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Article

New Conditions for Testing the Asymptotic and Oscillatory Behavior of Solutions of Neutral Differential Equations of the Fourth Order

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Abstract: In this work, in the noncanonical case, we find new properties for a class of positive solutions of fourth-order differential equations. These properties allow us to obtain iterative criteria that exclude positive decreasing solutions, and we then establish sufficient conditions to guarantee that all solutions to the examined equation oscillate. The importance of applying the results to a special case of the investigated equation is demonstrated.

Keywords: oscillation; differential equation; neutral delay; noncanonical case

MSC: 34C10; 34K11

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1. Introduction

This paper focuses on investigating the oscillation of solutions to the fourth-order neutral delay differential equation

$$(\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma})' + q(\mathrm{s})\omega^{\gamma}(\theta(\mathrm{s})) = 0,\ \mathrm{s} \geq \mathrm{s}_0, \tag{1}$$

where $\Phi(\mathrm{s}) = \omega(\mathrm{s}) + p(\mathrm{s})\omega(\tau(\mathrm{s}))$. Throughout this paper, we make the following assumptions:

(A_1) $\gamma > 0$ is a quotient of odd positive integers;
(A_2) $\varrho \in \mathbf{C}([\mathrm{s}_0,\infty),(0,\infty))$ satisfies

$$\pi(\mathrm{s}_0) = \int_{\mathrm{s}_0}^{\infty} \varrho^{-1/\gamma}(v)\mathrm{d}v < \infty; \tag{2}$$

(A_3) $\tau, \theta \in \mathbf{C}([\mathrm{s}_0,\infty),(0,\infty))$ satisfy $\tau(\mathrm{s}) < \mathrm{s}$, $\theta(\mathrm{s}) < \mathrm{s}$, $\lim_{\mathrm{s}\to\infty}\tau(\mathrm{s}) = \infty$ and $\lim_{\mathrm{s}\to\infty}\theta(\mathrm{s}) = \infty$;
(A_4) $p, q \in \mathbf{C}([\mathrm{s}_0,\infty),(0,\infty))$, $0 \leq p(\mathrm{s}) < 1$ and $p(\mathrm{s})\frac{\pi(\tau(\mathrm{s}))}{\pi(\mathrm{s})} < 1$.

By a solution of (1), we understand a four-time differentiable real-valued function ω which satisfies (1) for all s large enough. Our attention is restricted to those solutions of (1) that satisfy the condition $\sup\{|\omega(\mathrm{s})| : \mathrm{s} \geq L\} > 0$ for all $L \geq \mathrm{s}_0$. If a solution ω of (1) is essentially positive or negative, it is said to be nonoscillatory; otherwise, it is said to be oscillatory. The equation is referred to as oscillatory if all of its solutions oscillate.

The past of the system is taken into account via delay differential equations, allowing for a more efficient future prediction. This gave us a compelling reason to look into the qualitative properties of the solutions to these equations.

Neutral delay differential equations are a type of functional differential equation in which the delayed argument appears in the state variable's highest derivative. The qualitative analysis of such equations is quite beneficial in addition to its theoretical value. This form of the equation has fascinating applications in everyday life, for example, in networks with lossless transmission lines as in high-speed computers, in the vibration study of blocks connected to a flexible rod and for solving various problems with a time delay as well as in automated control theory and in aeromechanical systems in which inertia plays an important role. We refer the reader to Hale's monograph [1] for additional science and technological applications.

In the first half of the eighteenth century, with the problem of the vibrating rod, the study of fourth-order differential equations appeared. Such equations have been of great practical importance as they arise in the modeling of biological and physical phenomena such as the deformation of structures and problems of elasticity; see [2]. The qualitative theory of such equations has undergone an astonishing development as the oscillation theory of fourth-order differential equations has attracted much attention over the past decades; we refer the reader to [3,4].

The oscillatory and nonoscillatory properties of solutions are a focus of oscillation theory. Gyri and Ladas's book [5] summarizes some of the work in this field, particularly the relationship between the distribution of the characteristic equation's roots and the oscillation of all solutions, while Erbe et al. [6] and others covered several crucial topics such as determining the separation between zeros and the oscillation of nonlinear neutral equations.

Literature Review

As far as we know, the common case is the canonical case, with many publications exploring the oscillation of solutions to fourth-order neutral differential equations

$$\int_{s_0}^{s} \varrho^{-1/\gamma}(v)\mathrm{d}v \to \infty \text{ as } s \to \infty, \tag{3}$$

while the noncanonical case is

$$\int_{s_0}^{\infty} \varrho^{-1/\gamma}(v)\mathrm{d}v < \infty. \tag{4}$$

Assumption (3) has been widely used because the rating of positive solutions is lower, for example in an even order there are no positive decreasing solutions (see [7–10]). However, with assumption (4), we are faced with the problem that there are positive decreasing solutions, and this leads to an increased number of derivatives.

In studying the neutral delay equation in the canonical case, it is easy to find the relationship between the solution and the corresponding function $\omega(s) \geq (1 - p(s))\Phi(s)$. On the other hand, we note that the previous relationship is generally not satisfied when using assumption (4).

One of the most important goals in studying the oscillation of neutral delay differential equations in the noncanonical case is to find criteria that ensure the exclusion of positive decreasing solutions. This is because many of the frequently used relationships are invalid in that case. For the second order, Bohner et al. [11] tackled this issue in an intriguing work, finding the following constraint for the solution and a related function

$$\omega(s) \geq \left(1 - p(s)\frac{\pi(\tau(s))}{\pi(s)}\right)\Phi(s),$$

where $\pi(s)$ is defined as in (4). Due to this relationship, the authors were able to find several new criteria that simplified and improved their previous results in the literature.

As an extension of Bohner's results [11], Ramos et al. [12] found a new relationship between the solution and a corresponding function for the fourth order.

The topic of sufficient conditions for the oscillation of delay differential equation solutions has been extensively discussed in the literature; for more information, see [13–16] (and the references cited therein).

We present some related previous works.

Agarwal et al. [17] studied the oscillation of the fourth-order functional differential equation

$$\left[\varrho_3(s)\left(\left[\varrho_2(s)\left(\left[\varrho_1(s)(\omega'(s))^{\gamma_1}\right]'\right)^{\gamma_2}\right]'\right)^{\gamma_3}\right]' + q(s)f(\omega(\theta(s))) = 0,$$

where $\int^{\infty} \varrho_i^{-1/\gamma}(s)\mathrm{d}s < \infty$, $i = 1,2,3$.

Grace et al. [18] investigated the oscillatory behavior of all solutions of the fourth-order functional differential equation

$$(\varrho(s)(\omega'(s))^{\gamma})''' + q(s)f(\omega(\theta(s))) = 0,$$

where (4) holds. For neutral differential equations, we show the following.

