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On the Fixed Circle Problem on Metric Spaces and Related Results

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Abstract: The fixed-circle issue is a geometric technique that is connected to the study of geometric characteristics of certain points, and that are fixed by the self-mapping of either the metric space or of the generalized space. The fixed-disc problem is a natural resultant that arises as a direct outcome of this problem. In this study, our goal is to examine new classes of self-mappings that meet a new particular sort of contraction in a metric space. The common geometrical characteristic of the set of fixed points of any element of these classes is that a circle or even a disc, that is either termed the fixed circle or even the fixed disc of the appropriate self-map, is included within that set. In order to accomplish this, we establish two new classifications of contraction mapping: F_c-contractive mapping and F_c-expanding mapping. In the investigation of neural networks, activation functions with either fixed circles (or even fixed discs) are observed frequently. This demonstrates how successful our results with the fixed-circle (respectively, the fixed-disc) model were.

Keywords: fixed point; fixed circle; fixed disc

MSC: 47H10; 54H25

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1. Introduction

Over the course of the past few decades, the Banach contraction principle has been researched and expanded upon using a variety of methods. These methods include generalizing the contractive condition that was utilized (see [1–20] for more details) and to generalize the used metric space (see [21–29] for more details). A recent approach is to examine the geometric characteristics of the fixed point set of a self-mapping with the help of a special geometric shape, introduced by Özgür and Taş in [30]. For this purpose, several theorems on the fixed circle are derived as the geometric aspects of the generalization of fixed-point theorems (see [30–34], for further information).

Consider the metric space (X, d) and consider f as a self-mapping on X. First, we recall that circle $C_{x_0,\rho} = \{x \in X : \mathrm{d}(x, x_0) = \rho\}$ is a fixed circle of f if $fx = x$ for all $x \in C_{x_0,\rho}$ (see [30]). Similarly, disc $D_{x_0,\rho} = \{x \in X : \mathrm{d}(x, x_0) \leq \rho\}$ is called a fixed disc of f if $fx = x$ for all $x \in D_{x_0,\rho}$. There are several cases of self-mappings that demonstrate how the fixed point set of the self-mapping contains a circle (or a disc). Consider, for instance, the metric space $(\mathbb{C}, \mathrm{d})$ along with the metric d defined for the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, as follows:

$$\mathrm{d}(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2| + |x_1 - x_2 + y_1 - y_2|. \tag{1}$$

It is important to point out that the metric described in (1) is the same as the metric that is induced by the norm function

$$\|z\| = \|x + iy\| = |x| + |y| + |x + y|,$$

(notice the example given in 2.4 of [35]). In the accompanying illustration that was created with [36], you can see the circle denoted by $C_{0,1}$. Define the self-mapping f_1 on $\mathbb{C}$, as follows:

$$f_1 z = \begin{cases} z & ; \quad x \leq 0, y \geq 0 \text{ or } x \geq 0, y \leq 0 \\ -y + \frac{1}{2} + i\left(-x + \frac{1}{2}\right) & ; \quad x > 0, y > 0 \\ -y - \frac{1}{2} + i\left(-x - \frac{1}{2}\right) & ; \quad x < 0, y < 0 \end{cases},$$

for each $z = x + iy \in \mathbb{C}$, then clearly the fixed point set of f_1 contains circle $C_{0,1}$, that is, $C_{0,1}$ is a fixed circle of f_1 (see Figure 1). Therefore, the study of geometrical properties of the fixed points of a self-mapping seems to be an intriguing problem in the case where the fixed point is non unique.

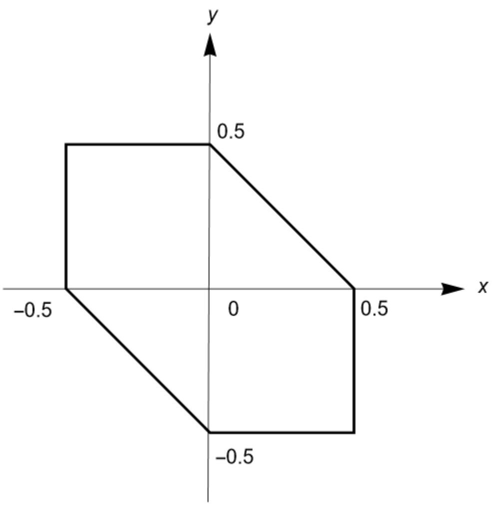

Figure 1. The graph of the circle $C_{0,1}$.

Moreover, self-mappings with fixed points are becoming more important to researchers studying neural networks. For instance, in [37], it was pointed out that the fixed points of a neural network can be determined by the fixed points of the activation function used. If the global input-output relationship in a neural network can be considered in the framework of Möbius transformations, then the existence of one or two fixed points throughout the neural network is guaranteed (see [38] for basic algebraic and geometric properties of Möbius transformations). Some possible applications of theoretical fixed-circle results to neural networks have been investigated in recent studies [30,32].

Next, we remind the readers of the following theorems on a fixed circle.

Theorem 1 ([30])**.** *Consider the metric space* (Y, d) *and let the mapping*

$$\phi : \mathrm{Y} \to [0, \infty) \text{ such that } \phi(y) = \mathrm{d}(y, y_0), \tag{2}$$

for every $y \in \mathrm{Y}$*. If there exists a self-mapping* $f : \mathrm{Y} \to \mathrm{Y}$ *meeting*

$$(\mathrm{C1}) \quad \mathrm{d}(y, fy) \leq \phi(y) - \phi(fy)$$

and

$$(\mathrm{C2}) \quad \mathrm{d}(fy, y_0) \geq \varrho,$$

for every $y \in C_{y_0,\varrho}$*, hence circle* $C_{y_0,\varrho}$ *is a fixed circle of* f*.*

Theorem 2 ([30])**.** *Assume that* (Y, d) *is a metric space and* ϕ *is a mapping described in* (2)*. If a self-mapping* $f : \mathrm{Y} \to \mathrm{Y}$ *fulfils the conditions*

$$(1)^{*} \;\; \mathrm{d}(y, fy) \leq \phi(y) + \phi(fy) - 2\varrho$$

and

$$(2)^{*} \;\; \mathrm{d}(fy, y_0) \leq \varrho,$$

for every $y \in C_{y_0,\varrho}$*, hence circle* $C_{y_0,\varrho}$ *is a fixed circle of* f*.*

Theorem 3 ([30])**.** *Assume that* (Y, d) *is a metric space and* ϕ *is the map defined in* (2)*. If a self-mapping* $f : \mathrm{Y} \to \mathrm{Y}$ *fulfils the following conditions*

$$(1)^{**} \;\; \mathrm{d}(y, fy) \leq \phi(y) - \phi(fy)$$

and

$$(2)^{**} \;\; k\mathrm{d}(y, fy) + \mathrm{d}(fy, y_0) \geq \varrho,$$

for every $y \in C_{y_0,\varrho}$ *and some* $k \in [0, 1)$*, then circle* $C_{y_0,\varrho}$ *is a fixed circle of* f*.*

