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Logarithmic Coefficients for Some Classes Defined by Subordination

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Abstract: In this paper, we obtain the sharp and accurate bounds for the logarithmic coefficients of some subclasses of analytic functions defined and studied in earlier works. Furthermore, we obtain the bounds of the second Hankel determinant of logarithmic coefficients for a class defined by subordination, such as the class of starlike functions $\mathcal{S}^*(\varphi)$. Some applications of our results, which are extensions of those reported in earlier papers are given here as special cases. In addition, the results given can be used for other popular subclasses.

Keywords: Hankel determinant; logarithmic coefficients; univalent functions; α-spiral-like functions; subordination; Ma–Minda-type function

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1. Introduction

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n,\ z \in \mathbb{D}, \tag{1}$$

and let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{D}$.

For $q, n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, the *q-th Hankel determinant* of a function $f \in \mathcal{A}$ having the form (1) is defined by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (\text{with } a_1 := 1).$$

More results for Hankel determinants of any degree with their applications can be seen in [1,2], and we emphasize that these papers contain some of the first main results for the different studies in this direction. Moreover, many problems in this field have been argued by many authors (see, for example, [3–5]), such as the bounds of the Hankel determinant for strongly starlike functions of some orders and estimations of the Hankel determinant for a class of bi-close-to-convex functions and for a certain subclass of bi-univalent functions.

Using the principle of subordination, Ma and Minda [6] introduced the class $\mathcal{S}^*(\varphi)$, and we make here weaker assumptions on the function φ. They considered that φ is univalent in the unit disk $\mathbb{D}$, it has positive real in $\mathbb{D}$ and satisfies the condition $\varphi(0) = 1$.

They considered the above-mentioned class defined by

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

where the symbol "$\prec$" stands for the usual subordination. It is well-known that $\mathcal{S}^*(\varphi) \subset \mathcal{S}$, and we emphasize that some special subclasses of the class $\mathcal{S}^*(\varphi)$ play a significant role in geometric function theory due to many interesting geometric reasons.

For example, taking $\varphi(z) = (1 + Az)/(1 + Bz)$, where $A \in \mathbb{C}$, $-1 \leq B \leq 0$ and $A \neq B$, we obtain the classes $\mathcal{S}^*[A, B]$. These classes with the restriction $-1 \leq B < A \leq 1$ reduce to the popular *Janowski starlike* functions. Certainly, for $B = -1$ and $A = e^{i\alpha}\left(e^{i\alpha} - 2\beta\cos\alpha\right)$, where $\beta \in [0, 1)$ and $\alpha \in (-\pi/2, \pi/2)$, the class $\mathcal{S}^*[A, B]$ reduces to the well-known class of α-spiral-like functions of order β defined by

$$\mathcal{S}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\mathrm{e}^{-i\alpha} \frac{zf'(z)}{f(z)} \right) > \beta\cos\alpha,\ z \in \mathbb{D} \right\}.$$

The functions belonging to $\mathcal{S}_\alpha(\beta)$ are univalent in $\mathbb{D}$ (see [7]). Furthermore, $\mathcal{S}_\alpha(\beta) \subset \mathcal{S}_\alpha(0) \subset \mathcal{S}$ for $\beta \in [0, 1)$, where the members of $\mathcal{S}_\alpha(0)$ (see [8]) are called α-spiral-like functions; however, they do not necessarily belong to the class of starlike functions $\mathcal{S}^*$. Moreover, $\mathcal{S}_0(\beta) =: \mathcal{S}^*(\beta)$ is the usual class of starlike functions of order β, and $\mathcal{S}^* := \mathcal{S}^*(0)$.

Raina and Sokół [9] considered the class $\mathcal{S}^*_{\leftmoon} := \mathcal{S}^*(h)$, where

$$h(z) = z + \sqrt{1 + z^2} = 1 + \sum_{n=1}^{+\infty} B_n z^n = 1 + z + \frac{z^2}{2} - \frac{z^4}{8} + \ldots,\ z \in \mathbb{D}. \tag{2}$$

They proved that $f \in \mathcal{S}^*_{\leftmoon}$ if and only if $zf'(z)/f(z) \in \mathcal{R}$, where

$$\mathcal{R} := \left\{ z \in \mathbb{C} : \left| w^2 - 1 \right| < 2|w| \right\}.$$

Furthermore, the function

$$\begin{aligned} H_n(z) &= z\exp\left(\int_0^z \frac{h\left(t^{n-1}\right) - 1}{t}\,\mathrm{d}t \right) \\ &= z + \frac{z^n}{n-1} + \frac{(n+1)z^{2n-1}}{4(n-1)^2} + \ldots,\ z \in \mathbb{D},\ n \in \mathbb{N} \setminus \{1\}, \end{aligned}$$

plays the role of extremal functions for various problems in the class for the class $\mathcal{S}^*_{\leftmoon}$.