Li et al. [19] investigated the oscillation of the even-order equation

$$\Phi^{(n)}(s) + q(s)\omega(\theta(s)) = 0.$$

In the case where α is the quotient of odd positive integers and $\alpha \geq \gamma$, Moaaz et al. [20] considered the fourth-order neutral differential equation of the form

$$(\varrho(s)(\Phi'''(s))^{\gamma})' + q(s)\omega^{\alpha}(\theta(s)) = 0,$$

which contributed to improve some well-known results which were published recently in the literature.

Ramos et al. [12] studied the oscillatory behavior of the solutions of the neutral delay differential equation

$$(\varrho(s)(\Phi'''(s))^{\gamma})' + q(s)\omega^{\gamma}(\theta(s)) = 0, \tag{5}$$

with (4) holding. They improved on previous results in the literature.

Theorem 1 ([12])**.** *Suppose that there exists some $s_1 \geq s_0$ such that $\pi_2(s) > p(s)\pi_2(\tau(s))$ for $s \geq s_1$. If there exists a function $\varkappa \in \mathbf{C}((s_0,\infty),(0,\infty))$ such that*

$$\limsup_{s\to\infty} \frac{\pi_2^{\gamma}(s)}{\varkappa(s)} \int_{s_1}^{s} \left(\varkappa(v)Q(v) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{(\varkappa'(v))^{\gamma+1}}{\varkappa^{\gamma}(v)\pi_1^{\gamma}(v)}\right)\mathrm{d}v > 1, \tag{6}$$

then (5) has no positive decreasing solutions.

Theorem 2 ([12])**.** *Suppose that there exists some $s_1 \geq s_0$ such that $\pi_2(s) > p(s)\pi_2(\tau(s))$, and that for $\lambda_0 \in (0,1)$, the first-order delay differential equation*

$$y'(s) + q(s)\left(\frac{\lambda_0}{6}\theta^3(s)\right)^{\gamma} \frac{Q(s)}{\varrho(\theta(s))} y(\theta(s)) = 0$$

is oscillatory, and that for some constant $\lambda_1 \in (0,1)$, it is

$$\limsup_{s\to\infty} \int_{s_1}^{s} \left(\frac{\lambda_1^{\gamma}}{(2!)^{\gamma}}\theta^{2\gamma}(v)Q(v)\pi_0^{\gamma}(v) - \frac{\gamma^{\gamma+1}\varrho^{-1/\gamma}(v)}{(\gamma+1)^{\gamma+1}\pi_0(v)}\right)\mathrm{d}v = \infty. \tag{7}$$

If (6) holds, then (5) is oscillatory.

Corollary 1 ([12])**.** *Suppose that there exists some* $s_1 \geq s_0$ *such that* $\pi_2(s) > p(s)\pi_2(\tau(s))$*, and that for some constant* $\lambda_0 \in (0,1)$*, (6) and (7) hold for some constant* $\lambda_1 \in (0,1)$ *and for* $s \geq s_1$*. If*

$$\liminf_{s\to\infty} \int_{\theta(s)}^{s} \left(\frac{\lambda_0}{6}\theta^3(v)\right)^{\gamma} \frac{\vartheta(v)}{\varrho(\theta(v))} dv > \frac{1}{e}, \tag{8}$$

then every solution of (5) is oscillatory.

Muhib et al. [21] derived new asymptotic properties of the positive solutions of the fourth-order neutral differential equation

$$(\varrho(s)(\Phi'''(s))^{\gamma})' + f(s,\omega(\theta(s))) = 0,$$

with (4), where $f(s,\omega) = h(s)\omega$ such that h is a continuous function.

Elabbasy et al. [22] studied the asymptotic and oscillatory behavior of the even-order neutral delay noncanonical differential equation

$$\left(\varrho(s)\left((\omega(s)+p(s)\omega(\tau(s)))^{(m-1)}\right)^{\gamma}\right)' + q(s)\omega^{\beta}(\theta(s)) = 0, \tag{9}$$

where $m \geq 4$ and β is a quotient of odd positive integers. They improved, simplified and complemented their new oscillation criteria with related results in the literature.

Lemma 1 ([23])**.** *Let* γ *be a ratio of two odd positive integers. Then,*

$$Lz^{(\gamma+1)/\gamma} - Mz \geq -\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{M^{\gamma+1}}{L^{\gamma}},\ L > 0. \tag{10}$$

and

$$A^{(\gamma+1)/\gamma} - (A-B)^{(\gamma+1)/\gamma} \leq \frac{1}{\gamma}B^{1/\gamma}[(1+\gamma)A - B],\quad \gamma \geq 1,\ AB \geq 0. \tag{11}$$

In this paper, we create new monotonic properties of a class from the positive solutions of Equation (1). We establish iterative criteria that exclude the existence of positive decreasing solutions by employing Riccati's general form and comparison method. By combining the results obtained in Section 2.3 with the known results reported in the literature, we create sufficient conditions to ensure that all solutions of the studied equation oscillate. Finally, we provide an example to demonstrate the effectiveness of our results. The article is concluded with a summary.

2. Main Results

2.1. Notations

We define the following to support our main results:

$$\pi_i(s) = \int_s^{\infty} \pi_{i-1}(v)dv,\ \text{ for } i = 1,2$$

and

$$\delta_* = \liminf_{s\to\infty} \frac{1}{\gamma} q(s)\pi_1^{-1}(s)\pi_2^{\gamma+1}(s).$$

In addition, we set

$$\mu_* = \liminf_{s\to\infty} \frac{\pi_2(\tau(s))}{\pi_2(s)}.$$

It is important to keep in mind that given (A2), $\mu_* \geq 1$. We frequently state in the proofs that there exists $s_1 \geq s_0$ sufficiently large such that, for arbitrary $\delta \in (0, \delta_*)$ and $\mu \in [1, \mu_*)$, we have

$$Q(s) = q(s)\left(1 - p(\theta(s))\frac{\pi_2(\tau(\theta(s)))}{\pi_2(\theta(s))}\right)^{\gamma},$$

$$Q(s)\pi_1^{-1}(s)\pi_2^{\gamma+1}(s) \geq \gamma\delta \tag{12}$$

and

$$\frac{\pi_2(\tau(s))}{\pi_2(s)} \geq \mu$$

on $[s_1, \infty)$.

