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Equitable and Paired Equitable Domination in Inflated Graphs and Their Complements

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Abstract: Domination plays an indispensable role in graph theory. Various types of domination explore various types of applications. Equal-status people work together and interlace with each other easily. In this paper, the paired equitable domination of a graph, its inflated graph, and its complement of an inflated graph were studied. The relationship between the domination number of the graph, the equitable domination number, and the paired equitable domination number of complements of the inflated graph were established. Furthermore, we proved the Nordhaus–Gaddum-type inequality, that is, $\gamma^e_{pr}(H) + \gamma^e_{pr}(H) \leq 6$ if H is a graph with m nodes where $m \equiv 0,\ 2(mod\ 8)$ and $d(a_i) = (m/2)$ for all a_i. The challenges and limitations of this parameter of paired equitable and equitable domination depends on the degree of the vertex of the graph. Practical applications are discussed in various fields and illustrated using the studied parameter.

Keywords: domination; inflated graph; complement graph; Nordhaus–Gaddam inequality

MSC: 05C69; 05C76

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1. Introduction

Let $G = (V, E)$ be a simple, undirected, and connected graph. $V(G)$ denotes the nonempty set of vertices of G, and $E(G)$ denotes the set of edges of G. For further graph theoretical terminology, we followed Bondy and Murthy [1].

Ore [2] was the first author to use the terms "dominating set" and "domination number" in his research. A subset $X' \subseteq V(G)$ is a dominating set if for every vertex a in $V - X'$, there exists a vertex b in X' such that a is adjacent to b. The minimum cardinality of a dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. We refer to a dominating set of minimum cardinality as $\gamma(G)$ set of G. For further theoretical domination terminology, we followed Haynes et al. [3]. Many types of domination parameters have been introduced and studied. Haynes et al. presented various types of domination and their relations with bounds. Total domination was studied by Cockayne et al. [4] and further investigated by Bujtas et al. and Favaron et al. [5,6]. Harary et al. studied double domination in their research work [7]. Paired domination was examined by Chellali et al., Gorzkowska et al., and Micheal et al. [8–10]. Chellali et al. presented the relationship between the total and paired domination of trees [8]. Edwards, Brigham et al., and Xinmin et al. studied the critical concepts of paired domination [11–13]. Meenakshi and Baskar babujee studied a complementary-tree paired-domination vertex critical graph [14]. Dunbar and Haynes [15] started the study of domination in inflated graphs, which was further continued by other researchers [16–19]. Further, Swaminathan and Dharmalingam studied equitable domination [20]. Paired equitable domination was introduced by Meenakshi and Baskar babujee in 2016 [21]. The equitable domination of the complement of inflated graphs was studied by Meenakshi [22]. Meenakshi elaborated on studies of the complement of inflated graphs of some standard graphs [23].

The motivation of this research is to discuss the relationship between domination numbers, equitable domination, and paired equitable domination numbers in inflated graphs. Section 2 focuses on the preliminary studies on equitable and paired equitable domination numbers of graphs. Section 3 deals with applications of equitable and paired equitable domination of graphs in various fields.

In Section 4, the effective results of the sum of the paired equitable domination numbers of certain types of graphs and their complements are discussed. Section 5 provides the relationship between domination number, equitable domination, and the paired equitable domination number of the complement of inflated graphs. Section 6 talks about the paired equitable domination of the complement of the inflated graph of the complement of G. In this paper, we considered only simple, finite, and connected graphs. Table 1 represents the symbols used in this research and their meaning.

Table 1. Symbols and their meanings.

Symbols	Meanings
$G(V,E)$	Simple graph G node set V and edge set E
$\deg(u)$	Degree of the node u
P_m	Path of length m
C_m	Cycle of length m
$K_{m,n}$	Complete bipartite graph with order m,n
$\gamma^e(G)$	Equitable domination number of a graph G
$G_{(l)}$	Inflated graph of G
${\gamma_{pr}}^e(G)$	Paired equitable domination number of a graph G
$\overline{G}$	The complement of a graph G
$\overline{(G_{(l)})}$	Complement of an inflated graph G

2. Equitable and Paired Equitable Domination in Inflated Graphs

2.1. Equitable Domination Number [20]

Let $G = (V, E)$ be a connected graph. A subset D of V is called an equitable dominating set of a graph G if for every $u \in V - G$ there exists a node $v \in V$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is called an equitable domination number of G and is denoted by $\gamma^e(G)$.

The complement of a graph G is denoted by $\overline{G}$, and its inflated graph is denoted by $G_{(l)}$.

2.2. Theorem 1 [20]

(i) For $G = P_m$ and C_m on m nodes, $\gamma^e(G) = \lceil \frac{m}{3} \rceil$.

(ii) For $G = K_m$ on m nodes, $\gamma^e(G) = 1$.

(iii) For $G = K_{r,s}$, $\gamma^e(K_{r,s}) = \begin{cases} 2, \ if|r-s| \leq 1 \\ r+s \ if \ |r-s| \geq 2, where \ r,s \geq 2 \end{cases}$.

