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Lyapunov Inequalities for Two Dimensional Fractional Boundary-Value Problems with Mixed Fractional Derivatives

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Abstract: We consider two types of partial fractional differential equations in two dimensions with mixed fractional derivatives. Appropriate Lyapunov-type inequalities are proved, and applications to the certain eigenvalue problems are presented. Moreover, some connections with the fractional variational problems are highlighted.

Keywords: Lyapunov-type inequality; mixed fractional derivatives; fractional partial differential equations; eigenvalue problems

MSC: 35P15; 26D20; 26B99

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1. Introduction

The Lyapunov inequality provides a necessary condition for the existence of a non-trivial positive solution to the certain ordinary second-order differential equation. Since it was proved in 1907 by A. M. Lyapunov, it has been generalized in many ways and found to play a remarkable role in the analysis of differential equations- available results concern, e.g., bounds for eigenvalues [1], estimates for intervals of disconjugacy [2], or criteria for stability of periodic differential equations [3]. Most results, however, refer to the single time case. Results for multitime are scarce and are considered, e.g., in [4], where the partial differential equation involving Grushin operator was investigated, or in [5,6], where problems with Laplace and p-Laplace operators were studied, respectively. For more details of Lyapunov and other type inequalities, we refer to the papers [7–11].

In the references cited above, the authors consider integer order derivatives. Nevertheless, in recent years, a lot of papers studied Lyapunov inequalities for the boundary-value problems involving fractional differential operators [12]. In contrary to the classical approach, fractional derivatives are operators with memory, i.e., they are defined non-locally and because of that they model time-dependent processes more accurately [13–18]. The first work on Lyapunov inequality for fractional boundary-value problems has been written by R. Ferreira and concerns a problem with a derivative of order in the interval $(1,2]$ [19]. In this paper, however, we find a class of fractional differential equations with mixed right and left fractional derivatives to be more interesting [15,20]. Mixed fractional differential operators have a significant property, specifically, they are symmetric in agreement with the fractional integration by parts formulas and because of that they naturally arise in the theory of fractional calculus of variations [13,15,16,21]. The following theorems, concerning Lyapunov inequality for boundary-value problems with mixed fractional derivatives, can be found in the literature and will be used further in the work.

Theorem 1 (Theorem 4 in [22])**.** *If the following boundary-value problem:*

$$
{}^{c}D^{\alpha}_{b-}\left(D^{\beta}_{a+}v(t)\right)+f(t)v(t)=0,\ \ t\in(a,b),\ \alpha,\beta\in(0,1],\ \alpha+\beta\in(1,2],
$$
$$
v(a)=D^{\beta}_{a+}v(b)=0,
$$

where $f : [a,b] \to \mathbb{R}_+$, $f \in C[a,b]$ has a continuous solution, which is nontrivial, then

$$\int\limits_a^b |f(s)|ds \geq \frac{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}. \tag{1}$$

Theorem 2 ([23])**.** *Suppose that $\alpha \in \left(\frac{1}{2}, 1\right)$ and $v \in C^1[0,1]$ is such that $I_{0+}^{1-\alpha\,c}D_{1-}^{\alpha}v \in AC[0,1]$. If v is a nonzero solution to the following boundary-value problem:*

$$D_{0+}^{\alpha}\left({}^{c}D_{1-}^{\alpha}v(t)\right) - f(t)v(t) = 0, \;\; t \in (0,1),$$
$$v(0) = v(1) = 0,$$

where $f : [0,1] \to \mathbb{R}$ is continuous. Then,

$$\int\limits_0^1 |f(s)|ds > \frac{(2\alpha-1)\Gamma^2(\alpha)}{h}, \tag{2}$$

where

$$h = \sup_{0<x<1}\left[(1-x)^{2\alpha-1} - (1-x^{2\alpha-1})^2\right].$$

The interest of this article lies in the derivation of the Lyapunov-type inequalities for two different types of fractional partial differential equations involving mixed fractional derivatives. Like in [4,24], our method is based first on reducing the analysis of considered fractional partial differential equations to the study of fractional ordinary differential equations and next to applying above theorems in the proofs of the Lyapunov–type inequalities. Summing up, the contributions of this paper are as follows:

- We obtain the Lyapunov-type inequalities, which provide the necessary conditions for the existence of nonzero positive solutions. Thanks to this, we can indicate when the nontrivial positive solution to the problem does not exist.
- Mixed fractional derivatives are considered, and because of that we can establish a connection to the fractional calculus of variations.