2.2. New Iterative Properties

Lemma 2. *Suppose that $\omega \in \mathbf{C}([s_0, \infty), (0, \infty))$ is a solution of (1). Then, $\Phi(s) > 0$, $\varrho(s)(\Phi''')^{\gamma} < 0$, and one of the following cases holds for $s \in [s_1, \infty)$, $s_1 \geq s_0$:*
(B_1) $\Phi'(s) > 0$, $\Phi'''(s) > 0$ and $\Phi^{(4)}(s) < 0$;
(B_2) $\Phi'(s) > 0$, $\Phi''(s) > 0$ and $\Phi'''(s) < 0$;
(B_3) $(-1)^i\Phi^{(i)}(s)$ are positive for $i = 1, 2, 3$ (note that in this case, Φ is a positive decreasing solution).

Proof. Let $\omega(t)$ be an eventually positive solution of (1). Then, there exists $s_1 \geq s_0$ such that $\omega(s) > 0$, $\omega(\tau(s)) > 0$ and $\omega(\theta(s)) > 0$ for all $s \geq s_1$. Hence, we see that $\Phi(s) > 0$ for $s \geq s_1$. From (1), we see that

$$\left(\varrho(s)(\Phi''')^{\gamma}\right)' \leq 0.$$

By using [24] (Lemma 2.2.1), cases (B_1) and (B_3) are easily accessible. □

Lemma 3. *Suppose that $\omega \in \mathbf{C}([s_0, \infty), (0, \infty))$ is a solution of (1) and $\Phi(s)$ is a positive decreasing solution. If*

$$\int_{s_0}^{\infty}\left(\frac{1}{\varrho(v)}\int_{s_1}^{v} Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v = \infty, \tag{13}$$

then
(i) $(\Phi(s)/\pi_2(s))' > 0$;
(ii) $\lim_{s\to\infty}\Phi(s) = 0$.

Proof. Let ω be an eventually positive solution of (1), taking into account that we are in case (B_3). Then, there exists $s_1 \geq s_0$ such that $\omega(\tau(s)) \geq 0$ for $s \geq s_1$; hence,

$$\left(\varrho(s)(\Phi'''(s))^{\gamma}\right)' = -q(s)\omega^{\gamma}(\theta(s)) \leq 0.$$

(i) Using the fact that $\varrho^{1/\gamma}(s)\Phi'''(s)$ is nonincreasing, we see that

$$\Phi''(s) \geq -\int_s^{\infty}\varrho^{-1/\gamma}(v)\varrho^{1/\gamma}(v)\Phi'''(v)\mathrm{d}v \geq -\varrho^{1/\gamma}(s)\Phi'''(s)\pi(s). \tag{14}$$

Now, we have

$$\left(\frac{\Phi''(s)}{\pi(s)}\right)' = \frac{\pi(s)\Phi'''(s) + \varrho^{-1/\gamma}(s)\Phi''(s)}{\pi^2(s)} \geq 0. \tag{15}$$

Thus, we get that

$$\Phi'(s) \leq -\int_s^{\infty}\pi(v)\frac{\Phi''(v)}{\pi(v)}\mathrm{d}v \leq -\frac{\Phi''(s)}{\pi(s)}\pi_1(s), \tag{16}$$

which implies

$$\left(\frac{\Phi'(\mathrm{s})}{\pi_1(\mathrm{s})}\right)' = \frac{\pi_1(\mathrm{s})\Phi''(\mathrm{s}) + \pi(\mathrm{s})\Phi'(\mathrm{s})}{\pi_1^2(\mathrm{s})} \leq 0. \tag{17}$$

This leads to

$$\Phi(\mathrm{s}) \geq -\int_{\mathrm{s}}^{\infty} \pi_1(v)\frac{\Phi'(v)}{\pi_1(v)}\mathrm{d}v \geq -\frac{\Phi'(\mathrm{s})}{\pi_1(\mathrm{s})}\pi_2(\mathrm{s}), \tag{18}$$

hence

$$\left(\frac{\Phi(\mathrm{s})}{\pi_2(\mathrm{s})}\right)' = \frac{\pi_2(\mathrm{s})\Phi'(\mathrm{s}) + \pi_1(\mathrm{s})\Phi(\mathrm{s})}{\pi_2^2(\mathrm{s})} \geq 0. \tag{19}$$

(ii) Since

$$\omega(\mathrm{s}) = \Phi(\mathrm{s}) - p(\mathrm{s})\omega(\tau(\mathrm{s})) \geq \Phi(\mathrm{s}) - p(\mathrm{s})\Phi(\tau(\mathrm{s})),$$

from (19), we have

$$\omega(\mathrm{s}) \geq \left(1 - p(\mathrm{s})\frac{\pi_2(\tau(\mathrm{s}))}{\pi_2(\mathrm{s})}\right)\Phi(\mathrm{s}).$$

Now, from (1), we get

$$\left(\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\right)' \leq -Q(\mathrm{s})\Phi^{\gamma}(\theta(\mathrm{s})).$$

Since, $\Phi'(\mathrm{s}) < 0$, we get that $\lim_{\mathrm{s}\to\infty}\Phi(\mathrm{s}) = \rho \geq 0$. Assume the contrary, $\rho > 0$, then there is a $\mathrm{s}_2 \geq \mathrm{s}_1$ with $\Phi(\mathrm{s}) \geq \rho$ for $\mathrm{s} \geq \mathrm{s}_2$. Thus,

$$\left(\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\right)' \leq -\rho^{\gamma}Q(\mathrm{s}).$$

Integrating the above inequality twice over $[\mathrm{s}_2, \mathrm{s})$, we obtain

$$\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s}) - \varrho(\mathrm{s}_2)\Phi'''(\mathrm{s}_2) \leq -\rho\left(\int_{\mathrm{s}_2}^{\mathrm{s}} Q(v)\mathrm{d}v\right)^{1/\gamma}.$$

Hence,

$$\Phi'''(\mathrm{s}) \leq -\rho\left(\frac{1}{\varrho(\mathrm{s})}\int_{\mathrm{s}_2}^{\mathrm{s}} Q(v)\mathrm{d}v\right)^{1/\gamma},$$

and then

$$\Phi''(\mathrm{s}) \leq \Phi''(\mathrm{s}_2) - \rho\int_{\mathrm{s}_2}^{\mathrm{s}}\left(\frac{1}{\varrho(v)}\int_{\mathrm{s}_2}^{v} Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v.$$

Letting $\mathrm{s} \to \infty$ and using (13), we obtain that $\lim_{\mathrm{s}\to\infty}\Phi''(\mathrm{s}) = -\infty$, which contradicts with $\Phi''(\mathrm{s}) > 0$. Thus, the proof is complete. □