Lately, in [10], the authors studied the new Ma–Minda-type function class $\mathcal{ST}_L(s)$ defined by

$$\mathcal{ST}_L(s) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \mathbb{L}_s(z) := (1 + sz)^2,\ 0 < s \leq \frac{1}{\sqrt{2}} \right\}$$

and investigated some outcomes regarding the behavior of the functions of this class.

In [11], the authors investigated another Ma–Minda-type function class $\mathcal{S}^*_{\mathrm{Ne}}$ and obtained some characteristic properties of this class defined by

$$\mathcal{S}^*_{\mathrm{Ne}} =: \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi_{\mathrm{Ne}}(z) := 1 + z - \frac{z^3}{3} \right\}.$$

Further, we mention that the functions

$$\Phi_{s,n}(z) := z\exp\left(\int_0^z \frac{\mathbb{L}_s(t^n)-1}{t}\,\mathrm{d}t\right)$$
$$= z + \frac{2s}{n}z^{n+1} + \frac{(n+4)s^2}{2n^2}z^{2n+1} + \ldots,\ n = 1,2,\ldots,\ z \in \mathbb{D},$$

and

$$\Omega_n(z) := z\exp\left(\int_0^z \frac{\varphi_{\mathrm{Ne}}(t^{n-1})-1}{t}\,\mathrm{d}t\right)$$
$$= z + \frac{z^n}{n-1} + \frac{z^{2n-1}}{2(n-1)^2} + \ldots,\ n = 2,3,\ldots,\ z \in \mathbb{D},$$

play the role of an extremal functions for several problems for the categories $\mathcal{ST}_L(s)$ and $\mathcal{S}^*_{\mathrm{Ne}}$, respectively.

The logarithmic coefficients γ_n of the function $f \in \mathcal{S}$ are defined with the aid of the following power series expansion

$$F_f(z) := \log\frac{f(z)}{z} = 2\sum_{n=1}^{+\infty}\gamma_n(f)z^n,\ z \in \mathbb{D}, \quad \text{where} \quad \log 1 = 0. \tag{3}$$

These coefficients play an important role for different estimates in the theory of univalent functions, and we use γ_n instead of $\gamma_n(f)$; in this regard, see [12,13] and [14] (Chapter 2).

The logarithmic coefficients γ_n of an arbitrary function $f \in \mathcal{S}$ (see [15] (Theorem 4)) satisfy the inequality

$$\sum_{n=1}^{+\infty}|\gamma_n|^2 \le \frac{\pi^2}{6},$$

and the equality is obtained for the Koebe function. For $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \le 1/n$ holds but it is not true for the whole class $\mathcal{S}$, even in order of magnitude (see [16] (Theorem 8.4)). However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for $n \ge 3$ is presumably still a concern.

Some authors have recently investigated the issues respecting the logarithmic coefficients and related problems, including [17–26]. In [27], the researchers obtained the (non-sharp) estimates for the logarithmic coefficients for the functions belonging to $\mathcal{ST}_L(s)$ and $\mathcal{S}^*_{\mathrm{Ne}}$ as follows:

Theorem 1 ([27] (Theorem 5))**.** *If the function $f \in \mathcal{ST}_L(s)$, then*

$$|\gamma_n| \le s,\ n \in \mathbb{N}.$$

This inequality is sharp for $n = 1$ for the function $\Phi_{s,1}$.

Theorem 2 ([27] (Theorem 6))**.** *If the function $f \in \mathcal{S}^*_{\mathrm{Ne}}$, then*

$$|\gamma_n| \le \frac{1}{2},\ n \in \mathbb{N}.$$

This inequality is sharp for $n = 1$ for the function Ω_2.

If f is given by (1), then, by equating the coefficients of z^n in (3) for $n = 1,2,3$, it follows that

$$2\gamma_1 = a_2, \quad 2\gamma_2 = a_3 - \frac{1}{2}a_2^2, \quad 2\gamma_3 = a_4 - a_2a_3 + \frac{1}{3}a_2^3. \tag{4}$$

The importance of the studies related to the logarithmic coefficients consists also in the fact that many studies have attempted to develop new and interesting methods for investigation that helped the researchers to obtain significant results in geometric function theory (GFT). Thus, in recent years, many studies related to logarithmic coefficients have been connected with bounded turning functions whose derivative is subordinated to the "three-leaf-shaped" function [28], to "with petal-shaped function" [29] or to starlike functions associated with the sine function [30] and other related topics.