Theorem 4 ([32])**.** *Assume that* (Y, d) *is a metric space and the mapping* $\phi_\varrho : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ *be defined by*

$$\phi_\varrho(y) = \begin{cases} y - \varrho & ; \quad y > 0 \\ 0 & ; \quad y = 0 \end{cases}, \tag{3}$$

for every $y \in \mathbb{R}^+ \cup \{0\}$*. If there exits a self-mapping* $f : \mathrm{Y} \to \mathrm{Y}$ *meeting*

1. $\mathrm{d}(fy, y_0) = \varrho$ *for every* $y \in C_{y_0,\varrho}$*,*
2. $\mathrm{d}(fy, fz) > \rho$ *for every* $y, z \in C_{y_0,\varrho}$ *and* $y \neq z$ *,*
3. $\mathrm{d}(fy, fz) \leq \mathrm{d}(y, z) - \phi_\varrho(\mathrm{d}(y, fy))$ *for every* $y, z \in C_{y_0,\varrho}$*,*

hence circle $C_{y_0,\varrho}$ *is a fixed circle of* f*.*

The following is the organization of this manuscript. In Section 2, we provide some generalizations of Theorems 1–3. In Section 3, we present the definitions of an "F_c-contraction" and an "F_c-expanding map" where we prove new theorems on a fixed circle. In Section 4, we consider the fixed point sets of some activation functions frequently used in the study of neural networks with a geometric viewpoint. This demonstrates how effective our results are, based on fixed circles. In Section 5, we present some open problems for future studies. When the fixed point we are looking at is not unique, our findings highlight the significance of the geometry of the other fixed points in a self-mapping.

2. New Fixed-Circle Theorems for Some Generalized Contractive Mappings

First, we provide a theorem for a fixed circle using an auxiliary function.

Theorem 5. *Assume that* (Y, d) *is a metric space,* f *is a self-mapping on* Y *and the mapping* $\theta_\varrho : \mathbb{R} \to \mathbb{R}$ *is described by*

$$\theta_\varrho(y) = \begin{cases} \varrho & ; \quad y = \varrho \\ y + \varrho & ; \quad y \neq \varrho \end{cases},$$

for every $y \in \mathbb{R}$ *and* $\varrho \geq 0$*. Suppose that*

1. $\mathrm{d}(fy, y_0) \leq \theta_\varrho(\mathrm{d}(y, y_0)) + L\mathrm{d}(y, fy)$ *for some* $L \in (-\infty, 0]$ *and every* $y \in \mathrm{Y}$*,*
2. $\varrho \leq \mathrm{d}(fy, y_0)$ *for every* $y \in C_{y_0,\varrho}$*,*
3. $\mathrm{d}(fy, fz) \geq 2\varrho$ *for every* $y, z \in C_{y_0,\varrho}$ *and* $y \neq z$*,*
4. $\mathrm{d}(fy, fz) < \varrho + \mathrm{d}(z, fy)$ *for every* $y, z \in C_{y_0,\varrho}$ *and* $y \neq z$*,*

then, f *fixes the circle* $C_{y_0,\rho}$*.*

Proof. Let $y \in C_{y_0,\varrho}$ be a point chosen at random. By using conditions (1) and (2), we obtain

$$d(fy, y_0) \leq \theta_\varrho(d(y, y_0)) + Ld(y, fy) = \varrho + Ld(y, fy)$$

and so

$$\varrho \leq d(fy, y_0) \leq \varrho + Ld(y, fy). \tag{4}$$

There are two distinct cases.

Case 1. If $L = 0$, then we find $d(fy, y_0) = \varrho$ by (4), that is, we have $fy \in C_{y_0,\varrho}$. Assume that $d(y, fy) \neq 0$ for $y \in C_{y_0,\varrho}$. Since $y \neq fy$, by using condition (3), we obtain

$$d(fy, f^2y) \geq 2\varrho. \tag{5}$$

Furthermore, using condition (4)

$$d(fy, f^2y) < \varrho + d(fy, fy)$$

and hence

$$d(fy, f^2y) < \varrho,$$

which contradicts inequality (5). Therefore, it should be $d(y, fy) = 0$ by which it implies that $fy = y$.

Case 2. Let $L \in (-\infty, 0)$. If $d(y, fy) \neq 0$, we obtain a contradiction by (4). Hence, it should be $d(y, fy) = 0$.

Thereby, we obtain $fy = y$ for every $y \in C_{y_0,\varrho}$, that is, $C_{y_0,\varrho}$ is a fixed circle of f. To put it another way, the fixed point set of f contains circle $C_{y_0,\varrho}$. □

Remark 1. *Notice that, if we consider the case* $L \in (-\infty, 0)$ *in condition* (1) *of Theorem 5 for* $y \in C_{x_0,\varrho}$, *then we obtain*

$$-Ld(y, fy) \leq \theta_\varrho(d(y, y_0)) - d(fy, y_0) = d(y, y_0) - d(fy, y_0) = \varphi(y) - \varphi(fy)$$

and hence

$$-Ld(y, fy) \leq \varphi(y) - \varphi(fy).$$

For $L = -1$, *we obtain*

$$d(y, fy) \leq \varphi(y) - \varphi(fy).$$

This means that condition (C1) (*resp. condition* $(1)^{**}$) *is satisfied for this case.*

Clearly, condition (2) *of Theorem 5 is the same as condition* (C2). *Moreover, if condition* (2) *of Theorem 5 is fulfilled, then condition* $(2)^{**}$ *is satisfied. Consequently, Theorem 5 is a generalization of Theorem 1 and 3 for the cases* $L \in (-\infty, 0) \setminus \{-1\}$. *For the case* $L = -1$, *Theorem 5 coincides with Theorem 1, and it is a particular case of Theorem 3.*

Next, we present some illustrative examples.

Example 1. *Consider the metric space* $(\mathbb{R}, d)$ *with the standard metric* $d(y_1, y_2) = |y_1 - y_2|$ *and circle* $C_{0,1} = \{-1, 1\}$. *If we describe the self-mapping* $f_1 : \mathbb{R} \to \mathbb{R}$ *as*

$$f_1 y = \begin{cases} 3y^2 + y - 3 & ; \quad y \in \{-1, 1\} \\ 0 & ; \quad \textit{elsewhere} \end{cases},$$

for each y *belongs to* $\mathbb{R}$, *so it is not difficult to see that* f_1 *meets the hypothesis of Theorem 5 for circle* $C_{0,1}$ *and* $L = \frac{-1}{2}$. *Indeed, conditions (2), (3), and (4) of Theorem 5 can be easily checked. For condition (1), we take into consideration the two cases below.*

Case 1. Let $y \in C_{0,1}$. *Hence, we have* $\theta_1(|y|) = 1$ *and so*

$$d(f_1 y, y_0) = \left|3y^2 - 3 + y\right| \leq 1 - \frac{1}{2}\left|3y^2 - 3\right| = \theta_\varrho(d(y, y_0)) + Ld(y, f_1 y).$$

Case 2. Let $y \notin C_{0,1}$. Then we have $|y| \neq 1$ and hence, $\theta_1(|y|) = |y| + 1$. Clearly, we have

$$d(f_1 y, y_0) = 0 \leq |y| + 1 - \frac{1}{2}|y| = \frac{1}{2}|y| + 1.$$

Consequently, $C_{0,1}$ is the fixed circle of f_1.

