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Second Hankel Determinant for a New Subclass of Bi-Univalent Functions Related to the Hohlov Operator

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Abstract: A new subclass of bi-univalent functions associated with the Hohlov operator is introduced. Certain properties such as the coefficient bounds, Fekete-Szegö inequality and the second Hankel determinant for functions in the subclass are obtained. In particular, several known results are generalized.

Keywords: analytic function; bi-univalent functions; subordination; Fekete-Szegö inequality; Hankel determinant; Hohlov operator

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1. Introduction

Let A denote the class of analytic functions in $U = \{z \in C : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U). \tag{1}$$

Furthermore, let $S \subset A$ denote the class of functions that are univalent in U.

Let f and g be two analytic functions in U. We say that the function f is subordinate to the function g and is written as follows:

$$f(z) \prec g(z) \quad (z \in U),$$

if there is a Schwarz function w such that

$$f(z) = g(w(z)).$$

Further, if the function g is univalent in U, then it follows that

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Denoted by P is the class of analytic functions φ having the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0)$$

and $\mathrm{Re}\varphi(z) > 0$ $(z \in U)$.

For functions $f \in A$ and $u \in A$ given by

$$u(z) = z + \sum_{n=2}^{\infty} u_n z^n \quad (z \in U),$$

the Hadamard product (or convolution) of f and u is defined by

$$(f * u)(z) = z + \sum_{n=2}^{\infty} a_n u_n z^n = (u * f)(z) \quad (z \in U).$$

For $a, b, c \in C$ and $c \neq 0, -1, -2, -3 \cdots$, the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as:

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \quad (z \in U), \end{aligned} \tag{2}$$

where $(\alpha)_n$ is the Pochhammer symbol, written in terms of the Gamma function Γ, by

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n = 0) \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1) & (n = 1, 2, 3, \cdots). \end{cases}$$

For positive real values a, b, c, using the Hadamard product and Gauss hypergeometric function, Hohlov (see [1,2]) proposed and studied a linear operator $J_{a,b;c}f \colon A \to A$ defined by

$$\begin{aligned} J_{a,b;c}f(z) &= z\, {}_2F_1(a, b; c; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \psi_n a_n z^n \quad (z \in U), \end{aligned} \tag{3}$$

where

$$\psi_n = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!}.$$

It is well known that every univalent function $f \in S$ has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega \quad \left(|\omega| < r_0(f);\ r_0(f) \geq \frac{1}{4} \right),$$

where

$$\begin{aligned} g(\omega) := f^{-1}(\omega) &= \omega - a_2 \omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2 a_3 + a_4) + \cdots \\ &= \omega + \sum_{n=2}^{\infty} b_n \omega^n. \end{aligned} \tag{4}$$

We say that a function $f \in A$ is bi-univalent in U if both f and f^{-1} are univalent in U and denote a class of normalized analytic and bi-univalent functions by $\Sigma(\subset S)$. Some elements of functions in Σ are presented below:

$$f_1(z) = \frac{z}{1-z}, \ \ f_2(z) = -\log(1-z) \ \text{ and } \ f_3(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z} \right),$$

and their corresponding inverses given by:

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega}, \ \ f_2^{-1}(\omega) = \frac{e^{\omega} - 1}{e^{\omega}} \ \text{ and } \ f_3^{-1}(\omega) = \frac{e^{2\omega} - 1}{e^{2\omega} + 1}.$$

Certain subclasses $S^*_\Sigma(\alpha)$ and $C_\Sigma(\alpha)$ of Σ introduced by Brannan and Taha [3] are similar to the subclasses $S^*(\alpha)$ and $C(\alpha)$ of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$, respectively. In [3], Brannan and Taha obtained the non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of $S^*_\Sigma(\alpha)$ and $C_\Sigma(\alpha)$. Recently, many scholars have defined various subclasses of bi-univalent functions (see [4–12]) and investigated the non-sharp estimates of the first two coefficients of the Taylor–Maclaurin series expansion.

The Hankel determinant is one of the important tools in the study of the theory of univalent functions. Noonan and Thomas [13] defined the q-th Hankel determinant of $f \in A$ as:

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1, n \geq 0, q \geq 1).$$

The Hankel determinants

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

are called the Fekete-Szegö functional and the second Hankel determinant functional, respectively. Further, Fekete and Szegö [14] considered the generalized functional $a_3 - \mu a_2^2$, where μ is a real number. Recently, several authors (see [15–19]) proved the upper bounds for the Hankel determinant for functions in various subclasses of the bi-univalent functions. On the other hand, Zaprawa [20] extended the study of the Fekete-Szegö inequality to several classes of bi-univalent functions. Deniz et al. [21] discussed the upper bounds of $H_2(2)$.

Now we introduce a new subclass of bi-univalent functions associated with the Hohlov operator.

Definition 1. *For $0 \leq \lambda \leq 1$ and $J_{a,b;c}$ given by (3), a function $f \in \Sigma$ given by (1) is said to be in the class $M^{a,b;c}_\Sigma(\lambda, \varphi)$ if it satisfies the following subordination conditions:*

$$\lambda\left\{1 + \frac{z(J_{a,b;c}f(z))''}{(J_{a,b;c}f(z))'}\right\} + (1-\lambda)\left\{\frac{z(J_{a,b;c}f(z))'}{J_{a,b;c}f(z)}\right\} \prec \varphi(z) \quad (z \in U)$$

and

$$\lambda\left\{1 + \frac{\omega(J_{a,b;c}g(\omega))''}{(J_{a,b;c}g(\omega))'}\right\} + (1-\lambda)\left\{\frac{\omega(J_{a,b;c}g(\omega))'}{J_{a,b;c}g(\omega)}\right\} \prec \varphi(\omega) \quad (\omega \in U),$$

where $\varphi \in P$ and the function g is the inverse of f given by (4).