The importance of these studies in GFT, among others, consist of the fact that the images of some usual expressions (such as the derivatives and starlikeness fraction) are subordinated to functions with a well-known range of the open unit disk, while the methods of the proofs are very elaborated. Further, due to the significant importance of the study of the logarithmic coefficients, in recent years, the problem of obtaining the sharp bounds of the second Hankel determinant of the logarithmic coefficients—that is, $H_{2,1}(F_f/2)$—was reported in the papers [31–33] for some subclasses of analytic functions, where the second Hankel determinant for $F_f/2$, by using the relations (4), is

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{5}$$

Note that $H_{2,1}(F_f/2)$ is invariant under rotations (see [32]).

The main purpose of this paper is to obtain the sharp bounds for the logarithmic coefficients of the functions belonging to $\mathcal{ST}_L(s)$ and $\mathcal{S}^*_{\mathrm{Ne}}$ and to find an upper bound of $H_{2,1}(F_f/2)$ for the subclass $\mathcal{S}^*(\varphi)$ of the starlike functions. The applications of our results, which are extensions of those reported in earlier papers, are given here as special results.

2. Main Results

First, we obtain the sharp bounds of the logarithmic coefficients for the functions belonging to the classes $\mathcal{ST}_L(s)$ and $\mathcal{S}^*_{\mathrm{Ne}}$, which are improvements of the results reported in Theorems A and B.

The first main result of the paper gives us the sharp bounds for the modules of the logarithmic coefficients for the function classes $\mathcal{ST}_L(s)$ as follows:

Theorem 3. *If the function $f \in \mathcal{ST}_L(s)$ with $0 < s \le \dfrac{1}{\sqrt{2}}$, then*

$$|\gamma_n| \le \frac{s}{n},\ n \in \mathbb{N}.$$

For a fixed $n_0 \in \mathbb{N}$, the above inequality is sharp for n_0 if $f = \Phi_{s,n_0}$, that is $|\gamma_{n_0}| = \dfrac{s}{n_0}$ for $f = \Phi_{s,n_0}$.

Proof. Supposing that $f \in \mathcal{ST}_L(s)$, then, by the definition of $\mathcal{ST}_L(s)$, it follows that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec 2sz + s^2z^2,$$

which, regarding to the logarithmic coefficients γ_n of f given by (3), leads to

$$\sum_{n=1}^{+\infty} 2n\gamma_n z^n \prec 2sz + s^2z^2.$$

For $k \ge 1$, the sequence $A_1 = 2s$, $A_2 = s^2$ and $A_k = 0$ for $k \ge 3$ is non-negative, non-increasing and convex; thus, $A_k - A_{k+1} \ge 0$, and $A_k - 2A_{k+1} + A_{k+2} \ge 0$ for $k \ge 1$. Therefore, from [34] (Theorem VII (i)), we find

$$2n|\gamma_n| \le A_1 = 2s,\ n \in \mathbb{N},$$

and our result is proven.

Since, for a fixed $n_0 \in \mathbb{N}$, we have

$$\log \frac{\Phi_{s,n_0}(z)}{z} = 2\sum_{k=1}^{+\infty} \gamma_k(\Phi_{s,n_0})z^k = \frac{2s}{n_0}z^{n_0} + \ldots, \; z \in \mathbb{D},$$

it follows that the bound is sharp for n_0 if $f = \Phi_{s,n_0}$, that is $|\gamma_{n_0}| = \dfrac{s}{n_0}$ for $f = \Phi_{s,n_0}$. $\square$

In the next theorem, we give the sharp bounds for the modules of the logarithmic coefficients for the function classes $\mathcal{S}^*_{\mathrm{Ne}}$:

Theorem 4. *If the function $f \in \mathcal{S}^*_{\mathrm{Ne}}$, then*

$$|\gamma_n| \leq \frac{1}{2n}, \; n \in \mathbb{N}.$$

For a fixed $n_0 \in \mathbb{N}$, the above inequality is sharp for n_0 if $f = \Omega_{n_0+1}$, that is $|\gamma_{n_0}| = \dfrac{s}{2n_0}$ for $f = \Omega_{n_0+1}$.

