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Quaternionic Fuzzy Sets

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Abstract: A novel concept of quaternionic fuzzy sets (QFSs) is presented in this paper. QFSs are a generalization of traditional fuzzy sets and complex fuzzy sets based on quaternions. The novelty of QFSs is that the range of the membership function is the set of quaternions with modulus less than or equal to one, of which the real and quaternionic imaginary parts can be used for four different features. A discussion is made on the intuitive interpretation of quaternion-valued membership grades and the possible applications of QFSs. Several operations, including quaternionic fuzzy complement, union, intersection, and aggregation of QFSs, are presented. Quaternionic fuzzy relations and their composition are also investigated. QFS is designed to maintain the advantages of traditional FS and CFS, while benefiting from the properties of quaternions. Cuts of QFSs and rotational invariance of quaternionic fuzzy operations demonstrate the particularity of quaternion-valued grades of membership.

Keywords: complex fuzzy sets; quaternionic fuzzy sets; quaternion-valued grades of membership; cuts; rotational invariance

MSC: 03E72

1. Introduction

In 1965, Zadeh [1] proposed the concept of fuzzy sets (FSs). In the past few decades, following Zadeh's pioneering work, various extensions of FSs have been given, enriching the contents of fuzzy theories and fuzzy methods. These extensions include interval-valued fuzzy sets [2], intuitionistic fuzzy sets (IFS) [3], Pythagorean fuzzy sets (PFS) [4], Fermatean fuzzy sets (FFS) [5,6], q-rung orthopair fuzzy sets (q-ROFS) [7], (2,1)-fuzzy sets [8], neutrosophic sets (NS) [9], hesitant fuzzy sets (HFS) [10], and complex fuzzy sets (CFS) [11,12]. In these extension of FSs, IFS is obtained by adding a non-membership value. PFS, q-ROFS, and (2,1)-FS have different restrictions on membership and non-membership values. Further, NS is obtained by adding an indeterminacy value. Another method is based on the algebraic extensions of number fields. The extension of crisp sets to FSs is mathematically analogous to the extension of integers $\mathbb{Z}$ to real numbers $\mathbb{R}$. In much the same way, Ramot et al's [11,12] extension of FSs to complex fuzzy sets (CFSs) is mathematically analogous to the extension of real numbers $\mathbb{R}$ to complex numbers $\mathbb{C}$. CFSs have numerous applications in signal processing [13,14], time series prediction [6,15–17], decision making [18–22], and complex fuzzy logic systems [23–25]. Note that the extension of fuzzy numbers to Buckley's [26] fuzzy complex numbers is also mathematically analogous to the extension of real numbers $\mathbb{R}$ to complex numbers $\mathbb{C}$.

Of course, the extensions of number fields do not end with complex numbers. As early as in 1843, Hamilton [27] discovered the quaternions as a generalization of complex numbers. Quaternion is an important mathematical tool in physics [28,29], quaternion neural networks [30,31], and computer science [32–34].

Interestingly, some scholars attempted to use quaternions in the fuzzy theories and applications. Ngan et al. [35] generalized and expanded the utility of complex intuitionistic fuzzy sets using the space of quaternion numbers. Pan et al. [36] proposed a quaternion model of a Pythagorean fuzzy set. These ideas are one step away from the innovative

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concept of quaternionic fuzzy sets (QFS). In this paper, as shown in Figure 1, we generalize the FS and CFS to QFS, which is similar to the generalization of real and complex numbers to quaternions. The QFS is characterized by a quaternion-valued membership function. The main advantage of the new concept of QFS is its representation of four composite features, which is more powerful than the representation of CFS.

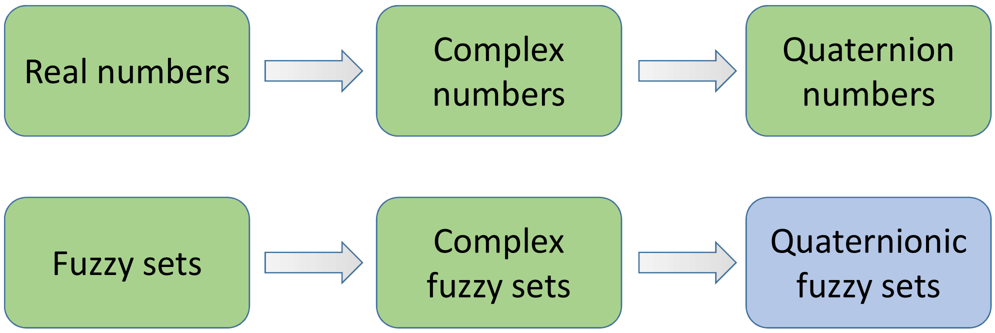

Figure 1. The relations among different concepts.

The motivation of this paper comes from the following two aspects:

1. As discussed above, the quaternion is an excellent mathematical tool in a number of different areas. Interestingly, some scholars used quaternions to handle complex intuitionistic fuzzy information and Pythagorean fuzzy information. Therefore, the quaternion is a novel mathematical tool to handle uncertain information.
2. CFS provides a way to extend the FS theory based on number fields. Moreover, CFS has been widely applied and is undergoing rapid progress, and it deserves further pursuit. The field of quaternions is another fundamental number field that we cannot ignore, so we continue to extend the CFS theory based on number fields.

Based on the aforementioned considerations, in this paper, as an extension of FS and CFS theories, we first introduce the novel QFS that have not been studied in the literature. Additionally, we introduce several fundamental operations. Comparatively, our proposed QFS and its operations have the following advantages.

1. The new concept of QFS is more comprehensive than CFS because the latter is a special case of the former. Both polar representation and Cartesian representation of QFS are given.
2. The proposed negation, join, and meet operations of QFSs are also extensions of Ramot et al's complex fuzzy negation, join, and meet operations. De Morgan's laws of quaternionic fuzzy negation, union, and intersection are studied. This means that these operations could be interconnected by an algebraic structure.

This article is structured as follows. In Section 2, we present some preliminary concepts of quaternions. In Section 3, we introduce the QFS. In Section 4, we study the cuts of QFS. In Section 5, we define several operations of QFS. In Section 6, we study the quaternionic fuzzy relations. In Section 7, we study the rotational invariance of quaternionic fuzzy operations. In Section 8, concluding remarks are offered.

2. Preliminaries

Quaternions

Quaternions were first proposed by Hamilton [27]. For an review of quaternions, we refer the reader to Refs. [33,34].

Let $\mathbb{H}$ be the set of quaternions. A quaternion $q \in \mathbb{H}$ is expressed as

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{1}$$

where $q_r \in \mathbb{R}$ ($r = 0, 1, 2, 3$) and i, j, and k are quaternion units, which obey

$$i^2 = j^2 = k^2 = ijk = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j. \quad (2)$$

The real and quaternionic imaginary parts of q are $Re(q) = q_0$ and $Qim(q) = q_1 i + q_2 j + q_3 k$, respectively.

For a quaternion h, its "quaternion conjugate" $\bar{h}$ is

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k, \quad (3)$$

and its modulus is $|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$.

A polar representation of q is

$$q = |q| e^{i\theta} e^{j\psi} e^{j\omega}, \quad (4)$$

where $(\theta, \psi, \omega) \in [-\pi, \pi) \times [-\pi/2, \pi/2) \times [-\pi/4, \pi/4]$ represent the three quaternionic phases.

