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On r-Compactness in Topological and Bitopological Spaces

Jamal Oudetallah [1], Rehab Alharbi [2] and Iqbal M. Batiha [3,4,*]

[1] Department of Mathematics, Irbid National University, Irbid 21110, Jordan
[2] Department of Mathematics, College of Science, Jazan University, New Campus, Jazan 2097, Saudi Arabia
[3] Department of Mathematics, Al-Zaytoonah University of Jordan, Amman 11733, Jordan
[4] Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, United Arab Emirates
* Correspondence: i.batiha@zuj.edu.jo

Abstract: This paper defines the so-called pairwise r-compactness in topological and bitopological spaces. In particular, several inferred properties of the r-compact spaces and their connections with other topological and bitopological spaces are studied theoretically. As a result, several novel theorems of the r-compact space are generalized on the pairwise r-compact space. The results established in this research paper are new in the field of topology.

Keywords: pairwise compact space; pairwise r-compact space; pairwise extremely disconnected space; bitopological space

MSC: 54D30; 54A05

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1. Introduction

Compactness possesses a really vital input in topology and so do a few of its lower and more grounded types. One of such types is H-close in which the hypothesis of these spaces was presented in 1929 by P. S. Alexandroff and his colleague [1]. In 1969, M. K. Singal and A. Aathur presented nearly compact spaces [2]. In 1976, T. Thompson presented a different kind of compact space called the s-compact space [3]. Once in a while, a few additional types of compactness have been investigated [4,5]. In recent time, V. V. Tkachuk provided in [6] a self-contained introduction to Cp-theory and general topology, including a unique problem-based introduction to the theory of function spaces and many results and methods related to the Cp-thoery. At the same time, H. H. Kadhem proposed in [7] a new type of compact spaces called an r-compact space, where a topological space (X, τ) is said to be an r-compact space if every regular open cover of X has a finite subfamily whose closures cover X. In this article, we introduce several novel theorems of weaker kind related to the compact spaces specified by the r-compact space, a generalization that concerns the pairwise r-compact space.

The bitopological space subject might be written as $Z = (Z, \gamma_1, \gamma_2)$, where γ_1 and γ_2 are two topologies defined on Z [8]. This concept is connected with a former investigation that was carried out on bitopological spaces so that each topology can be defined as a set of points that possesses nearby related points and satisfies specific axioms. In [9], Kelly defined each of the pairwise normal, pairwise Hausdorff and pairwise regular spaces with some conventional theorems indicated by Tietze's extension. An additional work in the bitopological space field was performed by Kim in [10]. In [11], the concept of $(\alpha - \beta)$-level spaces was defined by taking into account the fuzzy bitopological space concept. As a consequence of that work, a fuzzy $(\alpha - \beta)$ of a bitopological Hausdorff space was defined and the notion of a fuzzy $(\alpha - \beta)^*$ of a bitopological Hausdorff space was established using the $(\alpha - \beta)^*$ disjoint sets. In [12], with the help of an extended Pythagorean fuzzy topological space, the Pythagorean fuzzy bitopological space was defined, and several notions were accordingly inferred related to the pairwise Pythagorean fuzzy topological spaces coupled with several relations of their characteristics. In [13], the compact ultrametrics' range sets were described in regard to its order type. The expandability, near expandability and feeble expandability of a bitopological space were explained by Oudetallah in [14–16].

The primary objective of this work is to present and examine a novel kind of pairwise compact spaces, which is the so-called pairwise r-compact space (or simply the p-r-compact space). Accordingly, we derive several novel results related to the r-compact space that represent generalizations of their corresponding results from the pairwise r-compact space.

2. Preliminary

In this section, we aim to pave the way to our main results by recalling several significant definitions and facts.

Definition 1. *If $A \subseteq Z$ and (Z,γ) is a topological space, then it is said that:*

1. *A is a regular open set in Z if $A = \overline{A}^{o}$.*
2. *A is a regular closed set in Z iff $A = \overline{A^{o}}$.*
3. *There exists an open set U such that $U \subseteq A \subseteq \overline{U}$ iff A is a semiopen set in Z.*

Definition 2. *Suppose (Z, γ_1, γ_2) is a bitopological space so that $A \subseteq Z$, then it is said that:*

1. *A is a pairwise regular open set if $A = Int_{\gamma_1}(CL_{\gamma_1}(A))$ and $A = Int_{\gamma_2}(CL_{\gamma_2}(A))$.*
2. *A is a pairwise regular closed set if $A = CL_{\gamma_1}(Int_{\gamma_1}(A))$ and $A = CL_{\gamma_2}(Int_{\gamma_2}(A))$.*
3. *A is a pairwise semiopen set if $\exists$ an open set u such that $u_{\gamma_1} \subseteq A \subseteq CL_{\gamma_1}(u)$ and $u_{\gamma_2} \subseteq A \subseteq CL_{\gamma_2}(u)$.*

Remark 1. *Suppose (Z, γ_1, γ_2) is a bitopological space so that $A \subseteq Z$, then:*

1. *A is called a pairwise regular closed set if $Z - A$ is a pairwise regular open set.*
2. *A is called a pairwise regular open set if $Z - A$ is a pairwise regular closed set.*

Theorem 1. *Let (Z, γ_1, γ_2) be a bitopological space, then every pairwise open set is a pairwise semiopen set.*

Proof. Assume A is a pairwise regular open set, so $Int_{\gamma_i}(A) \subseteq A \subseteq CL_{\gamma_i}(A)$, $\forall i = 1, 2$. Consequently, $u \subseteq A \subseteq CL_{\gamma_i}(u)$, so A is a pairwise semiopen set, $\forall i = 1, 2$. □

Definition 3. *A space $Z = (Z, \gamma_1, \gamma_2)$ is called a pairwise r-compact if every regular open cover of Z possesses a finite subfamily for which its closures cover Z.*

Definition 4 ([17]). *Let (Z, γ) be a topological space, then Z is said to be extremely disconnected if for every open set A in Z, $\overline{A}$ is a clopen set in Z.*