Proof. If $f \in \mathcal{S}^*_{\mathrm{Ne}}$, from the definition of $\mathcal{S}^*_{\mathrm{Ne}}$, it follows that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log \frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec z - \frac{z^3}{3},$$

and using the logarithmic coefficients γ_n of f given by (3), we find

$$\sum_{n=1}^{+\infty} 2n\gamma_n z^n \prec z - \frac{z^3}{3}.$$

Now, we set in [34] (Theorem VI (i)) the sequence $A_1 = 1$, $A_2 = 0$, $A_3 = -\dfrac{1}{3}$, $A_n = 0$ for all $n \geq 4$, and $B_k = 0$ for all $k \geq n+1$, and then the function F_1 given by [34] ((1.10.1) p. 62) becomes

$$F_1(z) = \frac{1}{2} - \frac{z^2}{3}.$$

Since $A_1 = 1 > 0$, the function F_1 is analytic in $\mathbb{D}$ and satisfies $\operatorname{Re} F_1(z) > 0$, $z \in \mathbb{D}$; hence, all the assumptions of [34] (Theorem VI (i)) are satisfied. Therefore, we find

$$2n|\gamma_n| \leq A_1 = 1, \; n \in \mathbb{N},$$

which represents our result.

Since, for a fixed $n_0 \in \mathbb{N}$, we have

$$\log \frac{\Omega_{n_0+1}(z)}{z} = 2\sum_{k=1}^{+\infty} \gamma_k(\Omega_{n_0+1})z^k = \frac{1}{n_0}z^{n_0} + \ldots, \; z \in \mathbb{D},$$

it follows that the bound is sharp for n_0 if $f = \Omega_{n_0+1}$, that is $|\gamma_{n_0}| = \dfrac{1}{2n_0}$ for $f = \Omega_{n_0+1}$. $\square$

In order to obtain the upper bound of $H_{2,1}(F_f/2)$ for the class of starlike functions $\mathcal{S}^*(\varphi)$, we need the following definition and lemma.

Let Ω represent the category of all analytic functions ψ in $\mathbb{D}$ that satisfy the requirements $\psi(0) = 0$ and $|\psi(z)| < 1$ for all $z \in \mathbb{D}$.

Lemma 1 ([35] (Lemma 2.1))**.** *If $\psi(z)=\sum\limits_{n=1}^{+\infty}\psi_n z^n\in\Omega$, then*

$$\psi_2=x\left(1-\psi_1^2\right),$$
$$\psi_3=\left(1-\psi_1^2\right)\left(1-|x|^2\right)s-\psi_1\left(1-\psi_1^2\right)x^2,$$

for some x, s, with $|x|\leq 1$ and $|s|\leq 1$.

The following theorem represents our result related to the upper bound of $H_{2,1}(F_f/2)$ for the class $\mathcal{S}^*(\varphi)$:

Theorem 5. *If the function f belongs to the class $\mathcal{S}^*(\varphi)$, then*

$$\left|H_{2,1}(F_f/2)\right|\leq\frac{|B_1|}{48}\cdot\begin{cases}\dfrac{4PR-Q^2}{4P}, & \text{if}\quad P<0 \text{ and } P\leq-\dfrac{Q}{2}\leq 0,\\ \max\{R;P+Q+R\}, & \text{otherwise,}\end{cases}$$

where

$$P:=\left|4B_3-\frac{3B_2^2}{B_1}\right|-2|B_2|-|B_1|,\ Q:=2(|B_2|-|B_1|),\ R:=3|B_1|. \tag{6}$$

Proof. If $f\in\mathcal{S}^*(\varphi)$, then by the definition of the subordination there exists a function $\omega\in\Omega$, with $\omega(z)=\sum\limits_{n=1}^{+\infty}c_n z^n$, $z\in\mathbb{D}$, such that

$$\begin{aligned}\frac{zf'(z)}{f(z)}&=\varphi(\omega(z)) \\ &=1+B_1c_1z+\left(B_1c_2+B_2c_1^2\right)z^2+\left(B_1c_3+2c_1c_2B_2+B_3c_1^3\right)z^3+\ldots,\ z\in\mathbb{D},\end{aligned}\tag{7}$$

where $\varphi(z)=1+\sum\limits_{n=1}^{+\infty}B_n z^n$, $z\in\mathbb{D}$. Furthermore, since φ is univalent in $\mathbb{D}$, it follows that $B_1=\varphi'(0)\neq 0$.