The rest of the paper is organized as follows. In Section 2, we present preliminary definitions and properties of fractional calculus. Then, in Sections 3 and 4, we analyze two different types of problems involving mixed fractional derivatives—we prove Lyapunov-type inequalities in two dimensions and illustrate our results through some examples.

2. Preliminaries

In this section, we recall definitions and some elementary properties of the Riemann–Liouville and the Caputo fractional oprators. Throughout the work, we suppose that $a, b \in \mathbb{R}$, $a < b$ and by Γ we understand the Euler's gamma function.

Definition 1. *Let $\alpha \in \mathbb{R}$ ($\alpha > 0$) and $f \in L^1[a,b]$. The left Riemann–Liouville fractional integral I_{a+}^{α} of order α of function f is defined by*

$$I_{a+}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \;\; x \in (a,b],$$

while the right Riemann–Liouville fractional integral I_{b-}^{α} of order α ($\alpha > 0$) of function f is given by

$$I_{b-}^{\alpha}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(x)dt}{(t-x)^{1-\alpha}}, \;\; x \in [a,b).$$

With definitions of fractional integrals in hand, we are able to formulate the notions of the Riemann–Liouville and the Caputo fractional differential operators.

Definition 2. *Let $\alpha \in \mathbb{R}_+$ ($n-1 < \alpha \leq n$, $n \in \mathbb{N}$) and function $f \in L^1[a,b]$ be such that functions $I_{a+}^{n-\alpha} f$ and $I_{b-}^{n-\alpha} f$ are in $AC^n[a,b]$. The left Riemann–Liouville fractional derivative of order α of function f is given by*

$$\forall x \in (a,b], \;\; D_{a+}^{\alpha} f(x) := \left(\frac{d}{dx}\right)^n I_{a+}^{n-\alpha} f(x),$$

while the right Riemann–Liouville fractional derivative of order α of function f is defined by

$$\forall x \in [a,b), \;\; D_{b-}^{\alpha} f(x) := \left(-\frac{d}{dx}\right)^n I_{b-}^{n-\alpha} f(x).$$

Definition 3. *Suppose that $\alpha \in \mathbb{R}_+$ ($n-1 < \alpha \leq n, n \in \mathbb{N}$) and $f \in C^n[a,b]$. The left Caputo fractional derivative of order α of function f is given by*

$$\forall x \in (a,b], \;\; {}^cD_{a+}^{\alpha} f(x) := D_{a+}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k\right],$$

while the right Caputo fractional derivative of order α of function f is defined by

$$\forall x \in [a,b), \;\; {}^cD_{b-}^{\alpha} f(x) := D_{b-}^{\alpha}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(b-x)^k\right].$$

The next theorem presents a fractional counterpart of the integration by parts formula for the Riemann–Liouville-type and the Caputo-type differential operators.

Theorem 3 (cf. Lemma 2.19 [15])**.** *Let $\alpha \in (0,1)$, $f \in AC[a,b]$ and $g \in L^p[a,b]$, ($1 \leq p \leq \infty$). Then, the following integration by parts formulas are satisfied*

$$\int_a^b f(x) D_{a+}^{\alpha} g(x)\, dx = \int_a^b g(x) {}^cD_{b-}^{\alpha} f(x)\, dx + f(x) I_{a+}^{1-\alpha} g(x)\Big|_{x=a}^{x=b}, \tag{3}$$

$$\int_a^b f(x) D_{b-}^{\alpha} g(x)\, dx = \int_a^b g(x) {}^cD_{a+}^{\alpha} f(x)\, dx - f(x) I_{b-}^{1-\alpha} g(x)\Big|_{x=a}^{x=b}. \tag{4}$$