Example 2. *Consider $(\mathbb{R}, d)$ to be the standard metric space and circle $C_{0,2} = \{-2, 2\}$. Define $f_2 : \mathbb{R} \to \mathbb{R}$ by*

$$f_2 y = \begin{cases} 2 & ; \quad y = -2 \\ -2 & ; \quad y = 2 \\ 0 & ; \quad \textit{elsewhere} \end{cases},$$

for each $y \in \mathbb{R}$, then f_2 does not meet condition (1) of Theorem 5 for each $y \in C_{0,2}$ and for any $L \in (-\infty, 0)$. Furthermore, f_2 does not fulfil condition (4) for each $y \in C_{0,2}$ and for any $L \in (-\infty, 0]$. Clearly, f_2 does not fix $C_{0,2}$, and this example shows that condition (4) is crucial in Theorem 5.

Example 3. *Consider $(\mathbb{R}, d)$ to be the standard metric space and circles $C_{0,1} = \{-1, 1\}$ and $C_{0,2} = \{-2, 2\}$. If we define $f_3 : \mathbb{R} \to \mathbb{R}$ as*

$$f_3 y = \begin{cases} y & ; \quad y \in C_{0,1} \cup C_{0,2} \\ 0 & ; \quad \textit{otherwise} \end{cases},$$

for each $y \in \mathbb{R}$, then f_3 meets the hypothesis of Theorem 5 for circles $C_{0,1}$ and $C_{0,2}$ and for any $L \in [-1, 0]$. Clearly, $C_{0,1}$ and $C_{0,2}$ are the fixed circles of f_3.

Example 4. *Consider $(\mathbb{R}, d)$ to be the standard metric space and describe the self-mapping g: $\mathbb{R} \to \mathbb{R}$ as*

$$gy = \begin{cases} \frac{3}{2} & ; \quad y \geq 0 \\ -\frac{3}{2} & ; \quad y < 0 \end{cases},$$

for each $y \in \mathbb{R}$. Hence, g meets both conditions (2) and (4) of Theorem 5; however, it does not meet condition (1) for circle $C_{0,1}$. However, the fixed point set of g consists of points $-\frac{3}{2}$ and $\frac{3}{2}$. So, circle $C_{0,1}$ is not a fixed circle of g, and this example shows that condition (1) is required to obtain a fixed circle.

Moreover, $C_{0,\frac{3}{2}} = \{-\frac{3}{2}, \frac{3}{2}\}$ is the unique fixed circle of g. It is simple to verify that g satisfies conditions (2) and (4) of Theorem 5, but does not satisfy condition (1) for circle $C_{0,\frac{3}{2}}$. This demonstrates that the conclusion reached by applying the opposite of Theorem 5 does not hold true in most situations. Again, condition (1) is also crucial here.

We give another result of a fixed circle.

Theorem 6. *Let (Y, d) be a metric space, f be a self-mapping on Y and the mapping $\theta_\rho : \mathbb{R} \to \mathbb{R}$ be, as in Theorem 5. Suppose that*

1. $2d(y, y_0) - d(fy, y_0) \leq \theta_\rho(d(y, y_0)) + Ld(y, fy)$ *for some $L \in (-\infty, 0]$ and each $y \in Y$,*
2. $d(fy, y_0) \leq \rho$ *for each $y \in C_{y_0,\rho}$,*
3. $d(fy, fz) \geq 2\rho$ *for every $y, z \in C_{y_0,\rho}$ and $y \neq z$,*
4. $d(fy, fz) < \rho + d(z, fy)$ *for each $y, z \in C_{y_0,\rho}$ and $y \neq z$,*

hence, circle $C_{y_0,\rho}$ is fixed by the self-mapping f.

Proof. Consider $y \in C_{y_0,\rho}$ to be an arbitrary point. Using conditions (1) and (2), we obtain

$$2d(y, y_0) - d(fy, y_0) \leq d(y, y_0) + Ld(y, fy),$$

$$2\rho - d(fy, y_0) \leq \rho + Ld(y, fy)$$

and

$$\rho \leq \mathrm{d}(fy, y_0) + L\mathrm{d}(y, fy) \leq \rho + L\mathrm{d}(y, fy). \quad (6)$$

Similar to the arguments used in the proof of Theorem 5, a direct computation indicates that circle $C_{y_0,\rho}$ is fixed by f. $\square$

Remark 2. *Notice that, if we consider the case $L = -1$ in condition (1) of Theorem 6 for $y \in C_{y_0,\rho}$, then we obtain*

$$\mathrm{d}(y, fy) \leq \theta_\rho(\mathrm{d}(y, y_0)) + \mathrm{d}(fy, y_0) - 2\mathrm{d}(y, y_0) = \rho + \mathrm{d}(fy, y_0) - 2\rho = \varphi(y) + \varphi(fy) - 2\rho.$$

Hence, condition $(1)^*$ is satisfied. Furthermore, condition (2) of Theorem 6 is contained in condition $(2)^*$. Therefore, Theorem 6 is a particular case of Theorem 2 in this case. For the cases $L \in (-\infty, 0)$, Theorem 6 is an extension of Theorem 2.

Now, we look at some instances to help illustrate our point.

Example 5. *Consider the standard metric space $(\mathbb{R}, \mathrm{d})$ and circle $C_{0,1} = \{-1, 1\}$. Consider the map $f_4 : \mathbb{R} \to \mathbb{R}$ as*

$$f_4 y = \begin{cases} \frac{1}{y} & ; \quad y \in \{-1, 1\} \\ 2y & ; \quad \text{otherwise} \end{cases},$$

for each $y \in \mathbb{R}$, hence, f_4 satisfies the hypothesis of Theorem 6 for $L = -\frac{1}{2}$. Clearly, $C_{0,1}$ is a fixed circle of f_4. It is easy to check that f_4 does not fulfill condition (1) of Theorem 5 to any $L \in (-\infty, 0]$.

Example 6. *Consider the standard metric space $(\mathbb{R}, \mathrm{d})$ and circles $C_{0,1} = \{-1, 1\}$ and $C_{1,2} = \{-1, 3\}$. Consider the self-mapping $f_5 : \mathbb{R} \to \mathbb{R}$ defined by*

$$f_5 y = \begin{cases} y & ; \quad y \in C_{0,1} \cup C_{1,2} \\ \alpha y & ; \quad \text{otherwise} \end{cases},$$

for each $y \in \mathbb{R}$ and $\alpha \geq 2$, so f_5 satisfies the hypothesis of Theorem 6 for $L = 0$ and for circles $C_{0,1}$ and $C_{1,2}$. Clearly, $C_{0,1}$ and $C_{1,2}$ are the fixed circles of f_5. Notice that fixed circles $C_{0,1}$ and $C_{1,2}$ are not disjoint.

