; we make a copy n times $Sie(\Lambda,t)$ and attach the letter r at the beginning of every label of the vertices connecting to the copy of $Sie(\Lambda,t)$ corresponding to r. For every edge rs of Λ, attach an edge between $srr\dots r$ and $rss\dots s$. The vertices of this type $rrr\dots r$ are said to be extreme vertices of $Sie(\Lambda,t)$. If the graph $Sie(\Lambda,t)$ has n extreme vertices and if the vertex r connects with d_r edges, then the extreme vertex $rr\dots r$ of $Sie(\Lambda,t)$ also connects with d_r edges. The numbers d_x+1 and d_y+1 are the degrees of the vertices $srr\dots r$ and $rss\dots s$. These two vertices join the copies of $Sie(\Lambda,t-1)$.

The generalised Sierpinski graph $Sie(\Lambda,t)$ has total number of vertices $\frac{1}{6}\times 7^{t+1} - \frac{7}{6}$, in which there are three types of vertices, namely vertices of degree 2, vertices of degree 3, and vertices of degree 4, which are $\frac{343}{6}\times 7^{t-3} - \frac{1}{6}$, $\frac{539}{6}\times 7^{t-3} + \frac{7}{6}$ and $\frac{441}{6}\times 7^{t-3} - \frac{3}{2}$, respectively. Also, the total number of edges is $\frac{2401}{6}\times 7^{t-3} - \frac{7}{6}$.

For example, for graph $Sie(\Lambda,1) = \Lambda$ in Figure 1, the total number of vertices is 7 and we have given a formula for the total vertices $\frac{1}{6}\times 7^{t+1} - \frac{7}{6}$ where t is the number of iterations. If we put $t=1$ into this equation, $\frac{1}{6}\times 7^{1+1} - \frac{7}{6} = 49/6 - 7/6 = 42/6 = 7$. In this way, one can check for $t=2$ in Figure 2, $t>2$ and obtain an integer. Similarly, the formula for total number of edges is $\frac{2401}{6}\times 7^{t-3} - \frac{7}{6}$. If we put $t=1$ into this equation,

$\frac{2401}{6} \times 7^{t-3} - \frac{7}{6} = 2401/294 - 7/6 = 42/6 = 7$. In this way, one can check for $t \geq 2$ and obtain an integer.

To compute our main results, we partitioned the edges based on the degree of end vertices of each edge and represented in Table 1.

Table 1. The edge partition of graph Λ based on the degree of end vertices of each edge.

(d_h,d_j), where, pj∈ E(G)	$E_{h,j}$	Number of Edges
(1, 2)	$E_{(1,2)}$	$28 \times 7^{t-3}$
(1, 3)	$E_{(1,3)}$	$\frac{329}{6} \times 7^{t-3} + \frac{1}{6}$
(1, 4)	$E_{(1,4)}$	$\frac{119}{3} \times 7^{t-3} + \frac{1}{3}$
(2, 2)	$E_{(2,2)}$	$\frac{14}{3} \times 7^{t-3} + \frac{1}{3}$
(2, 3)	$E_{(2,3)}$	$\frac{98}{3} \times 7^{t-3} + \frac{1}{3}$
(2, 4)	$E_{(2,4)}$	$\frac{133}{3} \times 7^{t-3} - \frac{4}{3}$
(3, 3)	$E_{(3,3)}$	$\frac{105}{2} \times 7^{t-3} + \frac{1}{2}$
(3, 4)	$E_{(3,4)}$	$77 \times 7^{t-3} + 2$
(4, 4)	$E_{(4,4)}$	$\frac{133}{2} \times 7^{t-3} - \frac{7}{2}$

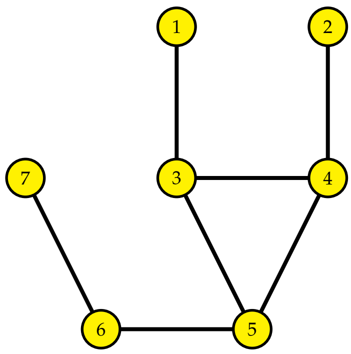

Figure 1. Seed graph $Sie(\Lambda, 1) = \Lambda$.

3. Results for M-Polynominal and Entropy

First, we will give some basic definitions and then will show our main results for topological indices with M-polynomials and entropy of generalized Sierpinski graphs.

Definition 1. *Let $\Lambda = (V, E)$ be a graph and $v \in V$, then let d_v be the degree of a vertex v. The M-polynomial of Λ, denoted by $M(\Lambda, l, q)$, is a polynomial of two variables "l" and "q", which was introduced by Klavzar and Deutsch [13] in* 2015 *and is defined as*

$$M(\Lambda, l, q) = \sum_{g \leq h} m_{gh}(\Lambda) l^g q^h. \tag{1}$$

where $m_{gh}(\Lambda), g, h \geq 1$ is the number of edges uv of Λ, such that $\{d_u, d_v\} = \{g, h\}$.

Theorem 1. *Consider the generalised Sierpinski graph $Sie(\Lambda, t)$, then its M-polynomial is*

$$\begin{aligned}
M(Sie(\Lambda,t),l,q) &= f(l,q) \\
&= \Big(28 \times 7^{n-3}\Big) l^1 q^2 + \Big(\frac{329}{6} \times 7^{n-3} + \frac{1}{6}\Big) l^1 q^3 + \Big(\frac{119}{3} \times 7^{n-3} + \frac{1}{3}\Big) l^1 q^4 \\
&+ \Big(\frac{98}{3} \times 7^{n-3} + \frac{1}{3}\Big) l^2 q^3 + \Big(\frac{133}{3} \times 7^{n-3} - \frac{4}{3}\Big) l^2 q^4 + \Big(\frac{105}{2} \times 7^{n-3} + \frac{1}{2}\Big) l^3 q^3 \\
&+ \Big(\frac{133}{2} \times 7^{n-3} - \frac{7}{2}\Big) l^4 q^4 + \Big(\frac{14}{3} \times 7^{n-3} + \frac{1}{3}\Big) l^2 q^2 + + \Big(77 \times 7^{n-3} + 2\Big) l^3 q^4
\end{aligned}$$