Remark 1. *In the Formula (3), through integration by parts, left-sided Riemann–Liouville fractional derivative is changed to the right-sided Caputo fractional derivative, while in the Formula (4) we do the opposite. Observe that, in both Formulas (3) and (4), we apply Riemmann–Liouville-type differentiation to the function g and the Caputo-type differentation to the function f. Assumptions on functions f and g correspond to the type of differentiation (not the side of the derivative) and because of that they can be the same for (3) and (4). In the book [15], only Formula (3) is given; however, the derivation of (4) is analogous.*

Note that definitions of the partial fractional integrals and derivatives, for functions of many variables, can be formulated in the similar manner when the order is integer. Precisely, we bring derivation of the partial fractional derivatives and integrals to the computation of a single-variable fractional operators (for more details see e.g., Section 24 of the book [17]).

3. Partial Differential Equation of the First Type

In this section, our goal is to prove the Lyapunov-type inequality for problems involving partial fractional derivatives. In contrast to the work [24], the derivative with respect to time is a composition of the right Caputo and the left Riemann–Liouville differential operators.

Suppose that $\alpha, \beta \in (0,1]$, $\alpha+\beta \in (1,2]$, $\gamma = \delta/2 \in (0,1]$, $\delta \in (0,2]$, $K \in \mathbb{R}_+$ and $w \in C[a,b]$. We are concerned with the following equation:

$$
\begin{aligned}
{}^cD^{\alpha}_{b-,t}\left(D^{\beta}_{a+,t}u(t,x)\right) - (1-x)^{\gamma}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K^cD^{\gamma}_{-1+,x}u(t,x)) \\
= w(t)u(t,x) \text{ for } (t,x) \in (a,b)\times(-1,1),
\end{aligned} \tag{5}
$$

under the boundary conditions

$$
u(t,-1) = 0, \quad I^{1-\gamma}_{1-,x}(K^cD^{\gamma}_{-1+,x}u(t,x))\Big|_{x=1} = 0, t \in (a,b), \tag{6}
$$

$$
u(a,x) = D^{\beta}_{a+,t}u(b,x) = 0, x \in (-1,1). \tag{7}
$$

By solution to the problem (5)–(7), we understand function $u \in C([a,b]\times[-1,1])$ satisfying conditions (5)–(7) such that the derivative $D^{\beta}_{a+,t}$ of u exists and is continuously differentiable for any $x \in [-1,1]$.

Lemma 1. *Let us consider the following boundary-value problem with mixed fractional derivatives:*

$$
{}^cD^{\alpha}_{b-}\left(D^{\beta}_{a+}v(t)\right) + f(t)v(t) = 0 \;\; for \;\; t \in (a,b),\; \alpha,\beta \in (0,1),\; \alpha+\beta \in (1,2], \tag{8}
$$

$$
v(a) = D^{\beta}_{a+}v(b) = 0, \tag{9}
$$

where

$$
f(t) = -\left(w(t) + \frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right), \;\; t \in [a,b]. \tag{10}
$$

If u is a positive solution to (5)–(7), which is not identically equal to zero, then function

$$
v(t) = \int_{-1}^{1} (1-x)^{-\gamma}u(t,x)\,dx, \;\; t \in [a,b] \tag{11}
$$

is a nonzero solution to the problem (8) and (9).