2.3. Paired Equitable Domination Number [23]

Let $G = (V, E)$ be a connected graph. An equitable dominating set S of a connected graph G is called the paired equitable dominating set (PEDS) if $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired equitable dominating set (denoted by $\gamma^e_{pr}\ set\ of\ G$) is called the paired equitable domination number of G and is denoted by $\gamma^e_{pr}(G)$. The existence of such a set is guaranteed only for a connected graph. An example of the existence of both an equitable and a paired equitable dominating set is shown in Figure 1.

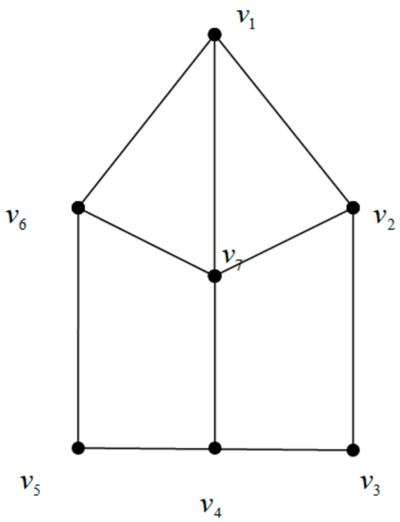


Figure 1. Graph G.

Example: Let G be a graph shown in Figure 1.

$S = \{v_1, v_7\}$ is an equitable and paired dominating set of G, since for every $v_i \in V - S$, there exists $v_j \in S$ such that $v_i v_j \in E(G)$ and $|\deg(v_i) - \deg(v_j)| \leq 1$. The minimum cardinality of the equitable dominating set is 2, hence the equitable domination number of G, $\gamma^e(G) = 2$. Furthermore, an induced subgraph of G has perfect matching. Hence, the paired equitable domination number of G is $\gamma^e_{pr}(G) = 2$.

2.4. Theorem 2 [22]

(i) For the path graph $G = P_n$, $\gamma^e_{pr}(G) = 2\left\lceil \frac{n}{4} \right\rceil$ $if\ n \geq 2$.
(ii) For the cycle graph $G = C_n$, $\gamma^e_{pr}(G) = 2\left\lceil \frac{n}{4} \right\rceil$.
(iii) For the complete graph $G = K_n$, $\gamma^e_{pr}(G) = 2$.
(iv) For the complete bipartite graph $G = K_{m,n}$,

$$\gamma^e_{pr}(G) = \begin{cases} 2,\ if\ |m-n| \leq 1 \\ m+n,\ otherwise \end{cases}.$$

(v) For any connected graph G, $\gamma(G) \leq \gamma_{pr}(G) \leq \gamma^e_{pr}(G)$.

2.5. Theorem 3 [23]

(i) For the path $G = P_n$ on n nodes, $\gamma^e_{pr}(G_{(l)}) = 2\left\lceil \frac{2n-2}{4} \right\rceil$.
(ii) For the cycle $G = C_n$ on n nodes, $\gamma^e_{pr}(G_{(l)}) = 2\left\lceil \frac{n}{2} \right\rceil$.
(iii) For the complete graph $G = K_n$ on n nodes, $\gamma^e_{pr}(G_{(l)}) \leq n+1$.
(iv) For the complete bipartite graph $G = K_{m,n}$, $\gamma^e_{pr}(G_{(l)}) \leq m+n+1\ if\ |m-n| \leq 1$.
(v) For the connected graph G of order n with $\delta\,(G) \geq 2$, then $\gamma^e_{pr}(G_{(l)}) \leq n+1$.

3. Applications

Equitable domination is a concept that arises in the field of graph theory. In graph theory, a graph consists of a set of vertices (also known as nodes) and a set of edges (also known as links) that connect pairs of vertices. Equitable domination focuses on the problem of assigning weights to the vertices of a graph in such a way that every vertex is dominated by the sum of its own weight and the weights of its neighbors. The concept of equitable domination has applications in various fields, including those outlined in the following subsections.

3.1. Network Design

Equitable domination can be applied in the design and analysis of communication networks, such as wireless networks or computer networks. By assigning appropriate weights to the vertices of the network graph, one can ensure a fair distribution of resources or coverage among the network nodes.

3.2. Social Networks

Equitable domination can be used to model and analyze social networks, where vertices represent individuals or entities and edges represent connections or relationships between them. Assigning equitable weights to the vertices can help to identify influential individuals or groups within the network, ensuring a balanced representation of influence or control.

3.3. Resource Allocation

Equitable domination can be utilized in scenarios where resources need to be allocated among a set of entities or regions. By assigning weights to the vertices representing entities or regions, one can ensure that the allocation is performed fairly and that no entity or region is significantly dominating or dominating excessively.

These are just a few examples of how equitable domination can be applied in different domains. The specific application depends on the context and problem at hand, but the underlying idea of achieving a fair distribution or coverage is common to all these applications.