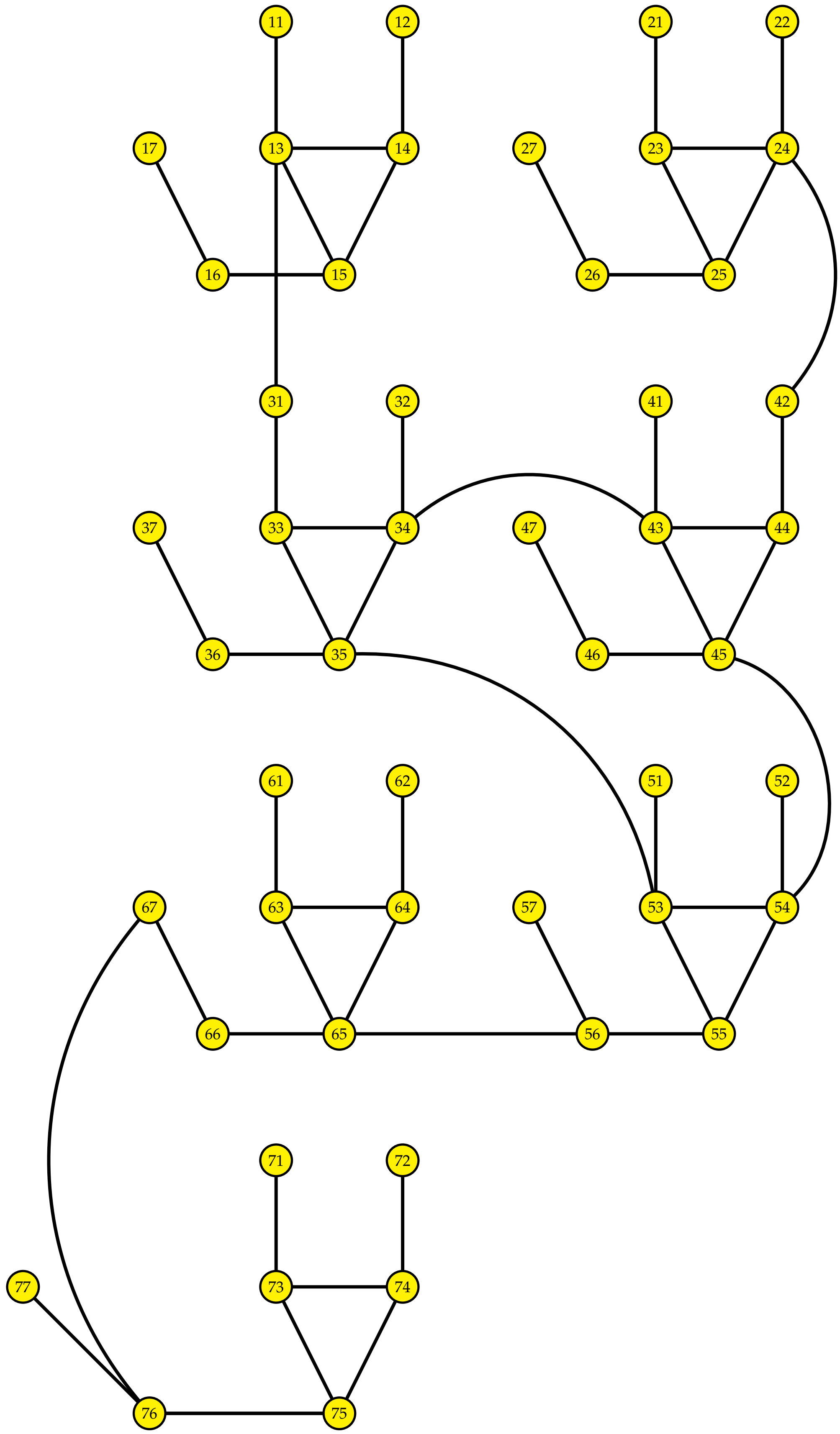

Figure 2. Generalised Sierpinski graph $Sie(\Lambda, 2)$ generated from its seed graph $Sie(\Lambda, 1) = \Lambda$.

Proof. Let $Sie(\Lambda, t)$ be a generalised Sierpinski graph, then, from above, discuss the total number of vertices, $\frac{1}{6} \times 7^{t+1} - \frac{7}{6}$, and total number of edges, $\frac{2401}{6} \times 7^{t-3} - \frac{7}{6}$. Now, the edges partition of a generalised Sierpinski graph $Sie(\Lambda, t)$ are as follows:
$|E_{\{1,2\}}| = |\{e = pj \in E(G|d_p = 1, d_j = 2\}| = 28 \times 7^{t-3}$,
$|E_{\{1,3\}}| = |\{e = pj \in E(G|d_p = 1, d_j = 3\}| = \frac{329}{6} \times 7^{t-3} + \frac{1}{6}$,
$|E_{\{1,4\}}| = |\{e = pj \in E(G|d_p = 1, d_j = 4\}| = \frac{119}{3} \times 7^{t-3} + \frac{1}{3}$,
$|E_{\{2,2\}}| = |\{e = pj \in E(G|d_p = 2, d_j = 2\}| = \frac{14}{3} \times 7^{t-3} + \frac{1}{3}$,
$|E_{\{2,3\}}| = |\{e = pj \in E(G|d_p = 2, d_j = 3\}| = \frac{98}{3} \times 7^{t-3} + \frac{1}{3}$,
$|E_{\{2,4\}}| = |\{e = pj \in E(G|d_p = 2, d_j = 4\}| = \frac{133}{3} \times 7^{t-3} - \frac{4}{3}$,
$|E_{\{3,3\}}| = |\{e = pj \in E(G|d_p = 3, d_j = 3\}| = \frac{105}{2} \times 7^{t-3} + \frac{1}{2}$,
$|E_{\{3,4\}}| = |\{e = pj \in E(G|d_p = 3, d_j = 4\}| = 77 \times 7^{t-3} + 2$
$|E_{\{4,4\}}| = |\{e = pj \in E(G|d_p = 4, d_j = 4\}| = \frac{133}{2} \times 7^{t-3} - \frac{7}{2}$

Now apply a definition of an M-polynomial

$$\begin{aligned}
M(Sie(\Lambda,t),l,q) &= \sum_{g\leq h} m_{gh} l^g q^h \\
M(Sie(\Lambda,t),l,q) &= \sum_{1\leq 2} m_{12} l^1 q^2 + \sum_{1\leq 3} m_{13} l^1 q^3 + \sum_{1\leq 4} m_{14} l^1 q^4 + \sum_{2\leq 2} m_{22} l^2 q^2 \\
&+ \sum_{2\leq 3} m_{23} l^2 q^3 + \sum_{2\leq 4} m_{24} l^2 q^4 + \sum_{3\leq 3} m_{33} l^3 q^3 + \sum_{3\leq 4} m_{34} l^3 q^4 + \sum_{4\leq 4} m_{44} l^4 q^4 \\
M(Sie(\Lambda,t),l,q) &= |E_{\{1,2\}}| l^1 q^2 + |E_{\{1,2\}}| l^1 q^3 + |E_{\{1,4\}}| l^1 q^4 + |E_{\{2,2\}}| l^2 q^2 + |E_{\{2,3\}}| l^2 q^3 \\
&+ |E_{\{2,4\}}| l^2 q^4 + |E_{\{3,3\}}| l^3 q^3 + |E_{\{3,4\}}| l^3 q^4 + |E_{\{4,4\}}| l^4 q^4 \\
&= \Big(28 \times 7^{t-3}\Big) l^1 q^2 + \Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) l^1 q^3 + \Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) l^1 q^4 \\
&+ \Big(\frac{98}{3}.7^{t-3} + \frac{1}{3}\Big) l^2 q^3 + \Big(\frac{133}{3} \times 7^{t-3} - \frac{4}{3}\Big) l^2 q^4 + \Big(\frac{105}{2} \times 7^{t-3} + \frac{1}{2}\Big) l^3 q^3 \\
&+ \Big(\frac{133}{2} \times 7^{t-3} - \frac{7}{2}\Big) l^4 q^4 + \Big(\frac{14}{3} \times 7^{t-3} + \frac{1}{3}\Big) l^2 q^2 + + \Big(77 \times 7^{t-3} + 2\Big) l^3 q^4
\end{aligned}$$
□

Proposition 1. *Let $Sie(\Lambda, t)$ be a generalised Sierpinski graph. Then, the topological indices are*