Remark 1. *For positive and finite δ_* and μ_* we define the sequence δ_n as follows:*

$$\delta_0 = \sqrt[\gamma]{\delta_*}$$

and

$$\delta_n = \frac{\delta_0\mu_*^{\delta_{n-1}}}{\sqrt[\gamma]{1-\delta_{n-1}}}, \quad n \in \mathbb{N}. \tag{20}$$

It is easy to see that, by induction, if for any $n \in \mathbb{N}$, $\delta_i < 1$, $i = 0, 1, 2, ..., n$, then δ_{n+1} exists and

$$\delta_{n+1} = \imath_n\delta_n > \delta_n, \tag{21}$$

where $\imath_n$ is defined by

$$\imath_0 = \frac{\mu_*^{\delta_0}}{\sqrt[\gamma]{1-\delta_0}},$$

and

$$\imath_{n+1} = \mu_*^{\delta_0(\imath_n - 1)} \sqrt[\gamma]{\frac{1-\delta_n}{1-\imath_n\delta_n}}, \quad n \in \mathbb{N}_0.$$

Lemma 4. *Let $\delta_* > 0$ and $\mu_* < \infty$. If $\omega \in \mathbf{C}([\mathrm{s}_0,\infty),(0,\infty))$ is a solution of (1) and $\Phi(\mathrm{s})$ is a positive decreasing solution, then for any $n \in \mathbb{N}_0$*

$$\left(\frac{\Phi(\mathrm{s})}{\pi_2^{\delta_n}(\mathrm{s})}\right)' < 0.$$

Proof. Let ω be a positive solution of (1), taking into account that we are in case (B_3) on $[\mathrm{s}_1,\infty)$ where $\mathrm{s}_1 \geq \mathrm{s}_0$ such that $\omega(\tau(\mathrm{s})) > 0$ and (12) holds for $\mathrm{s} \geq \mathrm{s}_1$. Integrating (1) from s_1 to s, we have

$$\begin{aligned} \varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma &\leq \varrho(\mathrm{s}_1)(\Phi'''(\mathrm{s}_1))^\gamma - \int_{\mathrm{s}_1}^{\mathrm{s}} Q(v)\Phi^\gamma(\theta(v))\mathrm{d}v \\ &\leq \varrho(\mathrm{s}_1)(\Phi'''(\mathrm{s}_1))^\gamma - \Phi^\gamma(\mathrm{s}) \int_{\mathrm{s}_1}^{\mathrm{s}} Q(v)\mathrm{d}v. \end{aligned}$$

By using (12) in the above inequality, we obtain

$$\begin{aligned} \varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma &\leq \varrho(\mathrm{s}_1)(\Phi'''(\mathrm{s}_1))^\gamma - \delta\Phi^\gamma(\mathrm{s}) \int_{\mathrm{s}_1}^{\mathrm{s}} \frac{\gamma}{\pi_1^{-1}(v)\pi_2^{\gamma+1}(v)}\mathrm{d}v \\ &\leq \varrho(\mathrm{s}_1)(\Phi'''(\mathrm{s}_1))^\gamma - \delta\frac{\Phi^\gamma(\mathrm{s})}{\pi_2^\gamma(\mathrm{s})} + \delta\frac{\Phi^\gamma(\mathrm{s})}{\pi_2^\gamma(\mathrm{s}_1)}. \end{aligned}$$

From Lemma 3, we have that $\lim_{\mathrm{s}\to\infty} \Phi(\mathrm{s}) = 0$. Hence, there is a $\mathrm{s}_2 \in [\mathrm{s}_1,\infty)$ such that

$$\varrho(\mathrm{s}_1)(\Phi'''(\mathrm{s}_1))^\gamma + \delta\frac{\Phi^\gamma(\mathrm{s})}{\pi_2^\gamma(\mathrm{s}_1)} < 0, \quad \mathrm{s} \geq \mathrm{s}_2.$$

Thus,

$$\varrho(\mathrm{s})\big(\Phi'''(\mathrm{s})\big)^\gamma < -\delta\frac{\Phi^\gamma(\mathrm{s})}{\pi_2^\gamma(\mathrm{s})}$$

or

$$\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_2(\mathrm{s}) < -\sqrt[\gamma]{\delta}\Phi(\mathrm{s}) = -v_0\delta_0\Phi(\mathrm{s}), \tag{22}$$

where $v_0 = \sqrt[\gamma]{\delta}/\delta_0$ and $v_0 \in (0,1)$. Note that,

$$\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi(\mathrm{s}) \geq \int_{\mathrm{s}}^{\infty} \varrho^{-1/\gamma}(v)\varrho^{1/\gamma}(v)\Phi'''(v)\mathrm{d}v \geq -\Phi''(\mathrm{s}),$$

then,

$$\Phi''(\mathrm{s}) \geq -\varrho^{1/\gamma}(\mathrm{s})\pi(\mathrm{s})\Phi'''(\mathrm{s}).$$

Repeating this step twice over $[\mathrm{s},\infty)$, we get

$$\Phi'(\mathrm{s}) \leq \varrho^{1/\gamma}(\mathrm{s})\pi_1(\mathrm{s})\Phi'''(\mathrm{s}) \tag{23}$$

and

$$\Phi(\mathrm{s}) \geq -\varrho^{1/\gamma}(\mathrm{s})\pi_2(\mathrm{s})\Phi'''(\mathrm{s}).$$

From (22) and (23), we obtain

$$\frac{\Phi'(s)}{\pi_1(s)} \leq \varrho^{1/\gamma}(s)\Phi'''(s)$$

and

$$\frac{\Phi'(s)}{\pi_1(s)} \leq -\sqrt[\gamma]{\delta}\frac{\Phi(s)}{\pi_2(s)},$$

hence

$$\pi_2(s)\Phi'(s) + \sqrt[\gamma]{\delta}\pi_1(s)\Phi(s) \leq 0.$$

Therefore,

$$\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)' = \frac{1}{\pi_2^{\sqrt[\gamma]{\delta}+1}(s)}\left(\sqrt[\gamma]{\delta}\pi_1(s)\Phi(s) + \pi_2(s)\Phi'(s)\right) \leq 0.$$

Integrating (1) from s_2 to s and using that $\Phi(s)/\pi_2^{\sqrt[\gamma]{\delta}}(s)$ is decreasing, we have

$$\begin{aligned} \varrho(s)(\Phi'''(s))^{\gamma} &\leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \int_{s_2}^{s} Q(v)\pi_2^{\gamma\sqrt[\gamma]{\delta}}(\tau(v))\frac{\Phi^{\gamma}(\tau(v))}{\pi_2^{\gamma\sqrt[\gamma]{\delta}}(\tau(v))}dv \\ &\leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \left(\frac{\Phi(\tau(s))}{\pi_2^{\sqrt[\gamma]{\delta}}(\tau(s))}\right)^{\gamma}\int_{s_2}^{s} Q(v)\pi_2^{\gamma\sqrt[\gamma]{\delta}}(\tau(v))dv, \end{aligned}$$

hence

$$\varrho(s)(\Phi'''(s))^{\gamma} \leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\int_{s_2}^{s} Q(v)\left(\frac{\pi_2(\tau(v))}{\pi_2(v)}\right)^{\gamma\sqrt[\gamma]{\delta}}\pi_2^{\gamma\sqrt[\gamma]{\delta}}(v)dv.$$