Remark 1. *For $a = c$ and $b = 1$ in the above definition, we have $M^{a,1;a}_\Sigma(\lambda, \varphi) = M_\Sigma(\lambda, \varphi)$, introduced and studied by Ali et al. [22].*

To prove our main results, the following lemmas are needed.

Lemma 1 ([23])**.** *Let a function $\nu(z) = \nu_1 z + \nu_2 z^2 + \nu_3 z^3 + \cdots$ be analytic in U, $\nu(0) = 0$ and $|\nu(z)| < 1$, then $|\nu_n| \leq 1$ $(n \in N)$.*

Lemma 2 ([24])**.** *Let $u(z) = \sum_{n=1}^{\infty} u_n z^n$ $(z \in U)$ be a Schwarz function, then*

$$u_2 = x(1 - u_1^2)$$

and

$$u_3 = (1 - u_1^2)(1 - |x|^2)s - u_1(1 - u_1^2)x^2$$

for some complex number x and s satisfying $|x| \leq 1$ and $|s| \leq 1$.

In this paper, we investigate some properties such as the coefficient bounds, Fekete-Szegö inequality and the second Hankel determinant for functions in the class $M_{\Sigma}^{a,b;c}(\lambda, \varphi)$. In particular, several previous results are generalized.

2. Main Results

In this section, we find estimates for the general Taylor–Maclaurin coefficients of the functions in the class $M_{\Sigma}^{a,b;c}(\lambda, \varphi)$.

Theorem 1. *Let $0 \leq \lambda \leq 1$ and the function $f \in \Sigma$ given by (1) belong to the class $M_{\Sigma}^{a,b;c}(\lambda, \varphi)$. Then*

$$|a_2| \leq \min\left\{ \begin{array}{l} \frac{B_1}{(1+\lambda)\psi_2} \\ \frac{B_1\sqrt{B_1}}{\sqrt{\left|\left[2(1+2\lambda)\psi_3-(1+3\lambda)\psi_2^2\right]B_1^2-(1+\lambda)^2B_2\right|}} \end{array} \right.$$

and

$$|a_3| \leq \min\left\{ \begin{array}{l} \frac{B_1}{2(1+2\lambda)\psi_3} + \frac{B_1^2}{(1+\lambda)^2\psi_2^2} \\ \frac{B_1}{2(1+2\lambda)\psi_3} + \frac{B_1^3}{\left|\left[2(1+2\lambda)\psi_3-(1+3\lambda)\psi_2^2\right]B_1^2-(1+\lambda)^2B_2\right|} \end{array} \right. .$$

Proof. Let $f \in \Sigma$ given by (1) belong to the class $M_{\Sigma}^{a,b;c}(\lambda, \varphi)$. There exist two Schwarz functions:

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots$$

and

$$\nu(\omega) = \nu_1\omega + \nu_2\omega^2 + \nu_3\omega^3 + \cdots,$$

such that

$$\lambda\left\{1 + \frac{z(J_{a,b;c}f(z))''}{(J_{a,b;c}f(z))'}\right\} + (1-\lambda)\left\{\frac{z(J_{a,b;c}f(z))'}{J_{a,b;c}f(z)}\right\} = \varphi(u(z)) \tag{5}$$

and

$$\lambda\left\{1 + \frac{\omega(J_{a,b;c}g(\omega))''}{(J_{a,b;c}g(\omega))'}\right\} + (1-\lambda)\left\{\frac{\omega(J_{a,b;c}g(\omega))'}{J_{a,b;c}g(\omega)}\right\} = \varphi(\nu(\omega)), \tag{6}$$

where

$$\varphi(u(z)) = 1 + B_1u_1z + (B_1u_2 + B_2u_1^2)z^2 + (B_1u_3 + 2B_2u_1u_2 + B_3u_1^3)z^3 + \cdots \tag{7}$$

and

$$\varphi(\nu(\omega)) = 1 + B_1\nu_1\omega + (B_1\nu_2 + B_2\nu_1^2)\omega^2 + (B_1\nu_3 + 2B_2\nu_1\nu_2 + B_3\nu_1^3)\omega^3 + \cdots. \tag{8}$$

Since f and $g = f^{-1}$ have the Taylor series expansion (1) and (4), respectively, we obtain

$$\begin{aligned} &\lambda\left\{1 + \frac{z(J_{a,b;c}f(z))''}{(J_{a,b;c}f(z))'}\right\} + (1-\lambda)\left\{\frac{z(J_{a,b;c}f(z))'}{J_{a,b;c}f(z)}\right\} \\ &= 1 + (1+\lambda)\psi_2a_2z + [2(1+2\lambda)\psi_3a_3 - (1+3\lambda)\psi_2^2a_2^2]z^2 \\ &+ [3(1+3\lambda)\psi_4a_4 - 3(1+5\lambda)\psi_2\psi_3a_2a_3 + (1+7\lambda)\psi_2^3a_2^3]z^3 + \cdots \end{aligned} \tag{9}$$

and

$$\lambda\left\{1+\frac{\omega(J_{a,b;c}g(\omega))''}{(J_{a,b;c}g(\omega))'}\right\}+(1-\lambda)\left\{\frac{\omega(J_{a,b;c}g(\omega))'}{J_{a,b;c}g(\omega)}\right\}$$
$$=1+(1+\lambda)\psi_2 b_2\omega+[2(1+2\lambda)\psi_3 b_3-(1+3\lambda)\psi_2^2 b_2^2]\omega^2$$
$$+[3(1+3\lambda)\psi_4 b_4-3(1+5\lambda)\psi_2\psi_3 b_2 b_3+(1+7\lambda)\psi_2^3 b_2^3]\omega^3+\cdots. \tag{10}$$