Proof. Let u be a positive solution to (5)–(7) such that $u \not\equiv 0$. If we multiply (5) by $y(x) = \dfrac{1}{(1-x)^{\gamma}}$ and integrate over (−1,1), then

$$
\begin{aligned}
\int_{-1}^{1}(1-x)^{-\gamma c}D^{\alpha}_{b-,t}\left(D^{\beta}_{a+,t}u(t,x)\right)dx - \int_{-1}^{1}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K^cD^{\gamma}_{-1+,x}u(t,x))\,dx \\
= \int_{-1}^{1}(1-x)^{-\gamma}w(t)u(t,x)\,dx, \;\; t \in (a,b).
\end{aligned}
$$

Since we integrate over x, we can exclude partial fractional derivatives with respect to t and obtain

$$
\begin{aligned}
{}^cD^{\alpha}_{b-,t}\left(D^{\beta}_{a+,t}\int_{-1}^{1}(1-x)^{-\gamma}u(t,x)\,dx\right) - \int_{-1}^{1}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K^cD^{\gamma}_{-1+,x}u(t,x))\,dx \\
= w(t)\int_{-1}^{1}(1-x)^{-\gamma}u(t,x)\,dx, \;\; t \in (a,b).
\end{aligned}
$$

Now, by applying (3) and (4) as well as the boundary conditions (6), we obtain

$$\int_{-1}^{1}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K^{c}D^{\gamma}_{-1+,x}u(t,x))\,dx=\int_{-1}^{1}D^{\gamma}_{1-,x}(K^{c}D^{\gamma}_{-1+,x}(1+x)^{\gamma})u(t,x)\,dx$$

Moreover, applying Theorem 3.4 from [20], because $y(x)=(1+x)^{\gamma}$ is an eigenfunction that corresponds to the eigenvalue $\lambda=\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}$, we obtain

$$\int_{-1}^{1}D^{\gamma}_{1-,x}(K^{c}D^{\gamma}_{-1+,x}(1+x)^{\gamma})u(t,x)\,dx=\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\int_{-1}^{1}(1-x)^{-\gamma}u(t,x)\,dx.$$

Therefore, for v being given by (11) and f being defined by (10), we have

$${}^{c}D^{\alpha}_{b-}\left(D^{\beta}_{a+}v(t)\right)+f(t)v(t)=0\ \text{ for } t\in(a,b),\ \alpha,\beta\in(0,1],\ \alpha+\beta\in(1,2].$$

Note that, because of the boundary conditions (7), we have $v(a)=v(b)=0$. Consequently, v is a solution to (8). Finally, since $u(t,x)>0$ for all $(t,x)\in(a,b)\times(-1,1)$ is a solution to (5)–(7) and because $(1-x)^{-\gamma}$ is positive for all $x\in(-1,1)$, we conclude that v is nonzero. □

Theorem 4. *If u is a positive solution to (5)–(7), which is not identically equal to zero, then the following Lyapunov-type inequality is satisfied*

$$\int_{a}^{b}\left|w(s)+\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\right|\,ds\geq\frac{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}.\tag{12}$$

Proof. If u is positive solution to (5)–(7), which is not identically equal to zero, then by Lemma 1, the function v given by (11) is a nontrivial solution to (8). Consequently, Theorem 1 implies (12). □

Example 1. *Now, let us analyze problem (5)–(7) with $a=0$, $b=1$, $K\in\mathbb{R}_{+}$, and $w(t)\equiv0$. Precisely, we consider the following homogeneous partial differential equation:*

$${}^{c}D^{\alpha}_{1-,t}\left(D^{\beta}_{0+,t}u(t,x)\right)-(1-x)^{\gamma}(1+x)^{\gamma}D^{\gamma}_{1-,x}(K^{c}D^{\gamma}_{-1+,x}u(t,x))=0,\ \ (t,x)\in(0,1)\times(-1,1),\tag{13}$$

with given boundary conditions

$$u(t,-1)=0,\ \ I^{1-\gamma}_{1-,x}(K^{c}D^{\gamma}_{-1+,x}u(t,x))\Big|_{x=1}=0\ \text{ for } t\in(0,1),\tag{14}$$

$$u(0,x)=D^{\beta}_{0+,t}u(1,x)=0\ \text{ for } x\in(-1,1).\tag{15}$$

Observe that, in this case, we deal with the finite-time fractional superdiffusion equation, where both the derivative with respect to time and the derivative with respect to space are compositions of the left-sided and the right-sided fractional derivatives.