It is clear that from (12), we have

$$\begin{aligned} \varrho(s)(\Phi'''(s))^{\gamma} &\leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \delta\left(\frac{\Phi(s)}{\pi^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\int_{s_2}^{s}\frac{\gamma\left(\frac{\pi_2(\tau(v))}{\pi_2(v)}\right)^{\gamma\sqrt[\gamma]{\delta}}}{\pi_1(v)\pi_2^{\gamma+1-\gamma\sqrt[\gamma]{\delta}}(v)}dv \\ &\leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\int_{s_2}^{s}\frac{\gamma\left(1-\sqrt[\gamma]{\delta}\right)}{\pi_1(v)\pi_2^{\gamma+1-\gamma\sqrt[\gamma]{\delta}}(v)}dv, \end{aligned}$$

which implies that

$$\varrho(s)(\Phi'''(s))^{\gamma} \leq \varrho(s_2)(\Phi'''(s_2))^{\gamma} - \frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\left(\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s)} - \frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s_2)}\right). \tag{24}$$

Now, we claim that $\lim_{s\to\infty}\Phi(s)/\pi_2^{\sqrt[\gamma]{\delta}}(s) = 0$. It is enough to show that there is $\epsilon > 0$ such that $\Phi(s)/\pi_2^{\sqrt[\gamma]{\delta}+\epsilon}(s)$ is eventually decreasing. Since $\pi_2(s)$ tends to zero, there is a constant

$$\imath \in \left(\frac{\sqrt[\gamma]{1-\sqrt[\gamma]{\delta}}}{\mu^{\sqrt[\gamma]{\delta}}}, 1\right)$$

and a $s_3 \geq s_2$ such that

$$\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s)} - \frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s_2)} > \imath^{\gamma}\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s)}, \quad s \geq s_3. \tag{25}$$

By using (25) in (24), we get

$$\varrho(s)\left(\Phi'''(s)\right)^{\gamma} \leq -\frac{\imath^{\gamma}\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2(s)}\right)^{\gamma},$$

which means

$$\varrho^{1/\gamma}(s)\Phi'''(s) \leq -\left(\sqrt[\gamma]{\delta}+\epsilon\right)\frac{\Phi(s)}{\pi_2(s)}, \tag{26}$$

where

$$\epsilon = \sqrt[\gamma]{\delta}\left(\frac{\imath\mu^{\sqrt[\gamma]{\delta}}}{\sqrt[\gamma]{1-\sqrt[\gamma]{\delta}}}-1\right) > 0.$$

Thus, from (26),

$$\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}+\epsilon}(s)}\right)' \leq 0, \quad s \geq s_3,$$

and hence the claim is valid. Therefore, for $s_4 \in [s_3,\infty)$,

$$-\varrho(s_2)\left(\Phi'''(s_2)\right)^{\gamma} - \frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s_2)} > 0, \quad s \geq s_4,$$

and using the above inequality in (24), we obtain

$$\begin{aligned}
\varrho(s)\left(\Phi'''(s)\right)^{\gamma} \leq{}& \varrho(s_2)\left(\Phi'''(s_2)\right)^{\gamma} - \frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s)} \\
&+\frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s_2)} \\
\leq{}& \varrho(s_2)\left(\Phi'''(s_2)\right)^{\gamma} - \frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2(s)}\right)^{\gamma} \\
&+\frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\left(\frac{\Phi(s)}{\pi_2^{\sqrt[\gamma]{\delta}}(s)}\right)^{\gamma}\frac{1}{\pi_2^{\gamma(1-\sqrt[\gamma]{\delta})}(s_2)},
\end{aligned}$$

hence

$$\varrho(s)\left(\Phi'''(s)\right)^{\gamma} < -\frac{\delta}{1-\sqrt[\gamma]{\delta}}\mu^{\gamma\sqrt[\gamma]{\delta}}\Phi^{\gamma}(s),$$

or

$$\varrho^{1/\gamma}(s)\Phi'''(s) < -\frac{\sqrt[\gamma]{\delta}}{\sqrt[\gamma]{1-\sqrt[\gamma]{\delta}}}\mu^{\sqrt[\gamma]{\delta}}\Phi(s) = -\epsilon_1\delta_1\Phi(s), \quad s \geq s_4,$$

where

$$\epsilon_1 = \sqrt[\gamma]{\frac{\delta\left(1-\sqrt[\gamma]{\delta_*}\right)}{\delta_*\left(1-\sqrt[\gamma]{\delta}\right)}\frac{\mu^{\sqrt[\gamma]{\delta}}}{\mu_*^{\sqrt[\gamma]{\delta_*}}}}, \quad \epsilon_1 \in (0,1);$$

we note that $\epsilon_1 \to 1$ at $\delta \to \delta_*$ and $\mu \to \mu_*$. Then,

$$\left(\frac{\Phi(s)}{\pi_2^{\epsilon_1\delta_1}(s)}\right) < 0, \quad s \geq s_4.$$

By using induction, for any $n \in \mathbb{N}_0$ and s large enough,

$$\left(\frac{\Phi(s)}{\pi_2^{\epsilon_n\delta_n}(s)}\right)' < 0,$$

where ϵ_n is given by

$$\epsilon_0 = \sqrt[\gamma]{\delta/\delta_*}$$

and

$$\epsilon_{n+1} = \epsilon_0 \sqrt[\gamma]{\frac{1-\delta_n}{1-\epsilon_n\delta_n} \frac{\mu^{\epsilon_n\delta_n}}{\mu_*^{\delta_n}}}, \quad n \in \mathbb{N}_0, \quad \epsilon_{n+1} \in (0,1);$$

we note that $\epsilon_{n+1} \to 1$ at $\delta \to \delta_*$ and $\mu \to \mu_*$. Finally, we claim that from any $n \in \mathbb{N}_0$

$$\left(\frac{\Phi(\mathrm{s})}{\pi_2^{\epsilon_{n+1}\delta_{n+1}}(\mathrm{s})} \right)' < 0.$$

