



Article Dynamics in an n-Species Lotka–Volterra Cooperative System with Delays

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Abstract: We studied a class of generalized n-species non-autonomous cooperative Lotka–Volterra (L-V) systems with time delays. We obtained new criteria on the dynamic properties of the systems. First, we obtained the boundedness and permanence of the system using the inequality analysis technique and comparison method. Then, the existence of positive periodic solutions was investigated using the coincidence degree theory. The global attractivity of the system was obtained by constructing suitable Lyapunov functionals and utilizing Barbalat's lemma. The existence and global attractivity of the periodic solutions were also obtained. Finally, we conducted two numerical simulations to validate the feasibility and practicability of our results.

Keywords: Lotka-Volterra cooperative delay system; global attractivity; permanence; periodic solution

MSC: 34A34; 34D23; 37B55



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1. Introduction

In recent decades, humans have caused the diversity of species on Earth to change at an unprecedented rate. Large-scale human disturbances endanger the survival and diversity of animal species. Predicting survival and diversity is one of the cutting-edge interdisciplinary research directions and hot issues in Earth science and ecology. Given the large timescales associated with population evolution and the inherent complexity of ecosystems, many important scientific questions such as these cannot be solved using traditional field observations and statistical analysis alone. The study and prediction of the dynamic behavior of relevant ecosystems are becoming increasingly popular [1].

A cooperative population system is an ecosystem formed by many individuals working together whose dynamic behavior is closely related to the cooperative relationship between the individuals. Cooperative populations can help us understand collaborative behavior in ecosystems. Cooperation is a common behavior in ecosystems; for example, bees gathering nectar together, or birds and fish migrating in large numbers [2].

Studying the dynamics of cooperative population systems can help us better understand these cooperative behaviors and how they function. Factors such as climate change and human activity can impact populations and collaborative behavior in ecosystems. It can also help us understand the effects of these changes on ecosystems and how to respond to the impacts of climate change. Understanding collaborative behavior in ecosystems can help us design more effective environmental management and conservation measures to protect the balance and stability of ecosystems and promote environmental management and protection [3]. The dynamic behavior of cooperative systems can provide a reference for the further research of new technologies, disease prevention, and regulation and control development trends of species [4–6]. The L-V system is a population dynamics model for representing the behavior of species in ecosystems. It is worth noting that understanding the survival of populations in biological systems is one of the most fundamental and important issues in biomathematics. Coexistence is a well-known way to achieve species survival. Many scholars have studied cooperative systems and obtained good results. For example, authors that considered the problem of determining the permanence or persistence of non-autonomous multi-species L-V systems with and without delays managed to obtain some criteria for the permanence or persistence of the systems they studied [7–13].

The problem of the uniform persistence (permanence) of autonomous and non-autonomous two-species L-V cooperative systems with or without delays has been studied, and the researchers derived some criteria for the uniform persistence of the systems [14–20].

In [7], the authors studied the following n-species cooperative system without delays and they obtained some criteria for the persistence of this system.

$$\frac{dr_j(\mu)}{d\mu} = r_j(\mu)[d_j(\mu) - c_{jj}y_j(\mu) + \sum_{i \neq j}^n c_{ji}r_i(\mu)], \qquad j = 1, 2, \dots, n.$$
(1)

However, in the real world, a population's environment often changes over time, leading to changes in population growth. Therefore, the impact of time delays on biological systems cannot be ignored.

Over the past few years, there have been many interesting results published on L-V cooperative systems with delays. For example, in [8], the authors discuss an autonomous two-species cooperative model with multiple discrete delays and they derived conditions for the permanence of system (2).

$$\dot{r}_{1}(\mu) = r_{1}(\mu) \left[b_{1}(\mu) - \sum_{l=1}^{l_{11}} a_{11l}(\mu) r_{1}(\mu - \tau_{11l}) + \sum_{l=1}^{l_{12}} a_{12l}(t) r_{2}(\mu - \tau_{12l}) \right],$$

$$\dot{r}_{2}(\mu) = r_{2}(\mu) \left[b_{2}(\mu) + \sum_{l=1}^{l_{21}} a_{21l}(\mu) r_{1}(\mu - \tau_{21l}) - \sum_{l=1}^{l_{22}} a_{22l}(\mu) r_{2}(\mu - \tau_{22l}) \right].$$
(2)

The authors also considered an autonomous two-species L-V cooperative model, based on system (2), with several discrete delays and established a new sufficient condition for its permanence. Furthermore, the authors presented an open problem on a class of n-species L-V cooperative systems with delays.