For any two quaternions $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$, their addition is

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k, \quad (5)$$

their product is

$$\begin{aligned} pq = {} & (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) \\ & + (p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2)i \\ & + (p_0 q_2 + p_2 q_0 - p_1 q_3 + p_3 q_1)j \\ & + (p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1)k. \end{aligned} \quad (6)$$

Obviously, the product is noncommutative, i.e., $pq \neq qp$. However, we have $|pq| = |p||q|$.

Their dot product is

$$p \bullet q = p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3. \quad (7)$$

Obviously, the dot product is commutative.

3. Introducing Quaternionic Fuzzy Sets

In this section, we give a formal definition of QFS and present two intuitive interpretations of quaternion-valued membership functions.

3.1. Definition of the Quaternionic Fuzzy Set

Definition 1. *Let U be a universe of discourse and $\mathbf{Q}$ be the set of quaternions whose modulus is less than or equal to 1, i.e.,*

$$\mathbf{Q} = \{q \in \mathbb{H} \,|\, |q| \leq 1\}, \quad (8)$$

a quaternionic fuzzy set A defined on U is a mapping: $U \to \mathbf{Q}$, which can be represented as the set of ordered pairs:

$$A = \{< x, \rho_A(x) > | x \in U\} \quad (9)$$

where quaternion-valued membership function $\rho_A(x)$ is of the form

$$a_A(x) + i_A(x)i + j_A(x)j + k_A(x)k \quad (10)$$

where all $a_A(x), i_A(x), j_A(x), k_A(x)$ are real-valued, and

$$\sqrt{(a_A(x))^2 + (i_A(x))^2 + (j_A(x))^2 + (k_A(x))^2} \leq 1. \tag{11}$$

The real and quaternionic imaginary parts of $\rho_A(x)$ are $Re(\rho_A(x)) = a_A(x)$ and $Qim(\rho_A(x)) = i_A(x)i + j_A(x)j + k_A(x)k$, respectively.

Note that $\rho_A(x)$ also could be of the form

$$r_A(x)e^{i\theta_A(x)}e^{j\psi_A(x)}e^{k\omega_A(x)} \tag{12}$$

where the amplitude term is $r_A(x) \in [0,1]$ and three quaternionic phase terms are $(\theta_A(x), \psi_A(x), \omega_A(x)) \in [-\pi, \pi) \times [-\pi/2, \pi/2) \times [-\pi/4, \pi/4]$.

Quaternionic fuzzy sets are generalizations of ordinary FSs and CFSs. If two quaternionic phase terms $\psi_A(x)$ and $\omega_A(x)$ are zero, then A is a CFS. If all three quaternionic phase terms are zero, then A is a conventional FS. Similarly, if two quaternionic imaginary parts $j_A(x)$ and $k_A(x)$ are zero, then A is a CFS. If all three quaternionic imaginary parts are zero and $a_A(x)$ is limited in [0, 1], then A is a conventional FS.

Remark 1. *Mathematically, a quaternionic fuzzy set is equivalent to the fuzzy set with the co-domain of*

$$\mathbf{S} = \left\{(a,b,c,d) \in \mathbb{R}^4 \middle| a^2 + b^2 + c^2 + d^2 \leq 1\right\}, \tag{13}$$

which is a unit four-dimensional sphere.

Remark 2. *Ngan et al. [35] defined the complex intuitionistic fuzzy set using quaternions $q_0 + q_1 i + q_2 j + q_3 k$, which satisfies the following conditions:*

$$q_0, q_1, q_2, q_3 \in [0,1], \tag{14}$$
$$q_0 + q_1 \leq 1, \tag{15}$$
$$q_2 + q_3 \leq 1, \tag{16}$$
$$q_0 + q_2 \leq 1, \tag{17}$$
$$q_1 + q_3 \leq 1, \tag{18}$$

and $q_0, q_1, q_2, q_3 \in [0,1]$ are the degrees of real membership, imaginary membership, real non-membership, and imaginary non-membership, respectively. Kyritsis [37] gave the idea of a quaternion fuzzy subset, but its co-domain is $\mathbb{H} \cup \{\infty\}$. Moura et al. [38] introduced the concept of fuzzy quaternion numbers, which is a mapping from the set of quaternions to [0, 1]. In the study of quaternion-valued fuzzy cellular neural networks [39,40], they used quaternions $\mathbb{H}$, not its subset $\mathbf{Q} = \{q \in \mathbb{H} | |q| \leq 1\}$. It is essential that they [39,40] did not give the idea of quaternion-valued grades of membership. This is entirely different from the QFS including quaternion-valued grades of membership. These works are indeed concerned with quaternions and fuzzy sets, but in a completely different manner than our study in this work.

3.2. Interpretation of the Quaternionic Fuzzy Set

From a mathematical viewpoint, QFS is natural. However, similar to complex fuzzy sets in [11], obtaining intuition of QFS is not a simple task. Both complex numbers and quaternions are not particularly intuitive.

The central issue is the meaning of phase term in membership function. Traditional membership functions may interfere with other membership functions. This interference is dependent on their phase terms. In practice, the amplitude term and phase term of CFSs are used to describe two different features. For example, Ma et al. [16] introduced a complex fuzzy product–sum aggregation operator in which the amplitude term is used to represent the periodicity in the data. Dai [20] used the amplitude term to represent the direction of an object. Different from the interpretations noted above, Ramot et al. [11] gave an interesting interpretation from the idea of quantum mechanics that uses complex-valued functions to describe the state of object. Note that Nguyen et al. [41] also considered the complex-valued "truth values" from the idea of quantum mechanics. Following this way, quaternionic quantum mechanics, as a generation of standard complex quantum mechanics, use quaternion-valued functions to describe the state of object. An analogy to this aspect of quaternionic quantum mechanics offers an interpretation: the interference between traditional membership functions may rely on the quaternionic phase terms.

From another point of view, QFSs are composed of a real part and a quaternionic imaginary part. In this case, the central issue is how to deal with the quaternionic imaginary part. An answer may be obtained from the application of quaternions in color image processing [42–44]. RGB images have the red, green, and blue components. Then, the image pixel may be converted to a quaternion pixel by placing the these three components into the three imaginary parts of the quaternion, leaving the real part zero [44]. An analogy to the use of quaternions offers an interpretation. That is, the interpretation of a quaternionic fuzzy proposition is a quaternion of truth value. For example, in a proposition of the form "x is *too white* " in which *too white* means that all the red, green, and blue components are *very high*. Thus, we can use the form "$x \cdots\ A \cdots\ B \cdots\ C \cdots$" for a proposition, then $i_A(x)$, $j_A(x)$, and $k_A(x)$ can be assigned to the terms A, B, and C, respectively.

4. Cuts of Quaternionic Fuzzy Sets

4.1. Method 1

In general, order relations such as "p is greater than q" are undefined for quaternions p and q. Based on the modulus of a quaternion, the ordering of $\mathbf{Q}$ is given by $p \leq q$ if $|p| \leq |q|$.

Theorem 1. *The order $\leq$ of $\mathbf{Q}$ given by the modulus of a quaternion is a pre-order, but not a partial order.*

Proof. We first prove that $\leq$ satisfies the reflexivity and transitivity conditions, i.e.,

(1) reflexivity: $p \leq p$;

(2) transitivity: $p \leq q$ and $q \leq o \Rightarrow p \leq o$.