Paired equitable domination is an extension of equitable domination that takes into account pairs of vertices in a graph. In this concept, the goal is to assign weights to both individual vertices and pairs of vertices in such a way that every vertex and every pair of vertices are dominated by the sum of their own weights and the weights of their neighbors.

While paired equitable domination is a relatively new concept in graph theory, it has the potential for various applications. A few possible applications are outlined in the following subsections.

3.4. Recommender Systems

Paired equitable domination can be used in recommender systems to improve the accuracy and fairness of recommendations. By assigning weights to both items and pairs of items, the system can take into account the similarity or compatibility between items when making recommendations. This can lead to more personalized and different recommendations that consider the preferences and relationships between users and items.

3.5. Collaborative Filtering

Paired equitable domination can be applied in collaborative filtering, a technique used in recommendation systems and information filtering. By assigning weights to both users and pairs of users, collaborative filtering algorithms can consider the similarity and compatibility between users' preferences when generating recommendations. This can lead to more accurate predictions and better user satisfaction.

3.6. Genetic Analysis

Paired equitable domination can be applied in genetic analysis to understand the interactions between genes or genetic variants. By assigning weights to both individual genes and pairs of genes, researchers can analyze the relationships and dependencies between genes more effectively. This can help in identifying gene interactions, predicting disease risk, or understanding complex genetic traits.

These are just a few examples of how paired equitable domination can potentially be applied. As the field of paired equitable domination evolves, more specific and tailored applications are likely to emerge in various domains in which understanding pairwise relationships is important.

Illustration of equitable domination: Assume that G is an organization with six employees. Each employee is equally qualified or may differ by a degree. Let the employees be denoted by EM1(J) (J represents junior), EM2(J), EM3(S) (S represents senior), EM4(J), EM5(S), and EM6(J). The aim is to find the minimum number of monitors or number of employees to share the information or knowledge to all. Select EM3(S) (which monitors EM2(J) and EM4(J)) and EM6(S) (which monitors EM1(J) and EM5(J)) to monitor others, which is subject to the constraint that the number of degrees of monitoring and nonmonitoring differs by at most one. To monitor the organization, a minimum of two employees is ample (see Figure 2).

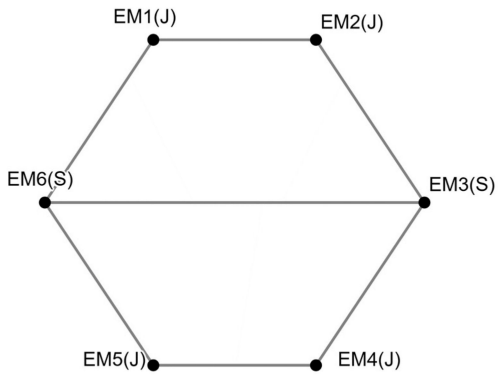


Figure 2. Organization model G.

4. Paired Equitable Domination of the Complement of the Graphs

The paired equitable domination of the complement of some classes of graphs was studied.

4.1. Theorem 1

(i) If $H = P_m$ is a path, then $\gamma^e_{pr}(\overline{H}) = 2 \; \forall \; m \geq 2$.

(ii) If $H = C_m$ is a path, then $\gamma^e_{pr}(\overline{H}) = 2 \; \forall \; m \geq 2$.

(iii) If $H = K_{r,s} - e$, then $\gamma^e_{pr}(\overline{H}) = 2 \; if \; |r - s| \leq 1$.

(iv) If $H = P_n \times C_m$, then $\gamma^e_{pr}(\overline{H}) = 2$.

Proof of (i): Let $\overline{H} = \overline{P}_m$ and A be a minimal *PEDS* of H. Let x and y be the end nodes of H, each of a degree of one. In $\overline{H}$, the degree of both x and y is $m - 2$ and the degree of the remaining nodes is $m - 3$. Since it is a biregular graph, choose the node x which is of degree $m - 2$ and any one node of degree $m - 3$, say w, then $A = \{u, w\}$ will be a dominating set of $\overline{H}$. Also, $|\deg(r) - \deg(s)| \leq 1$ for all $r \in A$ and $s \in V - A$. Therefore, A is a *PEDS* of $\overline{H}$. Hence, $\gamma^e_{pr}(\overline{P}_m) = 2 \; \forall \; m \geq 2$.

Proof of (ii): Let $\overline{H} = \overline{C}_m$ and A be a minimal *PEDS* of H. Since every node of H is of degree $m - 3$, arbitrarily choose any two nodes, say x and y, such that both are paired to each other; then $A = \{x, y\}$ will be a dominating set of H, since $|\deg(x) - \deg(y)| \leq 1$ for all $x \in A$ and $y \in V - A$. Hence $\gamma^e_{pr}(\overline{C}_m) = 2 \; \forall \; m \geq 2$.