The function $f\in\mathcal{S}^*(\varphi)$ has the power expansion series of the form (1), and by equating the coefficients of z^n in (7) for $n=1,2,3$, we obtain

$$\begin{cases}a_2=B_1c_1,\\ 2a_3-a_2^2=B_1c_2+B_2c_1^2,\\ 3a_4-3a_2a_3+a_2^3=B_1c_3+2c_1c_2B_2+B_3c_1^3,\end{cases}$$

and from these relations, it follows that

$$\begin{cases}a_2=B_1c_1,\\ 2a_3=B_1c_2+(B_2+B_1^2)c_1^2,\\ 3a_4=B_1\left[c_3+\left(\dfrac{3}{2}B_1+\dfrac{2B_2}{B_1}\right)c_1c_2+\left(\dfrac{3}{2}B_2+\dfrac{1}{2}B_1^2+\dfrac{B_3}{B_1}\right)c_1^3\right].\end{cases}$$

Therefore, after replacing the above values in (5), we obtain

$$\left|H_{2,1}(F_f/2)\right|=\frac{1}{4}\left|\left(\frac{B_1B_3}{3}-\frac{B_2^2}{4}\right)c_1^4+\frac{B_1B_2}{6}c_1^2c_2+\frac{B_1^2}{3}c_1c_3-\frac{B_1^2}{4}c_2^2\right|.$$

Using Lemma 1 for some x, s, with $|x|\leq 1$ and $|s|\leq 1$, from the above relation, it follows that

$$\begin{aligned}\left|H_{2,1}(F_f/2)\right| =&\frac{B_1}{4}\left|\left(\frac{4B_3-\frac{3B_2^2}{B_1}}{12}\right)c_1^4+\frac{B_2}{6}xc_1^2\left(1-c_1^2\right)\right.\\&\left.+\left(-\frac{B_1(1-c_1^2)}{4}-\frac{B_1}{3}c_1^2\right)x^2\left(1-c_1^2\right)+\frac{B_1}{3}c_1\left(1-c_1^2\right)\left(1-|x|^2\right)s\right|.\end{aligned}$$

Since it is well-known that $|c_1|\leq 1$ and as $H_{2,1}(F_f/2)$ and $\omega(z)$ are invariant under the rotations (see also [32]); therefore, to simplify the calculation, we may assume that $c:=c_1\in[0,1]$ (similar to the proof of [36] (Theorem 3. p. 80)). Hence, we have

$$\begin{aligned}\left|H_{2,1}(F_f/2)\right| =&\frac{|B_1|}{4}\left|\left(\frac{4B_3-\frac{3B_2^2}{B_1}}{12}\right)c^4+\frac{B_2}{6}xc^2\left(1-c^2\right)\right.\\&\left.+\left(-\frac{B_1(1-c^2)}{4}-\frac{B_1}{3}c^2\right)x^2\left(1-c^2\right)+\frac{B_1}{3}c_1\left(1-c_1^2\right)\left(1-|x|^2\right)s\right|\\&\leq\frac{|B_1|}{4}\left[\left|\frac{4B_3-\frac{3B_2^2}{B_1}}{12}\right|c^4+\frac{|B_2|}{6}|x|c^2\left(1-c^2\right)\right.\\&\left.+\left(\frac{|B_1|(1-c^2)}{4}+\frac{|B_1|}{3}c^2\right)|x|^2\left(1-c^2\right)+\frac{|B_1|}{3}c\left(1-c^2\right)\left(1-|x|^2\right)\right]\\&=\frac{|B_1|}{48}\left[\left|4B_3-\frac{3B_2^2}{B_1}\right|c^4+2|B_2|c^2\left(1-c^2\right)|x|\right.\\&\left.+|B_1|(1-c)(3-c)\left(1-c^2\right)|x|^2+4|B_1|c\left(1-c^2\right)\right]=:F_c(\lambda),\end{aligned}$$

where $\lambda:=|x|\in[0,1]$. A simple study shows that F_c is an increasing function of λ, and so it attains its maximum at $\lambda=1$, which is

$$\max\{F_c(\lambda):\lambda\in[0,1]\}=F_c(1)=:G(c),$$

where

$$G(c)=\frac{|B_1|}{48}\left[\left(\left|4B_3-\frac{3B_2^2}{B_1}\right|-2|B_2|-|B_1|\right)c^4+2(|B_2|-|B_1|)c^2+3|B_1|\right].$$

For simplicity, if we denote $u:=c^2\in[0,1]$ and set the values of P, Q and R, such as in (6), then

$$G(u)=\frac{|B_1|}{48}\left(Pu^2+Qu+R\right),\ u\in[0,1].$$

It is easy to show that

$$\max\left\{Pu^2+Qu+R:u\in[0,1]\right\}=\begin{cases}\dfrac{4PR-Q^2}{4P}, & \text{if}\quad P<0 \text{ and } P\leq-\dfrac{Q}{2}\leq 0,\\ \max\{R;P+Q+R\}, & \text{otherwise;}\end{cases}$$

hence, we conclude

$$\left|H_{2,1}(F_f/2)\right|\leq\frac{|B_1|}{48}\cdot\begin{cases}\dfrac{4PR-Q^2}{4P}, & \text{if}\quad P<0 \text{ and } P\leq-\dfrac{Q}{2}\leq 0,\\ \max\{R;P+Q+R\}, & \text{otherwise,}\end{cases}$$

where P, Q and R are given by (6). $\square$

Next, we emphasize some special cases of the previous main result by specializing the function φ with a few ones widely used in many other previous papers.