Considering Examples 3 and 6, we deduce that a fixed circle does not require the uniqueness in Theorems 5 and 6. If a fixed circle is non unique, then two fixed circles of a self-mapping can be disjoint or not. Next, we prove a theorem where f fixes a unique circle.

Theorem 7. *Let (Y, d) be a metric space and $f : Y \to Y$ be a self-mapping that fixes circle $C_{y_0,\varrho}$. If the following condition*

$$\mathrm{d}(fy, fz) < \max\{\mathrm{d}(z, fy), \mathrm{d}(z, fz)\}, \quad (7)$$

is satisfied by f for every $y \in C_{y_0,\varrho}$ and $z \in Y \setminus C_{y_0,\varrho}$, then $C_{y_0,\varrho}$ is the unique fixed circle of f.

Proof. Let $C_{y_1,\mu}$ be another fixed circle of f. If we take $y \in C_{y_0,\varrho}$ and $z \in C_{y_1,\mu}$ with $y \neq z$, from the inequality (7), we obtain

$$\begin{aligned} \mathrm{d}(y, z) &= \mathrm{d}(fy, fz) \\ &< \max\{\mathrm{d}(z, fy), \mathrm{d}(z, fz)\} = \mathrm{d}(y, z), \end{aligned}$$

that is a contradiction. We have $y = z$ for every $y \in C_{y_0,\varrho}$, $z \in C_{y_1,\mu}$ then f has only one fixed circle $C_{y_0,\varrho}$. $\square$

Example 7. *Consider the standard metric space $(\mathbb{C}, \mathrm{d})$ and circle $C_{0,\frac{1}{4}}$. Define f_6 on $\mathbb{C}$ as follows:*

$$f_6 t = \begin{cases} \frac{1}{16\bar{t}} & ; \quad t \neq 0 \\ 0 & ; \quad t = 0 \end{cases},$$

for $t \in \mathbb{C}$, where $\bar{t}$ represents the complex conjugate of t. It is not difficult to see that $C_{0,\frac{1}{4}}$ is the unique fixed circle of f_6, where f_6 does not fulfil the hypothesis of Theorem 7.

Example 7 demonstrates that the counterfactual of Theorem 7 is not correct in general. Now, in order to illustrate Theorem 7, we consider the following example.

Example 8. *Let $Y = \{-1, 0, 1\}$ and the metric $\mathrm{d} : Y \times Y \to [0, \infty)$ be described by*

$$\mathrm{d}(y,z) = \begin{cases} 0 & ; \quad y = z \\ |y| + |z| & ; \quad y \neq z \end{cases},$$

for every $y, z \in Y$. If we take the self-mapping $f_7 : Y \to Y$ is described by

$$f_7 y = 0,$$

for any $y \in Y$; hence, $C_{1,1} = \{0\}$ is the unique fixed circle of f_7.

Next, we present the following interesting theorem that involves the identity map $I_Y : Y \to Y$ described by $I_Y(y) = y$ for all $y \in Y$.

Theorem 8. *Consider the metric space (Y, d). Let the map f be from Y to itself with fixed circle $C_{y_0,\varrho}$. The self-mapping f fulfils the following condition*

$$\mathrm{d}(y, fy) \leq \alpha[\max\{\mathrm{d}(y, fy), \mathrm{d}(y_0, fy)\} - \mathrm{d}(y_0, fy)], \tag{8}$$

for every $y \in Y$ and some $\alpha \in (0,1)$, if and only when $f = I_Y$.

Proof. Take $y \in Y$ with $fy \neq y$. By inequality (8), if $\mathrm{d}(y, fy) \geq \mathrm{d}(y_0, fy)$, then we obtain

$$\mathrm{d}(y, fy) \leq \alpha[\mathrm{d}(y, fy) - \mathrm{d}(y_0, fy)] \leq \alpha \mathrm{d}(y, fy),$$

which leads us to a contradiction due to the fact that $\alpha \in (0,1)$. If $\mathrm{d}(y, fy) \leq \mathrm{d}(y_0, fy)$, then we obtain

$$\mathrm{d}(y, fy) \leq \alpha[\mathrm{d}(y_0, fy) - \mathrm{d}(y_0, fy)] = 0.$$

Hence, we have $fy = y$ and that is $f = I_Y$, since y is an arbitrary point in Y.
Conversely, I_Y satisfies condition (8) clearly. □

Corollary 1. *Let (Y, d) be a metric space and $f : Y \to Y$ be a self-mapping. If f satisfies the hypothesis of Theorem 5 (resp. Theorem 6) but condition (8) is not satisfied, then $f \neq I_Y$.*

Now, we rewrite the next theorem given in [30].

Theorem 9 ([30])**.** *Consider the metric space (Y, d) and let the map f be from Y to itself, which has a fixed circle $C_{y_0,\varrho}$ and ϕ, as in (2). Then f meets the condition*

$$\mathrm{d}(y, fy) \leq \frac{\phi(y) - \phi(fy)}{h}, \tag{9}$$

for every $y \in Y$ and $h > 1$, if and only when $f = I_Y$.

Theorem 10. *Consider the metric space (Y, d). Let the map f be from Y to itself, which has a fixed circle $C_{y_0,\varrho}$ and ϕ, as in (2). Then, f fulfils (8) if and only when f satisfies (9).*

Proof. The proof follows easily. □

3. New Classes of Contractive and Expanding Mappings in Metric Spaces

In this part of the article, we will apply a different strategy to acquire new results for the fixed circle. This new approach also ensures the existence of a fixed disc of a self-mapping. The following group of functions, which was first presented by Wardowski in [39], is our primary resource for accomplishing this.

Definition 1 ([39]). *Let $\mathbb{F}$ stand for the entire group of functions $\mathrm{F} : (0, \infty) \to \mathbb{R}$ in such a way that*

(F_1) F *is neither decreasing nor constant,*
(F_2) *For every sequence $\{\alpha_n\}$ in $(0, \infty)$ the below must be true.*

$$\lim_{n\to\infty} \alpha_n = 0 \text{ iff } \lim_{n\to\infty} \mathrm{F}(\alpha_n) = -\infty,$$

(F_3) *There exists $t \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^t \mathrm{F}(\alpha) = 0$.*

Several examples of functions that satisfy the axioms (F_1), (F_2) and (F_3) of Definition 1 are the following $\mathrm{F}(y) = \ln(y)$, $\mathrm{F}(y) = \ln(y) + y$, $\mathrm{F}(y) = -\frac{1}{\sqrt{y}}$ and $\mathrm{F}(y) = \ln(y^2 + y)$ (check [39] for further information).