Proof of (iii): Let $\overline{H} = K_{r,s} - e$ where $e = xy$ and S be a *PEDS* of $\overline{H}$. Let the degree of every node $x \in X(\overline{H})$ be s and the degree of every node $y \in Y(\overline{H})$ be r. Now every node in $X(\overline{H})$ is of degree $(r + s - 1) - s = r - 1$ except $x \in X(\overline{H})$ and $\deg(x) = r$. Similarly, every node in $Y(\overline{H})$ is of degree $(r + s - 1) - r = s - 1$ except $y \in Y(\overline{H})$ and $\deg(y) = s$. Clearly, x and y are paired in $\overline{H}$, and the pair dominates all the nodes, and also, $|\deg(u) - \deg(v)| \leq 1$ for every $u \in S$ and for every $v \in V(\overline{H})$. Hence, $S = \{x, y\}$ is a minimal *PEDS* of $\overline{H}$.

Proof of (iv): Let $H = P_n \times C_m$, which is shown in Figure 3, and let $V(H) = \{u_{11}, u_{12}, u_{13}, \ldots, u_{1n}; u_{21}, u_{22}, u_{23}, \ldots, u_{2n}; \ldots; u_{m1}, u_{m2}, u_{m3}, \ldots, u_{mn}\}$ be the vertex set of H. Now

$\deg(u_{1i}) = \deg(u_{2i}) = \deg(u_{3i}) = \ldots = \deg(u_{mi}) = 2, i = 2, 3, \ldots, n-1 : \deg(u_{11}) = \deg(u_{21}) = \deg(u_{31}) = \ldots = \deg(u_{m1}) = 3 : \deg(u_{1i}) = 3, i = 1 \text{ } to \text{ } m$ and $\deg(u_{in}) = 3, i = 1 \text{ } to \text{ } m$.

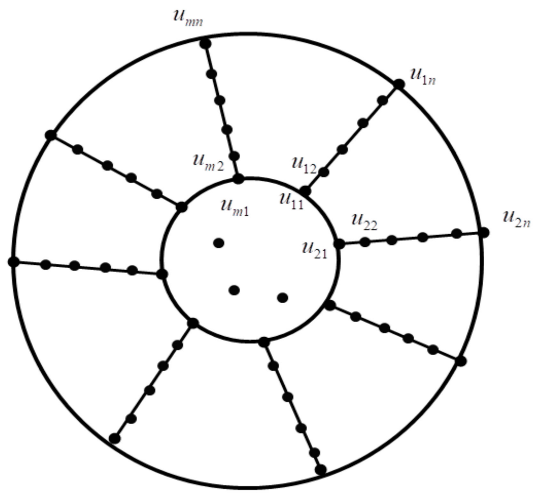


Figure 3. Graph $P_n \times C_m$.

Let $a \in V(H)$ and $b \in V(\overline{H})$ be the corresponding node of a; then deg(a) + deg(b) = $rs - 1$. Hence, $\deg(u_{j1}) = mn - 4,\ j = 1 \text{ } to \text{ } m$, $\deg(u_{1i}) = 3,\ i = 1 \text{ } to \text{ } m$, and $\deg(u_{in}) = 3$, $i = 1 \text{ } to \text{ } m$. Let X be a minimal *PEDS* of $\overline{H}$. Arbitrarily choose two nodes, say x and u; both are paired in $V(\overline{H})$, which dominates every node of $\overline{H}$. Also, $|\deg(x) - \deg(y)| \leq 0$ for all $x \in X$ and $y \in V(\overline{H}) - X$. Hence, $X = \{x, y\}$ is a *PEDS* of $\overline{H}$.

4.2. Theorem 2

If $\delta(G) \geq 1,\ \Delta(G) \leq 2$ and $\delta = \Delta \neq 1$, then prove that $\gamma^e_{pr}(H) \geq \gamma^e_{pr}(\overline{H})$.
Proof: It follows from Theorem 4.1(i) and 4.1(ii).

4.3. Theorem 3

If T' is an acyclic graph and $\overline{T}'$ is connected, then $\gamma^e_{pr}(\overline{T}') \leq 4$.
Proof: Using Theorem 4.1(i), this result is obvious.

4.4. Theorem 4

Prove that $\gamma^e_{pr}(H) + \gamma^e_{pr}(\overline{H}) \leq 6$ if

(i) H is a graph of order m where $m \equiv 0(\text{mod}8)$ and $\deg(a_i) = (m/2)$ for all a_i.
(ii) H is a graph of order m where $m \equiv 2(\text{mod}8)$ and $\deg(a_i) = (m/2)$ for all a_i

***Proof of (i)*:** Let A be a minimal *PEDS* of an $(m/2)$-regular graph H on m nodes. Construct the $(m/2)$-regular graph H such that each node b_i in H is joined to m/2 number of nodes to the left side of b_i and m/2 number of nodes to the right side of b_i. Since each vertex of H is joined to m/2 vertices, select any two paired vertices, say b_i and b_j, where $1 \leq i, j \leq m$; these two nodes are ample to dominate all the nodes in H. Further, $\left|\deg(b_i) - \deg(v_j)\right| \leq 0$ for all $b_i \in H$ and for all $v_j \in V - H$. Hence, $|A| = 2$.