Now, from (5), (7) and (9), we obtain

$$(1+\lambda)\psi_2 a_2=B_1 u_1 \tag{11}$$

and

$$2(1+2\lambda)\psi_3 a_3-(1+3\lambda)\psi_2^2 a_2^2=B_1 u_2+B_2 u_1^2. \tag{12}$$

Similarly, from (6), (8) and (10), we obtain

$$-(1+\lambda)\psi_2 a_2=B_1\nu_1 \tag{13}$$

and

$$2(1+2\lambda)\psi_3(2a_2^2-a_3)-(1+3\lambda)\psi_2^2 a_2^2=B_1\nu_2+B_2\nu_1^2. \tag{14}$$

It follows from (11) and (13) that

$$a_2=\frac{B_1 u_1}{(1+\lambda)\psi_2}=\frac{-B_1\nu_1}{(1+\lambda)\psi_2}. \tag{15}$$

Thus, we have

$$u_1=-\nu_1 \tag{16}$$

and

$$2(1+\lambda)^2\psi_2^2 a_2^2=B_1^2(u_1^2+\nu_1^2). \tag{17}$$

From (15) and Lemma 1, we obtain

$$|a_2|\leq\frac{B_1}{(1+\lambda)\psi_2}. \tag{18}$$

Adding (12) to (14), we obtain

$$a_2^2=\frac{B_1^3(u_2+\nu_2)}{\left[4(1+2\lambda)\psi_3-2(1+3\lambda)\psi_2^2\right]B_1^2-2(1+\lambda)^2 B_2}. \tag{19}$$

Therefore, by using Lemma 1, we have

$$|a_2|^2\leq\frac{B_1^3}{\left|\left[2(1+2\lambda)\psi_3-(1+3\lambda)\psi_2^2\right]B_1^2-(1+\lambda)^2 B_2\right|}. \tag{20}$$

It follows that

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|\left[2(1+2\lambda)\psi_3-(1+3\lambda)\psi_2^2\right]B_1^2-(1+\lambda)^2 B_2\right|}}.$$

Subtracting (14) from (12) and with some calculations, we obtain

$$a_3=\frac{B_1(u_2-\nu_2)}{4(1+2\lambda)\psi_3}+a_2^2. \tag{21}$$

By using Lemma 1, we obtain

$$|a_3| \leq \frac{B_1}{2(1+2\lambda)\psi_3} + |a_2|^2. \tag{22}$$

Putting (18) into (22), we have

$$|a_3| \leq \frac{B_1}{2(1+2\lambda)\psi_3} + \frac{B_1^2}{(1+\lambda)^2\psi_2^2}.$$

Similarly, putting (20) into (22), we obtain

$$|a_3| \leq \frac{B_1}{2(1+2\lambda)\psi_3} + \frac{B_1^3}{\left|\left[2(1+2\lambda)\psi_3 - (1+3\lambda)\psi_2^2\right]B_1^2 - (1+\lambda)^2 B_2\right|}.$$

□

This completes the proof of Theorem 1.

For $a = c$ and $b = 1$ in Theorem 1, we obtain a result of the class $M_\Sigma(\lambda, \varphi)$, considered by Ali et al. [22].

Corollary 1. *Let $0 \leq \lambda \leq 1$ and the function $f \in \Sigma$ given by (1) belong to the class $M_\Sigma(\lambda, \varphi)$. Then*

$$|a_2| \leq \min\left\{ \begin{array}{l} \frac{B_1}{1+\lambda} \\ \frac{B_1\sqrt{B_1}}{\sqrt{(1+\lambda)\left|B_1^2 - (1+\lambda)B_2\right|}} \end{array} \right.$$

and

$$|a_3| \leq \min\left\{ \begin{array}{l} \frac{B_1}{2(1+2\lambda)} + \frac{B_1^2}{(1+\lambda)^2} \\ \frac{B_1}{2(1+2\lambda)} + \frac{B_1^3}{(1+\lambda)\left|B_1^2 - (1+\lambda)B_2\right|} \end{array} \right. .$$

Theorem 2. *Let $0 \leq \lambda \leq 1$, $\sigma \in C$ and the function $f \in \Sigma$ given by (1) belong to the class $M_\Sigma^{a,b;c}(\lambda, \varphi)$. Then*

$$|a_3 - \sigma a_2^2| \leq \left\{ \begin{array}{ll} \frac{B_1}{2(1+2\lambda)\psi_3} & \left(0 \leq |h(\sigma)| \leq \frac{B_1}{4(1+2\lambda)\psi_3}\right) \\ 2|h(\sigma)| & \left(|h(\sigma)| > \frac{B_1}{4(1+2\lambda)\psi_3}\right), \end{array} \right.$$

where

$$h(\sigma) = \frac{(1-\sigma)B_1^3}{\left[4(1+2\lambda)\psi_3 - 2(1+3\lambda)\psi_2^2\right]B_1^2 - 2(1+\lambda)^2 B_2}. \tag{23}$$

Proof. From (21), we get

$$a_3 - \sigma a_2^2 = \frac{B_1(u_2 - v_2)}{4(1+2\lambda)\psi_3} + (1-\sigma)a_2^2. \tag{24}$$

Putting (19) into (24), we have

$$\begin{aligned} a_3 - \sigma a_2^2 &= \frac{B_1(u_2 - v_2)}{4(1+2\lambda)\psi_3} + \frac{B_1^3(1-\sigma)(u_2+v_2)}{\left[4(1+2\lambda)\psi_3 - 2(1+3\lambda)\psi_2^2\right]B_1^2 - 2(1+\lambda)^2 B_2} \\ &= \left(h(\sigma) + \frac{B_1}{4(1+2\lambda)\psi_3}\right)u_2 + \left(h(\sigma) - \frac{B_1}{4(1+2\lambda)\psi_3}\right)v_2, \end{aligned} \tag{25}$$

where $h(\sigma)$ is given by (23).