At this point, we are going to discuss a new sort of contraction that goes as follows.

Definition 2. *Consider the metric space (Y, d). Let f be a self-mapping on Y. If there exists $t > 0$, $\mathrm{F} \in \mathbb{F}$ and $y_0 \in \mathrm{Y}$ in such a way that*

$$\mathrm{d}(y, fy) > 0 \Rightarrow t + \mathrm{F}(\mathrm{d}(y, fy)) \leq \mathrm{F}(\mathrm{d}(y_0, y)),$$

for every $y \in \mathrm{Y}$, then f is called as an F_c-contraction.

It is important to notice that the point y_0, which is referred to in Definition 2, needs to be a fixed point in the mapping f. In fact, if y_0 is not a fixed point of f, we obtain $\mathrm{d}(y_0, fy_0) > 0$ and then

$$\mathrm{d}(y_0, fy_0) > 0 \Rightarrow t + \mathrm{F}(\mathrm{d}(y_0, fy_0)) \leq \mathrm{F}(\mathrm{d}(y_0, y_0)).$$

It is clear that it is a contradiction due to the fact that the domain of F is $(0, \infty)$. As a result, the next proposition may be stated as a direct consequence of Definition 2.

Proposition 1. *Consider the metric space (Y, d). If f is an F_c-contraction with $y_0 \in \mathrm{Y}$, then we obtain $fy_0 = y_0$.*

Using this new type of contraction, we will now state the following fixed-circle theorem.

Theorem 11. *Consider the metric space (Y, d). Let f be an F_c-contraction with $y_0 \in \mathrm{Y}$. Define the number γ by*

$$\gamma = \inf\{\mathrm{d}(y, fy) : y \neq fy, y \in \mathrm{Y}\}.$$

Then, $C_{y_0,\gamma}$ is a fixed circle of f. Particularly, f fixes each circle $C_{y_0,r}$ where $r < \gamma$.

Proof. If $\gamma = 0$, then clearly $C_{y_0,\gamma} = \{y_0\}$, and by using Proposition 1, we observe that $C_{y_0,\gamma}$ is a fixed circle of f. Assume $\gamma > 0$ and let $y \in C_{y_0,\gamma}$. If $fy \neq y$, then by the definition of γ, we have $\mathrm{d}(y, fy) \geq \gamma$. Since F is increasing, using the F_c-contractive property of f, we obtain

$$\begin{array}{rcl} \mathrm{F}(\gamma) & \leq & \mathrm{F}(\mathrm{d}(y, fy)) \\ & \leq & \mathrm{F}(\mathrm{d}(y_0, y)) - t \\ & < & \mathrm{F}(\mathrm{d}(y_0, y)) = \mathrm{F}(\gamma), \end{array}$$

which leads to a contradiction. Therefore, we have $\mathrm{d}(y, fy) = 0$, that is, $fy = y$. Consequently, $C_{y_0,\gamma}$ is a fixed circle of f.

Now, we prove that f also fixes any arbitrary circle $C_{y_0,r}$ with $r < \gamma$. Take $y \in C_{y_0,r}$ and assume that $\mathrm{d}(y, fy) > 0$. Again, using the F_c-contractive property of the self-mapping, we obtain

$$\mathrm{F}(\mathrm{d}(y, fy)) \leq \mathrm{F}(\mathrm{d}(y_0, y)) - t < \mathrm{F}(r).$$

Since F is increasing, we find

$$\mathrm{d}(y, fy) < r < \gamma.$$

However, $\gamma = \inf\{\mathrm{d}(y, fy) : \text{for every } y \neq fy\}$, which brings forth a contradiction. Hence, we obtain $\mathrm{d}(y, fy) = 0$, that is, $fy = y$. Accordingly, $C_{y_0,r}$, is a circle of f that is fixed. □

Remark 3. *(1) We note that in Theorem 11, the F_c-contraction f fixes the disc $D_{y_0,\gamma}$. Hence, the centre of any fixed circle is also fixed by f. In Theorem 4, the self-mapping f maps $C_{y_0,\rho}$ into (or onto) itself, but the centre of the fixed circle does not require to be fixed by f.*

(2) In relation to the number of points in the set Y*, the number of fixed circles of an F_c-contractive self-mapping f may be infinite (see Example 11).*

We give some illustrative examples.

Example 9. *We consider the set* $\mathrm{Y} = \{0, 1, e^2, -e^2, e^2 - 1, e^2 + 1\} \subset \mathbb{R}$ *with the usual metric and identify the self-mapping* $f_8 : \mathrm{Y} \to \mathrm{Y}$ *as*

$$f_8 y = \begin{cases} 1 & ; \quad y = 0 \\ y & ; \quad y \neq 0 \end{cases}.$$

The self-mapping f_8 is an F_c-contractive self-mapping, such as $\mathrm{F}(y) = \ln y$*, $t = 1$ and $y_0 = e^2$. Obviously, we have* $\mathrm{d}(y, f_8 y) > 0$ *only for the point $y = 0$ and*

$$t + \mathrm{F}(\mathrm{d}(y, f_8 y)) = 1 + \ln|0 - 1| \leq \mathrm{F}(\mathrm{d}(y_0, y)) = \ln\left|e^2 - 0\right| = 2\ln e = 2.$$

Clearly, we obtain $\gamma = 1$, and f_8 fixes the circle $C_{e^2,1} = \{e^2 - 1, e^2 + 1\}$. f_8 fixes also the disc $D_{e^2,1} = \{y \in \mathrm{Y} : d(y, e^2) \leq 1\} = \{e^2, e^2 - 1, e^2 + 1\}$. Notice that circle $C_{0,e^2} = \{-e^2, e^2\}$ is another fixed circle of f_8.

As may be observed in the following illustration, the converse assertion of Theorem 11 does not always hold true.

Example 10. *Take* (Y, d) *as a metric space, $y_0 \in \mathrm{Y}$ be any arbitrary point and $\mu > 0$ be any number. If we consider the self-mapping $f_9 : \mathrm{Y} \to \mathrm{Y}$ defined by*

$$f_9 y = \begin{cases} y & ; \quad \mathrm{d}(y, y_0) \leq \mu \\ y_0 & ; \quad \mathrm{d}(y, y_0) > \mu \end{cases},$$

Therefore, it is not hard to realize that f_9 is not an F_c-contractive self-mapping for the point y_0 but f_9 fixes each circle $C_{y_0,r}$ where $r \leq \mu$.