Definition 5 ([17]). *Let (Z, γ_1, γ_2) be a bitopological space, then Z is said to be pairwise extremely disconnected if for each γ_i-open set in Z, $\overline{A}$ is a γ_i-open set in Z, $\forall i = 1, 2$.*

Definition 6 ([17]). *A topological space (Z, γ) is called quasi H-closed if every open cover of Z possesses a finite subfamily for which its closures cover Z.*

Definition 7. *A bitopological space $Z = (Z, \gamma_1, \gamma_2)$ is called pairwise quasi H-closed if every γ_i-open cover of Z possesses a finite subfamily for which its closures cover Z, $\forall i = 1, 2$.*

Definition 8. *A space (Z, γ) is said to be a nearly compact space if every open cover possesses a finite subfamily such that the interiors of its closures cover Z.*

Definition 9. *A bitopological space $Z = (Z, \gamma_1, \gamma_2)$ is called a pairwise nearly compact space if every γ_i-open cover of Z possesses a finite subfamily such that the interiors of its closures cover Z, $\forall i = 1, 2$.*

Definition 10. *A space (Z, γ) is called S-closed if every semiopen cover of Z possesses a finite subfamily for which its closures cover Z.*

Definition 11. *A bitopological space $Z = (Z, \gamma_1, \gamma_2)$ is called pairwise S-closed if every γ_i-semiopen cover of Z possesses a finite subfamily for which its closures cover Z, $\forall i = 1, 2$.*

3. Main Results

In this part, different new theorems and properties of the r-compact spaces coupled with their relations with other topological and bitopological spaces are presented. In other words, we present in what follows the main results of this work.

Theorem 2. *If $Z = (Z, \gamma)$ is compact, then it is an r-compact space.*

Proof. Assume that Z is a compact space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a regular open cover of Z. Now, as Z is a compact space, $\exists\, \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \omega_{\alpha_k}$. Since ω_{α_k} is a regular open set in Z for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, then $\omega_{\alpha_k} = \omega_{\alpha_n}$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$. Hence, we have:

$$Z = \bigcup_{k=1}^{n} \omega_{\alpha_k} = \bigcup_{k=1}^{n} \overline{\omega}^{\,o}{}_{\alpha_k} \subseteq \bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k}. \tag{1}$$

Now, since $\omega_{\alpha_k} \subseteq Z$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, we obtain:

$$\bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k} \subseteq Z. \tag{2}$$

Consequently, by using (1) and (2), we obtain that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$, and therefore, Z is r-compact space. □

Theorem 3. *Every pairwise compact space is a pairwise r-compact space.*

Proof. Suppose $i = 1, 2$, Z is a pairwise compact space and $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i-regular open cover of Z. Since Z is a pairwise compact space, $\exists\, \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \omega_{\alpha_k}$. Since ω_{α_k} is a γ_i-regular open set in Z for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, $\omega_{\alpha_k} = \omega_{\alpha_n}$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$. Hence, we have:

$$Z = \bigcup_{k=1}^{n} \omega_{\alpha_k} = \bigcup_{k=1}^{n} \overline{\omega}^{\,o}{}_{\alpha_k} \subseteq \bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k}. \tag{3}$$

Since $\omega_\alpha \subseteq Z$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, we have:

$$\bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k} \subseteq Z. \tag{4}$$

Now, by using (3) and (4), we obtain that

$$Z = \bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k},$$

and therefore Z is a pairwise r-compact space. □

Theorem 4. *If $Z = (Z, \gamma)$ is nearly compact, then it is an r-compact space.*

Proof. Assume that Z is a nearly compact space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a regular open cover of Z. Then, $\{\omega_\alpha | \alpha \in \Omega\}$ is an open cover of Z. Now, as Z is a nearly compact space, $\exists\ \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}^{o}{}_{\alpha_k}$. Since $\bigcup\limits_{k=1}^{n} \overline{\omega}^{o}{}_{\alpha_k} \subseteq \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$, we can assert:

$$Z \subseteq \bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k}. \tag{5}$$

In the same regard, since $\omega_{\alpha_k} \subseteq Z$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, we have:

$$\bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k} \subseteq Z. \tag{6}$$

As a result, from (5) and (6), we obtain that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$, and thus Z is an r-compact space. □

Theorem 5. *Every pairwise nearly compact space is a pairwise r-compact space.*

Proof. Assume that $i = 1, 2$, Z is a pairwise nearly compact space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i regular open cover of Z. Thus, $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i open cover of Z. Now, as Z is a nearly compact space, $\exists\ \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}^{o}{}_{\alpha_k}$. Since $\bigcup\limits_{k=1}^{n} \overline{\omega}^{o}{}_{\alpha_k} \subseteq \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$, we can confirm:

$$Z \subseteq \bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k}. \tag{7}$$

In addition, since $\omega_{\alpha_k} \subseteq Z$ for each $\alpha_k \in \Omega$ and for each $k = 1, 2, \cdots, n$, we have:

$$\bigcup_{k=1}^{n} \overline{\omega}_{\alpha_k} \subseteq Z. \tag{8}$$

Now, based on (7) and (8), we obtain that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$, and hence Z is a pairwise r-compact space. □

Theorem 6. *If $Z = (Z, \gamma)$ is a quasi H-closed space, then it is an r-compact space.*

Proof. Assume that Z is a quasi H-closed space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a regular open cover of Z. Therefore, $\{\omega_\alpha | \alpha \in \Omega\}$ is an open cover of Z. Accordingly, as Z is quasi H-closed, $\exists\ \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$. Thus, Z is an r-compact space. □

Theorem 7. *Every pairwise quasi H-closed space is a pairwise r-compact space.*

Proof. Assume that Z is a pairwise quasi H-closed space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i regular open cover of Z. Hence, $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i open cover of Z. As Z is pairwise quasi H-closed, $\exists\ \omega_{\alpha_1}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{k=1}^{n} \overline{\omega}_{\alpha_k}$. Thus, Z is a pairwise r-compact space. $\square$