From (25) and Lemma 1, we derive

$$|a_3-\sigma a_2^2|\le\begin{cases}\frac{B_1}{2(1+2\lambda)\psi_3} & \left(0\le|h(\sigma)|\le\frac{B_1}{4(1+2\lambda)\psi_3}\right)\\ 2|h(\sigma)| & \left(|h(\sigma)|>\frac{B_1}{4(1+2\lambda)\psi_3}\right).\end{cases}$$

□

This completes the proof of Theorem 2.

For $a=c$ and $b=1$ in Theorem 2, we obtain a result of the class $M_\Sigma(\lambda,\varphi)$, introduced by Ali et al. [22].

Corollary 2. *Let $0\le\lambda\le1$, $\sigma\in C$ and the function $f\in\Sigma$ given by (1) be in the class $M_\Sigma(\lambda,\varphi)$. Then*

$$|a_3-\sigma a_2^2|\le\begin{cases}\frac{B_1}{2(1+2\lambda)} & \left(0\le|h(\sigma)|\le\frac{B_1}{4(1+2\lambda)}\right)\\ 2|h(\sigma)| & \left(|h(\sigma)|>\frac{B_1}{4(1+2\lambda)}\right),\end{cases}$$

where

$$h(\sigma)=\frac{(1-\sigma)B_1^3}{2(1+\lambda)(B_1^2-(1+\lambda)B_2)}.$$

Theorem 3. *Let $0\le\lambda\le1$ and the function $f\in\Sigma$ given by (1) belong to the class $M_\Sigma^{a,b;c}(\lambda,\varphi)$. Then*

$$|a_2a_4-a_3^2|\le B_1\begin{cases}Q_3 & (Q_2\le0,Q_1\le-Q_2)\\ Q_1+Q_2+Q_3 & \left(Q_2>0,Q_1>-\frac{Q_2}{2}\right)\text{ or }(Q_2\le0,Q_1>-Q_2)\\ \frac{4Q_1Q_3-Q_2^2}{4Q_1} & \left(Q_2>0,Q_1\le-\frac{Q_2}{2}\right),\end{cases}$$

where

$$Q_1=\left|\frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4}+\frac{[3(1+5\lambda)\psi_2\psi_3-(1+7\lambda)\psi_2^3-3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4}\right|$$
$$-\frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3}-\frac{2|B_2|+B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4}+\frac{B_1}{4(1+2\lambda)^2\psi_3^2},$$
$$Q_2=\frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3}+\frac{2|B_2|+B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4}-\frac{B_1}{2(1+2\lambda)^2\psi_3^2}$$

and

$$Q_3=\frac{B_1}{4(1+2\lambda)^2\psi_3^2}.$$

Proof. From (5), (7) and (9), we have

$$\begin{aligned}&3(1+3\lambda)\psi_4a_4-3(1+5\lambda)\psi_2\psi_3a_2a_3+(1+7\lambda)\psi_2^3a_2^3\\&=B_1u_3+2B_2u_1u_2+B_3u_1^3.\end{aligned}\tag{26}$$

Similarly, from (6), (8) and (10), we obtain

$$\begin{aligned}&-3(1+3\lambda)\psi_4(5a_2^3-5a_2a_3+a_4)+3(1+5\lambda)\psi_2\psi_3a_2(2a_2^2-a_3)-(1+7\lambda)\psi_2^3a_2^3\\&=B_1\nu_3+2B_2\nu_1\nu_2+B_3\nu_1^3.\end{aligned}\tag{27}$$

Subtracting (27) from (26) and with some calculations, we have

$$a_4 = \frac{B_1(u_3-\nu_3)}{6(1+3\lambda)\psi_4} + \frac{B_2u_1(u_2+\nu_2)}{3(1+3\lambda)\psi_4} + \frac{B_3u_1^3}{3(1+3\lambda)\psi_4} + \frac{5}{2}a_2a_3 + \frac{[6(1+5\lambda)\psi_2\psi_3 - 15(1+3\lambda)\psi_4 - 2(1+7\lambda)\psi_2^3]}{6(1+3\lambda)\psi_4}a_2^3. \tag{28}$$

From (15) and (21), we obtain

$$a_4 = \frac{B_1(u_3-\nu_3)}{6(1+3\lambda)\psi_4} + \frac{B_2u_1(u_2+\nu_2)}{3(1+3\lambda)\psi_4} + \frac{B_3u_1^3}{3(1+3\lambda)\psi_4} + \frac{5B_1^2u_1(u_2-\nu_2)}{8(1+\lambda)(1+2\lambda)\psi_2\psi_3} + \frac{[3(1+5\lambda)\psi_3 - (1+7\lambda)\psi_2^2]B_1^3u_1^3}{3(1+\lambda)^3(1+3\lambda)\psi_2^2\psi_4}. \tag{29}$$