As a particular case, if we take, in Theorem 5, the function

$$\begin{aligned}\varphi(z) &:= \frac{1+e^{i\alpha}(e^{i\alpha}-2\beta\cos\alpha)z}{1-z}\\ &= 1+2z(1-\beta)e^{i\alpha}\cos\alpha+2z^2(1-\beta)e^{i\alpha}\cos\alpha+2z^3(1-\beta)e^{i\alpha}\cos\alpha+\ldots,\ z\in\mathbb{D},\end{aligned}$$

where $\beta\in[0,1)$ and $\alpha\in(-\pi/2,\pi/2)$, since $P=-4(1-\beta)\cos\alpha<0$ and $P\le-\dfrac{Q}{2}=0\le0$ from the first part of the theorem, then we obtain the following corollary:

Corollary 1 ([33] (Theorem 3.1))**.** *If the function f belongs to the class $\mathcal{S}_\alpha(\beta)$, then*

$$\left|H_{2,1}(F_f/2)\right|\le\frac{(1-\beta)^2\cos^2\alpha}{4}.$$

Equality holds for the rotation of the function

$$\begin{aligned}f_*(z) &= \frac{z}{(1-z^2)^{(1-\beta)e^{i\alpha}\cos\alpha}}\\ &= z+(1-\beta)z^3e^{i\alpha}\cos\alpha+\frac{(1-\beta)e^{i\alpha}\cos\alpha\left[1+(1-\beta)e^{i\alpha}\cos\alpha\right]}{2}z^5+\ldots,\ z\in\mathbb{D}.\end{aligned}$$

Proof. To prove the second part of the corollary, a simple computation shows that

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf_*'(z)}{f_*(z)}\right)=\left[1+2(1-\beta)\operatorname{Re}\frac{z^2}{1-z^2}\right]\cos\alpha,\ z\in\mathbb{D},$$

and because

$$\operatorname{Re}\frac{z^2}{1-z^2}>-\frac{1}{2},\ z\in\mathbb{D},$$

it follows that

$$\operatorname{Re}\left(e^{-i\alpha}\frac{zf_*'(z)}{f_*(z)}\right)>\beta\cos\alpha,\ z\in\mathbb{D},$$

that is $f_*\in\mathcal{S}_\alpha(\beta)$.

It is easy to check that $f_*(z)=z+a_2z^2+a_3z^3+\ldots,\ z\in\mathbb{D}$, with $a_2=a_4=0$ and $a_3=(1-\beta)e^{i\alpha}\cos\alpha$; hence,

$$|H_{2,1}(F_{f_*}/2)|=\left|\frac{1}{4}\left(a_2a_4-a_3^2+\frac{1}{12}a_2^4\right)\right|=\frac{1}{4}\left|a_3^2\right|=\frac{(1-\beta)^2\cos^2\alpha}{4},$$

which shows that our estimation is sharp. $\square$

Remark 1. *For $\alpha=0$, Corollary 1 reduces to Theorem 2.1 from [31]. Moreover, taking $\alpha=\beta=0$ in Corollary 1, we obtain Theorem 2.1 from [32].*

Another special case can be obtained by taking, in Theorem 5, the function

$$\varphi_\alpha(z):=\left(\frac{1+z}{1-z}\right)^\alpha=1+2\alpha z+2\alpha^2z^2+\left(\frac{4}{3}\alpha^3+\frac{2}{3}\alpha\right)z^3+\ldots,\ z\in\mathbb{D},\ 0<\alpha\le1,$$

because $-\dfrac{Q}{2}\not\le0$ for $\alpha\in(0,1)$ and $P=-6\le-\dfrac{Q}{2}=0\le0$ for $\alpha=1$; thus, we obtain the following corollary:

Corollary 2. *Let us denote by,*

$$\widetilde{\mathcal{S}}^*(\alpha) := \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2},\ z \in \mathbb{D} \right\},\ 0 < \alpha \leq 1,$$

the class of strongly starlike functions of order α. *If the function* $f \in \widetilde{\mathcal{S}}^*(\alpha)$, *then*

$$\left| H_{2,1}(F_f/2) \right| \leq \frac{\alpha^2}{4},$$

while the equality holds for the function $f_\alpha \in \mathcal{A}$ *given by*

$$\frac{zf'_\alpha(z)}{f_\alpha(z)} = \left(\frac{1+z^2}{1-z^2} \right)^\alpha. \tag{8}$$