Clearly, $|p| = |p|$ for any $p \in \mathbf{Q}$. If $p \leq q$ and $q \leq o$, then we have $|p| \leq |q|$ and $|q| \leq |o|$ by the definition of $\leq$, then $|p| \leq |o|$.Thus we obtain $p \leq o$.

Second, we prove that $\leq$ does not satisfy the antisymmetry condition, i.e.,

(3) antisymmetry: $p \leq q$ and $q \leq p \Rightarrow p = q$.

Consider quaternions i and j, it is easy to check that $|i| = |j| = 1$, but $i \neq j$. □

Let U be a universe of discourse, $A = \{< x, \rho_A(x) > | x \in U\}$ be a quaternionic fuzzy set on U, a q-cut of A, for $q \in \mathbf{Q}$, is defined by

$$A^q = \{x \in U \big| |\rho_A(x)| \geq |q|\}. \tag{19}$$

Moreover, a variant of a q-cut is the strong q-cut defined as

$$A^{>q} = \{x \in U \big| |\rho_A(x)| > |q|\}. \tag{20}$$

The support of A, denoted by $supp(A)$, is defined as $A^{>0}$, i.e.,

$$supp(A) = \{x \in U \big| |\rho_A(x)| > 0\}. \tag{21}$$

In other words, the q-cut of A is the crisp set A^q that contains all the elements of U in which the moduli of quaternion-valued membership degrees are greater than or equal to the modulus of q.

Example 1. *Let $U = \{x_1, x_2, x_3, x_4\}$; consider the following quaternionic fuzzy set:*

$$A = \frac{0.3i + 0.4j}{x_1} + \frac{0.6j}{x_2} + \frac{0.6i - 0.8j}{x_3} + \frac{i}{x_4}, \tag{22}$$

then

$$A^q = \begin{cases} \{x_1, x_2, x_3, x_4\}, & \text{if } 0 \leq |q| \leq 0.5, \\ \{x_2, x_3, x_4\}, & \text{if } 0.5 < |q| \leq 0.6, \\ \{x_3, x_4\}, & \text{if } 0.6 < |q| \leq 1. \end{cases} \tag{23}$$

Figure 2 illustrates the quaternionic fuzzy set A and their q-cuts.

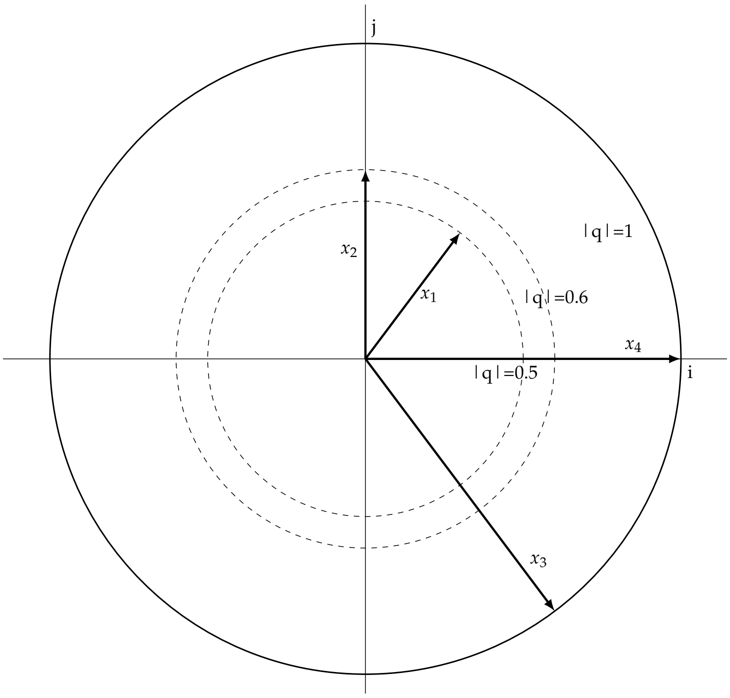

Figure 2. The q-cuts of QFSs in the i–j plane for Example 1.

The following theorem can be easily proved.

Theorem 2. *Let $A = \{< x, \rho_A(x) > | x \in U\}$ be a quaternionic fuzzy set on U, for any $p, q \in \mathbf{Q}$, if $|p| \leq |q|$, then $A^q \subseteq A^p$.*

Properties of cuts of quaternionic fuzzy sets are related to the order on $\mathbf{Q}$. Unfortunately, $\leq$ is not a partial order on $\mathbf{Q}$. In other words, $(\mathbf{Q}, \leq)$ is not a lattice, i.e., there exists $p, q \in \mathbf{Q}$ such that $p \vee q$ does not exist. Further, this leads to a special case that there exists two quaternionic fuzzy sets A and B such that $A \neq B$ but $A_q = B_q$ for any $q \in \mathbf{Q}$. See this in the following example.

Example 2. *Let $U = \{x_1, x_2, x_3, x_4\}$, consider the quaternionic fuzzy set A in Example 1 and the following quaternionic fuzzy set:*

$$B = \frac{0.4j + 0.3k}{x_1} + \frac{0.6k}{x_2} + \frac{0.6i + 0.8j}{x_3} + \frac{k}{x_4}, \tag{24}$$

then

$$B^q = \begin{cases} \{x_1, x_2, x_3, x_4\}, & \text{if } 0 \leq |q| \leq 0.5, \\ \{x_2, x_3, x_4\}, & \text{if } 0.5 < |q| \leq 0.6, \\ \{x_3, x_4\}, & \text{if } 0.6 < |q| \leq 1. \end{cases} \tag{25}$$

It is easy to check that $A \neq B$ *and* $A_q = B_q$ *for any* $q \in \mathbf{Q}$.

4.2. Method 2

As noted in Remark 1, a quaternionic fuzzy set is mathematically equivalent to the fuzzy set with the co-domain of $\mathbf{S} = \{(a,b,c,d) \in \mathbb{R}^4 | a^2 + b^2 + c^2 + d^2 \leq 1\}$.

Naturally, we have the following order $\preceq$ of $\mathbf{Q}$, for any two quaternions $q = q_0 + q_1 i + q_2 j + q_3 k, p = p_0 + p_1 i + p_2 j + p_3 k \in \mathbf{Q}$, we say $q \preceq p$ if

$$q_0 \leq p_0,\ q_1 \leq p_1,\ q_2 \leq p_2,\ \text{and } q_3 \leq p_3.$$

Theorem 3. *The order* $\leq$ *of* $\mathbf{Q}$ *is a partial order.*

Let U be a universe of discourse, $A = \{< x, \rho_A(x) > | x \in U\}$ be a quaternionic fuzzy set on U, a q-cut of A, for $q \in \mathbf{Q}$, is defined by

$$A^q = \{x \in U | \rho_A(x) \succeq q\}. \tag{26}$$

Moreover, a variant of a q-cut is the strong q-cut defined as

$$A^{\succ q} = \{x \in U | \rho_A(x) \succ q\}. \tag{27}$$

In this method, for any $q = q_0 + q_1 i + q_2 j + q_3 k$, q-cut of A means that for any $x \in U$ with membership grade $\rho_A(x) = a_A(x) + i_A(x)i + j_A(x)j + k_A(x)k$, we have $a_A(x) \geq q_0$, $i_A(x) \geq q_1$, $j_A(x) \geq q_2$, and $k_A(x) \geq q_3$.

The following theorem can be easily proved.