Thus, we obtain

$$\begin{aligned} a_2a_4 - a_3^2 = &\left\{ \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^4}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} + \frac{B_1B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4}\right\}u_1^4 \\ &+ \frac{B_1^3u_1^2(u_2-\nu_2)}{8(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} + \frac{B_1B_2u_1^2(u_2+\nu_2)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\ &+ \frac{B_1^2u_1(u_3-\nu_3)}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} - \frac{B_1^2(u_2-\nu_2)^2}{16(1+2\lambda)^2\psi_3^2}. \end{aligned} \tag{30}$$

By using Lemma 2, we derive

$$u_2 = x(1-u_1^2),\ \ \nu_2 = y(1-\nu_1^2),$$

$$u_3 = (1-u_1^2)(1-|x|^2)s - u_1(1-u_1^2)x^2$$

and

$$\nu_3 = (1-\nu_1^2)(1-|y|^2)h - \nu_1(1-\nu_1^2)y^2,$$

where $|x| \le 1$, $|y| \le 1$, $|s| \le 1$ and $|h| \le 1$. With some calculations, we obtain

$$u_2+\nu_2 = (1-u_1^2)(x+y),\ \ u_2-\nu_2 = (1-u_1^2)(x-y), \tag{31}$$

$$u_3-\nu_3 = (1-u_1^2)\Big\lfloor(1-|x|^2)s - (1-|y|^2)h\Big\rfloor - u_1(1-u_1^2)(x^2+y^2). \tag{32}$$

By substituting the relations (31) and (32) into (30), we have

$$\begin{aligned} a_2a_4 - a_3^2 = &\left\{ \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^4}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} + \frac{B_1B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4}\right\}u_1^4 \\ &+ \frac{B_1^3u_1^2(1-u_1^2)(x-y)}{8(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} + \frac{B_1B_2u_1^2(1-u_1^2)(x+y)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\ &+ \frac{B_1^2u_1(1-u_1^2)\{[(1-|x|^2)s - (1-|y|^2)h] - u_1(x^2+y^2)\}}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\ &- \frac{B_1^2(1-u_1^2)^2(x-y)^2}{16(1+2\lambda)^2\psi_3^2}. \end{aligned}$$

It follows that

$$
\begin{aligned}
|a_2a_4 - a_3^2| = \Bigg| \Bigg\{ & \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^4}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} \\
& + \frac{B_1B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg\} u_1^4 \\
& + \frac{B_1^2u_1(1-u_1^2)\left[(1-|x|^2)s - (1-|y|^2)h\right]}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\
& + \frac{B_1^3u_1^2(1-u_1^2)(x-y)}{8(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} + \frac{B_1B_2u_1^2(1-u_1^2)(x+y)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\
& - \frac{B_1^2u_1^2(1-u_1^2)(x^2+y^2)}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} - \frac{B_1^2(1-u_1^2)^2(x-y)^2}{16(1+2\lambda)^2\psi_3^2} \Bigg|.
\end{aligned}
$$

According to Lemmas 1 and 2, we assume without restriction that $u = u_1 \in [0,1]$. By applying the triangular inequality, we obtain

$$
\begin{aligned}
|a_2a_4 - a_3^2| \le B_1 \Bigg\{ & \Bigg| \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} \\
& + \frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg| u^4 + \frac{B_1^2u^2(1-u^2)(|x|+|y|))}{8(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} \\
& + \frac{|B_2|u^2(1-u^2)(|x|+|y|)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{B_1(u^2-u)(1-u^2)(|x|^2+|y|^2)}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} \\
& + \frac{B_1u(1-u^2)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{B_1(1-u^2)^2(|x|+|y|)^2}{16(1+2\lambda)^2\psi_3^2} \Bigg\}.
\end{aligned}
$$

Now, letting $\eta = |x| \le 1$ and $\gamma = |y| \le 1$, we have

$$
|a_2a_4 - a_3^2| \le B_1\Big[T_1 + (\eta+\gamma)T_2 + (\eta^2+\gamma^2)T_3 + (\eta+\gamma)^2T_4\Big] = B_1F(\eta,\gamma),
$$

where

$$
\begin{aligned}
T_1 = T_1(u) = & \Bigg\{ \Bigg| \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} \\
& + \frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg| u^4 + \frac{B_1u(1-u^2)}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg\} \ge 0, \\
T_2 = T_2(u) = & \frac{u^2(1-u^2)}{(1+\lambda)\psi_2}\left[\frac{B_1^2}{8(1+\lambda)(1+2\lambda)\psi_2\psi_3} + \frac{|B_2|}{3(1+3\lambda)\psi_4}\right] \ge 0, \\
T_3 = T_3(u) = & \frac{B_1(u^2-u)(1-u^2)}{6(1+\lambda)(1+3\lambda)\psi_2\psi_4} \le 0
\end{aligned}
$$

and

$$
T_4 = T_4(u) = \frac{B_1(1-u^2)^2}{16(1+2\lambda)^2\psi_3^2} \ge 0.
$$

Next, we need to maximize the function $F(\eta,\gamma)$ in the closed square

$$
\Delta = \{(\eta,\gamma) : \eta \in [0,1], \gamma \in [0,1]\}
$$

for $u \in [0,1]$. Since $F(\eta,\gamma)$ is the maximum with regard to u, we must investigate it according to $u = 0$, $u = 1$ and $u \in (0,1)$.