Since $\epsilon_{n+1} \to 1$, $\epsilon_{n+1}\delta_{n+1} > \delta_n$ and (21), we have

$$\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_2(\mathrm{s}) < -\epsilon_{n+1}\delta_{n+1}\Phi(\mathrm{s}) < -\delta_n\Phi(\mathrm{s}) \text{ for s large enough,}$$

and so

$$\left(\frac{\Phi(\mathrm{s})}{\pi_2^{\delta_n}(\mathrm{s})} \right)' < 0.$$

Now, the proof is complete. □

2.3. *Nonexistence of Positive Decreasing Solutions*

Theorem 3. *Suppose that* (A_1) *and* (A_2) *hold. If*

$$\limsup_{\mathrm{s}\to\infty} \int_{\mathrm{s}_0}^{\mathrm{s}} \left[\chi^\gamma(v)Q(v)\frac{\pi_2^{\gamma\delta_n}(\tau(v))}{\pi_2^{\gamma\delta_n}(v)} - \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{(\chi'(v))^{\gamma+1}}{\chi(v)\pi_1^\gamma(v)} \right] \mathrm{d}v = \infty, \tag{27}$$

then (1) has no positive decreasing solutions, where

$$\chi(\mathrm{s}) = \int_{\mathrm{s}}^{\infty} (v-\mathrm{s})\pi(v)\mathrm{d}v.$$

Proof. Assume that Φ is a positive decreasing solution of (1) on $[\mathrm{s}_1,\infty)$ where $\mathrm{s}_1 \geq \mathrm{s}_0$. Since $\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^\gamma$ is nonincreasing, we get

$$\varrho^{1/\gamma}(v)\Phi'''(v) \leq \varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s}), \quad v \geq \mathrm{s} \geq \mathrm{s}_1.$$

By dividing the previous inequality by $\varrho^{1/\gamma}(s)$, we have

$$\Phi'''(v) \leq \frac{\varrho^{1/\gamma}(\mathrm{s})}{\varrho^{1/\gamma}(v)}\Phi'''(\mathrm{s}).$$

By integrating the above inequality from c to s, we obtain

$$\Phi''(c) \leq \Phi''(\mathrm{s}) + \varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s}) \int_{\mathrm{s}}^{c} \varrho^{1/\gamma}(v)\mathrm{d}v.$$

Letting $c \to \infty$, we have

$$0 \leq \Phi''(\mathrm{s}) + \varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi(\mathrm{s}),$$

which produces

$$\Phi''(\mathrm{s}) \geq -\pi(\mathrm{s})\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s}).$$

Integrating the above inequality from s to ∞ yields

$$-\Phi'(\mathrm{s}) \geq -\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s}) \int_{\mathrm{s}}^{\infty} \pi(v)\mathrm{d}v. \tag{28}$$

Again, integrating (28) from s to ∞, we get

$$\Phi(\mathrm{s}) \geq -\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\int_{\mathrm{s}}^{\infty}(v-\mathrm{s})\pi(v)\mathrm{d}v.$$

Now, define the function F by

$$F(\mathrm{s}) := \frac{\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}}{(\Phi(\mathrm{s}))^{\gamma}} < 0, \tag{29}$$

for $\mathrm{s} \geq \mathrm{s}_1$. Differentiating (29), we get

$$F'(\mathrm{s}) = \frac{(\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma})'}{(\Phi(\mathrm{s}))^{\gamma}} - \gamma\frac{\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\Phi'(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma+1}}.$$

It follows from (1) and (28) that

$$F'(\mathrm{s}) \leq -Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma}}\frac{\Phi^{\gamma}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} - \gamma F^{1+1/\gamma}(\mathrm{s})\int_{\mathrm{s}}^{\infty}\pi(v)\mathrm{d}v.$$

From Lemma 4, we obtain

$$-\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{(\Phi(\tau(\mathrm{s})))^{\gamma}} \geq -\frac{\pi_2^{\gamma\delta_n}(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma}},$$

hence

$$F'(\mathrm{s}) \leq -Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} - \gamma F^{1+1/\gamma}(\mathrm{s})\int_{\mathrm{s}}^{\infty}\pi(v)\mathrm{d}v.$$

By multiplying the above inequality by $\chi^{\gamma}(\mathrm{s})$ and integrating the resulting inequality from s_1 to s, we have

$$\begin{aligned}&\chi^{\gamma}(\mathrm{s})F(\mathrm{s}) - \chi^{\gamma}(\mathrm{s}_1)F(\mathrm{s}_1) - \gamma\int_{\mathrm{s}_1}^{\mathrm{s}}\chi'(v)\chi^{\gamma-1}(v)F(v)\mathrm{d}v \\ &\quad + \int_{\mathrm{s}_1}^{\mathrm{s}}Q(v)\frac{\pi_2^{\gamma\delta_n}(\tau(v))}{\pi_2^{\gamma\delta_n}(v)}\chi^{\gamma}(v)\mathrm{d}v + \gamma\int_{\mathrm{s}_1}^{\mathrm{s}}F^{1+1/\gamma}(v)\pi_1(v)\chi^{\gamma}(v)\mathrm{d}v \leq 0.\end{aligned}$$

By using the inequality (10) with $M = -\chi'(v)\chi^{\gamma-1}(v)$, $L = \pi_1(v)\chi^{\gamma}(v)$ and $z = -F(v)$, we obtain

$$\int_{\mathrm{s}_1}^{\mathrm{s}}\left[Q(v)\frac{\pi_2^{\gamma\delta_n}(\tau(v))}{\pi_2^{\gamma\delta_n}(v)}\chi^{\gamma}(v) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{(\chi'(v))^{\gamma+1}}{\chi(v)\pi_1^{\gamma}(v)}\right]\mathrm{d}v \leq \chi^{\gamma}(\mathrm{s}_1)F(\mathrm{s}_1) + 1.$$

By taking the lim sup on both sides of this inequality, we obtain a contradiction with (27). Now, the proof is complete. □

Theorem 4. *Let $\gamma \geq 1$. If there exists a function $\varkappa \in \mathbf{C}^1([\mathrm{s}_0,\infty),(0,\infty))$ such that*

$$\limsup_{\mathrm{s}\to\infty}\int_{\mathrm{s}_0}^{\mathrm{s}}\left[\Psi(v) - \frac{\varkappa(v)}{(\gamma+1)^{(\gamma+1)}\pi_1^{\gamma}(v)}\left(\frac{\varkappa'(v)}{\varkappa(v)} + \frac{(1+\gamma)\pi_1(v)}{\pi_2(v)}\right)^{\gamma+1}\right]\mathrm{d}v = \infty, \tag{30}$$

where

$$\Psi(\mathrm{s}) = \varkappa(\mathrm{s})Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} + (1-\gamma)\varkappa(\mathrm{s})\pi_1(\mathrm{s})/\pi_2^{\gamma+1}(\mathrm{s}),$$

then (1) has no positive decreasing solutions.