Theorem 4. *Let* $A = \{< x, \rho_A(x) > | x \in U\}$ *be a quaternionic fuzzy set on* U*, for any* $p, q \in \mathbf{Q}$*, if* $p \preceq q$*, then* $A^q \subseteq A^p$.

Example 3. *Consider the quaternionic fuzzy set* A *in Example 1 in which* A *also could be represented as*

$$A = \frac{0 + 0.3i + 0.4j + 0k}{x_1} + \frac{0 + 0i + 0.6j + 0k}{x_2} + \frac{0 + 0.6i - 0.8j + 0k}{x_3} + \frac{0 + i + 0j + 0k}{x_4}. \tag{28}$$

For convenience, let $p = 0 + 0.3i + 0.4j + 0k$*,* $s = 0 + 0i + 0.6j + 0k$*,* $t = 0 + 0.6i - 0.8j + 0k$ *and* $v = 0 + i + 0j + 0k$*. Clearly, we only have the relation* $v \succ t$ *among* p, s, t, v*.*

Then we have

$$B^q = \begin{cases} \{x_1, x_2, x_3, x_4\}, & \text{if } q \preceq p, q \preceq s, q \preceq t, \\ \{x_1, x_3, x_4\}, & \text{if } q \preceq p, q \npreceq s, q \preceq t, \\ \{x_2, x_3, x_4\}, & \text{if } q \npreceq p, q \preceq s, q \preceq t, \\ \{x_1, x_2, x_4\}, & \text{if } q \preceq p, q \preceq s, q \npreceq t, q \preceq v, \\ \{x_1, x_2\}, & \text{if } q \preceq p, q \preceq s, q \npreceq t, q \npreceq v, \\ \{x_1\}, & \text{if } q \preceq p, q \npreceq s, q \npreceq t, q \npreceq v, \\ \{x_2\}, & \text{if } q \npreceq p, q \preceq s, q \npreceq t, q \npreceq v, \\ \{x_3, x_4\}, & \text{if } q \npreceq p, q \npreceq s, q \preceq t, \\ \{x_4\}, & \text{if } q = t. \end{cases} \tag{29}$$

For example, $B^q = \{x_1, x_2, x_3, x_4\}$ *for* $q \preceq p, q \preceq s$ *and* $q \preceq t$*, i.e.,* $q = a + bi + cj + dk \in$ $\mathbf{Q}$ *with*

$$a \leq 0,\ b \leq 0,\ c \leq -0.8,\ d \leq 0.$$

Figure 3 illustrates the quaternionic fuzzy set A *and their q-cuts in the i–j plane.*

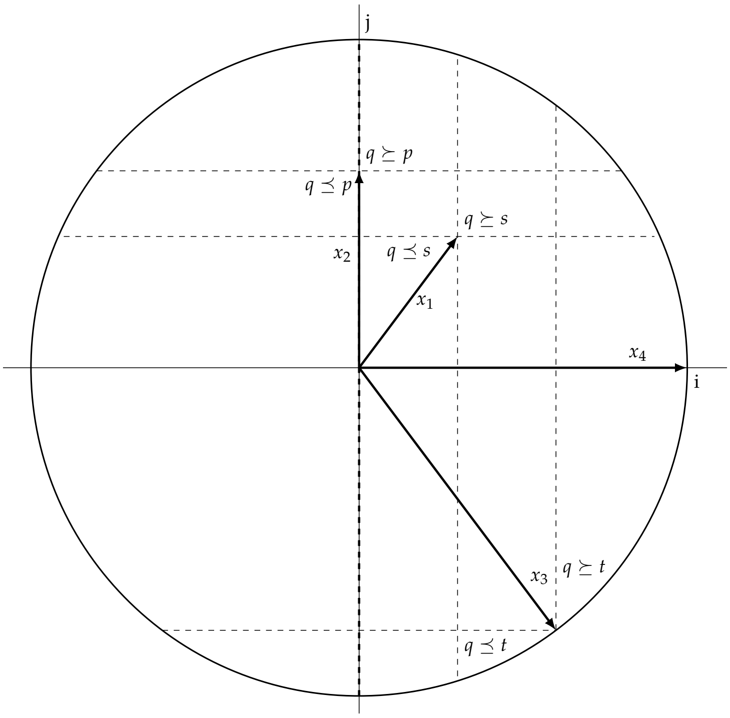

Figure 3. The q-cuts of QFSs in the i–j plane for Example 3.

We say $(\mathbf{Q}, \preceq)$ has a bottom element $\perp \in \mathbf{Q}$ if $\perp \preceq q$ for all $q \in \mathbf{Q}$. Unfortunately, $(\mathbf{Q}, \preceq)$ does not have a bottom element. For example, there does not exist a $q \in \mathbf{Q}$ such that both $q \preceq -i$ and $q \preceq -j$ hold.

Further, this leads to a special case that there exists a quaternionic fuzzy set A such that there does not exist $q \in \mathbf{Q}$ such that A^q includes all elements of A. See this in the following example.

Example 4. *Let* $U = \{x_1, x_2\}$*. Consider the following quaternionic fuzzy set:*

$$B = \frac{-i}{x_1} + \frac{-k}{x_2}, \tag{30}$$

If $B^q = \{x_1, x_2\}$ *for some* $q = a + bi + cj + dk \in \mathbf{Q}$*, then* $-i \geq q$ *and* $-k \geq q$*, i.e.,* $a \leq 0, b \leq -1, c \leq 0, d \leq 0$ *and* $a \leq 0, b \leq 0, c \leq -1, d \leq 0$*. However, if* $b \leq -1$ *and* $c \leq -1$*, then* $a^2 + b^2 + c^2 + d^2 \geq b^2 + c^2 \geq 2$*. Thus* $q = a + bi + cj + dk \notin \mathbf{Q}$*. This is a contradiction.*

5. Set Theoretic Operation of the Quaternionic Fuzzy Set

In this section, the operations of quaternionic fuzzy complement, quaternionic fuzzy union, and quaternionic fuzzy intersection are defined. Then, De Morgan's laws of quaternionic fuzzy union and intersection are discussed. Next, quaternionic fuzzy aggregation is introduced. Finally, rotational invariance is proposed.

A quaternionic grade of membership $\rho_A(x)$ is restricted to the subset of quaternions $\mathbf{Q}$, i.e., $|\rho_A(x)|$ is limited to [0, 1]. For convenience, we only consider the quaternionic fuzzy operation over $\mathbf{Q}$.

5.1. Quaternionic Fuzzy Complement

Definition 2. *A function $\neg : \mathbf{Q} \to \mathbf{Q}$ is called a quaternionic fuzzy complement if it satisfies the following two conditions: $\forall q, p \in \mathbf{Q}$*

(1) *Amplitude boundary conditions:*

$$q = 0 \Rightarrow |\neg q| = 1, \quad |q| = 1 \Rightarrow \neg q = 0;$$

(2) *Amplitude monotonicity:*

$$|q| \leq |p| \Rightarrow |\neg p| \leq |\neg q|.$$

In addition, in some cases, $\neg$ should satisfy also the following conditions:

(3) Continuity: $\neg$ is a continuous function;
(4) Amplitude involutivity:

$$|\neg(\neg q)| = |q|.$$

This definition is a generation of crisp, traditional, and Ramot et al's complex fuzzy complement.