Theorem 8. *If $Z = (Z, \gamma)$ is an S-closed space, then it is an r-compact space.*

Proof. Assume that Z is an S-closed space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a regular open cover of Z. Thus, ν_α is a semiopen set, and $\{\omega_\alpha | \alpha \in \Omega\}$ is a semiopen cover of Z. As Z is an S-closed space, $\exists\ \gamma$-$\omega_{\alpha_1}, \omega_{\alpha_2}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{i=1}^{n} \overline{\nu}_{\alpha_i}$. Thus, Z is an r-compact space. $\square$

Theorem 9. *Every pairwise S-closed space is a pairwise r-compact space.*

Proof. Assume that $i = 1, 2$, Z is a pairwise S-closed space and suppose that $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i-regular open cover of Z. Thus, ν_α is γ_i-semiopen set, and $\{\omega_\alpha | \alpha \in \Omega\}$ is a γ_i semiopen cover of Z. Now, since Z is a pairwise S-closed, there exist γ_i-$\omega_{\alpha_1}, \omega_{\alpha_2}, \cdots, \omega_{\alpha_n}$ such that $Z = \bigcup\limits_{i=1}^{n} \overline{\nu}_{\alpha_i}$, and thus, Z is a pairwise r-compact space. $\square$

Next, in light of the extremely disconnected space definition, we intend to state and prove the following results.

Theorem 10. *If $Z = (Z, \gamma)$ is an extremely disconnected space, then the statements below are equivalent:*

1. *Z is r-compact.*
2. *Z is nearly compact.*
3. *Z is quasi-H-closed.*

Proof. $1 \to 2:$ Suppose $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a open cover of Z. As Z is a pairwise r-compact space, $\exists\ \underset{\sim}{B} = \{u_{\alpha_k} : k = 1, 2, \cdots, n\}$ for which $Z = \bigcup\limits_{k=1}^{n} \overline{u_{\alpha_k}}$ and $u_{\alpha_k} \in U$, $\forall i = 1, \cdots, n$. Moreover, since Z is an extremely disconnected space, $\overline{u_{\alpha_k}}$ is an open set, $\forall k = 1, \cdots, n$. Therefore, $\overline{u_{\alpha_k}}^{o} = \overline{u_{\alpha_k}}$, and so $Z = \bigcup\limits_{k=1}^{n} \overline{u}^{o}_{\alpha_k}$. Thus, $\left\{\overline{u}^{o}_{\alpha_k}\right\}_{k=1}^{n}$ forms a subfamily of interior sets covered by Z, and therefore Z is nearly compact.

$2 \to 3:$ Assume that $Z = (Z, \gamma)$ is nearly compact. Suppose $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a open cover of Z. Now, as Z is nearly compact, $\exists$ a finite subfamily $\{u_{\alpha_k} : k = 1, \cdots, n\}$ and $Z \subseteq \bigcup\limits_{k=1}^{n} \overline{u_{\alpha_k}}^{o}$. Nevertheless, $\forall k = 1, \cdots, n$ and $\forall\ \alpha \in \Lambda$, we have $\overline{u_{\alpha_k}}^{o} \in \overline{u_{\alpha_k}}$. Thus, $Z \in \overline{u_{\alpha_k}}^{o} \in \overline{u_{\alpha_k}}$, and so there exists a subfamily $\{u_{\alpha_k} : k = 1, \cdots, n\}$ of $\underset{\sim}{U}$ whose closures cover Z. Therefore, Z is a quasi-H-closed space.

$3 \to 1:$ Suppose Z is quasi-H-closed. Assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a cover of Z, where u_α is a regular open set. Now, since Z is a quasi-H-closed space, $\underset{\sim}{U}$ has a finite subfamily $\{u_{\alpha_k} : k = 1, \cdots, n\}$ such that $Z \in \bigcup\limits_{k=1}^{n} \overline{u_{\alpha_k}}$. Hence, Z is an r-compact space. $\square$

Theorem 11. *If $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise extremely disconnected space, then the statements below are equivalent:*

1. *Z is pairwise r-compact.*
2. *Z is pairwise nearly compact.*
3. *Z is pairwise quasi-H-closed.*

Proof. $1 \to 2$: Assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a γ_i-open cover of Z, $\forall\ i = 1,2$. Since Z is a pairwise r-compact space, $\exists\ \underset{\sim}{B} = \{u_{\alpha_k} : k = 1,2,\cdots,n\}$ such that $Z = \bigcup_{k=1}^{n} \overline{u_{\alpha_k}}$ and $u_{\alpha_k} \in U, \forall k = 1,\cdots,n$. Furthermore, since Z is a pairwise extremely disconnected space, $\overline{u_{\alpha_k}}$ is an open set, $\forall k = 1,\cdots,n$. Consequently, we obtain $\overline{u_{\alpha_k}}^{o} = \overline{u_{\alpha_k}}$, and so $Z = \bigcup_{k=1}^{n} \overline{u}^{o}_{\alpha_k}$. This implies $\left\{\overline{u}^{o}_{\alpha_k}\right\}_{k=1}^{n}$ form a subfamily of interior sets that cover Z, and therefore Z is pairwise nearly compact.

$2 \to 3$: Suppose that $Z = (Z, \gamma_1, \gamma_2)$ is pairwise nearly compact. Assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a γ_i-open cover of Z, $\forall\ i = 1,2$. Since Z is pairwise nearly compact, there exists a γ_i-finite subfamily $\{u_{\alpha_k} : \forall k = 1,\cdots,n\}$ and $Z \subseteq \bigcup_{k=1}^{n} \overline{u_{\alpha_k}}^{o}$. Nevertheless, $\forall\ k = 1,\cdots,n$ and $\forall\ \alpha \in \Lambda$, we have $\overline{u_{\alpha_k}}^{o} \in \overline{u_{\alpha_k}}$. Thus, $Z \in \overline{u_{\alpha_k}}^{o} \in \overline{u_{\alpha_k}}$, and so $\exists$ a subfamily $\{u_{\alpha_k} : \forall k = 1,\cdots,n\}$ of $\underset{\sim}{U}$ whose closures cover Z. Hence, Z is a pairwise quasi-H-closed space.