Proof. Assume that $\Phi(t)$ is a positive decreasing solution of (1). Since $\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}$ is nonincreasing, we get

$$\begin{aligned}\Phi''(v)-\Phi''(\mathrm{s}) &= \int_{\mathrm{s}}^{v}\frac{1}{\varrho^{1/\gamma}(\zeta)}\left(\varrho(\zeta)(\Phi'''(\zeta))^{\gamma}\right)^{1/\gamma}\mathrm{d}\zeta \\ &\leq \varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\int_{\mathrm{s}}^{v}\frac{1}{\varrho^{1/\gamma}(\zeta)}\mathrm{d}\zeta.\end{aligned}$$

Letting $v\to\infty$, we have

$$\Phi''(\mathrm{s})\geq-\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi(\mathrm{s}).$$

Integrating the above inequality from s to ∞ yields

$$-\Phi'(\mathrm{s})\geq-\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_1(\mathrm{s}). \tag{31}$$

Again, integrating (31) from s to ∞, we get

$$\Phi(\mathrm{s})\geq-\varrho^{1/\gamma}(\mathrm{s})\Phi'''(\mathrm{s})\pi_2(\mathrm{s}).$$

Now, define the function F_1 by

$$F_1(\mathrm{s})=\varkappa(\mathrm{s})\left(\frac{\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}}{(\Phi(\mathrm{s}))^{\gamma}}+\frac{1}{\pi_2^{\gamma}(\mathrm{s})}\right),\quad \mathrm{s}\geq\mathrm{s}_1. \tag{32}$$

Then, we see that $F_1(\mathrm{s})>0$ for $\mathrm{s}\geq\mathrm{s}_1$. Therefore, we have

$$\begin{aligned}F_1'(\mathrm{s}) &= \frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})}F_1(\mathrm{s})+\varkappa(\mathrm{s})\frac{\left(\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\right)'}{(\Phi(\mathrm{s}))^{\gamma}} \\ &\quad -\gamma\varkappa(\mathrm{s})\frac{\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\Phi'(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma+1}}-\gamma\varkappa(\mathrm{s})\frac{\pi_2'(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})}.\end{aligned}$$

It follows from (1) that

$$\begin{aligned}F_1'(\mathrm{s}) &= \frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})}F_1(\mathrm{s})-Q(\mathrm{s})\varkappa(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma}}\frac{\Phi^{\gamma}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} \\ &\quad -\gamma\varkappa(\mathrm{s})\frac{\varrho(\mathrm{s})(\Phi'''(\mathrm{s}))^{\gamma}\Phi'(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma+1}}-\gamma\varkappa(\mathrm{s})\frac{\pi_2'(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})}.\end{aligned}$$

From (31) and (32), we find

$$\begin{aligned}F_1'(\mathrm{s}) &\leq \frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})}F_1(\mathrm{s})-Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\mathrm{s})}{(\Phi(\mathrm{s}))^{\gamma}}\frac{\Phi^{\gamma}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} \\ &\quad -\gamma\varkappa(\mathrm{s})\pi_1(\mathrm{s})\left(\frac{F_1(\mathrm{s})}{\varkappa(\mathrm{s})}-\frac{1}{\pi_2^{\gamma}(\mathrm{s})}\right)^{1+1/\gamma}+\gamma\varkappa(\mathrm{s})\frac{\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})},\end{aligned}$$

hence

$$\begin{aligned}F_1'(\mathrm{s}) &\leq \frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})}F_1(\mathrm{s})-\varkappa(\mathrm{s})Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})}+\gamma\varkappa(\mathrm{s})\frac{\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})} \\ &\quad -\gamma\varkappa(\mathrm{s})\pi_1(\mathrm{s})\left(\frac{F_1(\mathrm{s})}{\varkappa(\mathrm{s})}-\frac{1}{\pi_2^{\gamma}(\mathrm{s})}\right)^{1+1/\gamma}.\end{aligned}$$

By the inequality (11) with $A = F_1(\mathrm{s})/\varkappa(\mathrm{s})$ and $B = 1/\pi_2^{\gamma}(\mathrm{s})$, we obtain

$$\begin{aligned} F_1'(\mathrm{s}) &\leq \frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})}F_1(\mathrm{s}) - \varkappa(\mathrm{s})Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} + \gamma\varkappa(\mathrm{s})\frac{\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})} \\ &\quad -\gamma\varkappa(\mathrm{s})\pi_1(\mathrm{s})\left\{\left(\frac{F_1(\mathrm{s})}{\varkappa(\mathrm{s})}\right)^{1+1/\gamma} - \frac{1}{\pi_2(\mathrm{s})}\left((1+\gamma)\frac{F_1(\mathrm{s})}{\varkappa(\mathrm{s})} - \frac{1}{\pi_2^{\gamma}(\mathrm{s})}\right)\right\}, \end{aligned}$$

hence

$$\begin{aligned} F_1'(\mathrm{s}) &\leq \left(\frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})} + \frac{(1+\gamma)\pi_1(\mathrm{s})}{\pi_2(\mathrm{s})}\right)F_1(\mathrm{s}) - \varkappa(\mathrm{s})Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} - \frac{\gamma\pi_1(\mathrm{s})}{\varkappa^{1/\gamma}(\mathrm{s})}F_1^{1+1/\gamma}(\mathrm{s}) \\ &\quad -\frac{\varkappa(\mathrm{s})\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})} + \frac{\gamma\varkappa(\mathrm{s})\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})}. \end{aligned}$$

Using the inequality (10) with $M = \varkappa'(\mathrm{s})/\varkappa(\mathrm{s}) + (1+\gamma)\pi_1(\mathrm{s})/\pi_2(\mathrm{s})$, $L = \gamma\pi_1(\mathrm{s})/\varkappa^{1/\gamma}(\mathrm{s})$ and $z = F_1(\mathrm{s})$, we obtain

$$\begin{aligned} F_1'(\mathrm{s}) &\leq -\varkappa(\mathrm{s})Q(\mathrm{s})\frac{\pi_2^{\gamma\delta_n}(\tau(\mathrm{s}))}{\pi_2^{\gamma\delta_n}(\mathrm{s})} + (\gamma-1)\frac{\varkappa(\mathrm{s})\pi_1(\mathrm{s})}{\pi_2^{\gamma+1}(\mathrm{s})} \\ &\quad + \frac{\varkappa(\mathrm{s})}{(\gamma+1)^{(\gamma+1)}\pi_1^{\gamma}(\mathrm{s})}\left(\frac{\varkappa'(\mathrm{s})}{\varkappa(\mathrm{s})} + \frac{(1+\gamma)\pi_1(\mathrm{s})}{\pi_2(\mathrm{s})}\right)^{\gamma+1}. \end{aligned}$$