Example 11. *Consider the standard metric space* $(\mathbb{C}, \mathrm{d})$ *and describe the self-mapping* $f_{10} : \mathbb{C} \to \mathbb{C}$ *as*

$$f_{10}z = \begin{cases} z & ; \quad |z| < 2 \\ z+1 & ; \quad |z| \geq 2 \end{cases},$$

for all $z \in \mathbb{C}$. *We have* $\gamma = \min\{\mathrm{d}(z, f_{10}z) : z \neq f_{10}z\} = 1$ *and* f_{10} *is an* F_c*-contractive self-mapping with* $\mathrm{F}(x) = \ln y$, $t = \ln 2$ *and* $z_0 = 0 \in \mathbb{C}$. *Clearly, the self-mapping* f_{10} *has infinitely many circles that are fixed.*

Presently, we have settled on a new theorem of fixed circles based on the following well-known fact, that if a self-mapping f on Y is surjective, then there exists a self mapping $f^* : \mathrm{Y} \to \mathrm{Y}$, in such a way that the map $(f \circ f^*)$ is the identity map for Y. First, we give a new type of expanding map.

Definition 3. *A self-mapping* f *on a metric space* Y *is referred to as an* F_c*-expanding map, if there exist* $t < 0$, $\mathrm{F} \in \mathbb{F}$ *and* $y_0 \in \mathrm{Y}$ *in such a way that*

$$\mathrm{d}(y, fy) > 0 \Rightarrow \mathrm{F}(\mathrm{d}(y, fy)) \leq \mathrm{F}(\mathrm{d}(y_0, fy)) + t,$$

for every $y \in \mathrm{Y}$.

Theorem 12. *Consider the metric space* (Y, d). *If* $f : \mathrm{Y} \to \mathrm{Y}$ *is a surjective* F_c*-expanding map with* $y_0 \in \mathrm{Y}$, *then* f *has a circle that is fixed in* Y.

Proof. Since f is surjective, there exists a self-mapping $f^* : \mathrm{Y} \to \mathrm{Y}$, such that the map $(f \circ f^*)$ is the identity map for Y. Take $y \in \mathrm{Y}$ be such that $\mathrm{d}(y, f^*y) > 0$ and $z = f^*y$. First, notice the following fact

$$fz = f(f^*y) = (f \circ f^*)y = y.$$

Since

$$\mathrm{d}(z, fz) = \mathrm{d}(fz, z) > 0,$$

by using F_c-expanding property of f, we obtain

$$\mathrm{F}(\mathrm{d}(z, fz)) \leq \mathrm{F}(\mathrm{d}(y_0, fz)) + t$$

and

$$\mathrm{F}(\mathrm{d}(f^*y, y)) \leq \mathrm{F}(\mathrm{d}(y_0, y)) + t.$$

Therefore, we obtain

$$-t + \mathrm{F}(\mathrm{d}(f^*y, y)) \leq \mathrm{F}(\mathrm{d}(y_0, y)).$$

Consequently, f^* is an F_c-contraction of Y with y_0 as $-t > 0$. Then, using Theorem 11, f^* has a fixed circle $C_{y_0,\gamma}$. Let $z \in C_{y_0,\gamma}$ be any point. Using the fact that

$$fz = f(f^*z) = z,$$

we deduce that $fz = z$, then z is a fixed point of f, which implies that f also fixes $C_{y_0,\gamma}$, as required. □

Example 12. *Let us take the set* $\mathrm{Y} = \{1, 2, 3, 4, 5\}$ *with the standard metric and define the self-mapping* $f_{11} : \mathrm{Y} \to \mathrm{Y}$ *by*

$$f_{11}y = \begin{cases} 1 & ; \quad y = 2 \\ 2 & ; \quad y = 1 \\ y & ; \quad y \in \{3, 4, 5\} \end{cases}.$$

f_{11} *is a surjective* F_c*-expanding map with* $y_0 = 4$, $\mathrm{F}(y) = \ln y$ *and* $t = -\ln 2$. *We obtain*

$$\gamma = \min\{\mathrm{d}(y, fy) : y \neq fy, y \in \mathrm{Y}\} = 1$$

and circle $C_{4,1} = \{3, 5\}$ is the fixed circle of f.

Example 13. *Let $(\mathbb{C}, \mathrm{d})$ be the standard metric space and consider the self-mapping $f_{12} : \mathbb{C} \to \mathbb{C}$ defined by*

$$f_{12}z = \begin{cases} z & ; \quad |z| \le 1 \\ \frac{2}{3}z & ; \quad |z| > 1 \end{cases},$$

for all $z \in \mathbb{C}$. We have

$$\begin{aligned} \gamma &= \inf\{\mathrm{d}(z, f_{12}z) : z \neq f_{12}z\} \\ &= \inf\left\{ \left| z - \frac{2}{3}z \right| = \frac{|z|}{3} : |z| > 1 \right\} \\ &= \frac{1}{3}. \end{aligned}$$

f_{12} is a surjective F_c-expanding map with $y_0 = 0$, $\mathrm{F}(y) = \ln y$ and $t = \ln\left(\frac{3}{4}\right)$. Indeed, we obtain

$$\ln\left(\frac{1}{2}\right) < \ln\left(\frac{3}{4}\right) \Rightarrow \ln\left(\frac{|z|}{3}\right) < \ln\left(\frac{2|z|}{3}\right) + \ln\left(\frac{3}{4}\right)$$

for each z with $|z| > 1$. Circle $C_{0,\frac{1}{3}} = \left\{ z : |z| = \frac{1}{3} \right\}$ is a fixed circle of f.

Remark 4. *The conclusion for Theorem 12 does not hold true in all cases if f is not a surjective map. For instance, if we consider the set $\mathrm{Y} = \{1, 2, 3, 4\}$ using the standard metric d and let the self-mapping be defined as $f_{12} : \mathrm{Y} \to \mathrm{Y}$ where*

$$f_{13}y = \begin{cases} 1 & ; \quad y = 2 \\ 2 & ; \quad y \in \{1, 3\} \\ 4 & ; \quad y = 4 \end{cases}.$$

It is not hard to verify that f_{13} fulfils the condition

$$\mathrm{d}(y, fy) > 0 \Rightarrow \mathrm{F}(\mathrm{d}(y, fy)) \le \mathrm{F}(\mathrm{d}(y_0, fy)) + t$$

for all $y \in \mathrm{Y}$, with $\mathrm{F}(y) = \ln y$, $y_0 = 4$ and $t = -\ln 2$. Therefore, f_{13} meets all of the axioms of Theorem 12, except that f_{13} is not surjective. Notice that $\gamma = 1$ and f_{13} does not fix circle $C_{4,1}$.

4. Fixed Point Sets of Activation Functions

Activation functions are the primary neural networks' decision-making units in a neural network; and hence, it is important to choose the most appropriate activation function for the neural network analysis [40,41]. Characteristic properties of activation functions play an important role in learning and stability issues of a neural network. A comprehensive analysis of different activation functions with individual real-world applications was given in [40]. We note that the fixed point sets of commonly used activation functions (e.g., Ramp function, ReLU function, Leaky ReLU function) contain some fixed discs and fixed circles.