$3 \to 1$: Suppose Z is pairwise quasi-H-closed. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$, where u_α is a regular open set, be a γ_i-cover of Z, $\forall\ i = 1,2$. Now, since Z is a pairwise quasi-H-closed space, $\underset{\sim}{U}$ has a γ_i finite subfamily $\{u_{\alpha_k} : k = 1,\cdots,n\}$ such that $Z \in \bigcup_{k=1}^{n} \overline{u_{\alpha_k}}$. So, Z is a pairwise r-compact space.
□

Theorem 12. *Let $Z = (Z, \gamma)$ be an r-compact and extremely disconnected space, then every closed subspace of Z is r-compact.*

Proof. Suppose $Z = (Z, \gamma)$ is an r-compact and extremely disconnected space and let (A, γ_A) be a subspace of Z. First, to show A is r-compact, we assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a regular open cover of A. This implies $\overline{u_\alpha}^{o} = u_\alpha$, $\forall\ \alpha \in \Lambda$ and $A \subseteq \bigcup_{\alpha \in \Lambda} u_\alpha$. Now, $Z = A \cup (Z - A)$, and so $Z \subseteq \bigcup_{\alpha \in \Lambda} u_\alpha \cup (Z - A)$. In this regard, $Z - A$ is open in Z and Z is extremely disconnected. Then, $Z - A$ is a clopen set in Z, and therefore, $\overline{Z - A}^{o} = Z - A$. Consequently, $Z - A$ is a regular open set, hence $\underset{\sim}{U}^* = \{u_\alpha : \alpha \in \Lambda, Z - A\}$ forms a regular open cover of Z. In addition, since Z is an r-compact space, then $\underset{\sim}{U}^*$ has a finite subfamily of the form $\{u_{\alpha_k} : k = 1,\cdots,n, Z - A\}$ such that $Z \subseteq \bigcup_{k=1}^{n} \overline{u_{\alpha_k}} \cup \overline{Z - A}$. Now, $\overline{Z - A}$ covers $Z - A$, and so $A \subseteq \bigcup_{k=1}^{n} \overline{u_{\alpha_k}}$. Thus, $\{u_{\alpha_k} : k = 1,\cdots,n\}$ is a finite subfamily of $\underset{\sim}{U}$ whose closures cover A. Hence, A is r-compact. □

Theorem 13. *Let $Z = (Z, \gamma_1, \gamma_2)$ be a pairwise r-compact and a pairwise extremely disconnected space, then every closed subspace of Z is pairwise r-compact.*

Proof. Suppose $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise r-compact and pairwise extremely disconnected space and let $(A, \gamma_{1_A}, \gamma_{2_A})$ be a subspace of Z. Herein, to show A is pairwise r-compact, we assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a γ_i-regular open cover of A, $\forall\ i = 1,2$. This gives $\overline{u_\alpha}^{o} = u_\alpha$, $\forall\ \alpha \in \Lambda$ and $A \subseteq \bigcup_{\alpha \in \Lambda} u_\alpha$. Now, $Z = A \cup (Z - A)$, and so $Z \subseteq \bigcup_{\alpha \in \Lambda} u_\alpha \cup (Z - A)$. In this respect, $Z - A$ is γ_i-open in Z and Z is pairwise extremely disconnected. Therefore, $Z - A$ is a γ_i-clopen set in Z, and then $\overline{Z - A}^{o} = Z - A$. Thus, $Z - A$ is a γ_i-regular open set, and $\underset{\sim}{U}^* = \{u_\alpha : \alpha \in \Lambda, Z - A\}$ form a regular γ_i-open cover of Z. Moreover, as Z is a γ_i-r-compact space, $\underset{\sim}{U}^*$ has a γ_i-finite subfamily of the type $\{u_{\alpha_k} : k = 1, \cdots, n\}$ such that $Z \subseteq \bigcup_{k=1}^{n} \overline{u_{\alpha_k}} \cup \overline{Z - A}$. Now, $\overline{Z - A}$ covers $Z - A$, so $A \subseteq \bigcup_{k=1}^{n} \overline{u_{\alpha_k}}$. Thus, $\{u_{\alpha_k} : k = 1, \cdots, n\}$ is a γ_i-finite subfamily of $\underset{\sim}{U}$ whose closures cover A. Hence, A is pairwise r-compact. □

Definition 12. *Assume that $Z = (Z, \gamma)$ is a topological space, then Z is called:*

1. *An r-T_o-space if $\forall\ x \neq y$ in Z, $\exists$ a regular open set that includes one of them but not both.*
2. *An r-T_1-space if $\forall\ x \neq y$ in Z, $\exists$ two regular open sets u_x and v_y so that $x \in u_x$, $y \notin u_x$ and $x \notin v_y$, $y \in v_y$.*
3. *An r-T_2-space if $\forall\ x \neq y$ in Z, $\exists$ two disjoint regular open sets u_x and v_y so that $x \in u_x$ and $v \in v_y$.*
4. *An r-regular space if $\forall\ x \notin A$ and A a closed set in Z, $\exists$ two disjoint regular open sets u_x and v_y so that $x \in u_x$ and $A \subseteq v_y$.*
5. *An r-T_3-space if Z is an r-T_1-space and an r-regular space.*
6. *An r-normal space if for every two disjoint closed sets A and B in Z, $\exists$ two disjoint regular open sets u_A and v_B so that $A \subseteq u_A$ and $B \subseteq v_B$.*
7. *An r-T_4-space if Z is an r-normal space and an r-T_1 space.*

Definition 13. *Assume that $Z = (Z, \gamma_1, \gamma_2)$ is a bitopological space, then Z is called:*