Some examples are as follows: Let $q = |q|e^{i\theta}e^{j\psi}e^{k\omega}$, define

$$\neg_1 q = (1 - |q|)e^{i\theta}e^{j\psi}e^{k\omega}, \tag{31}$$

$$\neg_2 q = (1 - |q|)e^{-i\theta}e^{-j\psi}e^{-k\omega}. \tag{32}$$

Two functions satify the above four conditions.

Example 5. *Let $q = 0.6e^{-i\pi/3}e^{j\pi/4}e^{k\pi/8} \in \mathbf{Q}$, then*

$$\neg_1 q = 0.4e^{-i\pi/3}e^{j\pi/4}e^{k\pi/8}, \tag{33}$$

$$\neg_2 q = 0.4e^{i\pi/3}e^{-j\pi/4}e^{-k\pi/8}. \tag{34}$$

Using the standard fuzzy complement, i.e., $f(x) = 1 - x, \forall x \in [0,1]$, we obtained two functions,

$$q = q_0 + q_1 i + q_2 j + q_3 k \Rightarrow f_1(q) = 1 - q; \tag{35}$$

$$q = q_0 + q_1 i + q_2 j + q_3 k \Rightarrow f_2(q) = (1 - q_0) + (1 - q_1)i + (1 - q_2)j + (1 - q_3)k. \tag{36}$$

Unfortunately, both functions are not closed over $\mathbf{Q}$.

5.2. Quaternionic Fuzzy Union

Definition 3. *A function $\cup : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ is called a quaternionic fuzzy union if it satisfies the following four conditions: $\forall q, p, o, s \in \mathbf{Q}$*

(1) *Boundary condition:*

$$p \cup 0 = p;$$

(2) *Amplitude monotonicity:*

$$|q| \leq |p| \Rightarrow |p \cup o| \leq |q \cup o|;$$

(3) *Commutativity:*

$$p \cup q = q \cup p;$$

(4) *Associativity:*

$$p \cup (q \cup o) = (p \cup q) \cup o.$$

In addition, in some cases, $\cup$ should satisfy also the following conditions:

(5) Continuity: $\cup$ is a continuous function;

(6) Amplitude superidempotency:
$$|p \cup p| \geq |p|;$$

(7) Amplitude strict monotonicity:
$$|q| \leq |p|, |o| \leq |s| \Rightarrow |q \cup o| \leq |p \cup s|.$$

Two examples are as follows: Let $q = |q|e^{i\theta_q}e^{j\psi_q}e^{k\omega_q}$ and $p = |p|e^{i\theta_p}e^{j\psi_p}e^{k\omega_p}$, then

$$q \cup_1 p = (|q| \star |p|)e^{i(\theta_q \vee \theta_p)}e^{j(\psi_q \vee \psi_p)}e^{k(\omega_q \vee \omega_p)}, \tag{37}$$
$$q \cup_2 p = (|q| \star |p|)e^{i(\theta_q \ddot{+} \theta_p)}e^{j(\psi_q \ddot{+} \psi_p)}e^{k(\omega_q \ddot{+} \omega_p)} \tag{38}$$

where $\star$ represents a t-conorm, and

$$i(\theta_q \ddot{+} \theta_p) = \begin{cases} i(\theta_q + \theta_p + \pi), & \text{if } \theta_q + \theta_p \leq -\pi, \\ i(\theta_q + \theta_p), & \text{if } -\pi \leq \theta_q + \theta_p < \pi, \\ i(\theta_q + \theta_p - \pi), & \text{if } \theta_q + \theta_p \geq \pi. \end{cases} \tag{39}$$

$$j(\theta_q \ddot{+} \theta_p) = \begin{cases} i(\theta_q + \theta_p + \pi/2), & \text{if } \theta_q + \theta_p \leq -\pi/2, \\ i(\theta_q + \theta_p), & \text{if } -\pi/2 \leq \theta_q + \theta_p < \pi/2, \\ i(\theta_q + \theta_p - \pi/2), & \text{if } \theta_q + \theta_p \geq \pi/2. \end{cases} \tag{40}$$

$$k(\theta_q \ddot{+} \theta_p) = \begin{cases} i(\theta_q + \theta_p + \pi/4), & \text{if } \theta_q + \theta_p \leq -\pi/4, \\ i(\theta_q + \theta_p), & \text{if } -\pi/4 \leq \theta_q + \theta_p < \pi/4, \\ i(\theta_q + \theta_p - \pi/4), & \text{if } \theta_q + \theta_p \geq \pi/4. \end{cases} \tag{41}$$

Both $\cup_1$ and $\cup_2$ satisfy the above four conditions.

Example 6. *Let* $q = 0.6e^{-i\pi/3}e^{j\pi/4}e^{k\pi/8}, p = 0.5e^{i\pi/2}e^{j\pi/8}e^{-k\pi/8} \in \mathbf{Q}$ *and* $\star = \vee$*, then*

$$\begin{aligned} q \cup_1 p &= (0.6 \vee 0.5)e^{i(-\pi/3)\vee(\pi/2)}e^{j(\pi/4)\vee(\pi/8)}e^{k(-\pi/8)\vee(\pi/8)} \\ &= 0.6e^{i\pi/2}e^{j\pi/4}e^{k\pi/8}, \end{aligned} \tag{42}$$

$$\begin{aligned} q \cup_2 p &= (0.6 \vee 0.5)e^{i(-\pi/3 \ddot{+} \pi/2)}e^{j(\pi/4 \ddot{+} \pi/8)}e^{k(-\pi/8 \ddot{+} \pi/8)} \\ &= 0.6e^{i\pi/6}e^{j3\pi/8}e^{k0}. \end{aligned} \tag{43}$$

5.3. Quaternionic Fuzzy Intersection

Definition 4. *A function* $\cap : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ *is called a quaternionic fuzzy intersection if it satisfies the following four conditions:* $\forall q, p, o, s \in \mathbf{Q}$

(1) *amplitude boundary condition:*
$$|q| = 1 \Rightarrow |p \cap q| = |p|;$$

(2) *amplitude monotonicity:*
$$|q| \leq |p| \Rightarrow |p \cap o| \leq |q \cap o|;$$

(3) *commutativity:*
$$p \cap q = q \cap p;$$

(4) *associativity:*
$$p \cap (q \cap o) = (p \cap q) \cap o.$$

In addition, in some cases, $\cap$ should satisfy also the following conditions:

(5) continuity: $\cap$ is a continuous function;

(6) amplitude superidempotency:
$$|p \cap p| \leq |p|;$$

(7) amplitude strict monotonicity:
$$|q| \leq |p|, |o| \leq |s| \Rightarrow |q \cap o| \leq |p \cap s|.$$

Two examples are as follows: Let $q = |q|e^{i\theta_q}e^{j\psi_q}e^{k\omega_q}$ and $p = |p|e^{i\theta_p}e^{j\psi_p}e^{k\omega_p}$

$$q \cap_1 p = (|q| * |p|)e^{i(\theta_q \wedge \theta_p)}e^{j(\psi_q \wedge \psi_p)}e^{k(\omega_q \wedge \omega_p)}, \tag{44}$$
$$q \cap_2 p = (|q| * |p|)e^{i(\theta_q \ddot{+} \theta_p)}e^{j(\psi_q \ddot{+} \psi_p)}e^{k(\omega_q \ddot{+} \omega_p)} \tag{45}$$

where $*$ represents a t-norm. Both $\cap_1$ and $\cap_2$ satisfy the above four conditions.