The Lyapunov–type inequality for problem (13)–(15) is given by

$$\frac{K\Gamma(1+\gamma)}{\Gamma(1-\gamma)}\geq(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta).\tag{16}$$

Moreover, if we take $\alpha=\frac{1}{2}$, $\beta=\frac{3}{4}$ and $\gamma=\frac{1}{2}$, then (16) is satisfied if and only if $K>\frac{1}{2}\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)$, which means that for $K\leq\frac{1}{2}\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)$ nonzero positive solution to (5)–(7) does not exist.

4. Partial Differential Equation of the Second Type

This section is dedicated to the following partial differential equation with mixed partial fractional derivatives defined on the set $(0,1)\times(-1,1)$

$$D^{\alpha}_{0+,t}\left({}^{c}D^{\alpha}_{1-,t}u(t,x)\right)-(1-x)^{\beta}(1+x)^{\beta}D^{\beta}_{1-,x}(K{}^{c}D^{\beta}_{-1+,x}u(t,x))=w(t)u(t,x), \quad (17)$$

with boundary conditions

$$u(t,-1)=0,\quad I^{1-\beta}_{1-,x}(K{}^{c}D^{\beta}_{-1+,x}u(t,x))\Big|_{x=1}=0,\quad t\in(0,1), \quad (18)$$
$$u(0,x)=u(1,x)=0,\quad x\in(-1,1), \quad (19)$$

where $\alpha\in(\frac{1}{2},1)$, $\beta=\delta/2\in(0,1]$, $\delta\in(0,2]$, $K\in\mathbb{R}_{+}$, and $w\in C[0,1]$.

By solution to the problem (17)–(19) we mean function $u\in C^{1}([0,1]\times[-1,1])$ such that $I^{1-\alpha}_{0+}{}^{c}D^{\alpha}_{1-}u(\cdot,x)\in AC[0,1]$ for any $x\in[-1,1]$.

Note that, using a similar method as in Theorem 4 proved in the work [25], we can deduce that Equation (17) is an Euler–Lagrange equation associated with some fractional variational problem. Precisely, let us note that Euler–Lagrange equation for the problem of minimizing the functional

$$J(u)=\iint\limits_{(a,b)\times(c,d)}F(t,x,u(t,x),{}^{c}D^{\alpha}_{b-,t}u(t,x),{}^{c}D^{\beta}_{c+,x}u(t,x))\,dt\,dx,$$

where $F\in C^{1}([a,b]\times[c,d]\times\mathbb{R}^{3})$, $(t,x,u,v,z)\mapsto F(t,x,u,v,z)$, subject to the boundary conditions

$$u(t,c)=0,\quad I^{1-\beta}_{d-,x}\left(\frac{\partial F}{\partial z}\right)\Big|_{x=d}=0,\quad t\in(a,b),$$
$$u(a,x)=u(b,x)=0,\quad x\in(c,d),$$

is given by

$$\frac{\partial F}{\partial u}+D^{\alpha}_{a+,t}\left(\frac{\partial F}{\partial v}\right)-D^{\beta}_{d-,x}\left(\frac{\partial F}{\partial z}\right)=0. \quad (20)$$

In particular, for problem of minimizing the functional

$$J(u)=\iint\limits_{(0,1)\times(-1,1)}\left(\left({}^{c}D^{\alpha}_{1-,t}u(t,x)\right)^{2}+K\left({}^{c}D^{\beta}_{-1+,x}u(t,x)\right)^{2}-(1-x)^{-\beta}(1+x)^{-\beta}w(t)(u(t,x))^{2}\right)dt\,dx$$

subject to

$$u(t,-1)=0,\quad I^{1-\beta}_{d-,x}(K{}^{c}D^{\beta}_{c+,x}u(t,x))\Big|_{x=1}=0,\quad t\in(0,1),$$
$$u(1,x)=u(1,x)=0,\quad x\in(-1,1),$$

We have $F(t,x,u,v,z)=v^{2}+Kz^{2}-(1-x)^{-\beta}(1+x)^{-\beta}w(t)u^{2}$, and from Equation (20) we deduce (17).