1. $M_1(Sie(\Lambda, t)) = \frac{7805}{3} \times 7^{t-3} - \frac{83}{3}$
2. $M_2(Sie(\Lambda, t)) = \frac{6521}{2} \times 7^{t-3} - 33$
3. ${}^m M_2(Sie(\Lambda, t)) = \frac{20,377}{288} \times 7^{t-3} + \frac{11}{96}$
4. $$\begin{aligned}
R_\alpha(Sie(\Lambda,t)) &= 2^\alpha\Big(28 \times 7^{t-3}\Big) + 3^\alpha\Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) + 4^\alpha\Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) \\
&+ 2^{2\alpha}\Big(\frac{14}{3} \times 7^{t-3} + \frac{1}{3}\Big) + 2^\alpha 3^\alpha\Big(\frac{98}{3} \times 7^{t-3} + \frac{1}{3}\Big) + 2^\alpha 4^\alpha\Big(\frac{133}{3} \times 7^{t-3} - \frac{4}{3}\Big) \\
&+ 3^{2\alpha}\Big(\frac{105}{2} \times 7^{t-3} + \frac{1}{2}\Big) + 3^\alpha 4^\alpha\Big(77 \times 7^{t-3} + 2\Big) + 4^{2\alpha}\Big(\frac{133}{2} \times 7^{t-3} + \frac{7}{2}\Big)
\end{aligned}$$

5.

$$
\begin{aligned}
RR_{\alpha}(Sie(\Lambda,t)) &= \frac{28}{2^{\alpha}}\times 7^{t-3}+\frac{1}{3^{\alpha}}\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)+\frac{1}{4^{\alpha}}\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\\
&+ \frac{1}{2^{2\alpha}}\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)+\frac{1}{3^{\alpha}2^{\alpha}}\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\\
&+ \frac{1}{2^{\alpha}4^{\alpha}}\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)+\frac{1}{3^{2\alpha}}\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\\
&+ \frac{1}{3^{\alpha}4^{\alpha}}\Big(77\times 7^{t-3}+2\Big)+\frac{1}{4^{2\alpha}}\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)
\end{aligned}
$$

6. $SDD(Sie(\Lambda,t))=\frac{18{,}445}{18}\times 7^{t-3}-\frac{65}{36}$

7. $H(Sie(\Lambda,t))=\frac{9461}{120}\times 7^{t-3}-\frac{247}{5040}$

8. $I(Sie(\Lambda,t))=\frac{68{,}137}{120}\times 7^{t-3}-\frac{1751}{504}$

9. $A(Sie(\Lambda,t))=\frac{5{,}287{,}567{,}733}{1{,}296{,}000}\times 7^{t-3}-\frac{47{,}962{,}067}{1{,}296{,}000}$

Proof. According to Theorem 1, we have

$$
\begin{aligned}
M(Sie(\Lambda,t),l,q) &= f(l,q)\\
&= \Big(28\times 7^{t-3}\Big)l^1q^2+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)l^1q^3+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^1q^4\\
&+ \Big(\frac{98}{3}.7^{t-3}+\frac{1}{3}\Big)l^2q^3+\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^2q^4+\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^3\\
&+ \Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^4q^4+\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^2++\Big(77\times 7^{t-3}+2\Big)l^3q^4
\end{aligned}
$$

which is now as shown in Table 1.
First, we find D_l and D_q

$$
D_l=l\Big[\frac{\partial f(l,q)}{\partial l}\Big] \qquad\qquad D_q=q\Big[\frac{\partial f(l,q)}{\partial q}\Big] \tag{2}
$$

$$
\begin{aligned}
f(l,q) &= \Big(28\times 7^{t-3}\Big)l^1q^2+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)l^1q^3+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^1q^4\\
&+ \Big(\frac{98}{3}.7^{t-3}+\frac{1}{3}\Big)l^2q^3+\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^2q^4+\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^3\\
&+ \Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^4q^4+\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^2++\Big(77\times 7^{t-3}+2\Big)l^3q^4
\end{aligned}
$$

After partial differentiation with respect to l, we have

$$
\begin{aligned}
\frac{\partial f(l,q)}{\partial l} &= \Big(28\times 7^{t-3}\Big)q^2+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)q^3+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)q^4\\
&+ \Big(\frac{196}{3}.7^{t-3}+\frac{2}{3}\Big)lq^3+\Big(\frac{266}{3}\times 7^{t-3}-\frac{8}{3}\Big)lq^4+\Big(\frac{315}{2}\times 7^{t-3}+\frac{3}{2}\Big)l^2q^3\\
&+ \Big(\frac{532}{2}\times 7^{t-3}-\frac{28}{2}\Big)l^3q^4+\Big(\frac{28}{3}\times 7^{t-3}+\frac{2}{3}\Big)lq^2+\Big(231\times 7^{t-3}+6\Big)l^2q^4
\end{aligned}
$$

Now, multiplying the above equation by l

$$\begin{aligned}
D_l &= l \times \frac{\partial f(l,q)}{\partial l} \\
D_l &= \Big(28 \times 7^{t-3}\Big) l q^2 + \Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) l q^3 + \Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) l q^4 \\
&+ \Big(\frac{196}{3}.7^{t-3} + \frac{2}{3}\Big) l^2 q^3 + \Big(\frac{266}{3} \times 7^{t-3} - \frac{8}{3}\Big) l^2 q^4 + \Big(\frac{315}{2} \times 7^{t-3} + \frac{3}{2}\Big) l^3 q^3 \\
&+ \Big(\frac{532}{2} \times 7^{t-3} - \frac{28}{2}\Big) l^4 q^4 + \Big(\frac{28}{3} \times 7^{t-3} + \frac{2}{3}\Big) l^2 q^2 + \Big(231 \times 7^{t-3} + 6\Big) l^3 q^4
\end{aligned}$$

Similarly, we find

$$\begin{aligned}
D_q &= q \times \frac{\partial f(l,q)}{\partial q} \\
D_q &= \Big(56 \times 7^{t-3}\Big) l^1 q^2 + \Big(\frac{329}{2} \times 7^{t-3} + \frac{1}{2}\Big) l^1 q^3 + \Big(\frac{476}{3} \times 7^{t-3} + \frac{4}{3}\Big) l^1 q^4 \\
&+ \Big(\frac{28}{3} \times 7^{t-3} + \frac{2}{3}\Big) l^2 q^2 + \Big(\frac{294}{3}.7^{t-3} + 1\Big) l^2 q^3 + \Big(\frac{532}{3}.7^{t-3} - \frac{16}{3}\Big) l^2 q^4 \\
&+ \Big(\frac{315}{2}.7^{t-3} + \frac{3}{2}\Big) l^3 q^3 + \Big(308 \times 7^{t-3} + 8\Big) l^3 q^4 + \Big(266 \times 7^{t-3} - 14\Big) l^4 q^4
\end{aligned}$$

Now, we have

$$M_1(Sie(\Lambda,t)) = (D_l + D_q)(f(l,q))|_{l=q=1}$$

substituting the values D_l and D_q in $M_1(Sie(\Lambda,t))$.

$$\begin{aligned}
M_1(Sie(\Lambda,t)) &= \Big|\Big((28 \times 7^{t-3}\Big) l^1 q^2 + \Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) l^1 q^3 + \Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) l^1 q^4 \\
&+ \Big(\frac{28}{3} \times 7^{t-3} + \frac{2}{3}\Big) l^2 q^2 + \Big(\frac{196}{3} \times 7^{t-3} + \frac{2}{3}\Big) l^2 q^3 + \Big(\frac{266}{3} \times 7^{t-3} - \frac{8}{3}\Big) l^2 q^4 \\
&+ \Big(\frac{315}{2} \times 7^{t-3} + \frac{3}{2}\Big) l^3 q^3 + \Big(231 \times 7^{t-3} + 2\Big) l^3 q^4 + \Big(\frac{532}{2} \times 7^{t-3} - \frac{28}{2}\Big) l^4 q^4) \\
&+ (56 \times 7^{t-3}) l^1 q^2 + \Big(\frac{329}{2} \times 7^{t-3} + \frac{1}{2}\Big) l^1 q^3 + \Big(\frac{476}{3} \times 7^{t-3} + \frac{4}{3}\Big) l^1 q^4 \\
&+ \Big(\frac{28}{3} \times 7^{t-3} + \frac{2}{3}\Big) l^2 q^2 + \Big(\frac{294}{3} \times 7^{t-3} + 1\Big) l^2 q^3 + \Big(\frac{532}{3} \times 7^{t-3} - \frac{16}{3}\Big) l^2 q^4 \\
&+ \Big(\frac{315}{2} \times 7^{t-3} + \frac{3}{2}\Big) l^3 q^3 + \Big(308 \times 7^{t-3} + 8\Big) l^3 q^4 \\
&+ \Big(266 \times 7^{t-3} - 14\Big) l^4 q^4 \Big|_{l=q=1}
\end{aligned}$$