Since H is an $(m/2)$-regular graph on m nodes, deg(a_i) = $m/2$ for every $v_i \in V(\overline{H})$. Let A' be a minimal PEDS of $\overline{H}$. Since $\overline{H}$ is an $(m/2)$-regular graph, every node in $\overline{H}$ dominates $(m/2)$ nodes. Hence, we can find exactly two paired nodes, say a_i and a_j where $1 \leq i \leq (m/2)$ and $(m/2) \leq j \leq m$, that pairwise dominate $\overline{H}$. Further, $|\deg(a_i) - \deg(a_s)| \leq 0$ for every $a_i \in A'$ and for every $a_s \in V - A'$. Hence, $|A'| = 2$. Hence, $\gamma^e_{pr}(H) + \gamma^e_{pr}(\overline{H}) = 4$.

***Proof of (ii)*:** Let A be a *PEDS* of $(m/2)$-regular graph H on m nodes. Using Theorem 4.4(i) for the graph H, $(m/2)$-regular graph on m nodes, $\gamma^e_{pr}(H) + \gamma^e_{pr}(\overline{H}) = 4 + 2 = 6$. Hence, $\gamma^e_{pr}(H) + \gamma^e_{pr}(\overline{H}) \leq 6$.

5. Relation between Domination Number and Equitable and Paired Equitable Domination Number of Complement of Inflated Graphs

In this section, we study the equitable domination of the complement of the inflated graph of some standard graphs like the path, cycle, complete graph, and complete bipartite graph. Let G be a connected graph, and its complement of an inflated graph is denoted by $\overline{(G_{(l)})}$.

5.1. Theorem 1

If G is a path of order n where $n \geq 4$, then (i) $\gamma^e\overline{(G_{l)})} \leq \gamma(G)$, and (ii) $\gamma^e_{pr}\overline{(G_{(l)})} \leq \gamma(G)$.

Proof of (i): Let $G = P_n$. The number of nodes in P_n is n, and $\sum_{i-1}^{n} \deg(v_i) = 2(n-2) + 2 = 2n - 2$ is the number of nodes in $G_{(l)}$. Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. It consists of $2n - 2$ nodes among exactly two nodes, say r and s, of degree $2n - 4$. The node r is not adjacent to exactly one node, say y of degree $2n - 5$; select arbitrarily any two nodes, say r and s, to dominate $2n - 2$ nodes. Further, $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists a node $v \in X'$ such that $xv \in E(G)$, and hence, $\gamma^e\overline{(G_{(l)})} = 2$.

The domination number of P_n is $\gamma(P_n) = \begin{cases} \lfloor \frac{n-3}{3} \rfloor + 2, n \equiv 1,2(\text{mod}3) \\ \frac{n}{3}, n \equiv 0(\text{mod}3) \end{cases}$.

Clearly, $\gamma^e\overline{(G_{(l)})} \leq \gamma(G)$; this equality holds good when n = 4.

Proof of (ii): Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. Since the nodes r and s are paired in the case of path and cycle and also both nodes are of degree $2n - 4$, therefore for every $x \in V - X\prime$, $|\deg(r) - \deg(x)| \leq 1$ such that $rx \in E(G)$ and hence $\gamma^e\overline{(G_{(l)})} \leq 2$, and this completes the proof.

This theorem is true for a cycle graph of order n where $n \geq 4$.

5.2. Theorem 2

If $G = K_n$ is a complete graph of order n, $n \geq 4$, then (i) $\gamma^e\overline{(G_{(l)})} > \gamma(G)$, and (ii) $\gamma^e_{pr}\overline{(G_{(l)})} > \gamma(G)$.

Proof of (i): Let $G = K_n$ and $V(G) = \{a_1, a_2, \ldots, a_n\}$. Since G is complete, $\sum_{i=1}^{n} \deg(a_i) = n(n-1)$. Hence, $\left|V(G_{(l)})\right| = n(n-1)$. Each vertex u_i in $V(G)$ produces a clique K_{n-1} in $G_{(l)}$ whose nodes are $a_ia_{i-1}, a_ia_{i-2}, \ldots, a_ia_{i+1}, a_ia_{i+2,\ldots,}a_ia_{n^2-n-1,}a_ia_{n^2-n}$. Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. It consists of $n^2 - n$ nodes; $\deg(x_i) = n(n-2)$ for every I = 1 to $n^2 - n$. Select a node, say x, in $\overline{(G_{(l)})}$ that is adjacent to exactly $n(n-2)$ nodes. To dominate $n^2 - n$ nodes, we need at least two nodes. Arbitrarily select the nodes, say r and s, to dominate $2n$ nodes. Further, $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists node $v \in X'$ such that $xv \in E(G)$. Hence, $\gamma^e\overline{(G_{(l)})} = 2$. Clearly, $\gamma(G) = 1$, hence $\gamma^e\overline{(G_{(l)})} > \gamma(G)$.