Lemma 2. *Let us consider the following boundary-value problem with mixed fractional derivatives:*

$$D^{\alpha}_{0+}\left({}^{c}D^{\alpha}_{1-}v(t)\right)-f(t)v(t)=0,\quad t\in(0,1), \quad (21)$$
$$v(0)=v(1)=0, \quad (22)$$

where

$$f(t)=-\left(w(t)+\frac{K\Gamma(1+\beta)}{\Gamma(1-\beta)}\right),\quad t\in[0,1]. \quad (23)$$

If u is a positive solution to (17)–(19), such that $u \neq 0$, then function

$$v(t) = \int_{-1}^{1} (1-x)^{-\beta} u(t,x)\, dx, \quad t \in [0,1] \tag{24}$$

is a nontrivial solution to (21)–(22).

Proof. Let u be a positive solution to (17)–(19) such that $u \neq 0$. If we multiply (17) by $y(x) = (1-x)^{-\beta}$ and integrate with respect to x over $(-1,1)$, then we obtain

$$D^{\alpha}_{0+,t}\left({}^{c}D^{\alpha}_{1-,t} \int_{-1}^{1} (1-x)^{-\beta} u(t,x)\, dx \right) - \int_{-1}^{1} (1+x)^{\beta} D^{\beta}_{1-,x}(K {}^{c}D^{\beta}_{-1+,x} u(t,x))\, dx$$
$$= w(t) \int_{-1}^{1} (1-x)^{-\beta} u(t,x)\, dx,$$

for all $t \in (0,1)$. Following arguments analogous to the ones used in the proof of Lemma 1, for v being given by (24) and f being given by (23), applying integration by parts formula stated in Theorem 3, and boundary conditions (18) and Theorem 3.4 from [20], we have

$$D^{\alpha}_{0+}\left({}^{c}D^{\alpha}_{1-} v(t)\right) - f(t)v(t) = 0, \quad t \in (0,1).$$

In addition, bearing in mind boundary conditions (19), we have $v(a) = v(b) = 0$. Therefore, we deduce that v is a solution to (21)–(22). Finally, because u is a positive solution to (17)–(19) such that $u \neq 0$ and

$$(1-x)^{-\beta} > 0, \quad x \in (-1,1),$$

We see that v is nontrivial. $\square$

The following theorem provides the Lyapunov–type inequality to problem (17)–(19).

Theorem 5. *Let $\alpha \in (\frac{1}{2},1)$, $\beta = \delta/2 \in (0,1]$, $\delta \in (0,2]$ and w be continuous on $[0,1]$. If u is a positive solution to (17)–(19) such that $u \neq 0$, then*

$$\int_{0}^{1} |w(s) + \frac{K\Gamma(1+\beta)}{\Gamma(1-\beta)}| ds > \frac{(2\alpha-1)\Gamma^2(\alpha)}{h}, \tag{25}$$

where

$$h = \sup_{0<x<1}\left[(1-x)^{2\alpha-1} - (1-x^{2\alpha-1})^2\right].$$

Proof. Inequality (25) can be easily proved using Lemma 2 and Theorem 2. $\square$

Example 2. *In this example, we analyze (17)–(19) with $w(t) = \mu$, $\mu \in \mathbb{R}$, $K = 1$. Precisely, we study the following eigenvalue problem:*

$$D^{\alpha}_{0+,t}\left({}^{c}D^{\alpha}_{1-,t} u(t,x)\right) - (1-x)^{\beta}(1+x)^{\beta} D^{\beta}_{1-,x}\left({}^{c}D^{\beta}_{-1+,x} u(t,x)\right) = \mu u(t,x), \quad (t,x) \in (0,1)\times(-1,1), \tag{26}$$

$$u(t,-1) = 0, \quad I^{1-\beta}_{1-,x}\left({}^{c}D^{\beta}_{-1+,x} u(t,x)\right)\Big|_{x=1} = 0, \quad t \in (0,1), \tag{27}$$