1. *A pairwise r-T_o-space if $\forall\ x \neq y$ in Z, $\exists$ a γ_i-regular open set that includes one of them but not both.*
2. *A pairwise r-T_1-space if $\forall\ x \neq y$ in Z, $\exists$ two γ_i-regular open sets u_x and v_y so that $x \in u_x$, $y \notin u_x$ and $x \notin v_y$, $y \in v_y$.*
3. *A pairwise r-T_2-space if $\forall\ x \neq y$ in Z, $\exists$ two disjoint γ_i-regular open sets u_x and v_y so that $x \in u_x$ and $v \in v_y$.*
4. *A pairwise r-regular space if $\forall\ x \notin A$ and A a γ_i-closed set in Z, $\exists$ two disjoint γ_i-regular open sets u_x and v_y so that $x \in u_x$ and $A \subseteq v_y$.*
5. *A pairwise r-T_3-space if Z is a pairwise r-T_1 space and a pairwise r-regular space.*
6. *A pairwise r-normal space if for every two disjoint γ_i-closed sets A and B in Z, $\exists$ two disjoint γ_i-regular open sets u_A and v_B so that $A \subseteq u_A$ and $B \subseteq v_B$.*
7. *A pairwise r-T_4-space if Z is a pairwise r-normal space and a pairwise r-T_1 space.*

Theorem 14. *If A is an r-compact subset in a r-T_2-space, then $\forall\ x \notin A$, $\exists$ two disjoint r-open sets u_x and v_A so that $x \in u_x$ and $A \subseteq v_A$.*

Proof. For all $a \in A$, we have $a \neq x$. Now, since $x \notin A$ and Z is an r-T_2-space, $\exists$ two r-open sets $u_x(a)$ and $v(a)$ for which $x \in u_x(a)$, $a \in v(a)$ and $u_x(a) \cap v(a) = \phi$. Now, $\underset{\sim}{V} = \{v(a) : a \in A\}$ forms an r-open cover of A. However, A is r-compact, and thus $A \subseteq \bigcup_{k=1}^{n} v(a_k)$. So, for all v-open sets $v(a_k)$, $k = 1, \cdots, n$, $\exists$ a corresponding open set $u_{a_k}(x)$, such that $x \in u_{a_k}(x)$ and $u_{a_k}(x) \cap v(a_k) = \phi$ as Z is an r-T_2-space. Now, let $\underset{\sim}{U} = \bigcap_{k=1}^{n} u_{a_k}(x)$. Then, u is an r-open set in Z whereby $u \cup v(a_k) = \phi$, $\forall\ k = 1, \cdots, n$. Hence, $u \subseteq v(a_k)$ implies that $u \cap v(a_k) \subseteq u_{a_k}(x) \cap v(a_k) = \phi$, and therefore, $(u \cap v(a_1)) \cup (u \cap v(a_2)) \cup \cdots \cup (u \cap v(a_n)) = \phi$. Thus, $u \cap \bigcup_{k=1}^{n} v(a_k) = \phi$. Assume that $r = \bigcup_{k=1}^{n} v(a_k)$. Thus, v is an r-open set in Z with $A \in r$, and so the result holds. □

Theorem 15. *If A is a γ_i-r-compact subset in a pairwise r-T_2-space, then $\forall\ x \notin A$, $\exists$ two disjoint γ_i-r-open sets u_x and v_A such that $x \in u_x$ and $A \subseteq v_A$.*

Proof. Let $i = 1, 2$. For all $a \in A$, we get $a \neq x$. Since $x \notin A$ and Z is a pairwise r-T_2-space, $\exists$ two γ_i-r-open sets $u_x(a)$ and $v(a)$ for which $x \in u_x(a)$, $a \in v(a)$ and $u_x(a) \cap v(a) = \phi$.

Now, $\underset{\sim}{V} = \{v(a) : a \in A\}$ forms a γ_i-r-open cover of A. However, A is γ_i-r-compact, and thus $A \subseteq \bigcup_{k=1}^{n} v(a_k)$. Thus, $\forall$ γ_i-r-open sets $v(a_k)$, $k = 1, \cdots, n$, $\exists$ a corresponding γ_i-r-open set $u_{a_k}(x)$ for which $x \in u_{a_k}(x)$ and $u_{a_k}(x) \cap v(a_k) = \phi$ because Z is a pairwise r-T_2-space. Now, let $\underset{\sim}{U} = \bigcap_{k=1}^{n} u_{a_k}(x)$. Then, u is a γ_i-r-open set in Z with $u \cup v(a_k) = \phi$, $\forall$ $k = 1, \cdots, n$. Hence, $u \subseteq v(a_k)$ implies that $u \cap v(a_k) \subseteq u_{a_k}(x) \cap v(a_k) = \phi$, and therefore, $(u \cap v(a_1)) \cup (u \cap v(a_2)) \cup \cdots \cup (u \cap v(a_n)) = \phi$. Thus, $u \cap \bigcup_{k=1}^{n} v(a_k) = \phi$. Now, if one lets $r = \bigcup_{k=1}^{n} v(a_k)$, then r is a γ_i-r-open set in Z and $A \in r$, and so the result holds. □

Theorem 16. *If A and B are two disjoint r-compact subsets of an r-T_2-space $Z = (Z, \gamma)$, then $\exists$ two disjoint r-open sets u_A and v_B so that $A \subseteq u_A$ and $B \subseteq v_B$.*