Proof of (ii): Let $G = K_n$. Let X' be a minimum equitable dominating set of $\left(\overline{G_{(l)}}\right)$.

Since every node is of the same degree $n(n-2)$, consider any two nodes, say r and s, are paired with each other; further, $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists a node $G \in X'$ such that $rx \in E(\overline{G_{(l)}})$ and hence $\gamma^e\overline{(G_{(l)})} \leq 2$, and this completes the proof.

5.3. Theorem 3

If $G = K_{m,n}$, then (i) $\gamma^e\overline{(G_{(l)})} = \gamma(G)$, and (ii) $\gamma^e_{pr}\overline{(G_{(l)})} = \gamma(G)$ if $|m-n| \leq 1$.

Proof of (i): Let $G = K_{m,n}$. Now $V(G) = A \cup B$ where $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b, b_3, \ldots, b_m\}$. Since each node $a_i \in A(G)$ *where* $1 < i < n$ is of degree n and each $b_j \in B(G)$ *where* $1 < j < m$ is of degree m, the inflated graph $G_{(l)}$ consists of m complete graphs of order n and n complete graphs of order m. Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. The number of nodes in $K_{m,n}$ is mn, and $\sum_{i-1}^{n} \deg(x_i) = 2mn$ is the number of nodes in $G_{(l)}$. Let $V_{(1)}$ be the set of nodes of degree $n(2m-1)-1$, which contains mn nodes, and $V_{(2)}$ be the set of nodes of degree $m(2n-1)-1$, which contains mn nodes. Since it contains $2mn$ nodes, $deg(x_i) = n(2m-1)-1$ for every $i = 1$ to n and $deg(y_i) = m(2n-1)-1$ for every $i = 1$ to m. To dominate $2mn$ nodes, arbitrarily select one node r from $V_{(1)}$ of the same degree and one more, say s, from the remaining mn nodes of the same degree from $V_{(2)}$. Further, $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists a node $v \in X'$ such that $xv \in E(G)$ and hence $\gamma^e\overline{(G_{(l)})} = 2$. Clearly, $\gamma(G) = 2$, and therefore $\gamma^e\overline{(G_{(l)})} = \gamma(G)$.

Proof of (ii): Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. If $|m-n| = 0$, then this follows from Section 5.2. Suppose $|m-n| = 1$. Let $n = m + 1$, then $deg(x_i) = 2m^2 + m - 2$ for every $i = 1$ to n and $deg(y_i) = 2m^2 + m - 1$ for every $i = 1$ to m. $\overline{(G_{(l)})}$ contains $m(n+1)$ nodes, and we need two nodes to dominate it. Arbitrarily choose paired nodes r and s; both are ample to dominate the nodes $m(n+1)$. Further, $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists a node $v \in X'$ such that $rx \in E(\overline{G_{(l)}})$ and hence $\gamma^e_{pr}\overline{(G_{(l)})} = 2$. Clearly, $\gamma(G) = 2$, which completes the proof.

5.4. Theorem 4: If $G = P_n \times C_m$ Then $\gamma^e\overline{(G_{(l)})} \leq 2$

Proof: Let $G = P_n \times C_m$, which is shown in Figure 3, and let $V(G) = \{u_{11}, u_{12}, u_{13}, \ldots, u_{1n}; u_{21}, u_{22}, u_{23}, \ldots, u_{2n}; \ldots; u_{m1}, u_{m2}, u_{m3}, \ldots, u_{mn}\}$ be the node set of G_{ed}. Now $\deg(u_{1i}) = \deg(u_{2i}) = \deg(u_{3i}) = \ldots = \deg(u_{mi}) = 2, i = 2, 3, \ldots, n-1 : \deg(u_{11}) = \deg(u_{21}) = \deg(u_{31}) = \ldots = \deg(u_{m1}) = 3 : \deg(u_{1i}) = 3, i = 1 \textit{ to } m$ and $\deg(u_{in}) = 3$, $i = 1 \textit{ to } m$.

The number of nodes in G is $2m + m(n-2)$, and $\sum_{i-1}^{n} \deg(x_i) = 4mn - 4m$ is the number of nodes in $\overline{(G_{(l)})}$. Let X' be a minimum equitable dominating set of $\overline{(G_{(l)})}$. Since it contains $4mn - mn$ nodes, $deg(x_i) = 3$ for every $i = 1$ to $2m$ and $deg(y_j) = 2$ for every $j = 1$ to $m(n-2)$. To dominate $4mn - 4m$ nodes, we need at least two nodes. Arbitrarily select the nodes, say r and s, to dominate these nodes such that $|\deg(r) - \deg(x)| \leq 1$; for every $x \in V - X'$, there exists a node $v \in X'$, such that $xv \in E(\overline{G_{(l)}})$, hence the proof.