$$u(0,x) = u(1,x) = 0, \quad x \in (-1,1). \tag{28}$$

Theorem 5 implies that, if $\mu \in \mathbb{R}$ is an eigenvalue of problem (26)–(28), then

$$\left|\mu + \frac{\Gamma(1+\beta)}{\Gamma(1-\beta)}\right| > \frac{(2\alpha-1)\Gamma^2(\alpha)}{h},$$

where

$$h = \sup_{0<x<1}\left[(1-x)^{2\alpha-1} - (1-x^{2\alpha-1})^2\right].$$

Note that, using similar method as in the proof of Theorem 3.4 from [26], one can deduce that Equation (26) is an Euler–Lagrange equation for the following fractional isoperimetric problem:

$$J(u) = \iint_{(0,1)\times(-1,1)} \left(\left({}^cD^{\alpha}_{1-,t}u(t,x)\right)^2 + \left({}^cD^{\beta}_{-1+,x}u(t,x)\right)^2\right) dt\,dx$$

subject to the boundary conditions

$$u(t,-1) = 0,\ \ I^{1-\beta}_{1-,x}({}^cD^{\beta}_{-1+,x}u(t,x))\Big|_{x=1} = 0,\ \ t \in (0,1),$$
$$u(0,x) = u(1,x) = 0,\ \ x \in (-1,1),$$

and an isoperimetric constraint

$$I(u) = \iint_{(0,1)\times(-1,1)} (u(t,x))^2(1-x)^{-\beta}(1+x)^{-\beta}\,dt\,dx = 1.$$

5. Conclusions

In this work, two types of partial differential equations involving mixed fractional derivatives are studied. We provide simple conditions (Lyapunov-type inequlities) to allow one to check whether these highly complicated nonlocal problems possess positive nontrivial solutions. Furthermore, contrary to the previous works, we link our boundary-value problems with the fractional calculus of variations theory. Our results are illustrated through two examples: in Example 1, we study the equation that could be interpreted as the fractional counterpart of the finite-time superdiffusion equation, while in Example 2, our results allow us to give bounds on eigenvalues for some eigenvalue problems. Note that, unfortunately, the Lyapunov-type inequalities are necessary conditions, which means that if they fail we can say that the solution does not exist, but if they are satisfied, the solution may or may not exist. This is an important issue, and in the forthcoming works we will study this problem.

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References

1. Ha, C.-W. Eigenvalues of a Sturm–Liouville problem and inequalities of Lyapunov-type. *Proc. AMS* **1998**, *126*, 3507–3511. [CrossRef]
2. Brown, R.C.; Hinton, D.B. Lyapunov inequalities and their applications. In *Survey on Mathematical Inequalities, Mathematics and Its Applications*; Rassias, T.M., Ed.; Springer: Berlin/Heidelberg, Germany, 2000; pp. 1–25.
3. Medriveci, A.F.; Guseinov, G.S.; Kaymakcalan, B. On Lyapunov inequality in stability theory for Hill's equation on time scales. *J. Inequal. Appl.* **2000**, 5, 603–620.
4. Jleli, M.; Kirane, M.; Samet, B. On Lyapunov–type inequalities for a certain class of partial differential equations. *Appl. Anal.* **2020**, *99*, 40–49. [CrossRef]