After substituting the values of $l = 1$ and $q = 1$, we have the following result:

$$M_1(Sie(\Lambda,t)) = \frac{7805}{3} \times 7^{t-3} - \frac{83}{3}$$

Similarly, in all remaining parts from 2 to 9, we have used the values of D_l, D_q, δ_l, δ_q to obtain our results.

$$M_2(Sie(\Lambda,t)) = (D_l D_q)(f(l,q))|_{l=q=1}$$

Multiplying D_l and D_q, we obtain the following result:

$$
\begin{aligned}
D_l D_q &= \Big(56 \times 7^{t-3}\Big) l^1 q^2 + (\frac{329}{2} \times 7^{t-3} + \frac{1}{2}) l^1 q^3 + (\frac{476}{3} \times 7^{t-3} + \frac{4}{3}) l^1 q^4 \\
&+ \Big(\frac{56}{3} \times 7^{t-3} + \frac{4}{3}\Big) l^2 q^2 + \Big(\frac{588}{3} \times 7^{t-3} + 2) l^2 q^3 + (\frac{1064}{3} \times 7^{t-3} - \frac{32}{3}) l^2 q^4 \\
&+ \Big(\frac{945}{2} \times 7^{t-3} + \frac{9}{2}\Big) l^3 q^3 + \Big(924 \times 7^{t-3} + 24) l^3 q^4 + (1064 \times 7^{t-3} - 56) l^4 q^4
\end{aligned}
$$

$$
\begin{aligned}
M_2(Sie(\Lambda, t)) &= \Big|\Big(56 \times 7^{t-3}\Big) l^1 q^2 + \Big(\frac{329}{2} \times 7^{t-3} + \frac{1}{2}\Big) l^1 q^3 + (\frac{476}{3} \times 7^{t-3} + \frac{4}{3}) l^1 q^4 \\
&+ \Big(\frac{56}{3} \times 7^{t-3} + \frac{4}{3}\Big) l^2 q^2 + \Big(\frac{588}{3} \times 7^{t-3} + 2\Big) l^2 q^3 + \Big(\frac{1064}{3} \times 7^{t-3} - \frac{32}{3}\Big) l^2 q^4 \\
&+ \Big(\frac{945}{2} \times 7^{t-3} + \frac{9}{2}\Big) l^3 q^3 + \Big(924 \times 7^{t-3} + 24\Big) l^3 q^4 + \Big(1064 \times 7^{t-3} - 56\Big) l^4 q^4 \Big|_{l=q=1} \\
M_2(Sie(\Lambda, t)) &= \frac{6521}{2} \times 7^{t-3} - 33
\end{aligned}
$$

$$
\begin{aligned}
\delta_l &= \int_0^l \Big[\frac{f(p,q)}{p} dp\Big] \\
\delta_q &= \int_0^q \Big[\frac{f(l,p)}{p} dp\Big]
\end{aligned}
$$

First, we find $f(p,q)$ and $\frac{f(p,q)}{p}$, then

$$
\begin{aligned}
\delta_l &= \int_0^l \Big[\frac{f(p,q)}{p} dp\Big] \\
\delta_q &= \int_0^q \Big[\frac{f(l,p)}{p} dp\Big]
\end{aligned}
$$

$$
\begin{aligned}
f(p,q) &= \Big(28 \times 7^{t-3}\Big) p^1 q^2 + \Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) p^1 q^3 + \Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) p^1 q^4 \\
&+ \Big(\frac{98}{3} \times 7^{t-3} + \frac{1}{3}\Big) p^2 q^3 + \Big(\frac{133}{3} \times 7^{t-3} - \frac{4}{3}\Big) p^2 q^4 + \Big(\frac{105}{2} \times 7^{t-3} + \frac{1}{2}\Big) p^3 q^3 \\
&+ \Big(\frac{133}{2} \times 7^{t-3} - \frac{7}{2}\Big) p^4 q^4 + \Big(\frac{14}{3} \times 7^{t-3} + \frac{1}{3}\Big) p^2 q^2 + \Big(77 \times 7^{t-3} + 2\Big) p^3 q^4
\end{aligned}
$$

$$
\begin{aligned}
\frac{f(p,q)}{p} &= \Big(28 \times 7^{t-3}\Big) q^2 + \Big(\frac{329}{6} \times 7^{t-3} + \frac{1}{6}\Big) q^3 + \Big(\frac{119}{3} \times 7^{t-3} + \frac{1}{3}\Big) q^4 \\
&+ \Big(\frac{98}{3} \times 7^{t-3} + \frac{1}{3}\Big) p^1 q^3 + \Big(\frac{133}{3} \times 7^{t-3} - \frac{4}{3}\Big) p^1 q^4 + \Big(\frac{105}{2} \times 7^{t-3} + \frac{1}{2}\Big) p^2 q^3 \\
&+ \Big(\frac{133}{2} \times 7^{t-3} - \frac{7}{2}\Big) p^3 q^4 + \Big(\frac{14}{3} \times 7^{t-3} + \frac{1}{3}\Big) p^1 q^2 + \Big(77 \times 7^{t-3} + 2\Big) p^2 q^4
\end{aligned}
$$

Now, upon integrating, we have δ_l and δ_q

$$
\begin{aligned}
\delta_l &= \int_0^l \Big[\frac{f(p,q)}{p} dp\Big] \\
\delta_q &= \int_0^q \Big[\frac{f(l,p)}{p} dp\Big]
\end{aligned}
$$

Multiplying δ_l and δ_q, we obtain the following result:

$$\begin{aligned}
{}^{m}M_2(Sie(\Lambda,t)) &= (\delta_l\delta_q)(f(l,q))|_{l=q=1} \\
\delta_q f(l,q) &= \Big(14\times 7^{t-3}\Big)l^1q^2 + \Big(\frac{329}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^1q^3 + \Big(\frac{119}{12}\times 7^{t-3}+\frac{1}{12}\Big)l^1q^4 \\
&+ \Big(\frac{7}{3}\times 7^{t-3}+\frac{1}{6}\Big)l^2q^2 + \Big(\frac{98}{9}\times 7^{t-3}+\frac{1}{9}\Big)l^2q^3 + \Big(\frac{133}{12}\times 7^{t-3}-\frac{1}{3}\Big)l^2q^4 \\
&+ \Big(\frac{105}{6}\times 7^{t-3}+\frac{1}{6}\Big)l^3q^3 + \Big(\frac{77}{4}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^4 + \Big(\frac{133}{8}\times 7^{t-3}-\frac{7}{8}\Big)l^4q^4 \\
\delta_l\delta_q &= \Big(14\times 7^{t-3}\Big)l^1q^2 + \Big(\frac{329}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^1q^3 + \Big(\frac{119}{12}\times 7^{t-3}+\frac{1}{12}\Big)l^1q^4 \\
&+ \Big(\frac{7}{6}\times 7^{t-3}+\frac{1}{12}\Big)l^2q^2 + \Big(\frac{98}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^2q^3 + \Big(\frac{133}{24}\times 7^{t-3}-\frac{1}{6}\Big)l^2q^4 \\
&+ \Big(\frac{105}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^3q^3 + \Big(\frac{77}{12}\times 7^{t-3}+\frac{1}{6}\Big)l^3q^4 + \Big(\frac{133}{32}\times 7^{t-3}-\frac{7}{32}\Big)l^4q^4 \\
{}^{m}M_2(Sie(\Lambda,t)) &= \Big|\Big(14\times 7^{t-3}\Big)l^1q^2 + \Big(\frac{329}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^1q^3 + \Big(\frac{119}{12}\times 7^{t-3}+\frac{1}{12}\Big)l^1q^4 \\
&+ \Big(\frac{7}{6}\times 7^{t-3}+\frac{1}{12}\Big)l^2q^2 + \Big(\frac{98}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^2q^3 + \Big(\frac{133}{24}\times 7^{t-3}-\frac{1}{6}\Big)l^2q^4 \\
&+ \Big(\frac{105}{18}\times 7^{t-3}+\frac{1}{18}\Big)l^3q^3 + \Big(\frac{77}{12}\times 7^{t-3}+\frac{1}{6}\Big)l^3q^4 + \Big(\frac{133}{32}\times 7^{t-3}-\frac{7}{32}\Big)l^4q^4\Big|_{l=q=1} \\
{}^{m}M_2(Sie(\Lambda,t)) &= \frac{20,377}{288}\times 7^{t-3}+\frac{11}{96}
\end{aligned}$$

$$\begin{aligned}
R_\alpha(Sie(\Lambda,t)) &= (D_l^\alpha D_q^\alpha)(f(l,q))|_{l=q=1} \\
D_q^\alpha f(l,q) &= \Big(2^\alpha 28\times 7^{t-3}\Big)l^1q^2 + 3^\alpha\Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^1q^3 + 4^\alpha\Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^1q^4 \\
&+ 2^\alpha\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^2 + 3^\alpha\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^3 + 4^\alpha\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^2q^4 \\
&= 3^\alpha\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^3 + 4^\alpha\Big(77\times 7^{t-3}+2\Big)l^3q^4 + 4^\alpha\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^4q^4 \\
D_l^\alpha D_q^\alpha f(l,q) &= \Big(2^\alpha 28\times 7^{t-3}\Big)l^1q^2 + 3^\alpha\Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^1q^3 + 4^\alpha\Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^1q^4 \\
&+ 2^{2\alpha}\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^2 + 2^\alpha 3^\alpha\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^3 + 2^\alpha 4^\alpha\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^2q^4 \\
&+ 3^{2\alpha}\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^3 + 3^\alpha 4^\alpha\Big(77\times 7^{t-3}+2\Big)l^3q^4 + 4^{2\alpha}\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^4q^4
\end{aligned}$$

substituting the values $D_l^\alpha D_q^\alpha f(l,q)$

$$\begin{aligned}
R_\alpha(Sie(\Lambda,t)) &= \Big|\Big(2^\alpha 28\times 7^{t-3}\Big)l^1q^2 + 3^\alpha\Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^1q^3 + 4^\alpha\Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^1q^4 \\
&+ 2^{2\alpha}\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^2 + 2^\alpha 3^\alpha\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^2q^3 + 2^\alpha 4^\alpha\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^2q^4 \\
&+ 3^{2\alpha}\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^3q^3 + 3^\alpha 4^\alpha\Big(77\times 7^{t-3}+2\Big)l^3q^4 + 4^{2\alpha}\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^4q^4\Big|_{l=q=1} \\
R_\alpha(Sie(\Lambda,t)) &= 2^\alpha\times 28\times 7^{t-3} + 3^\alpha\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big) + 4^\alpha\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big) \\
&+ 2^{2\alpha}\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big) + 2^\alpha 3^\alpha\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big) + 2^\alpha 4^\alpha\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big) \\
&+ 3^{2\alpha}\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big) + 3^\alpha 4^\alpha\Big(77\times 7^{t-3}+2\Big) + 4^{2\alpha}\Big(\frac{133}{2}\times 7^{t-3}+\frac{7}{2}\Big)
\end{aligned}$$

$$
\begin{aligned}
RR_\alpha(Sie(\Lambda,t)) &= (\delta_l^\alpha \delta_q^\alpha)(f(l,q))|_{l=q=1} \\
\delta_q^\alpha f(l,q) &= \Big(28\times 7^{t-3}\Big)\frac{l^1q^2}{2^\alpha} + \Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^1q^3}{3^\alpha} + \Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^1q^4}{4^\alpha} \\
&+ \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2^\alpha} + (\frac{98}{3}\times 7^{t-3}+\frac{1}{3})\frac{l^2q^3}{3^\alpha} + \Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^2q^4}{4^\alpha} \\
&+ \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^3q^3}{3^\alpha} + (77\times 7^{t-3}+2)\frac{l^3q^4}{4^\alpha} + \Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^4q^4}{4^\alpha} \\
\delta_l^\alpha\delta_q^\alpha f(l,q) &= \Big(28\times 7^{t-3}\Big)\frac{l^1q^2}{2^\alpha} + \Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^1q^3}{3^\alpha} + \Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^1q^4}{4^\alpha} \\
&+ \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2^{2\alpha}} + \Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^3}{2^\alpha 3^\alpha} + \Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^2q^4}{2^\alpha 4^\alpha} \\
&+ \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^3q^3}{3^{2\alpha}} + \Big(77\times 7^{t-3}+2\Big)\frac{l^3q^4}{3^\alpha 4^\alpha} + \Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^4q^4}{4^{2\alpha}}
\end{aligned}
$$

substituting the value in $RR_\alpha(Sie(\Lambda,t))$

$$
\begin{aligned}
RR_\alpha(Sie(\Lambda,t)) &= \Big|\Big(28\times 7^{t-3}\Big)\frac{l^1q^2}{2^\alpha} + \Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^1q^3}{3^\alpha} + \Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^1q^4}{4^\alpha} \\
&+ \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2^{2\alpha}} + \Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^3}{2^\alpha 3^\alpha} + \Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^2q^4}{2^\alpha 4^\alpha} \\
&+ \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^3q^3}{3^{2\alpha}} + \Big(77\times 7^{t-3}+2\Big)\frac{l^3q^4}{3^\alpha 4^\alpha} + \Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^4q^4}{4^{2\alpha}}\Big|_{l=q=1} \\
RR_\alpha(Sie(\Lambda,t)) &= \frac{28}{2^\alpha}\times 7^{t-3} + \frac{1}{3^\alpha}\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big) + \frac{1}{4^\alpha}\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big) + \frac{1}{2^{2\alpha}}\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big) \\
&+ \frac{1}{3^\alpha 2^\alpha}\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big) + \frac{1}{2^\alpha 4^\alpha}\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big) + \frac{1}{3^{2\alpha}}\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big) \\
&+ \frac{1}{3^\alpha 4^\alpha}\Big(77\times 7^{t-3}+2\Big) + \frac{1}{4^{2\alpha}}\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
SDD(Sie(\Lambda,t)) &= (\delta_q D_l + \delta_l D_q)(f(l,q))|_{l=q=1} \\
\delta_q D_l f(l,q) &= \Big(28\times 7^{t-3}\Big)\frac{l^1q^2}{2} + \Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^1q^3}{3} + \Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^1q^4}{4} \\
&+ 2\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2} + 2\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^3}{3} + 2\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^2q^4}{4} \\
&+ 3\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^3q^3}{3} + 3(77\times 7^{t-3}+2)\frac{l^3q^4}{4} + 4\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^4q^4}{4} \\
\delta_l D_q f(l,q) &= \Big(56\times 7^{t-3}\Big)l^1q^2 + \Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^1q^3 + \Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^1q^4 \\
&= 2\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2} + 2\Big(\frac{294}{3}\times 7^{t-3}+1\Big)\frac{l^2q^3}{2} + \Big(\frac{532}{3}\times 7^{t-3}-\frac{16}{3}\Big)\frac{l^2q^4}{2} \\
&+ \Big(\frac{315}{2}\times 7^{t-3}+\frac{3}{2}\Big)\frac{l^3q^3}{3} + \Big(308\times 7^{t-3}+8\Big)\frac{l^3q^4}{3} + \Big(266\times 7^{t-3}-14\Big)\frac{l^4q^4}{4}
\end{aligned}
$$