Example 14. *Let us consider the Leaky ReLU function defined by*

$$f(y) = \max(ky, y) = \begin{cases} ky & ; \quad y \le 0 \\ y & ; \quad y > 0 \end{cases},$$

where $k \in [0, 1]$. In [42], the Leaky–Reluplex algorithm was proposed to verify deep neural networks (DNNs) with the Leaky ReLU activation function (see [42] for more details). Now, we consider the fixed point set of the Leaky ReLU activation function by a geometric viewpoint. Clearly, the fixed point set of f is $Fix(f) = [0, \infty)$. Let $\rho = y_0 \in (0, \infty)$ be a fixed point and consider the circle

$C_{y_0,\rho} = \{0, 2y_0\}$. *Then, it is easy to verify that the function* $f(y)$ *meets the criteria of Theorem 5 for circle* $C_{y_0,\rho}$ *with* $L = 0$. *Clearly, circle* $C_{y_0,\rho}$ *is a fixed circle of* f *and the centre of the fixed circle is also fixed by* f.

Most of the known fixed point theorems (e.g., Banach fixed point theorem, Brouwer's fixed point theorem) have been used in the theoretic studies of neural networks. For example, in [43], the existence of a fixed point for every recurrent neural network was shown, and a geometric approach was used to locate the fixed points. Brouwer's fixed point theorem was used to maintain the existence of a fixed point. This study shows the importance of the geometric viewpoint and theoretic fixed point results in applications.

5. Conclusions and Prospective Initiatives

In this section, we point out the investigation of some open questions. Concerning the geometry of non unique fixed points of a self-mapping on a metric space, novel geometric (fixed-circle or fixed-disc) findings have been found. To do this, we use two different approaches. One of them is to measure whether a given circle is fixed or not by a self-mapping. Another approach is to find which circle is fixed by a self-mapping under some contractive or expanding conditions. The investigation of new conditions which ensure that a circle or a disc is fixed by a self-mapping can be considered as a future problem. For a self-mapping in which the fixed point set contains a circle or a disc, new contractive or expanding conditions can also be investigated.

Additionally, there are several examples of self-mappings that have a common fixed circle. Here is an example, let $(\mathbb{R}, d)$ be the usual metric space and consider circle $C_{0,1} = \{-1, 1\}$. We define the self-mappings $f_{13}\colon \mathbb{R} \to \mathbb{R}$ and $f_{14}\colon \mathbb{R} \to \mathbb{R}$ by

$$f_{13}y = \begin{cases} \frac{1}{y} & ; \quad y \in \{-1,1\} \\ 0 & ; \quad \text{otherwise} \end{cases} \quad \text{and } f_{14}y = \frac{5y+3}{3y+5},$$

for each $y \in \mathbb{R}$, respectively. Both self-mappings f_{13} and f_{14} fix circle $C_{0,1} = \{-1, 1\}$, then, circle $C_{0,1} = \{-1, 1\}$ is a common fixed circle of the self-mappings f_{13} and f_{14}. At this point, the following question can be left for future study.

Question 13. *What condition(s) must exist for any circle* $C_{x_0,\rho}$ *to be the common fixed circle for two or more self-mappings?*

In conclusion, the problems discussed in this study can also be investigated on various generalized metric spaces. For instance, the notion of an M_s-metric space was introduced in [44].

Notation 14. *We use the following notations.*

1. $m_{s_{a,b,c}} := min\{m_s(a,a,a), m_s(b,b,b), m_s(c,c,c)\}$
2. $M_{s_{a,b,c}} := max\{m_s(a,a,a), m_s(b,b,b), m_s(c,c,c)\}$

Definition 4. *An* M_s*-metric on a set* Y *that contains at least one point is function* $m_s : Y^3 \to \mathbb{R}^+$, *if for all* $a, b, c, t \in Y$ *we have*

1. $m_s(a,a,a) = m_s(b,b,b) = m_s(c,c,c) = m_s(a,b,c) \Longleftrightarrow a = b = c$,
2. $m_{s_{a,b,c}} \leq m_s(a,b,c)$,
3. $m_s(a,a,b) = m_s(b,b,a)$,
4.
$$\begin{aligned}(m_s(a,b,c) - m_{s_{a,b,c}}) \leq &(m_s(a,a,t) - m_{s_{a,a,t}}) \\ &+ (m_s(b,b,t) - m_{s_{b,b,t}}) + (m_s(c,c,t) - m_{s_{c,c,t}}).\end{aligned}$$

Then, the pair (Y, m_s) *is called an* M_s*-metric space.*

One can consult [44] for some examples and basic notions of an M_s-metric space.

In M_s-metric spaces, we define a circle as follows:

$$C_{a_0,\rho} = \{a \in \mathrm{Y} \mid m_s(a_0, a, a) - m_{s_{a_0,a,a}} = \rho\}.$$

Question 15. *Consider the M_s-metric space (Y, m_s) where $k > 1$, and let f be a surjective self-mapping on* Y*. Yet, we obtain*

$$m_s(a, fa, f^2a) \leq k m_s(a_0, a, fa),$$

for every $a \in \mathrm{Y}$ and some $a_0 \in \mathrm{Y}$. Does f have a point circle on Y*?*

Question 16. *Let (Y, m_s) be an M_s-metric space, $t > 0$, $\mathrm{F} \in \mathbb{F}$, and f be a surjective self-mapping on* Y*. Yet, we have*

$$m_s(a, fa, f^2a) > 0 \Rightarrow \mathrm{F}(m_s(a, fa, f^2a)) \geq \mathrm{F}(m_s(a_0, a, fa)) + t,$$

for every $a \in \mathrm{Y}$ and some $a_0 \in \mathrm{Y}$. Does f have a fixed circle on Y*?*

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References

1. Altun, I.; Aslantaş, M.; Şahin, H. Best proximity point results for p-proximal contractions. *Acta Math. Hung.* **2020**, *162*, 393–402. [CrossRef]
2. Caristi, J. Fixed point theorems for mappings satisfying inwardness conditions. *Trans. Am. Math. Soc.* **1976**, *215*, 241–251. [CrossRef]
3. Chatterjea, S.K. Fixed point theorem. *C. R. Acad. Bulgare Sci.* **1972**, *25*, 727–730. [CrossRef]
4. Ćirić, L.B. A generalization of Banach's contraction principle. *Proc. Am. Math. Soc.* **1974**, *45*, 267–273. [CrossRef]
5. Djafari Rouhani, B.; Moradi, S. On the existence and approximation of fixed points for Ćirić type contractive mappings. *Quaest. Math.* **2014**, *37*, 179–189. [CrossRef]
6. Edelstein, M. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.* **1962**, *37*, 74–79. [CrossRef]
7. Kannan, R. Some results on fixed points II. *Am. Math. Mon.* **1969**, *76*, 405–408.
8. Kim, H. Fixed point theorems for Chandrabhan type maps in abstract convex uniform spaces. *J. Nonlinear Funct. Anal.* **2021**, *2021*, 1–8 .
9. Latif, A.; Al Subaie, R.F.; Alansari, M.O. Fixed points of generalized multi-valued contractive mappings in metric type spaces. *J. Nonlinear Var. Anal.* **2022**, *6*, 123–138.
10. Nemytskii, V.V. The fixed point method in analysis. *Usp. Mat. Nauk* **1936**, *1*, 141–174. (In Russian)
11. Olgun, M.; Biçer, Ö.; Alyıldız, T.; Altun, İ. A related fixed point theorem for F-contractions on two metric spaces. *Hacet. J. Math. Stat.* **2019**, *48*, 150–156. [CrossRef]
12. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Amer. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
13. Shatanawi, W. On w-compatible mappings and common coupled coincidence point in cone metric spaces. *Appl. Math. Lett.* **2012**, *25*, 925–931. [CrossRef]
14. Shatanawi, W.; Ćojbašić Rajić, R.V.; Radenović, S.; Al-Rawashdeh, A. Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces. *Fixed Point Theory Appl.* **2012**, *106*, 7. [CrossRef]
15. Al-Rawashdeh, A.; Aydi, H.; Felhi, A.; Sehmim, S.; Shatanawi, W. On common fixed points for $\alpha - F$-contractions and applications. *J. Nonlinear Sci. Appl.* **2016**, *9*, 3445–3458. [CrossRef]