For $u = 0$,

$$F(\eta,\gamma) = \frac{B_1(\eta+\gamma)^2}{16(1+2\lambda)^2\psi_3^2},$$

we can easily obtain

$$\max\{F(\eta,\gamma) : (\eta,\gamma) \in [0,1]\times[0,1]\} = F(1,1) = \frac{B_1}{4(1+2\lambda)^2\psi_3^2}.$$

For $u = 1$,

$$F(\eta,\gamma) = \left|\frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4}\right|,$$

we have

$$\begin{aligned}&\max\{F(\eta,\gamma) : (\eta,\gamma) \in [0,1]\times[0,1]\}\\ &= \left|\frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4}\right|.\end{aligned}$$

For $0 < u < 1$, by letting $\eta + \gamma = \varsigma$ and $\eta \cdot \gamma = \xi$, we obtain

$$F(\eta,\gamma) = T_1 + T_2\varsigma + (T_3+T_4)\varsigma^2 - 2T_3\xi = J(\varsigma,\xi),$$

where $\varsigma \in [0,2]$ and $\xi \in [0,1]$. Then we need to maximize the function:

$$J(\varsigma,\xi) \in \Lambda = \{(\varsigma,\xi) : \varsigma \in [0,2], \xi \in [0,1]\}.$$

By differentiating $J(\varsigma,\xi)$, we let

$$\begin{cases} \frac{\partial J}{\partial \varsigma} = T_2 + 2(T_3+T_4)\varsigma = 0 \\ \frac{\partial J}{\partial \xi} = -2T_3 = 0. \end{cases}$$

The above results show that $J(\varsigma,\xi)$ does not have a critical point in Λ. Therefore, the function $F(\eta,\gamma)$ does not have a critical point in Δ. As a result, the function $F(\eta,\gamma)$ cannot have a local maximum value in the interior of the square Δ. Next, we find the maximum of $F(\eta,\gamma)$ on the boundary of the square Δ.

For $\eta = 0$ and $0 \le \gamma \le 1$ (or $\gamma = 0$ and $0 \le \eta \le 1$), we have

$$F(0,\gamma) = H(\gamma) = T_1 + \gamma T_2 + \gamma^2(T_3+T_4).$$

In order to investigate the maximum of $H(\gamma)$, the situation of $H(\gamma)$ as increasing or decreasing is discussed below. By deriving the function $H(\gamma)$, we have

$$H'(\gamma) = T_2 + 2\gamma(T_3+T_4).$$

(i) Let $T_3 + T_4 \ge 0$, then $H'(\gamma) > 0$, such that $H(\gamma)$ is an increasing function. Thus, the maximum of $H(\gamma)$ occurs at $\gamma = 1$ and

$$\max\{H(\gamma) : \gamma \in [0,1]\} = H(1) = T_1 + T_2 + T_3 + T_4.$$

(ii) Let $T_3 + T_4 < 0$. We need to consider the critical point $\gamma = \frac{T_2}{-2(T_3+T_4)} = \frac{T_2}{2\theta}$, where $\theta = -(T_3+T_4) > 0$. Now the following two cases arise:

Case 1. Suppose that $\gamma = \frac{T_2}{2\theta} > 1$. Then $\theta < \frac{T_2}{2} \leq T_2$ and $T_2 + T_3 + T_4 \geq 0$. We have

$$H(0) = T_1 \leq T_1 + T_2 + T_3 + T_4 = H(1).$$

Case 2. Suppose that $\gamma = \frac{T_2}{2\theta} \leq 1$. Since $\frac{T_2}{2} \geq 0$ and $\frac{T_2^2}{4\theta} \leq \frac{T_2}{2} \leq T_2$, we obtain

$$H(0) = T_1 \leq T_1 + \frac{T_2^2}{4\theta} = H\left(\frac{T_2}{2\theta}\right) \leq T_1 + T_2$$

and

$$H(1) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2.$$

Therefore, the maximum of $H(\gamma)$ occurs when $T_3 + T_4 \geq 0$:

$$\max\{H(\gamma) : \gamma \in [0,1]\} = H(1) = T_1 + T_2 + T_3 + T_4.$$

For $\eta = 1$ and $0 \leq \gamma \leq 1$ (or $\gamma = 1$ and $0 \leq \eta \leq 1$), we have

$$F(1,\gamma) = D(\gamma) = T_1 + T_2 + T_3 + T_4 + \gamma(T_2 + 2T_4) + \gamma^2(T_3 + T_4).$$

In order to investigate the maximum of $D(\gamma)$, the scenario of when $D(\gamma)$ in increasing or decreasing is discussed. By deriving the function $D(\gamma)$, we have

$$D'(\gamma) = T_2 + 2T_4 + 2\gamma(T_3 + T_4).$$

(iii) Let $T_3 + T_4 \geq 0$ then $D'(\gamma) > 0$. This shows that $D(\gamma)$ is an increasing function. Thus, the maximum of $D(\gamma)$ occurs at $\gamma = 1$:

$$\max\{D(\gamma) : \gamma \in [0,1]\} = D(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

(iv) Let $T_3 + T_4 < 0$. We need to consider the critical point $\gamma = \frac{T_2+2T_4}{-2(T_3+T_4)} = \frac{T_2+2T_4}{2\theta}$, where $\theta = -(T_3 + T_4) > 0$. The following two cases arise:

Case 3. Suppose that $\gamma = \frac{T_2+2T_4}{2\theta} > 1$. Then $\theta < \frac{T_2+2T_4}{2} \leq T_2 + 2T_4$ and $T_2 + T_3 + 3T_4 \geq 0$. We have

$$D(0) = T_1 + T_2 + T_3 + T_4 \leq T_1 + T_2 + T_3 + T_4 + (T_2 + T_3 + 3T_4) = D(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Case 4. Suppose that $\gamma = \frac{T_2+2T_4}{2\theta} \leq 1$. Since $\frac{T_2+T_4}{2} \geq 0$ and $\frac{(T_2+2T_4)^2}{4\theta} \leq \frac{T_2+2T_4}{2} \leq T_2 + 2T_4$, we obtain

$$\begin{aligned} D(0) = T_1 + T_2 + T_3 + T_4 &\leq D\left(\frac{T_2 + 2T_4}{2\theta}\right) \\ &\leq T_1 + T_2 + T_3 + T_4 + T_2 + 2T_4 \\ &= T_1 + 2T_2 + T_3 + 3T_4 \end{aligned}$$

and

$$D(1) = T_1 + 2T_2 + 2T_3 + 4T_4 \leq T_1 + 2T_2 + T_3 + 3T_4.$$

Therefore, the maximum of $D(\gamma)$ occurs when $T_3 + T_4 \geq 0$:

$$\max\{D(\gamma) : \gamma \in [0,1]\} = D(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

Since $H(1) \leq D(1)$ for $u \in (0,1)$, we have

$$\max\{F(\eta,\gamma) : (\eta,\gamma) \in [0,1] \times [0,1]\} = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4. \tag{33}$$

Let $K : [0,1] \to R$,

$$\begin{aligned} K(u) &= B_1 \max\{F(\eta,\gamma) : (\eta,\gamma) \in [0,1] \times [0,1]\} \\ &= B_1 F(1,1) = B_1(T_1 + 2T_2 + 2T_3 + 4T_4). \end{aligned}$$

Now, inserting T_1, T_2, T_3 and T_4 into the function K, we obtain

$$\begin{aligned} K(u) = B_1 \Bigg\{ \Bigg[\Bigg| & \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} \\ & + \frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg| - \frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} \\ & - \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{B_1}{4(1+2\lambda)^2\psi_3^2} \Bigg] u^4 \\ & + \Bigg[\frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} - \frac{B_1}{2(1+2\lambda)^2\psi_3^2} \\ & + \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} \Bigg] u^2 + \frac{B_1}{4(1+2\lambda)^2\psi_3^2} \Bigg\}. \end{aligned} \tag{34}$$

Letting $u^2 = t$, we have

$$K(t) = B_1(Q_1 t^2 + Q_2 t + Q_3) \quad (t \in [0,1]),$$

where

$$\begin{aligned} Q_1 = & \left| \frac{B_3}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{[3(1+5\lambda)\psi_2\psi_3 - (1+7\lambda)\psi_2^3 - 3(1+3\lambda)\psi_4]B_1^3}{3(1+\lambda)^4(1+3\lambda)\psi_2^4\psi_4} \right| \\ & - \frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} - \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} + \frac{B_1}{4(1+2\lambda)^2\psi_3^2}, \end{aligned}$$

$$Q_2 = \frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)\psi_2^2\psi_3} + \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)\psi_2\psi_4} - \frac{B_1}{2(1+2\lambda)^2\psi_3^2}$$

and

$$Q_3 = \frac{B_1}{4(1+2\lambda)^2\psi_3^2}.$$

Since

$$\begin{aligned} & \max_{0 \le t \le 1} (Q_1 t^2 + Q_2 t + Q_3) \\ & = \begin{cases} Q_3 & (Q_2 \le 0, Q_1 \le -Q_2) \\ Q_1 + Q_2 + Q_3 & \left(Q_2 > 0, Q_1 > -\frac{Q_2}{2}\right) \text{ or } (Q_2 \le 0, Q_1 > -Q_2) \\ \frac{4Q_1Q_3 - Q_2^2}{4Q_1} & \left(Q_2 > 0, Q_1 \le -\frac{Q_2}{2}\right), \end{cases} \end{aligned}$$

it shows that

$$|a_2 a_4 - a_3^2| \le B_1 \begin{cases} Q_3 & (Q_2 \le 0, Q_1 \le -Q_2) \\ Q_1 + Q_2 + Q_3 & \left(Q_2 > 0, Q_1 > -\frac{Q_2}{2}\right) \text{ or } (Q_2 \le 0, Q_1 > -Q_2) \\ \frac{4Q_1Q_3 - Q_2^2}{4Q_1} & \left(Q_2 > 0, Q_1 \le -\frac{Q_2}{2}\right). \end{cases}$$

□

This completes the proof of Theorem 3.

For $a = c$ and $b = 1$ in Theorem 3, we derive a result of the class $M_\Sigma(\lambda, \varphi)$, studied by Ali et al. [22].

Corollary 3. *Let $0 \leq \lambda \leq 1$ and the function $f \in \Sigma$ given by (1) be in the class $M_\Sigma(\lambda, \varphi)$. Then*

$$|a_2a_4 - a_3^2| \leq B_1 \begin{cases} Q_3 & (Q_2 \leq 0, Q_1 \leq -Q_2) \\ Q_1 + Q_2 + Q_3 & \left(Q_2 > 0, Q_1 > -\frac{Q_2}{2}\right) \text{ or } (Q_2 \leq 0, Q_1 > -Q_2) \\ \frac{4Q_1Q_3 - Q_2^2}{4Q_1} & \left(Q_2 > 0, Q_1 \leq -\frac{Q_2}{2}\right). \end{cases}$$

where

$$Q_1 = \left|\frac{B_3}{3(1+\lambda)(1+3\lambda)} - \frac{B_1^3}{3(1+\lambda)^3(1+3\lambda)}\right| - \frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)} - \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)} + \frac{B_1}{4(1+2\lambda)^2},$$