Example 7. *Let* $q = 0.6e^{-i\pi/3}e^{j\pi/4}e^{k\pi/8}$, $p = 0.5e^{i\pi/2}e^{j\pi/8}e^{-k\pi/8} \in \mathbf{Q}$ *and* $\star = \wedge$, *then*

$$\begin{aligned} q \cap_1 p &= (0.6 \wedge 0.5)e^{i(-\pi/3)\wedge(\pi/2)}e^{j(\pi/4)\wedge(\pi/8)}e^{k(-\pi/8)\wedge(\pi/8)} \\ &= 0.5e^{-i\pi/3}e^{j\pi/8}e^{-k\pi/8}. \end{aligned} \tag{46}$$

$$\begin{aligned} q \cap_2 p &= (0.6 \wedge 0.5)e^{i(-\pi/3 \ddot{+} \pi/2)}e^{j(\pi/4 \ddot{+} \pi/8)}e^{k(-\pi/8 \ddot{+} \pi/8)} \\ &= 0.5e^{i\pi/6}e^{j3\pi/8}e^{k0}. \end{aligned} \tag{47}$$

Now we consider two famous operations in quaternion theory: quaternionic dot product and quaternionic product.

Lemma 1. *Let* $p, q \in \mathbf{Q}$, *then* $p \bullet q \in [-1, 1]$ *and* $pq \in \mathbf{Q}$.

Proof. Let $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$ with $\sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2} \leq 1$ and $\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \leq 1$. We know p_n, q_n, $(n = \{0, 1, 2, 3\})$ are real numbers. By the Cauchy–Schwarz inequality, we have

$$(p_0 q_0 + p_1 q_1 + p_2 q_2 + p_3 q_3)^2 \leq (p_0^2 + p_1^2 + p_2^2 + p_3^2)(q_0^2 + q_1^2 + q_2^2 + q_3^2) \leq 1. \tag{48}$$

Thus $p \bullet q \in [-1, 1]$; $pq \in \mathbf{Q}$ because of $|pq| = |p||q| \leq 1$. □

Quaternionic dot product is closed over $\mathbf{Q}$. However, it does not satisfy condition (1) above.

Quaternionic product is also closed over $\mathbf{Q}$. It satisfies above conditions (1), (2), (4)–(7). In order to bring the quaternionic product into our study, we introduce the concept of quaternionic fuzzy non-commutative intersection.

Definition 5. *A function* $\cap : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ *is called a quaternionic fuzzy non-commutative intersection if it satisfies above conditions (1), (2), and (4) in Definition 4.*

Obviously, we have the following result.

Theorem 5. *Quaternionic product over* $\mathbf{Q}$ *is a quaternionic fuzzy non-commutative intersection.*

5.4. De Morgan's Laws of Quaternionic Fuzzy Union and Intersection

Theorem 6. *If t-norm* $*$ *and t-conorm* $\star$ *are dual, i.e.,* $\forall a, b \in [0, 1]$

$$a * b = 1 - (1 - a) \star (1 - b), \tag{49}$$
$$a \star b = 1 - (1 - a) * (1 - b), \tag{50}$$

then the quaternionic fuzzy union and intersection, respectively defined in Equations (42) and (46), satisfy $\forall p, q \in \mathbf{Q}$

$$\neg_2(p \cup_1 q) = (\neg_3 p) \cap_1 (\neg_2 q), \tag{51}$$

$$\neg_2(p \cap_1 q) = (\neg_3 p) \cup_1 (\neg_2 q). \tag{52}$$

Proof. Let $q = |q|e^{i\theta_q}e^{j\psi_q}e^{k\omega_q}$, $p = |p|e^{i\theta_p}e^{j\psi_p}e^{k\omega_p}$. Recall the definition $\neg_2$, we have

$$\begin{aligned}
\neg_2(p \cup_1 q) &= \neg\Big(|q| * |p|e^{i(\theta_q \wedge \theta_p)}e^{j(\psi_q \wedge \psi_p)}e^{k(\omega_q \wedge \omega_p)}\Big) \\
&= (1 - |q| * |p|)e^{-i(\theta_q \wedge \theta_p)}e^{-j(\psi_q \wedge \psi_p)}e^{-k(\omega_q \wedge \omega_p)} \\
&= (1 - |q|) \star (1 - |p|)e^{(-i\theta_q) \vee (-i\theta_p)}e^{(-j\psi_q) \vee (-j\psi_p)}e^{-(-k\omega_q) \vee (-k\omega_p)} \\
&= (\neg_2 p) \cap_1 (\neg_2 q),
\end{aligned}$$

and

$$\begin{aligned}
\neg_2(p \cap_1 q) &= \neg\Big(|q| \star |p|e^{i(\theta_q \vee \theta_p)}e^{j(\psi_q \vee \psi_p)}e^{k(\omega_q \vee \omega_p)}\Big) \\
&= (1 - |q| \star |p|)e^{-i(\theta_q \vee \theta_p)}e^{-j(\psi_q \vee \psi_p)}e^{-k(\omega_q \vee \omega_p)} \\
&= (1 - |q|) * (1 - |p|)e^{(-i\theta_q) \wedge (-i\theta_p)}e^{(-j\psi_q) \wedge (-j\psi_p)}e^{-(-k\omega_q) \wedge (-k\omega_p)} \\
&= (\neg_2 p) \cup_1 (\neg_2 q).
\end{aligned}$$

□

Note that Equations (40) and (41) do not hold for $\neg_1$.

5.5. Quaternionic Fuzzy Aggregation

Quaternionic fuzzy aggregation is specified by a function $f : \mathbf{Q}^n \to \mathbf{Q}$. Here we define the quaternionic fuzzy weighted arithmetic (QFWA) aggregation operator as

$$QFWA(q_1, q_2, \cdots, q_n) = \sum_{l=1}^{n} (w_l q_l) \tag{53}$$

where $w_l \in \mathbf{Q}$ for all l, and $= \sum_{l=1}^{n} |w_l| = 1$.

Note: The purpose of quaternionic weights is to make the definition as general as possible. In ordinary circumstances, weights are real-valued, i.e., $w_l \in [0,1]$ with $= \sum_{l=1}^{n} w_l = 1$. In the following, we only consider the real-valued weights.

We show that QFWA aggregation operator is closed over $\mathbf{Q}$.

Theorem 7. *If* $q_1, q_2, \cdots, q_n \in \mathbf{Q}$, *then* $QFWA(q_1, q_2, \cdots, q_n) \in \mathbf{Q}$ *for any real-valued weights, i.e.,* $w_l \in [0,1]$ *with* $= \sum_{l=1}^{n} w_l = 1$.

Proof. For any real-valued weights $w_l \in [0,1]$ with $= \sum_{l=1}^{n} w_l = 1$, since $|q_l| \leq 1$ for all l, we have

$$\begin{aligned}
|QFWA(q_1, q_2, \cdots, q_n)| &= |w_1 q_1 + w_2 q_2 + \cdots + w_n q_n| \\
&\leq |w_1 q_1| + |w_2 q_2| + \cdots + |w_n q_n| \\
&= w_1|q_1| + w_2|q_2| + \cdots + w_n|q_n| \\
&\leq w_1 + w_2 + \cdots + w_n \\
&= 1.
\end{aligned}$$

Thus $QFWA(q_1, q_2, \cdots, q_n) \in \mathbf{Q}$. □

If $w_l = 1/n$ for all l, then the QFWA aggregation operator is the arithmetic average of quaternions $q_1, q_2, \cdots, q_n$, denoted by quaternionic fuzzy arithmetic average (QFAA) operator, i.e.,

$$QFAA(q_1, q_2, \cdots, q_n) = \sum_{l=1}^{n} \frac{q_l}{n}. \tag{54}$$

Obviously, the QFWA aggregation operator is a generalization of the complex fuzzy weighted arithmetic aggregation operator in [12,45].