Proof. Note that $x \in A$ and $x \notin B$ because $A \cap B = \phi$. Therefore, by Theorem 15, $\exists$ two r-open sets u_x and v_y in Z for which $x \in u_\alpha$, $B \subseteq v_\alpha$ and $u_\alpha \cap v_\alpha = \phi$. Now, $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ forms an r-open cover of A. However, A is an r-compact space, and so $A \subseteq \bigcup_{k=1}^{n} u_{\alpha_k}$. Now, if one lets $u = \bigcup_{k=1}^{n} u_{\alpha_k}$, then u is r-open in Z, and hence we have:

$$A \subseteq u. \tag{9}$$

Now, $\forall$ $k = 1, \cdots, n$, we have an r-open set u_{α_k} that corresponds to an r-open set v_{α_k} so that $B \subseteq v_{\alpha_k}$. Thus, $B \subseteq \bigcap_{k=1}^{n} v_{\alpha_k}$. If one lets $v = \bigcap_{k=1}^{n} v_{\alpha_k}$, then v is an r-open set such that

$$B \subseteq v. \tag{10}$$

Now, $\forall$ $k = 1, \cdots, n$, we have $u_{\alpha_k} \cap v_{\alpha_k} = \phi$. Thus, $(\bigcup_{k=1}^{n} u_{\alpha_k}) \cap v_{\alpha_k} = \phi$. Consequently, since $\bigcap_{k=1}^{n} v_{\alpha_k} \subseteq v_{\alpha_k}$, $u \cap \bigcap_{k=1}^{n} v_{\alpha_k} \subseteq (u \cap v_{\alpha_k}) = \phi$. Therefore, $u \cap v \subseteq \phi$, hence we have:

$$u \cap v = \phi. \tag{11}$$

Consequently, by (9)–(11), the result holds. □

Theorem 17. *If A and B are two disjoint γ_i-r-compact subsets of a pairwise r-T_2-space $Z = (Z, \gamma_1, \gamma_2)$, then $\exists$ two disjoint γ_i-r-open sets u_A and v_B so that $A \subseteq u_A$ and $B \subseteq v_B$.*

Proof. Let $i = 1, 2$. For all $x \in A$, $x \notin B$ because $A \cap B = \phi$. Therefore, by Theorem 15, $\exists$ two r-open sets u_x and v_y in Z for which $x \in u_\alpha$, $B \subseteq v_\alpha$ and $u_\alpha \cap v_\alpha = \phi$. Now, $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ forms an pairwise r-open cover of A. However, A is an r-compact space, and so $A \subseteq \bigcup_{k=1}^{n} u_{\alpha_k}$. If one lets $u = \bigcup_{k=1}^{n} u_{\alpha_k}$, then u is γ_i-r-open in Z, and hence we have:

$$A \subseteq u. \tag{12}$$

Now, $\forall$ $k = 1, \cdots, n$, we have a γ_i-r-open set u_{α_k} that corresponds to a γ_i-r-open set v_{α_k} for which $B \subseteq v_{\alpha_k}$. Thus, $B \subseteq \bigcap_{k=1}^{n} v_{\alpha_k}$. Now, if one lets $v = \bigcap_{k=1}^{n} v_{\alpha_k}$, then v is a γ_i-r-open set, and hence we have:

$$B \subseteq v. \tag{13}$$

Now, $\forall\, k = 1, \cdots, n$, we have $u_{\alpha_k} \cap v_{\alpha_k} = \phi$. So, $(\bigcup_{k=1}^{n} u_{\alpha_k}) \cap v_{\alpha_k} = \phi$. Immediately, since $\bigcap_{k=1}^{n} v_{\alpha_k} \subseteq v_{\alpha_k}$, $u \cap \bigcap_{k=1}^{n} v_{\alpha_k} \subseteq (u \cap v_{\alpha_k}) = \phi$. Therefore, $u \cap v \subseteq \phi$, and hence we have:

$$u \cap v = \phi. \tag{14}$$

Consequently, by (12), (13) and (14), the result holds. □

Theorem 18. *If $Z = (Z, \gamma)$ is an r-compact, r-T_2- and r-extremely disconnected space, then Z is an r-T_4-space.*

Proof. Assume that $Z = (Z, \gamma)$ is an r-compact and r-T_2-space, then it is clearly r-T_1-space. Now, assume that A and B are two r-closed subsets in Z, in which $A \cap B = \phi$. As Z is an r-compact space, then, by Theorem 13, A and B are r-compact subsets in the r-T_2-space Z. Thus, by Theorem 17, $\exists$ two r-open sets u_A and v_B for which $A \subseteq u_A$, $B \subseteq v_B$ and $u_A \cap v_B = \phi$. Thus, Z is an r-T_4-space. □

Theorem 19. *If $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise r-compact, pairwise r-T_2- and pairwise r-extremely disconnected space, then Z is a pairwise r-T_4-space.*

Proof. Suppose $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise r-compact and pairwise r-T_2-space, then it is clearly pairwise r-T_1-space. Now, assume that A and B are two γ_i-r-closed subsets in Z in which $A \cap B = \phi$, $\forall\, i = 1, 2$. Since Z is a pairwise r-compact space, by Theorem 13, A and B are two γ_i-r-compact subsets in the pairwise r-T_2-space Z. Thus, by Theorem 17, $\exists$ two γ_i-r-open sets u_A and v_B for which $A \subseteq u_A$, $B \subseteq v_B$ and $u_A \cap v_B = \phi$. Hence, Z is a pairwise r-T_4-space. □

Theorem 20. *Let $Z = (Z, \gamma)$ be an r-T_2- and r-extremely disconnected space, then every subset in Z is a closed set.*

Proof. Assume that A is an r-compact subset of Z. Now, if one lets $x \notin A$, then by Theorem 14, $\exists$ two r-open sets u_x and v_A such that $x \in u_x$, $A \subseteq v_A$ and $u_x \cap v_A = \phi$. Thus, $u \subseteq (v_A)^{\complement}$. Moreover, since $A \subseteq v$ implies that $v^{\complement} \subseteq A^{\complement}$, then $x \in u_x \subseteq v^{\complement} \subset A^{\complement}$, and so u is an r-open set. Hence, $A^{\complement}$ is an r-open set, and so A is an r-closed set. □

Theorem 21. *Let $Z = (Z, \gamma_1, \gamma_2)$ be a pairwise r-T_2- and a pairwise r-extremely disconnected space, then every subset in Z is a γ_i-closed set.*