Proof. To prove the second part of our result, we easily see that there exists a function $f_\alpha \in \mathcal{A}$ that satisfies the differential Equation (8), and it is given by

$$f_\alpha(z) = z \exp\left(\int_0^z \frac{\varphi_\alpha(t^2) - 1}{t}\, \mathrm{d}t \right).$$

On the other hand, since

$$\left(\frac{1+z^2}{1-z^2} \right)^\alpha = 1 + 2\alpha z^2 + 2\alpha^2 z^4 + \ldots,\ z \in \mathbb{D},$$

from (8), we have

$$\sum_{n=1}^{+\infty} 2n\gamma_n z^n = z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log \frac{f_\alpha(z)}{z} \right) = \frac{zf'_\alpha(z)}{f_\alpha(z)} - 1 = 2\alpha z^2 + 2\alpha^2 z^4 + \ldots,\ z \in \mathbb{D}.$$

Therefore, $2\gamma_1 = 0$, $4\gamma_2 = 2\alpha$, and $6\gamma_3 = 0$; thus, $\gamma_1 = \gamma_3 = 0$, and $\gamma_2 = \frac{\alpha}{2}$. Consequently,

$$\left| H_{2,1}(F_{f_\alpha}/2) \right| = \left| \gamma_1\gamma_3 - \gamma_2^2 \right| = \left| \gamma_2^2 \right| = \frac{\alpha^2}{4};$$

hence, the estimation is sharp. □

Setting, in Theorem 5, the function

$$\varphi(z) := G(z) = \frac{2}{1+\mathrm{e}^{-z}} = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 + \ldots,\ z \in \mathbb{D},$$

because $-\frac{Q}{2} = \frac{1}{2} \nleq 0$, and $P + Q < 0$, we obtain the following result:

Corollary 3. *Let* $\mathcal{S}^*_{SG}$ *be the class defined in [37] by*

$$\mathcal{S}^*_{SG} =: \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec G(z) := \frac{2}{1+e^{-z}} \right\}.$$

If $f \in \mathcal{S}^*_{SG}$, *then*

$$\left| H_{2,1}(F_f/2) \right| \leq \frac{1}{64},$$

and the equality holds for the function

$$\Omega_3(z) = z \exp\left(\int_0^z \frac{G(t^2) - 1}{t}\, \mathrm{d}t \right) = z + \frac{z^3}{4} + \frac{z^5}{32} + \ldots\ z \in \mathbb{D}.$$

Proof. In order to prove the second part of this corollary, since the function $\Omega_3(z) = z + a_2 z^2 + a_3 z^3 + \ldots, z \in \mathbb{D}$ belongs to the class $\mathcal{S}^*_{SG}$ (see [38]) with $a_2 = a_4 = 0$ and $a_3 = \dfrac{1}{4}$, it follows that

$$\left|H_{2,1}(F_{\Omega_3}/2)\right| = \left|\frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right)\right| = \frac{1}{4}\left|a_3^2\right| = \frac{1}{64},$$

and the sharpness of our estimation is proven. □

If we take, in Theorem 5, the function

$$\begin{aligned}\varphi(z) := \mathfrak{q}_s(z) &= \frac{1}{(1-z)^s} = e^{s\log(1-z)} \\ &= 1 + sz + \frac{s(s+1)}{2}z^2 + \frac{s(s+1)(s+2)}{6}z^3 + \ldots, \; z \in \mathbb{D},\end{aligned}$$

because $-\dfrac{Q}{2} = \dfrac{s(1-s)}{2} \not\leq 0$ for $s \in (0,1)$ with $P + Q + R < R$ and $P = -2 \leq -\dfrac{Q}{2} = 0 \leq 0$ for $s = 1$, we obtain the following special case:

Corollary 4. *Let $\mathcal{ST}_{hpl}(s)$ be the class defined in [39] by*

$$\mathcal{ST}_{hpl}(s) =: \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \mathfrak{q}_s(z) := \frac{1}{(1-z)^s}, \; 0 < s \leq 1 \right\}.$$

If the function $f \in \mathcal{ST}_{hpl}(s)$, then

$$\left|H_{2,1}(F_f/2)\right| \leq \frac{s^2}{16},$$

and equality holds for the function

$$\Phi_{s,2}(z) = z\exp\left(\int_0^z \frac{\mathfrak{q}_s(t^2) - 1}{t}\,\mathrm{d}t\right) = z + \frac{s}{2}z^3 + \frac{4s^2 + 2s}{16}z^5 + \ldots, \; z \in \mathbb{D}.$$