6. Quaternionic Fuzzy Relations

In this section, we introduce the concepts of quaternionic fuzzy relations.

Definition 6. *Let U and V be two universes of discourse. A quaternionic fuzzy relation $Q(U, V)$ is a quaternionic fuzzy subset of the product space $U \times V$; $Q(U, V)$ is characterized by the quaternion-valued membership function $\mu_Q(u, v)$, where $u \in U$ and $v \in V$.*

Then we define the compositions of quaternionic fuzzy relations as follows.

Definition 7. *Let $Q(U, V)$ and $S(V, W)$ be two quaternionic fuzzy relations over $U \times V$ and $V \times W$, respectively. Their composition is $Q \circ S(U, W)$ whose membership function is*

$$\mu_{Q \circ S}(u, w) = \bigcup_{v \in V} \Big(\mu_Q(u, v) \cap \mu_Q(v, w) \Big) \tag{55}$$

where $\cup$ and $\cap$ are quaternionic fuzzy union and intersection, respectively.

Let U and V be two universes of discourse. A quaternionic fuzzy relation $Q(U, V)$ is a quaternionic fuzzy subset of the product space $U \times V$; $Q(U, V)$ is characterized by the quaternion-valued membership function $\mu_Q(u, v)$, where $u \in U$ and $v \in V$.

Example 8. *Let Q and S be two quaternionic fuzzy relations defined as*

$$Q = \begin{bmatrix} 0.4e^{i0.1}e^{-j0.2}e^{-k0.3} & 0.8e^{i0.5}e^{-j0.8}e^{-k0.8} \\ 0.4e^{-i0.5}e^{-j0.5}e^{-k0.5} & 0.7e^{-i0.1}e^{-j0.5}e^{-k0.4} \\ 0.9e^{i0.5}e^{j0.6}e^{k0.6} & 0.7e^{i0.5}e^{-j0.1}e^{k0.2} \end{bmatrix},$$

$$S = \begin{bmatrix} 0.3e^{i0}e^{-j0.6}e^{-k0.5} & 0.6e^{i0.5}e^{-j0.5}e^{-k0.5} \\ 0.1e^{i0.5}e^{j0.4}e^{k0.3} & 0.7e^{i0.5}e^{-j0.5}e^{k0.5} \end{bmatrix}.$$

Let $* = \vee$ and $\star = \wedge$, by using $\cup_1$ and $\cap_1$, then

$$Q \circ S = \begin{bmatrix} 0.3e^{i0.5}e^{-j0.6}e^{-k0.5} & 0.7e^{i0.5}e^{-j0.5}e^{-k0.5} \\ 0.3e^{-i0.1}e^{-j0.5}e^{-k0.4} & 0.7e^{-i0.1}e^{-j0.5}e^{-k0.4} \\ 0.3e^{i0.5}e^{-j0.1}e^{k0.2} & 0.7e^{i0.5}e^{-j0.5}e^{k0.2} \end{bmatrix}$$

7. Rotational Invariance

In the case of complex fuzzy logic, rotational invariance of complex fuzzy operations is studied in [23,46]. For a complex number $c \in \mathbb{C}$, $e^{i\theta}c$ is referred to as the rotated vector of c. We consider c as a two-dimensional vector, and then we just rotate this vector about the origin counterclockwise by θ radians and obtain $e^{i\theta}c$.

Now we consider a quaternion $p \in \mathbb{H}$; let $q \in \mathbb{H}$ and $|q| = 1$, then $|p \cdot q| = |q \cdot p| = |p|$. Because $p \cdot q \neq q \cdot p$, so $p \cdot q$ and $q \cdot p$ maybe are two different rotated vectors of p. We write $p \cdot q$ as the right-rotated vector of p and $q \cdot p$ as the left-rotated vector of p.

In this section, we investigate the rotational invariance of quaternionic fuzzy operations.

Definition 8. *Let $f : \mathbf{Q}^n \to \mathbf{Q}$ be an n-order function; f is right-rotationally invariant if*

$$f(p_1 \cdot q, p_2 \cdot q, \cdots, p_n \cdot q) = f(p_1, p_2, \cdots, p_n) \cdot q, \tag{56}$$

for any $p_1, p_2, \cdots, p_n \in \mathbf{Q}$ and $q \in \mathbf{Q}$ with $|q| = 1$.

Definition 9. *Let $f : \mathbf{Q}^n \to \mathbf{Q}$ be an n-order function; f is left-rotationally invariant if*

$$f(q \cdot p_1, q \cdot p_2, \cdots, q \cdot p_n) = q \cdot f(p_1, p_2, \cdots, p_n), \tag{57}$$

for any $p_1, p_2, \cdots, p_n \in \mathbf{Q}$ and $q \in \mathbf{Q}$ with $|q| = 1$.

Right-rotational invariance and left-rotational invariance are two different concepts since $pq \neq qp$ for some $p, q \in \mathbf{Q}$. Right-rotational invariance and left-rotational invariance are equivalent when we limit the values $p_1, p_2, \cdots, p_n$ and q to complex numbers $\mathbb{C}$. Clearly, right-rotational invariance and left-rotational invariance are generalizations of Dick's rotational invariance [23] in the case of complex fuzzy logic.

Theorem 8. *If $f : \mathbf{Q} \to \mathbf{Q}$ is defined as $f(p) = -p$, then it is both right- and left-rotationally invariant.*

Proof. For any quaternion $|q| = 1$, we have $f(pq) = -pq = (-p)q = f(p)q$ and $f(qp) = -qp = (-q)p = f(q)p$. □

Theorem 9. *$\neg_1$ is both right-rotationally invariant and left-rotationally invariant.*

Proof. For any quaternion $p = |p|e^{i\theta}e^{j\psi}e^{k\omega}$, let $q = e^{i\theta'}e^{j\psi'}e^{k\omega'}$ because $|q| = 1$, then $pq = |p|e^{i\theta}e^{j\psi}e^{k\omega}e^{i\theta'}e^{j\psi'}e^{k\omega'}$. Therefore, we have $\neg_1(pq) = (1 - |p|)e^{i\theta}e^{j\psi}e^{k\omega}e^{i\theta'}e^{j\psi'}e^{k\omega'} = \big((1 - |p|)e^{i\theta}e^{j\psi}e^{k\omega}e^{i\theta'}\big)e^{j\psi'}e^{k\omega'} = \neg_1(p)q$. Similarly, we have $\neg_1(qp) = q\neg_1(p)$. □

Theorem 10. *$\neg_2$ is neither right-rotationally invariant nor left-rotationally invariant.*

Proof. Consider $p = 0.6e^{-i\pi/2}$, $\neg_2 p = (1 - 0.6)e^{-i(-\pi/2)} = 0.4e^{i\pi/2}$. Let $q = e^{-i\pi/4}$ then $\neg_2(pq) = 0.4e^{i(-\pi/2-\pi/4)} = 0.4e^{-i3\pi/4}$, but $\neg_2(p)q = 0.4e^{i\pi/2}e^{-i\pi/4} = 0.4e^{i(\pi/4)} \neq \neg_2(pq)$. Similarly, we have $q\neg_2(p) \neq \neg_2(qp)$. □