$$Q_2 = \frac{B_1^2}{4(1+\lambda)^2(1+2\lambda)} + \frac{2|B_2| + B_1}{3(1+\lambda)(1+3\lambda)} - \frac{B_1}{2(1+2\lambda)^2}$$

and

$$Q_3 = \frac{B_1}{4(1+2\lambda)^2}.$$

3. Conclusions

In the study of bi-univalent functions, estimates on the first two Taylor–Maclaurin coefficients are usually considered. In this paper, we introduce a new subclass of bi-univalent functions associated with the Hohlov operator. Some properties such as the coefficient bounds, Fekete-Szegö inequality and the second Hankel determinant for functions in $M_\Sigma^{a,b;c}(\lambda, \varphi)$ are derived. In particular, several previous results are generalized.

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References

1. Hohlov, Y.E. Hadamard convolutions, hypergeometric functions and linear operators in the class of univalent functions. *Doki. Akad. Nauk Ukrain. SSR Ser. A* **1984**, *7*, 25–27.
2. Hohlov, Y.E. Convolution operators that preserve univalent functions. *Ukrain. Mat. Zh.* **1985**, *37*, 220–226.
3. Brannan, D.A.; Taha, T.S. On some classes of bi-unvalent functions. *Math. Anal. Appl.* **1988**, 53–60. [CrossRef]
4. Aldawish, I.; Al-Hawary, T.; Frasin, B.A. Subclasses of bi-univalent functions defined by Frasin differential operator. *Mathematics* **2020**, *8*, 783. [CrossRef]
5. Khan, B.; Srivastava, H.M.; Tahir, M.; Darus, M.; Ahmad, Q.Z.; Khan, N. Applications of a certain q-integral operator to the subclasses of analytic and bi-univalent functions. *AIMS Math.* **2020**, *6*, 1024–1039. [CrossRef]
6. Murugusundaramoorthy, G.; Cotîrlă, L.I. Bi-univalent functions of complex order defined by Hohlov operator associated with Legendrae polynomial. *AIMS Math.* **2022**, *7*, 8733–8750. [CrossRef]
7. Srivastava, H.M.; Motamednezhad, A.; Adegan, E.A. Faber polynomial coefficient estimates for bi-univalent functions defined by using differential subordination and a certain fractional derivative operator. *Mathematics* **2020**, *8*, 172. [CrossRef]
8. Shaba, T.G. Subclasses of bi-univalent functions satisfying subordinate conditions defined by Frasin differential operator. *Turk. J. Ineq.* **2020**, *4*, 50–58.

9. Zhai, J.; Srivastava, R.; Liu, J.-L. Faber polynomial coefficient estimates of bi-close-to-convex functions associated with generalized hypergeometric functions. *Mathematics* **2022**, *10*, 3073. [CrossRef]
10. Páll-Szabó, Á.O.; Oros, G.I. Coefficient related studies for new classes of bi-univalent functions. *Mathematics* **2020**, *8*, 1110. [CrossRef]
11. Cotîrlă, L.I. New classes of analytic and bi-univalent functions. *AIMS Math.* **2021**, *6*, 10642–10651. [CrossRef]
12. Yousef, F.; Amourah, A.; Frasin, B.A.; Bulboacă, T. An avant-Garde construction for subclasses of analytic bi-univalent functions. *Axioms* **2022**, *11*, 267. [CrossRef]
13. Noonan, J.W.; Thomas, D.K. On the Second Hankel Determinant of a really mean p-valent functions. *Trans. Am. Math. Soc.* **1976**, *223*, 337–346.
14. Fekete, M.; Szegö, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1933**, *2*, 85–89. [CrossRef]
15. Ali, R.M.; Lee, S.K.; Ravichandran, V.; Supramaniam, S. The Fekete-Szegö coefficient functional for transforms of analytic functions. *Bull. Iranian Math. Soc.* **2009**, *35*, 119–142.
16. Altinkaya, S.; Yalcin, S. Fekete-Szegö inequalities for classes of bi-univalent functions defined by subordination. *Adv. Math. Sci. J.* **2014**, *3*, 63–71.
17. Deekonda, V.K.; Thoutreedy, R. An upper bound to the second Hankel determinant for functions in Mocanu class. *Vietnam J. Math.* **2015**, *43*, 541–549. [CrossRef]
18. Janteng, A.; Halim, S.A.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math. Anal.* **2007**, *1*, 619–625.
19. Lee, S.K.; Ravichandran, V.; Supramaniam, S. Bounds for the second Hankel determinant of certain univalent functions. *J. Ineq. Appl.* **2013**, *281*, 1–17. [CrossRef]
20. Zaprawa, P. On the Fekete-Szegö problems for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin* **2014**, *21*, 169–178. [CrossRef]
21. Deniz, E.; Caglar, M.; Orhan, H. Second Hankel determinant for bi-starlike and bi-convex functions of order β. *Appl. Math. Comput.* **2015**, *271*, 301–307. [CrossRef]
22. Ali, R.M.; Lee, S.K.; Ravichandran, V.; Subramaniam, S. Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Appl. Math. Lett.* **2012**, *25*, 344–351. [CrossRef]
23. Duren, P.L. *Univalent Functions. Grundlehren der Mathematischen Wissenschaften, Band 259*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983.
24. Kanas, S.; Adegani, E.A.; Zireh, A. An unified approach to second Hankel determinant of bi-subordinate functions. *Mediterr. J. Math.* **2017**, *14*, 233. [CrossRef]