Proof. Assume that A is a γ_i-r-compact subset of Z. Let $x \notin A$. Then, by Theorem 15, $\exists$ two γ_i-r-open sets u_x and v_A for which $x \in u_x$, $A \subseteq v_A$ and $u_x \cap v_A = \phi$. Thus, $u \subseteq (v_A)^{\complement}$ and since $A \subseteq v$ implies that $v^{\complement} \subseteq A^{\complement}$, $x \in u_x \subseteq v^{\complement} \subset A^{\complement}$. Now, since u is a γ_i-r-open set, $A^{\complement}$ is a γ_i-r-open set as well. Therefore, A is a γ_i-r-closed set, $\forall\, i = 1, 2$. □

Theorem 22. *If $Z = (Z, \gamma)$ is an r-compact, r-T_2- and r-extremely disconnected space, then every subset of Z is r-compact if and only if it is an r-closed set.*

Proof. $\Rightarrow$) Assume that A is an r-compact subset of Z, then by Theorem 20, A is an r-closed set.
$\Leftarrow$) Assume that A is an r-closed set in an r-compact r-T_2-extremely disconnected space, then, by Theorem 12, A is r-compact. □

Theorem 23. *If $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise r-compact, pairwise r-T_2- and pairwise r-extremely disconnected space, then every γ_i-subset of Z is pairwise r-compact if and only if it is a γ_i-r-closed set, $\forall\, i = 1, 2$.*

Proof. $\Rightarrow$) Assume that A is a pairwise r-compact subset of Z, then by Theorem 21, A is a γ_i-r-closed set, $\forall i = 1, 2$.
$\Leftarrow$) Assume that A is γ_i-r-closed in a pairwise r-compact, pairwise r-T_2-extremely disconnected space, then, by Theorem 12, A is γ_i-r-compact, $\forall\, i = 1, 2$. □

Theorem 24. *Let $Z = (Z, \gamma)$ be an r-T_3-extremely disconnected space. If A is a subset in Z such that $A \subseteq u_A$, for some r-open set u_A in Z, then $\exists$ an r-open set v_A in Z in which:*

$$A \subseteq v_A \subseteq \overline{v_A} \subseteq u_A.$$

Proof. For all $x \in A$ and $x \in u_A$, we have Z is r-regular space. This implies that $\exists$ an r-open set v_x for which:

$$x \in v_x \subseteq \overline{v_x} \subseteq u_A. \tag{15}$$

Now, $\underset{\sim}{V} = \{v(x) : x \in \Lambda\}$ forms an r-open cover of A, and because A is r-compact, $A \subseteq \bigcup\limits_{k=1}^{n} v_{x_k}$. Thus, from (15), we have:

$$A \subseteq \bigcup_{k=1}^{n} v_{x_k} \subseteq \bigcup_{k=1}^{n} \overline{v_{x_k}} \subseteq u_A.$$

Now, if one lets $v_A = \bigcup\limits_{k=1}^{n} v_{x_k}$, then we get:

$$A \subseteq v_A \subseteq \overline{v_A} \subseteq u_A.$$

□

Theorem 25. *Let $Z = (Z, \gamma_1, \gamma_2)$ be a pairwise r-T_3-extremely disconnected space. If A is a subset in Z such that $A \subseteq u_A$, for some γ_i-r-open set u_A in Z, then $\exists$ a γ_i-r-open set v_A in Z in which:*

$$A \subseteq v_A \subseteq \overline{v_A} \subseteq u_A,$$

$\forall\, i = 1, 2$.

Proof. For all $x \in A$ and $x \in u_A$, we have Z is a pairwise r-regular space. This consequently implies that $\exists$ a γ_i-r-open set v_x in which:

$$x \in v_x \subseteq \overline{v_x} \subseteq u_A. \tag{16}$$

Now, $\underset{\sim}{V} = \{v(x) : x \in \Lambda\}$ forms a γ_i-r-open cover of A, and because A is γ_i-r-compact, $A \subseteq \bigcup\limits_{k=1}^{n} v_{x_k}$, $\forall\, i = 1, 2$. Thus, based on (16), we have:

$$A \subseteq \bigcup_{k=1}^{n} v_{x_k} \subseteq \bigcup_{k=1}^{n} \overline{v_{x_k}} \subseteq u_A.$$

Now, if one lets $v_A = \bigcup\limits_{k=1}^{n} v_{x_k}$, then we get:

$$A \subseteq v_A \subseteq \overline{v_A} \subseteq u_A.$$

□

Definition 14. *Let $Z = (Z, \gamma_1, \gamma_2)$ be a bitopological space and $\underset{\sim}{A}$ be a family of γ_i subsets of Z. Then, it is said that $\underset{\sim}{A}$ has a finite intersection property (F.I.P.) if the intersection of a finite number of members of $\underset{\sim}{A}$ is not empty, $\forall\ i = 1, 2$.*

Theorem 26. *A topological space $Z = (Z, \gamma)$ is an r-extremely disconnected compact space if and only if every family of closed subsets of Z with the F.I.P has a nonempty intersection.*

Proof. $\Rightarrow$) Suppose Z is an r-extremely disconnected compact space. Suppose that $\exists$ a family $\underset{\sim}{F} = \{F_\alpha : \alpha \in \Lambda\}$ of r-closed subsets of Z with the F.I.P. and $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$. Thus, we have:

$$\bigcup_{\alpha\in\Lambda} {F_\alpha}^{\complement} = (\bigcap_{\alpha\in\Lambda} F_\alpha)^{\complement} = Z.$$

However, F_α is r-closed in Z, $\forall\ \alpha \in \Lambda$. Therefore, ${F_\alpha}^{\complement}$ is r-open in Z, $\forall\ \alpha \in \Lambda$. Thus, $\underset{\sim}{V} = {F_\alpha}^{\complement}$ such that $\alpha \in \Lambda$, for an r-open cover of Z. Therefore, by the assumption $Z = \bigcup_{k=1}^{n} {F_{\alpha_k}}^{\complement} = (\bigcap_{k=1}^{n} F_{\alpha_k})^{\complement}$, we can have $\phi = \bigcap_{k=1}^{n} F_{\alpha_k}$, which contradicts with $\underset{\sim}{F}$ that has the F.I.P., and so the result holds.