5. Canada, A.C.; Monteiro, J.A.; Villegas, S. Lyapunov inequalitites for partial differential equations. *J. Funct. Anal.* **2006**, *237*, 176–193. [CrossRef]
6. de Nápoli, P.L.; Pinasco, J.P. Lyapunov–type inequalities for partial differential equations. *J. Funct. Anal.* **2016**, *270*, 1995–2018. [CrossRef]
7. Agarwal, R.P.; Bohner, M.; Özbekler, A. *Lyapunov Inequalities and Applications*; Springer: Cham, Switzerland, 2021.
8. Agarwal, R.P.; Jleli, M.; Samet, B. On De La Vallée Poussin–type inequalities in higher dimension and applications. *Appl. Math. Lett.* **2018**, *86*, 264–269. [CrossRef]
9. Bohner, M.; Clark, S.; Ridenhour, J. Lyapunov inequalities for time scales. *J. Inequal. Appl.* **2002**, *7*, 61–67. [CrossRef]
10. De La Vallée Poussin, C. Sur l'équation différentielle linéqire du second order. Détermination d'une intégrale par deux valuers assignés. Extension aux équasions d'ordre n. *J. Math. Pures Appl.* **1929**, *8*, 125–144.
11. Saker, S.H.; Tunç, C.; Mahmoud, R.R. New Carlson–Bellman and Hardy–Littlewood dynamic inequalities. *Math. Inequal. Appl.* **2018**, *21*, 967–983. [CrossRef]
12. Ntouyas, S.K.; Ahmad, B.; Horikis, T.P. Recent developments of Lyapunov-type inequalities for fractional differential equations. In *Differential and Integral Inequalities*; Springer Optimization and Its Applications; Andrica, D., Rassias, T.M., Eds.; Springer: Berlin/Heidelberg, Germany, 2019; pp. 619–686.
13. Almeida, R.; Pooseh, S.; Torres, D.F.M. *Computational Methods in the Fractional Calculus of Variations*; Imperial College Press: Singapure, 2015.
14. Bohner, M.; Hristova, S. Stability for generalized Caputo proportional fractional delay integro–differential equations. *Bound. Value Probl.* **2022**, *14*, 1–15. [CrossRef]
15. Klimek, M. *On Solutions of Linear Fractional Differential Equations of a Variational Type*; The Publishing Office of Czestochowa University of Technology: Czestochowa, Poland, 2009.
16. Malinowska, A.B.; Odzijewicz, T.; Torres, D.F.M. *Advanced Methods in the Fractional Calculus of Variations*; Springer Briefs in Applied Sciences and Technology; Springer International Publishing: Cham, Switzerland, 2015.
17. Samko, S.G.; Kilbas A.A.; Marichev, Ȯ.I. *Fractional Integrals and Derivatives*; Translated from the 1987 Russian original; Gordon and Breach: Yverdon, Switzerland, 1993.
18. Tunç, O.; Tunxcx, C. Solution estimates to Caputo proportional fractional derivative delay integro—Differential equations. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **2023**, *12*, 117. [CrossRef]
19. Ferreira, R.A.C. A Lyapunov-type inequality for a fractional boundary-value problem. *Fract. Calc. Appl. Anal.* **2013**, *16*, 978–984. [CrossRef]
20. Zayernouri, M.; Karniadakis, G.E. Fractional Sturm–Liouville eigen–problems: Theory and numerical approximation. *J. Comput. Phys.* **2013**, *252*, 495–517. [CrossRef]
21. Klimek, M.; Odzijewicz, T.; Malinowska, A.B. Variational methods for the fractional Sturm–Liouville problem. *J. Math. Anal. Appl.* **2014**, *416*, 402–426. [CrossRef]
22. Guezane-Lakoud, A.; Khaldi, R.; Torres, D.F.M. Lyapunov–type inequality for a fractional boundary-value problem with natural conditions. *SeMA J.* **2018**, *75*, 157–162. [CrossRef]
23. Eneeva, L.M. Lyapunov inequality for an equation with fractional derivatives with different origins. *Vestnik KRAUNC. Fiz.-Mat. Nauki* **2019**, *28*, 32–39.
24. Odzijewicz, T. Inequality criteria for existence of solutions to some fractional partial differential equations. *Appl. Math. Lett.* **2020**, *101*, 106075. [CrossRef]
25. Odzijewicz, T.; Malinowska, A.B.; Torres, D.F.M. Fractional calculus of variations of several independent variables. *Eur. Phys. J.* **2013**, *222*, 1813–1826. [CrossRef]
26. Odzijewicz, T. *Variable o Fractional Isoperimetric Problem of Several Variables, Advances in the Theory and Applications of Non-integer Order Systems*; Springer: Berlin/Heidelberg, Germany, 2013; Volume 257, pp. 133–139.