Proof. The sharpness of our result follows easily because the function $\Phi_{s,2}(z) = z + a_2 z^2 + a_3 z^3 + \ldots, z \in \mathbb{D}$ belongs to $\mathcal{ST}_{hpl}(s)$ (see [38]) with $a_2 = a_4 = 0$ and $a_3 = \dfrac{s}{2}$. Hence, we obtain

$$\left|H_{2,1}(F_{\Phi_{s,2}}/2)\right| = \left|\frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right)\right| = \frac{1}{4}\left|a_3^2\right| = \frac{s^2}{16},$$

and consequently the given estimation is sharp. □

Considering, in Theorem 5, the particular case

$$\varphi(z) := h(z) = z + \sqrt{1 + z^2} = 1 + z + \frac{z^2}{2} - \frac{z^4}{8} + \ldots, \; z \in \mathbb{D},$$

because $-\dfrac{Q}{2} = \dfrac{1}{2} \not\leq 0$ with $P + Q < 0$, we obtain the following outcome:

Corollary 5. *If the function $f \in \mathcal{S}^*_{\mathfrak{C}}$, then*

$$\left|H_{2,1}(F_f/2)\right| \leq \frac{1}{16},$$

and the equality holds for the function

$$H_3(z) = z\exp\left(\int_0^z \frac{h(t^2)-1}{t}\,\mathrm{d}t\right) = z + \frac{z^3}{2} + \frac{z^5}{4} + \dots,\ z \in \mathbb{D}.$$

Proof. The second part can be easily proven because the function $H_3(z) = z + a_2 z^2 + a_3 z^3 + \dots$, $z \in \mathbb{D}$ belongs to the class $\mathcal{S}^*_{\mathbb{C}}$ (see [38]) with $a_2 = a_4 = 0$ and $a_3 = \frac{1}{2}$. Therefore,

$$\left|H_{2,1}(F_{H_3}/2)\right| = \left|\frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right)\right| = \frac{1}{4}\left|a_3^2\right| = \frac{1}{16}.$$

Hence, the estimation is sharp. □

If we set, in Theorem 5, the function

$$\varphi(z) := \varphi_{\mathrm{Ne}}(z) = 1 + z - \frac{z^3}{3},\ z \in \mathbb{D},$$

because $-\frac{Q}{2} = 1 \not\leq 0$ with $P + Q < 0$, we obtain the following particular case:

Corollary 6. *If the function $f \in \mathcal{S}^*_{\mathrm{Ne}}$, then*

$$\left|H_{2,1}(F_f/2)\right| \leq \frac{1}{16},$$

and the equality holds for the function

$$\Omega_3(z) := z\exp\left(\int_0^z \frac{\varphi_{\mathrm{Ne}}(t^2)-1}{t}\,\mathrm{d}t\right) = z + \frac{z^3}{2} + \frac{z^5}{8} + \dots,\ z \in \mathbb{D}.$$

Proof. For proving the sharpness of the above result, we see that the function $\Omega_3(z) = z + a_2 z^2 + a_3 z^3 + \dots$, $z \in \mathbb{D}$ belongs to the class $\mathcal{S}^*_{\mathrm{Ne}}$ (see [27]) with $a_2 = a_4 = 0$ and $a_3 = \frac{1}{2}$. Hence,

$$\left|H_{2,1}(F_{\Omega_3}/2)\right| = \left|\frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right)\right| = \frac{1}{4}\left|a_3^2\right| = \frac{1}{16},$$

which proves the sharpness of our estimation. □

3. Conclusions

In the final section, we emphasize that all the bounds of logarithmic coefficients for the classes $\mathcal{ST}_L(s)$ and $\mathcal{S}^*_{\mathrm{Ne}}$ that we obtained were sharp. Moreover, by Theorem 5, we obtained sharp bounds of the second Hankel determinant of logarithmic coefficients for many well-known subclasses (known or new) as consequences. In addition, the result given in Theorem 5 can be used for determining the upper bound of $\left|H_{2,1}(F_f/2)\right|$ for other popular subclasses.

We also mention that, as a new direction in this area, in the recent article [40], the authors posed a question that can be interpreted as "an inverse Fekete–Szegő problem", and this was linked to the so-called filtration of infinitesimal generators. The authors first defined new filtration classes using the non-linear differential operator, and then they obtained certain properties of these classes. Sharp upper bounds of the modulus of the Fekete–Szegő functional over some filtration classes were found, and some open problems for further study concluded the work.

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