$$
\begin{aligned}
SDD(Sie(\Lambda,t)) \quad = \quad & \Big|\Big(28\times 7^{t-3}\Big)\frac{l^1q^2}{2}+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^1q^3}{3}+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^1q^4}{4} \\
+ \quad & 2\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2}+2(\frac{98}{3}\times 7^{t-3}+\frac{1}{3})\frac{l^2q^3}{3}+2(\frac{133}{3}\times 7^{t-3}-\frac{4}{3})\frac{l^2q^4}{4} \\
+ \quad & 3\Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^3q^3}{3}+3\Big(77\times 7^{t-3}+2\Big)\frac{l^3q^4}{4}+4\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^4q^4}{4} \\
+ \quad & \Big(56\times 7^{t-3}\Big)l^1q^2+\Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^1q^3+\Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^1q^4 \\
+ \quad & 2\Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^2q^2}{2}+2\Big(\frac{294}{3}\times 7^{t-3}+1\Big)\frac{l^2q^3}{2}+\Big(\frac{532}{3}\times 7^{t-3}-\frac{16}{3}\Big)\frac{l^2q^4}{2} \\
+ \quad & \Big(\frac{315}{2}\times 7^{t-3}+\frac{3}{2}\Big)\frac{l^3q^3}{3}+\Big(308\times 7^{t-3}+8\Big)\frac{l^3q^4}{3}+\Big(266\times 7^{t-3}-14\Big)\frac{l^4q^4}{4}\Big|_{l=q=1} \\
SDD(Sie(\Lambda,t)) \quad = \quad & \frac{18,445}{18}\times 7^{t-3}-\frac{65}{36}
\end{aligned}
$$

$$
\begin{aligned}
H(Sie(\Lambda,t)) \quad = \quad & 2\delta_l J(f(l,q))|_{l=1} \\
J(f(l,q)) \quad = \quad & \Big(28\times 7^{t-3}\Big)l^3+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)l^4+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^5 \\
+ \quad & \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^4+\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)l^5+\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)l^6 \\
+ \quad & \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^6+\Big(77\times 7^{t-3}+2\Big)l^7+\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)l^8 \\
\delta_l J(f(l,q)) \quad = \quad & \Big(28\times 7^{t-3}\Big)\frac{l^3}{3}+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^4}{4}+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^5}{5} \\
+ \quad & \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^4}{4}+\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^5}{5}+\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^6}{6} \\
+ \quad & \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^6}{6}+\Big(77\times 7^{t-3}+2\Big)\frac{l^7}{7}+\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^8}{8} \\
H(Sie(\Lambda,t)) \quad = \quad & \Big|2\Big(\Big(28\times 7^{t-3}\Big)\frac{l^3}{3}+\Big(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\Big)\frac{l^4}{4}+\Big(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^5}{5} \\
+ \quad & \Big(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^4}{4}+\Big(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\Big)\frac{l^5}{5}+\Big(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\Big)\frac{l^6}{6} \\
+ \quad & \Big(\frac{105}{2}\times 7^{t-3}+\frac{1}{2}\Big)\frac{l^6}{6}+\Big(77\times 7^{t-3}+2\Big)\frac{l^7}{7}+\Big(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\Big)\frac{l^8}{8}\Big)\Big|_{l=1} \\
H(Sie(\Lambda,t)) \quad = \quad & \frac{9461}{120}\times 7^{t-3}-\frac{247}{5040}
\end{aligned}
$$

$$
\begin{aligned}
I(Sie(\Lambda,t)) \quad = \quad & \delta_l JD_lD_q(f(l,q))|_{l=1} \\
JD_lD_qf(l,q) \quad = \quad & \Big(56\times 7^{t-3}\Big)l^3+\Big(\frac{329}{2}\times 7^{t-3}+\frac{1}{2}\Big)l^4+\Big(\frac{476}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^5 \\
+ \quad & \Big(\frac{56}{3}\times 7^{t-3}+\frac{4}{3}\Big)l^4+\Big(\frac{588}{3}\times 7^{t-3}+2\Big)l^5+\Big(\frac{1064}{3}\times 7^{t-3}-\frac{32}{3}\Big)l^6 \\
+ \quad & \Big(\frac{945}{2}\times 7^{t-3}+\frac{9}{2}\Big)l^6+\Big(924\times 7^{t-3}+24\Big)l^7+\Big(1064\times 7^{t-3}-56\Big)l^8
\end{aligned}
$$

$$
\begin{aligned}
\delta_l JD_l D_q f(l,q) &= \Big(56\times 7^{t-3}\Big)\frac{l^3}{3} + (\frac{329}{2}\times 7^{t-3} + \frac{1}{2})\frac{l^4}{4} + (\frac{476}{3}\times 7^{t-3} + \frac{4}{3})\frac{l^5}{5} \\
&+ \Big(\frac{56}{3}\times 7^{t-3} + \frac{4}{3}\Big)\frac{l^4}{4} + (\frac{588}{3}\times 7^{t-3} + 2)\frac{l^5}{5} + (\frac{1064}{3}\times 7^{t-3} - \frac{32}{3})\frac{l^6}{6} \\
&+ \Big(\frac{945}{2}\times 7^{t-3} + \frac{9}{2}\Big)\frac{l^6}{6} + \Big(924\times 7^{t-3} + 24\Big)\frac{l^7}{7} + \Big(1064\times 7^{t-3} - 56\Big)\frac{l^8}{8} \\
I(Sie(\Lambda,t)) &= \Big|\Big(56\times 7^{t-3}\Big)\frac{l^3}{3} + \Big(\frac{329}{2}\times 7^{t-3} + \frac{1}{2}\Big)\frac{l^4}{4} + \Big(\frac{476}{3}\times 7^{t-3} + \frac{4}{3}\Big)\frac{l^5}{5} \\
&+ \Big(\frac{56}{3}\times 7^{t-3} + \frac{4}{3}\Big)\frac{l^4}{4} + \Big(\frac{588}{3}\times 7^{t-3} + 2\Big)\frac{l^5}{5} + \Big(\frac{1064}{3}\times 7^{t-3} - \frac{32}{3}\Big)\frac{l^6}{6} \\
&+ \Big(\frac{945}{2}\times 7^{t-3} + \frac{9}{2}\Big)\frac{l^6}{6} + \Big(924\times 7^{t-3} + 24\Big)\frac{l^7}{7} + \Big(1064\times 7^{t-3} - 56\Big)\frac{l^8}{8}\Big|_{l=1} \\
I(Sie(\Lambda,t)) &= \frac{68,137}{120}\times 7^{t-3} - \frac{1751}{504}
\end{aligned}
$$