5.5. Theorem 5

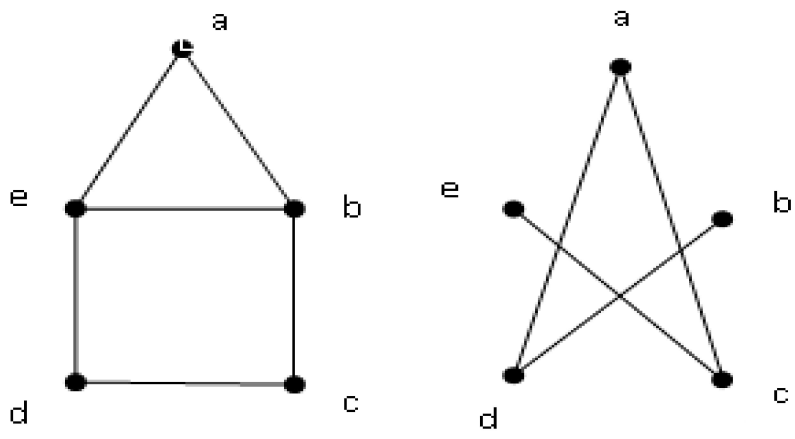
If G is a (k_1, k_2) biregular graph with m as the number of nodes of degree k_1 and n as the number of nodes of degree $k_2 = k_1 + 1$, then $\gamma^e\overline{(G(l))} = 2$.

Proof: Since G is a biregular graph, the result is obvious.

6. Paired Equitable Domination of the Complement of the Inflated Graph of Complements

In this section, we discuss the paired equitable domination of the complement of $(\overline{G})_l$. The complement of $(\overline{G})_l$ is denoted by $\overline{(\overline{G})_l}$.

Example: Let G be the graph shown in Figure 4a and its complement is $\overline{G}$; the inflated graph of $\overline{G}$ and complement of $(\overline{G})_l$ are shown in Figure 4b, Figure 5, and Figure 6, respectively.



(**a**) (**b**)

Figure 4. (**a**) Graph G; (**b**) complement of G.

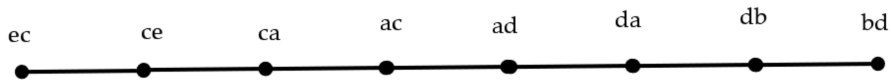


Figure 5. Graph $(\overline{G})_l$.

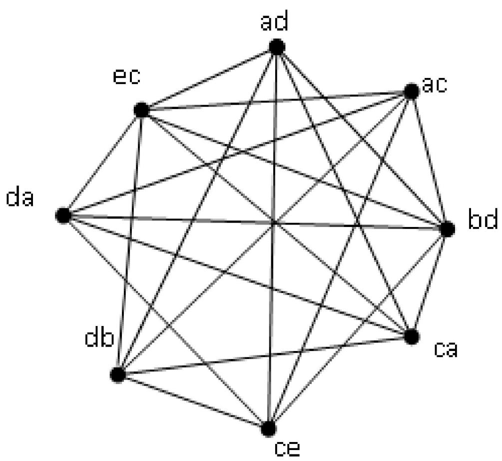


Figure 6. Graph $\overline{(\overline{G})_l}$.

The paired equitable dominating set of $\overline{(\overline{G})_l}$ = {ec, db}. Hence, $\gamma^e_{pr}(\overline{(\overline{G})_l}) = 2$.

6.1. Theorem 1

Let G be either a regular graph or a biregular graph of order $m \geq 5$ with $\delta(G) \geq 2$ and $\Delta(G) \leq m - 2$; then $\gamma^e_{pr}(\overline{(\overline{G})_l}) \leq 2$.

Proof: Let G be a connected graph of order m with $\delta(G) \geq 2$ and $\Delta(G) \leq m - 2$ and $(\overline{G})_l$ be an inflated graph of complement of G. Let D be a minimal equitable dominating set of $(\overline{G})_l$. Construct the graph $(\overline{G})_l$.

Case(i): Let G be an s-regular graph where $s \leq m - 2$. The degree of every node in $\overline{G}$ is $m - (1 + r)$. Now there are $m[m - (1 + s)]$ number of nodes in $(\overline{G})_l$. Every node in $(\overline{G})_l$ is adjacent to exactly $m[m - (1 + s)] - [m - (1 + s)] = (n - 1 - s)(m - 1)$ number of nodes. Since $(m - 1 - s)(m - 1) < m[m - (1 + s)]$, choose at least two nodes, enough to dominate $m[m - (1 + s)]$ number of nodes in $(\overline{G}_{ped})_l$. Also, $\left|\deg(u_i u_j) - \deg(u_i u_s)\right| \leq 1$, $u_i u_j u_i u_r \in E(\overline{G})_l$ and $\forall u_i u_j \in D$ and $u_i u_s \in V_{ped} - D$. Hence, $\gamma^e_{pr}(\overline{(\overline{G})_l}) \leq 2$.

Case(ii): Let G be a biregular graph (k_1, k_2) with n_1 number of nodes of degree k_1 and n_2 number of nodes of degree k_2 such that $n_1 + n_2 = m$. Construct the graphs $\overline{G}$ and $(\overline{G})_l$.