Theorem 11. *The quaternionic dot product is neither right-rotationally invariant nor left-rotationally invariant.*

Proof. Consider $p_1 = 0.5i$ and $p_2 = 0.5i$. By definition, their dot product is -0.25. Now, let $q = i$, consider their right-rotated values $p_1 q = -0.5$ and $p_1 q = -0.5$, their dot product is $(p_1 q) \bullet (p_2 q) = 0.25$, but $(p_1 \bullet p_2)q = -0.25$. Similarly, we have $(qp_1) \bullet (qp_2) \neq q(p_1 \cdot p_2)$. □

Theorem 12. *The quaternionic product is neither right-rotationally invariant nor left-rotationally invariant.*

Proof. Consider i and j. By definition, their product is k. Now, consider their right-rotated values $ij = k$ and $jj = -1$, their product is $-k$, but $kj = -i \neq -k$. Similarly, consider their left-rotated values $ji = -k$ and $jj = -1$, their product is k, but $jk = i \neq k$. □

Theorem 13. *$\cup_1$, $\cup_2$, $\cap_1$, and $\cap_2$ are neither right-rotationally invariant nor left-rotationally invariant.*

Proof. Here, we just give the proof of that $\cap_1$ is not right-rotationally invariant. Consider complex numbers $ae^{i\theta_1}$ and $be^{i\theta_2}$.

$$ae^{i\theta_1}e^{i\theta_3} \cap_1 be^{i\theta_2}e^{i\theta_3} = (a \wedge b)e^{i(\theta_1 \ddot{+} \theta_3 \ddot{+} \theta_2 \ddot{+} \theta_3)} \neq (a \wedge b)e^{i(\theta_1 \ddot{+} \theta_2 \ddot{+} \theta_3)} = (ae^{i\theta_1} \cap_1 be^{i\theta_2})e^{i\theta_3}.$$

Other cases can be proved in a similar way. □

Theorem 14. *The QFWA aggregation operator is both right- and left-rotationally invariant.*

Proof. For any $p_1, p_2, \cdots, p_n \in \mathbf{Q}$ and $q \in \mathbf{Q}$ with $|q| = 1$, we have

$$\begin{aligned} QFWA(p_1 \cdot q, p_2 \cdot q, \cdots, p_n \cdot q) &= w_1 p_1 \cdot q + w_2 p_2 \cdot q + \cdots + w_n p_n \cdot q \\ &= (w_1 p_1 + w_2 p_2 + \cdots + w_n p_n) \cdot q \\ &= QFWA(q_1, q_2, \cdots, q_n) \cdot q. \end{aligned}$$

Thus, the QFWA aggregation operator is right-rotationally invariant.
For any real-valued weights $w_l \in [0, 1]$, we have $w_l q = q w_l$. Then

$$\begin{aligned} QFWA(q \cdot p_1, q \cdot p_2, \cdots, q \cdot p_n) &= w_1 q \cdot p_1 + w_2 q \cdot p_2 + \cdots + w_n q \cdot p_1 \\ &= q w_1 \cdot p_1 + q w_2 \cdot p_2 + \cdots + q w_n \cdot 1 \\ &= q \cdot (w_1 p_1 + w_2 p_2 + \cdots + w_n p_n) \\ &= q \cdot QFWA(q_1, q_2, \cdots, q_n). \end{aligned}$$

Thus, the QFWA aggregation operator is left-rotationally invariant. □

Let $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ be a quaternionic vector; $\mathbf{p}q = (p_1 \cdot q, p_2 \cdot q, \cdots, p_n \cdot q)$ is the right-rotated vector of $\mathbf{p}$ and $q\mathbf{p} = (q \cdot p_1, q \cdot p_2, \cdots, q \cdot p_n)$ is the left-rotated vector of $\mathbf{p}$. Rotational invariance in the above theorem states that the aggregated result $QFWA(\mathbf{p}q)$ is the right-rotated vector of the aggregated result $QFWA(\mathbf{p})$, and $QFWA(q\mathbf{p})$ is the left-rotated vector of the aggregated result $QFWA(\mathbf{p})$.

The rotational invariance of quaternionic fuzzy operations are summarized as in Table 1. As can be seen, $\neg_1$ could be a both right- and left-rotationally invariant complement; $f(p) = -p$ is both right- and left-rotationally invariant, but on the other hand, is not a quaternionic fuzzy complement of Definition 2. Quaternionic product, quaternionic dot product, and $\cup_1, \cup_2, \cap_1, \cap_2$ are neither right-rotationally invariant nor left-rotationally invariant. We need a more comprehensive concept of rotational invariance for quaternionic fuzzy operations. Interestingly, the QFWA aggregation operator is a both right- and left-rotationally invariant operator.

Table 1. Rotational invariance of quaternionic fuzzy operations

	Right-Rotationally Invariant	**Left-Rotationally Invariant**
$-$	$\surd$	$\surd$
Quaternionic product	$\times$	$\times$
Quaternionic dot product	$\times$	$\times$
$\neg_1$	$\surd$	$\surd$
$\neg_2$	$\times$	$\times$
$\cup_1, \cup_2, \cap_1, \cap_2$	$\times$	$\times$
QFWA	$\surd$	$\surd$

8. Concluding Remarks

A new concept of QFS was introduced in this paper. QFS allows quaternion-valued membership grade with four representative parameters. We gave a discussion of the intuitive interpretation of quaternion-valued membership grade. Several quaternionic fuzzy operations, including complement, union, intersection, and aggregation, were presented. Rotational invariance of these quaternionic fuzzy operations was also studied.

QFS is a promising novel concept. Obviously, many theoretical studies and application development are possible topics for future consideration. We present our views on theories and potential applications.

(1) Geometric properties of complex fuzzy operations are often studied and analyzed by scholars, such as continuity [47] and preserving orthogonality [14]. These properties are important for both complex fuzzy operations and quaternionic fuzzy operations. Moreover, we should consider some special properties only for quaternionic fuzzy operations but not for complex fuzzy operations.
(2) We should consider the quaternionic fuzzy logic for logical reasoning based on QFS. Obviously, a more detailed discussion of the axiomatization of quaternionic fuzzy logic is necessary.
(3) CFS is often used to construct complex-valued neuro-fuzzy systems to solve practical problems [48]. Recently, quaternion-valued neural networks have received an increasing amount of interest [30]. It will be meaningful to construct quaternion-valued neuro-fuzzy systems to solve practical problems.
(4) Quaternions are a powerful tool for describing the orientation of an object in 3D space; as a result, they are highly efficient and well-suited for solving rotation and orientation problems in the areas of computer graphics, robotics, and animation [33,34]. These areas are also potential applications of QFSs.

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Abbreviations

The following abbreviations are used in this manuscript:

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
FFS	Fermatean fuzzy set
q-ROFS	q-rung orthopair fuzzy set
NS	Neutrosophic set
HFS	Hesitant fuzzy set
CFS	Complex fuzzy set
QFS	Quaternionic fuzzy set
QFWA	Quaternionic fuzzy weighted arithmetic
QFAA	Quaternionic fuzzy arithmetic average

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