$$
\begin{aligned}
A(Sie(\Lambda,t)) &= \delta_l^3 Q_2 JD_l^3 D_q^3 (f(l,q))|_{l=1} \\
D_q^3 f(l,q) &= \Big(2^3 28\times 7^{t-3}\Big)l^1q^2 + 3^3\Big(\frac{329}{2}\times 7^{t-3} + \frac{1}{2}\Big)l^1q^3 + 4^3\Big(\frac{476}{3}\times 7^{t-3} + \frac{4}{3}\Big)l^1q^4 \\
&+ 2^3\Big(\frac{14}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^2q^2 + 3^3\Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^2q^3 + 4^3\Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)l^2q^4 \\
&+ 3^3\Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{2}\Big)l^3q^3 + 4^3\Big(77\times 7^{t-3} + 2\Big)l^3q^4 + 4^3\Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)l^4q^4 \\
JD_l^3 D_q^3 f(l,q) &= \Big(2^3 28\times 7^{t-3}\Big)l^3 + 3^3\Big(\frac{329}{2}\times 7^{t-3} + \frac{1}{2}\Big)l^4 + 4^3\Big(\frac{476}{3}\times 7^{t-3} + \frac{4}{3}\Big)l^5 \\
&+ 2^6\Big(\frac{14}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^4 + 2^3 3^3\Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^5 + 2^3 4^3\Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)l^6 \\
&+ 3^6\Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{2}\Big)l^6 + 3^3 4^3\Big(77\times 7^{t-3} + 2\Big)l^7 + 4^6\Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)l^8 \\
Q_2 JD_l^3 D_q^3 &= 2^3\times 28\times 7^{t-3}l^1 + 3^3\Big(\frac{329}{6}\times 7^{t-3} + \frac{1}{6}\Big)l^2 + 4^3\Big(\frac{119}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^3 \\
&+ 2^6\Big(\frac{14}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^2 + 2^3\times 3^3\Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)l^3 + 2^3\times 4^3\Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)l^4 \\
&+ 3^6\Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{2}\Big)l^4 + 3^3\times 4^3\Big(77\times 7^{t-3} + 2\Big)l^5 + 4^6\Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)l^6 \\
\delta_l^3 Q_2 JD_l^3 D_q^3 &= 2^3\times 28\times 7^{t-3}l^1 + 3^3\Big(\frac{329}{6}\times 7^{t-3} + \frac{1}{6}\frac{l^2}{2^3}\Big) + 4^3\Big(\frac{119}{3}\times 7^{t-3} + \frac{1}{3}\frac{l^3}{3^3}\Big) \\
&+ 2^6\Big(\frac{14}{3}\times 7^{t-3} + \frac{1}{3}\Big)\frac{l^2}{2^3} + 2^3\times 3^3\Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)\frac{l^3}{3^3} + 2^3\times 4^3\Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)\frac{l^4}{4^3} \\
&+ 3^6\Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{2}\Big)\frac{l^4}{4^3} + 3^3\times 4^3\Big(77\times 7^{t-3} + 2\Big)\frac{l^5}{5^3} + 4^6\Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)\frac{l^6}{6^3} \\
A(Sie(\Lambda,t)) &= \Big|2^3\times 28\times 7^{t-3}l^1 + 3^3\Big(\frac{329}{6}\times 7^{t-3} + \frac{1}{6}\Big)\frac{l^2}{2^3} + 4^3\Big(\frac{119}{3}\times 7^{t-3} + \frac{1}{3}\Big)\frac{l^3}{3^3} \\
&+ 2^6\Big(\frac{14}{3}\times 7^{t-3} + \frac{1}{3}\Big)\frac{l^2}{2^3} + 2^3\times 3^3\Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)\frac{l^3}{3^3} + 2^3\times 4^3\Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)\frac{l^4}{4^3} \\
&+ 3^6\Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{2}\Big)\frac{l^4}{4^3} + 3^3\times 4^3\Big(77\times 7^{t-3} + 2\Big)\frac{l^5}{5^3} + 4^6\Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)\frac{l^6}{6^3}\Big|_{l=1} \\
A(Sie(\Lambda,t)) &= \frac{5,287,567,733}{1,296,000}\times 7^{t-3} - \frac{47,962,067}{1,296,000}
\end{aligned}
$$

□

Entropy, Shannon Entropy, and Graph Entropy

The measure of the unreliability of a framework is the entropy of a probability distribution; this idea was presented in Shannon's acclaimed paper [14]. This graph entropy is applied in different fields of sciences such as chemistry, biology, nature, and human science [15,16]. There are numerous types of entropy measures, such as the graph entropy measure related to the distribution of components (vertices, edges) and probability distribution. However, the least examined graphs are entropies for weighted graphs [17].

Shannon entropy and graph entropy are two distinct concepts used in different domains of mathematics and science. Here are the key differences between the two:

Shannon entropy, often referred to as information entropy, is a concept from information theory. It quantifies the uncertainty or randomness associated with a random variable or a probability distribution. It measures the average amount of information contained in a random variable. Shannon entropy is widely used in information theory, data compression, and communication theory. It is used to analyze and measure the uncertainty or randomness in data, such as in coding theory, cryptography, and data compression algorithms.

Graph entropy is a concept used in network theory and graph theory. It quantifies the structural properties and information content of a graph (a collection of nodes and edges). It characterizes the complexity or diversity of a network's topology. Graph entropy is applied in the analysis of complex networks, such as social networks, biological networks, and transportation networks. It is used to understand the organization and properties of these networks.

Various methods exist to compute graph entropy, depending on the aspects of the network one wants to capture. Common measures include degree entropy, clustering coefficient entropy, and betweenness centrality entropy. In summary, Shannon entropy measures the information content or randomness of data, while graph entropy quantifies the structural properties and complexity of a network or graph. They serve different purposes and are used in distinct fields of mathematics and science.

Definition 2. *In [18], the entropy of G is an edge-weighted graph $G(V,E,W)$, and is described as follows:*

$$I(G,W) = -\sum_{pj\in E(G)} (p_{pj})log(p_{pj})$$

where $p_{pj} = \frac{w(pj)}{\sum\limits_{pj\in E(G)}(w_{pj})}$

Definition 3. *Define as $w(e) = (d_p \times d_j)^{\alpha}$, where the degree of p is described as d_p, an edge $E = pj$ and any real number α, one the Randic index in [19–21] is defined as:*

$$R_{\alpha}(G) = \sum_{pj\in E(G)} (d_p \times d_j)^{\alpha}$$

According to ref. [22], the Randić' index has exhibited usefulness for assessing the degree of extension of a carbon-atom skeleton of saturated hydrocarbons.
Let $I(G,\alpha)$ be the entropy of the graph G. Now, the relationship between $I(G,\alpha)$ and $R_{\alpha}(G)$ is

$$I(G,\alpha) = logR_{\alpha}(G) - \frac{\alpha}{R_{\alpha}(G)} \sum_{pj\in E(G)} ((d_p \times d_j)^{\alpha} log(d_p \times d_j))$$

where α is a real number. Now, we will give our computations on the entropy of generalized Sierpinski graphs $Sie(\Lambda,t)$.