There are n_1 number nodes of degree $(m-1-k_1)$ and n_2 number nodes of degree $(m-1-k_2)$ in $\overline{G}$ and $n_1(m-1-k_1)+n_2(m-1-k_2)$ number nodes in $(\overline{G})_l$. The nodes of degree $(m-1-k_2)$ are adjacent to exactly $\{n_1(m-1-k_1)+n_2(m-1-k_2)-(m-1-k_2)\}-1=\{n_1(m-1-k_1)+(m-1-k_2)(n_2-1)\}-1$ nodes, and the nodes of degree $(m-1-k_1)$ are adjacent to exactly $\{n_1(m-1-k_1)+n_2(n-1-k_2)-(m-1-k_1)\}-1=\{m-1-k_1)(n_1-1)+n_2(m-1-k_2)\}-1$ nodes.

Since the number of nodes $n_1(m-1-k_1)+n_2(m-1-k_2)$ in $(\overline{G})_l$ is adjacent to either $\{n_1(m-1-k_1)+(m-1-k_2)(n_2-1)\}-1$ or $\{(m-1-k_1)(n_1-1)+n_2(n-1-k_2)\}-1$, two nodes are ample to dominate all the nodes. Select two nodes, say x and y, such that both are paired to each other. Furthermore, $\left|\deg(u_iu_j)-\deg(u_iu_s)\right|\leq 1$, $u_iu_ju_iu_r\in$ E $((\overline{G})_l)$ $(\overline{G}_{ped})_l$ and $\forall u_iu_j\in S$ and $u_iu_{.s}\in V-S$. Hence, $\gamma^e_{pr}(\overline{(\overline{G})_l})\leq 2$.

6.2. Theorem 2

For any connected graph G of order $m\geq 5$, then $\gamma^e_{pr}(\overline{(\overline{G})_l})\leq M$ where M is the number of different degrees present in $(\overline{G})_l$.

Proof: Let n_i be the number of nodes of degree m_i, $i=1,2,\ldots,m$, where $n_1+n_2+\ldots n_m=m$. Construct the graphs $\overline{G}$ and $(\overline{G})_l$. Let X be the minimal dominating set of $(\overline{G})_l$. There are n_i nodes of degree $(m-1-m_i)$, $i=1,2,\ldots,m$ in $\overline{G}$ and $n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_m)$ vertices in $(\overline{G})_l$. The nodes of degree $(m-1-k_1)$ are adjacent to exactly $\{[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_m)]-(m-1-k_1)\}-1=[(m-1-k_1)(n_1-1)+n_2(m-1-k_2)+\ldots+n_{m(}m-1-k_m)]-1$ number of nodes. The nodes of degree $(m-1-k_2)$ are adjacent to exactly $\{[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_m)]-(m-1-k_2)\}-1=[n_1(m-1-k_1)+(m-1-k_2)(n_2-1)\ldots+n_m(m-1-k_n)]-1$ number of nodes. Likewise, the nodes of degree $(m-1-k_m)$ are adjacent to exactly $\{[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_n)]-(m-1-k_m)\}-1=[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+\mathrm{m}-1-k_m)(n_m-1)]-1$ number of nodes.

Since the nodes of degree $(m-1-k_1)$ are adjacent to $\{[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_m)]-(m-1-k_1)\}-1$ number of nodes, $(m-1-k_i)$ are paired with $\{[n_1(m-1-k_1)+n_2(m-1-k_2)+\ldots+n_m(m-1-k_m)]-(m-1-k_i)\}-1$ for i = 1, 2, …, m number of nodes. To pairwise dominate the nodes in $(\overline{G})_l$, select one of the nodes from each distinct degree, say $(m-1-k_i)$, i = 1, 2, …, m; we obtain m number of nodes to pairwise dominate the nodes in $(\overline{G})_l$. Since one node, say r_i from different degrees, then $\left|\deg(r_iu_j)-\deg(r_iu_s)\right|\leq 1$, $r_iu_jr_iu_r\in E((\overline{G})_l)$ and $\forall r_iu_j\in X$ and $r_iu_{.s}\in V-X$. Hence, $\gamma^e_{pr}(\overline{(\overline{G})_l})\leq 1+1+1+\ldots+1(m\ times)=M$.

7. Conclusions

The goal of this study is to investigate the interaction between domination numbers, equitable domination numbers, and paired equitable domination numbers in inflated graphs. A significant result is that the sum of the paired equitable domination numbers for certain types of graphs and their complement is less than or equal to 6. This study also presents the relationship between the domination number, equitable domination, and the paired equitable domination number of the complement of inflated graphs. Since equitable and paired equitable domination are based on the degrees of the vertices of the graph, this study combining inflated graphs is a more effective and informative way to examine the effectiveness of the graphs that are based on different domination parameters. In the future, the plan is to explore different operations applied to the graph and the effectiveness of the graph.

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