16. Shatanawi, W.; Mustafa, Z.; Tahat, N. Some coincidence point theorems for nonlinear contraction in ordered metric spaces. *Fixed Point Theory Appl.* **2011**, *68*, 15. [CrossRef]
17. Shatanawi, W. Some fixed point results for generalized ψ -weak contraction mappings in orbitally metric spaces. *Chaos Solitons Fractals* **2012**, *45*, 520–526. [CrossRef]
18. Shatanawi, W.; Shatnawi, T.A.M. New fixed point results in controlled metric type spaces based on new contractive conditions. *AIMS Math.* **2023**, *8*, 9314–9330. [CrossRef]
19. Rezazgui, A.-Z.; Tallafha, A.A.; Shatanawi, W. Common fixed point results via $A\nu - \alpha-$contractions with a pair and two pairs of self-mappings in the frame of an extended quasi b-metric space. *AIMS Math.* **2023**, *8*, 7225–7241. [CrossRef]
20. Joshi, M.; Tomar, A.; Abdeljawad, T. On fixed points, their geometry and application to satellite web coupling problem in $S-$metric spaces. *AIMS Math.* **2023**, *8*, 4407–4441. [CrossRef]
21. An, T.V.; Dung, N.V.; Kadelburg, Z.; Radenović, S. Various generalizations of metric spaces and fixed point theorems. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math.* **2015**, *109*, 175–198. [CrossRef]
22. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal. Gos. Ped. Inst. Unianowsk* **1989**, *30*, 26–37.
23. Gähler, S. 2-metrische Räume und ihre topologische Struktur. *Math. Nachr.* **1963**, *26*, 115–148. [CrossRef]
24. Karayılan, H.; Telci, M. Caristi type fixed point theorems in fuzzy metric spaces. *Hacet. J. Math. Stat.* **2019**, *48*, 75–86. [CrossRef]
25. Mustafa, Z.; Sims, B. A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.* **2006**, *7*, 289–297.
26. Mitrović, Z.D.; Radenović, S. A common fixed point theorem of Jungck in rectangular b-metric spaces. *Acta Math. Hung.* **2017**, *153*, 401–407. [CrossRef]
27. Pasicki, L. A strong fixed point theorem. *Topol. Appl.* **2020**, *282*, 107300. [CrossRef]
28. Sedghi, S.; Shobe, N.; Aliouche, A. A generalization of fixed point theorems in S-metric spaces. *Mat. Vesn.* **2012**, *64*, 258–266.
29. Sihag, V.; Vats, R.K.; Vetro, C. A fixed point theorem in G-metric spaces via α-series. *Quaest. Math.* **2014**, *37*, 429–434. [CrossRef]
30. Özgür, N.Y.; Taş, N. Some fixed-circle theorems on metric spaces. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 1433–1449. [CrossRef]
31. Özgür, N.Y.; Taş, N. Fixed-circle problem on S-metric spaces with a geometric viewpoint. *Facta Univ. Ser. Math. Inform.* **2019**, *34*, 459–472. [CrossRef]
32. Özgür, N.Y.; Taş, N. Some fixed-circle theorems and discontinuity at fixed circle. *AIP Conf.* **2018**, *1926*, 020048.
33. Özgür, N.Y.; Taş, N.; Çelik, U. New fixed-circle results on S-metric spaces. *Bull. Math. Anal. Appl.* **2017**, *9*, 10–23.
34. Özgür, N. Fixed-disc results via simulation functions . *Turk. J. Math.* **2019**, *43*, 2794–2805. [CrossRef]
35. Özgür, N.Y. Ellipses and similarity transformations with norm functions. *Turk. J. Math.* **2018**, *42*, 3204–3210. [CrossRef]
36. Wolfram Research, Inc. *Mathematica*; Version 12.0; Wolfram Research, Inc.: Champaign, IL, USA, 2019.
37. Mandic, D.P. The use of Möbius transformations in neural networks and signal processing. In *Neural Networks for Signal Processing X, Proceedings of the 2000 IEEE Signal Processing Society Workshop (Cat. No. 00TH8501), Sydney, Australia, 11–13 December 2000*; IEEE: Piscataway, NJ, USA; Volume 1, pp. 185–194.
38. Jones, G.A.; Singerman, D. *Complex Functions: An Algebraic and Geometric Viewpoint*; Cambridge University Press: Cambridge, UK, 1987.
39. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 94. [CrossRef]
40. Szandała, T. Review and Comparison of Commonly Used Activation Functions for Deep Neural Networks, In *Bio-Inspired Neurocomputing. Studies in Computational Intelligence*; Bhoi, A., Mallick, P., Liu, C.M., Balas, V., Eds.; Springer: Singapore, 2021; Volume 903. [CrossRef]
41. Zhang, L. Implementation of fixed-point neuron models with threshold, ramp and sigmoid activation functions. *IOP Conf. Ser. Mater. Sci. Eng.* **2017**, *224*, 012054. [CrossRef]
42. Xu, J.; Li, Z.; Du, B.; Zhang, M.; Liu, J. Reluplex made more practical: Leaky ReLU. In Proceedings of the 2020 IEEE Symposium on Computers and Communications (ISCC), Rennes, France, 7–10 July 2020; IEEE: Piscataway, NJ, USA, 2020; pp. 1–7.
43. Li, L.K. Fixed point analysis for discrete-time recurrent neural networks. In Proceedings of the 1992 IJCNN International Joint Conference on Neural Networks, Baltimore, MD, USA, 15–19 January 1992; Volume 4, pp. 134–139.
44. Mlaiki, N.; Souayah, N.; Abodayeh, K.; Abdeljawad, T. Contraction principles in M_s-metric spaces. *J. Nonlinear Sci. Appl.* **2017**, *10*, 575–582. [CrossRef]

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