Theorem 2. *Let $Sie(\Lambda,t)$ be a generalized Sierpinski graph. Then, the entropy of $Sie(\Lambda,t)$ is*

$$\begin{aligned}
I(Sie(\Lambda,t),\alpha) &= \log\Bigg[28\times 7^{n-3}\times 2^{\alpha}+\left(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\right)3^{\alpha}+\left(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}\\
&+ \left(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}+\left(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\right)6^{\alpha}+\left(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\right)8^{\alpha}\\
&+ \left(\frac{105}{2}\times 7^{t-3}+\frac{1}{3}\right)9^{\alpha}+\left(77\times 7^{t-3}+2\right)12^{\alpha}+\left(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\right)16^{\alpha}\Bigg]\\
&- \alpha\left[\frac{A}{B}\right]
\end{aligned}$$

$$\begin{aligned}
A &= 28\times 7^{t-3}\times 2^{\alpha}\log(2^{\alpha})+\left(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\right)3^{\alpha}\log(3^{\alpha})\\
&+ \left(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}\log(4^{\alpha})\\
&+ \left(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}\log(4^{\alpha})\\
&+ \left(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\right)6^{\alpha}\log(6^{\alpha})\\
&+ \left(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\right)8^{\alpha}\log(8^{\alpha})+\left(\frac{105}{2}\times 7^{t-3}+\frac{1}{3}\right)9^{\alpha}\log(9^{\alpha})\\
&+ \left(77\times 7^{t-3}+2\right)12^{\alpha}\log(12^{\alpha})+\left(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\right)16^{\alpha}\log(16^{\alpha})
\end{aligned}$$

$$\begin{aligned}
B &= 28\times 7^{t-3}\times 2^{\alpha}+\left(\frac{329}{6}\times 7^{t-3}+\frac{1}{6}\right)3^{\alpha}\\
&+ \left(\frac{119}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}+\left(\frac{14}{3}\times 7^{t-3}+\frac{1}{3}\right)4^{\alpha}\\
&+ \left(\frac{98}{3}\times 7^{t-3}+\frac{1}{3}\right)6^{\alpha}\\
&+ \left(\frac{133}{3}\times 7^{t-3}-\frac{4}{3}\right)8^{\alpha}\\
&+ \left(\frac{105}{2}\times 7^{t-3}+\frac{1}{3}\right)9^{\alpha}\\
&+ \left(77\times 7^{t-3}+2\right)12^{\alpha}+\left(\frac{133}{2}\times 7^{t-3}-\frac{7}{2}\right)16^{\alpha}
\end{aligned}$$

where α is a real number.

Proof. The generalised Sierpinski graph is shown in Figure 2. Let $E_{p,j}$ denote the number of edges connecting the vertices of degree d_p and d_j. In this graph $Sie(\Lambda,t)$, the total number of vertices are $\frac{2401}{6}\times 7^{t-3}-\frac{7}{6}$. The number of vertices of degree two, three, and four are $\frac{343}{6}\times 7^{t-3}-\frac{1}{6}$, $\frac{539}{6}\times 7^{t-3}+\frac{7}{6}$, and $\frac{441}{6}\times 7^{t-3}-\frac{3}{2}$, respectively. The total number of edges of the generalised Sierpinski graph is $\frac{1}{6}\times 7^{t+1}-\frac{7}{6}$. The edge partition based on the degree of the end vertices of each edge is shown in Table 2. The formula for the general Randić index is

$$R_{\alpha}(G)=\sum_{pj\in E(G)}(d_pd_j)^{\alpha}$$

Table 2. M-polynomial topological indices.

Topological Indices	Derivation from M(Sie(Λ,t),l,q)
First Zagreb index	$M_1(Sie(\Lambda,t)) = (D_l + D_q)(f(l,q))\|_{l=q=1}$
Second Zagreb index	$M_2(Sie(\Lambda,t)) = (D_l D_q)(f(l,q))\|_{l=q=1}$
Second modified Zagreb index	${}^m M_2(Sie(\Lambda,t)) = (\delta_l \delta_q)(f(l,q))\|_{l=q=1}$
General Randić index, $\alpha \neq 0$	$R_\alpha(Sie(\Lambda,t)) = (D_l^\alpha D_q^\alpha)(f(l,q))\|_{l=q=1}, \alpha \in N$
Inverse general Randić index, $\alpha \neq 0$	$RR_\alpha(Sie(\Lambda,t)) = (\delta_l^\alpha \delta_q^\alpha)(f(l,q))\|_{l=q=1}, \alpha \in N$
Symmetric division index	$SDD(Sie(\Lambda,t)) = (D_l \delta_q + \delta_l D_q)(f(l,q))\|_{l=q=1}$
Harmonic index	$H(Sie(\Lambda,t)) = 2\delta_l J(f(l,q))\|_{l=1}$
Inverse sum index	$I(Sie(\Lambda,t)) = \delta_l J D_l D_q(f(l,q))\|_{l=1}$
Augmented Zagreb	$A(Sie(\Lambda,t)) = \delta_l^3 Q_{-2} J D_l^3 D_q^3(f(l,q))\|_{l=1}$

where

$$D_l = l\Big[\frac{\partial f(l,q)}{\partial l}\Big] \qquad D_q = q\Big[\frac{\partial f(l,q)}{\partial q}\Big] \tag{3}$$

$$\delta_l = \int_0^l \Big[\frac{f(t,q)}{t}dt\Big] \qquad \delta_q = \int_0^q \Big[\frac{f(l,t)}{t}dt\Big] \tag{4}$$

$$J(f(l,q)) = f(l,l) \qquad Q_\alpha(f(l,q)) = l^\alpha f(l,q) \tag{5}$$

This implies that

$$\begin{aligned}
R_\alpha() &= E_{1,2}\,(1\times 2)^\alpha + E_{1,3}\,(1\times 3)^\alpha + E_{1,4}\,(1\times 4)^\alpha + E_{2,2}\,(2\times 2)^\alpha + E_{2,3}\,(2\times 3)^\alpha \\
&+ E_{2,4}\,(2\times 4)^\alpha + E_{3,3}\,(3\times 3)^\alpha + E_{3,4}\,(3\times 4)^\alpha + E_{4,4}\,(4\times 4)^\alpha \\
R_\alpha(G) &= 28\times 7^{t-3}2^\alpha + \Big(\frac{329}{6}7^{t-3} + 1/6\Big)3^\alpha + \Big(\frac{119}{3}\times 7^{t-3} + \frac{1}{3}\Big)4^\alpha \\
&+ \Big(14/3\times 7^{t-3} + \frac{1}{3}\Big)4^\alpha + \Big(\frac{98}{3}\times 7^{t-3} + \frac{1}{3}\Big)6^\alpha \\
&+ \Big(\frac{133}{3}\times 7^{t-3} - \frac{4}{3}\Big)8^\alpha + \Big(\frac{105}{2}\times 7^{t-3} + \frac{1}{3}\Big)9^\alpha \\
&+ \Big(77\times 7^{t-3} + 2\Big)12^\alpha + \Big(\frac{133}{2}\times 7^{t-3} - \frac{7}{2}\Big)16^\alpha
\end{aligned}$$

The formula for the entropy is given as follows:

$$I(Sie(\Lambda,t),\alpha) = logR_\alpha(Sie(\Lambda,t)) - \frac{\alpha}{R_\alpha(Sie(\Lambda,t))}\sum_{pj\in E(Sie(\Lambda,t))}((d_p.d_j)^\alpha log(d_p.d_j))$$

after inserting the value of R_α, we obtain the desired results. □

4. Conclusions

Polynomials of any order always show some applications in daily life, such as quadratic polynomials that are used to find the maximum and minimum values of the function and also to discuss the parabolic behavior of the function. In this paper, we have studied the M-Polynomial and entropy of a generalised Sierpinski graph and extracted many topological indices out of it. The advantage of an M-polynomial is the ability to obtain more than 10 degree-based topological indices at once. These results will be helpful and will open new horizons for the readers.

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