$\Leftarrow$) Suppose every family of r-closed subsets of Z with the F.I.P. has a nonempty intersection. Assume that Z is not an r-compact space. Then, $\exists$ an r-open cover of Z, say $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$, that cannot be reducible to a finite subcover of Z. Thus, we have:

$$Z = \bigcup_{\alpha\in\Lambda} u_\alpha. \tag{17}$$

Consequently, $\phi = (\bigcup_{\alpha\in\Lambda} u_\alpha)^{\complement} = \bigcap_{\alpha\in\Lambda} {u_\alpha}^{\complement}$, and because u_α is r-open $\forall\ \alpha \in \Lambda$, $\alpha \in \Lambda^{\complement}$ is r-closed $\forall \alpha \in \Lambda$. Consequently, $\underset{\sim}{F} = \left\{{u_\alpha}^{\complement}\right\}_{\alpha\in\Lambda}$ is a family of r-closed subsets of Z. Now, to finish this proof, we should concern ourselves with the following claim:

Claim: $\underset{\sim}{F}$ has the F.I.P.

To prove this claim, we suppose by contrary that the above statement does not hold. Then, $\exists\ u_1, \cdots, u_n$ for which $\bigcap_{k=1}^{n} {u_k}^{\complement} = \phi$. Thus, $Z = \bigcup_{k=1}^{n} u_k$, and so $\underset{\sim}{U}$ has a finite subcover of Z, which is a contradiction. Therefore, $\underset{\sim}{F} = \left\{{u_\alpha}^{\complement} : \alpha \in \Lambda\right\}$ has the F.I.P., and so by the assumption $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$, we get $\phi \neq \bigcap_{\alpha\in\Lambda} {u_\alpha}^{\complement}$, which implies $Z \neq Z - (\bigcap_{\alpha\in\Lambda} {u_\alpha}^{\complement})$. This means $Z \neq \bigcup_{\alpha\in\Lambda} u_\alpha$, which contradicts (17). Hence, Z is an r-compact space. $\square$

Theorem 27. *A bitopological space $Z = (Z, \gamma_1, \gamma_2)$ is a pairwise r-extremely disconnected compact space if and only if every family of γ_i-closed subsets of Z with the F.I.P. has a nonempty intersection, $\forall\ i = 1, 2$.*

Proof. $\Rightarrow$) Assume that $i = 1, 2$, and Z is a pairwise r-extremely disconnected compact space. Suppose that $\exists$ a family $\underset{\sim}{F} = \{F_\alpha : \alpha \in \Lambda\}$ of γ_i-r-closed subsets of Z with the F.I.P. and $\bigcap_{\alpha\in\Lambda} F_\alpha = \phi$. This implies $\bigcup_{\alpha\in\Lambda} {F_\alpha}^{\complement} = (\bigcap_{\alpha\in\Lambda} F_\alpha)^{\complement} = Z$. However, F_α is γ_i-r-closed in Z, $\forall\ \alpha \in \Lambda$, and so ${F_\alpha}^{\complement}$ is γ_i-r-open in Z, $\forall\ \alpha \in \Lambda$. Consequently, we have $\underset{\sim}{V} = \{{F_\alpha}^{\complement} : \alpha \in \Lambda\}$, for a γ_i-r-open cover of Z. Then, by the assumption $Z = \bigcup_{k=1}^{n} {F_{\alpha_k}}^{\complement} = (\bigcap_{k=1}^{n} F_{\alpha_k})^{\complement}$, we can have $\phi = \bigcap_{k=1}^{n} F_{\alpha_k}$, which is a contradiction with $\underset{\sim}{F}$ that has the F.I.P.. Hence, the result holds.

$\Leftarrow$) Conversely, suppose every family of γ_i-r-closed subsets of Z with the F.I.P. has a nonempty intersection. Assume that Z is not a pairwise r-compact space. Then, $\exists$ a γ_i-r-open cover of Z, say $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ that cannot be reducible to a finite subcover of Z, and hence we have:

$$Z = \bigcup_{\alpha \in \Lambda} u_\alpha. \tag{18}$$

Consequently, we obtain $\phi = (\bigcup_{\alpha \in \Lambda} u_\alpha)^{\complement} = \bigcap_{\alpha \in \Lambda} {u_\alpha}^{\complement}$. Since u_α is γ_i-r-open $\forall\ \alpha \in \Lambda$, then $\alpha \in \Lambda^{\complement}$ is γ_i-r-closed $\forall\ \alpha \in \Lambda$. This leads to assert that $\underset{\sim}{F} = \left\{{u_\alpha}^{\complement}\right\}_{\alpha \in \Lambda}$ is a family of γ_i-r-closed subsets of Z. Now, to finish this proof, we should concern with the following claim:

Claim: $\underset{\sim}{F}$ has the F.I.P.

To prove this claim, we suppose on the contrary that the above statement does not hold. Then, there exist $u_1, \cdots, u_n$ such that $\bigcap_{k=1}^{n} {u_k}^{\complement} = \phi$. Then, $Z = \bigcup_{k=1}^{n} u_k$, and so $\underset{\sim}{U}$ has a finite subcover of Z, which is a contradiction. Thus, $\underset{\sim}{F} = \left\{{u_\alpha}^{\complement} : \alpha \in \Lambda\right\}$ has the F.I.P., and so by the assumption $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \phi$, we have $\phi \neq \bigcap_{\alpha \in \Lambda} {u_\alpha}^{\complement}$, which implies $Z \neq Z - (\bigcap_{\alpha \in \Lambda} {u_\alpha}^{\complement})$. This means $Z \neq \bigcup_{\alpha \in \Lambda} u_\alpha$, which contradicts (18). Hence, Z is a pairwise r-compact space. $\square$

4. Conclusions

In this work, the so-called pairwise r-compactness was well-defined in topological and bitopological spaces. Several properties of these spaces with their relations with other topological and bitopological spaces were consequently established theoretically. The inferred results can pave the way to deriving other novel theorems related to the finite product and mappings of pairwise expandable spaces, feebly pairwise expandable spaces and fuzzy bitopological spaces, which is left to future considerations.

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