Integrating the above inequality from s_1 to s, we have

$$\int_{\mathrm{s}_1}^{\mathrm{s}}\left[\Psi(v) - \frac{\varkappa(v)}{(\gamma+1)^{(\gamma+1)}\pi_1^{\gamma}(v)}\left(\frac{\varkappa'(v)}{\varkappa(v)} + \frac{(1+\gamma)\pi_1(v)}{\pi_2(v)}\right)^{\gamma+1}\right]\mathrm{d}v \leq F_1(\mathrm{s}_1).$$

By taking the lim sup on both sides of this inequality, we obtain a contradiction with (30). Now, the proof is complete. □

Theorem 5. *Suppose that $\omega \in \mathbf{C}((\mathrm{s}_0,\infty),(0,\infty))$ is a solution of (1). If the differential equation*

$$\Phi'(\mathrm{s}) + \frac{1}{\pi_2(\tau(\mathrm{s}))}\left(\int_{\mathrm{s}}^{\infty}\int_{\zeta}^{\infty}\frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{\mathrm{s}_1}^{v} q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v\mathrm{d}\zeta\right)\Phi(\tau(\mathrm{s})) = 0. \tag{33}$$

is oscillatory, then (1) has no positive decreasing solutions.

Proof. Assume that case (B_3) holds. From (1) and integrating from s_1 to s, we get

$$\varrho(\mathrm{s})\left(\Phi'''(\mathrm{s})\right)^{\gamma} \leq -\Phi^{\gamma}(\tau(\mathrm{s}))\int_{\mathrm{s}_1}^{\mathrm{s}} Q(v)\mathrm{d}v. \tag{34}$$

As in the proof of Lemma 3, we get that (15), (17) and (19) hold. By integrating (34) from s to ∞ and using (19), we obtain

$$-\Phi''(\mathrm{s}) \leq -\int_{\mathrm{s}}^{\infty}\frac{\Phi(\tau(v))}{\pi_2(\tau(v))}\frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{\mathrm{s}_1}^{v} Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v.$$

From Lemma 3, note that $\Phi(\mathrm{s})/\pi_2(\mathrm{s})$ is nondecreasing, which yields

$$-\Phi''(\mathrm{s}) \leq -\frac{\Phi(\tau(\mathrm{s}))}{\pi_2(\tau(\mathrm{s}))}\int_{\mathrm{s}}^{\infty}\frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{\mathrm{s}_1}^{v} Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v. \tag{35}$$

Integrating (35) from s to ∞, we find

$$\begin{aligned}\Phi'(s) &\leq -\int_s^\infty \frac{\Phi(\tau(\zeta))}{\pi_2(\tau(\zeta))}\int_\zeta^\infty \frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{s_1}^v Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v\mathrm{d}\zeta \\ &\leq -\frac{\Phi(\tau(s))}{\pi_2(\tau(s))}\int_s^\infty\int_\zeta^\infty \frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{s_1}^v Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v\mathrm{d}\zeta.\end{aligned}$$

It is obvious that Φ is a positive solution of the first-order delay differential inequality

$$\Phi'(s)+\frac{1}{\pi_2(\tau(s))}\left(\int_s^\infty\int_\zeta^\infty \frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{s_1}^v Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v\mathrm{d}\zeta\right)\Phi(\tau(s))\leq 0.$$

According to [25], (33) also has a positive solution, which is a contradiction. This completes the proof. □

Corollary 2. *Suppose that $\omega\in\mathbf{C}((s_0,\infty),(0,\infty))$ is a solution of (1). If*

$$\liminf_{s\to\infty}\int_{\tau(s)}^s \frac{1}{\pi_2(\tau(\varsigma))}\left(\int_\varsigma^\infty\int_\zeta^\infty \frac{\pi_2(\tau(v))}{\varrho^{1/\gamma}(v)}\left(\int_{s_1}^v Q(u)\mathrm{d}u\right)^{1/\gamma}\mathrm{d}v\mathrm{d}\zeta\right)\mathrm{d}\varsigma>\frac{1}{e}, \tag{36}$$

then (1) has no positive decreasing solutions.

Proof. Using [25], we observe that condition (36) guarantees the oscillation of (33). The proof is now finished. □

2.4. Oscillation Theorem

Here, we combine the known criteria in the literature that exclude cases (B_1) and (B_2) with the criteria we obtained that excludes case (B_3) to determine the oscillation of (1).

The proof of the case when (B_1) or (B_2) holds in the following theorems is identical to [12] [Theorem 2.1, Theorem 2.2]. Finally, conditions (27), (30) and (36), whichever of them excludes the case (B_3).

Theorem 6. *Assume that (27), (30) or (36) holds. If (8) and (7) hold for some $\lambda_1\in(0,1)$, then (1) oscillates.*

Example 1. *We consider*

$$\left(s^4(\omega(s)+p_0\omega(\lambda s))'''\right)'+q_0\omega(\xi s)=0,$$

where $\lambda,\xi\in(0,1)$, $p\in(0,\lambda)$ and $q_0>0$. It is easy to verify that $\pi_0(s)=1/3s_0^3$, $\pi_1(s)=1/6s^2$ and $\pi_2(s)=1/6s$. Note that $n=0$, $\delta_*=q_0/6$ and $\delta_0=\sqrt{q_0/6}$. By applying condition (6), we obtain

$$q_0>\frac{6}{4(1-p_0/\lambda)}. \tag{37}$$

Moreover, by applying condition (30), we obtain

$$q_0>\frac{6(\lambda)^{\sqrt{q_0/6}}}{4(1-p_0/\lambda)}. \tag{38}$$

In the special case where $\lambda=1/2$ and $p_0=1/4$, the conditions (37) and (38) become $q_0>3.1844$ and $q_0>2.0088$, respectively. As a result, our new results offer more accurate criteria for the exclusion of positive decreasing solutions.

3. Conclusions

To establish the oscillation criterion, the exclusion conditions for each case of the solution derivatives must be found separately. The criterion that frequently has the greatest impact on the oscillation test of the equation is the exclusion of positive decreasing solutions. In this study, we used the noncanonical case to examine the asymptotic properties of fourth-order differential equation solutions. We created new properties that helped us obtain more efficient terms for the oscillation of Equation (1). Then, using the results from Section 2.3 and known results, we created new criteria for the oscillation of the investigated equation. Finally, we offered a special case study to highlight the novelty and importance of our results.

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