In [9], the authors discussed a class of n-species L-V systems with time delays, and they established some criteria for the permanence of system.

$$\dot{r}_{1}(t) = r_{1}(\mu) \left[c_{1}(\mu) - \sum_{j=1, j \neq 2}^{n} \sum_{l=0}^{m} a_{1j}^{l}(\mu) r_{j}(\mu - l\tau) + a_{12}^{1}(\mu) r_{2}(\mu - \tau) \right],$$

$$\dot{r}_{i}(\mu) = r_{i}(\mu) \left[c_{i}(\mu) - \sum_{j=1, j \neq i+1}^{n} \sum_{l=0}^{m} a_{ij}^{l}(\mu) r_{j}(\mu - l\tau) + \sum_{l=0}^{m} a_{ii+1}^{l}(\mu) r_{i+1}(\mu - l\tau) \right],$$

$$\dot{r}_{n}(t) = r_{n}(\mu) \left[c_{n}(\mu) - \sum_{j=2}^{n} \sum_{l=0}^{m} a_{nj}^{l}(\mu) r_{j}(\mu - l\tau) + \sum_{l=0}^{m} a_{n1}^{l}(\mu) r_{i+1}(\mu - l\tau) \right],$$

$$i = 2, \cdots, n-1.$$
(3)

The following general L-V cooperative system with both varying and distributed time delays was studied in [11]. They obtained sufficient conditions for the extinction and permanence of model (4).

$$\frac{dr_i(\mu)}{d\mu} = r_i(\mu) \Big[d_i(\mu) - e_{ii}(\mu)r_i(\mu) + \sum_{j \neq i}^n e_{ij}(\mu)r_j(\mu - \tau_{ij}(\mu)) + \sum_{j \neq i}^n \int_{-\sigma_{ij}}^0 f_{ij}(\mu, s)r_j(\mu + s)ds \Big],$$
(4)
$$i = 1, 2, \dots, n.$$

As we all know, periodic phenomena are commonplace in nature. In an ecosystem, environmental factors such as the season, availability of food, and physiological characteristics always show periodic changes, which implies that these system parameters are periodic functions of time and that the population has multiple dynamic equilibrium states. Therefore, studying the periodic solutions and its stability in dynamic population systems has great practical significance. In [12], the author studied the following periodic cooperative systems (5) and derived several conditions of periodic solutions for them.

$$\dot{r}_{i}(\mu) = r_{i}(t) \Big[c_{i}(\mu) - p_{ii}(\mu)r_{i}(\mu) + \sum_{k \neq i}^{n} p_{ik}(\mu)r_{k}(\mu) - \sum_{j=1}^{n} q_{ij}(\mu)r_{i}(\mu - \tau_{ij}(\mu)) \\ + \sum_{k \neq i}^{n} \sum_{j=1}^{n} d_{ikj}(\mu)r_{k}(\mu - \gamma_{ikj}(\mu)) \Big], \qquad (i = 1, 2, \dots, n).$$
(5)

In [8–11,13,15–19], the authors derived conditions for the persistence, permanence, and extinction of the considered systems. However, while studying the dynamics of those systems, they ignore the periodic solutions and the global attractivity of the cooperative models. Furthermore, there have been few relevant studies on the permanence, periodic solutions, and the global attractivity of multispecies cooperative systems with delays.

Considering these factors, and inspired by the studies cited above, we investigated the following delayed n-species L-V cooperative systems:

$$\dot{z}_{i}(t) = z_{i}(t) \Big[s_{i}(t) - b_{i0}(t) z_{i}(t) - \sum_{k=1}^{m} b_{ik}(t) z_{i}(t-k\epsilon) + \sum_{j=1, j\neq i}^{n} c_{ij}(t) z_{j}(t) \Big],$$

$$i = 1, 2, \dots, n.$$
(6)

where $z_i(t)$ corresponds to the density of *n* cooperative species, respectively; $c_{ij}(t)$ denotes the cooperative coefficients between *n* species, respectively; $b_{ij}(t)$ represents the intra-patch restriction density, respectively; $s_i(t)$ represents the intrinsic growth rate, respectively; and ϵ is a time delay.

The purpose of this study is to obtain further conditions on global attractivity, permanence, and periodic solutions. We will achieve this by constructing appropriate Lyapunov functionals, using inequality techniques and a comparison method, coincidence degree theory, and knowledge of differential equations.

2. Preliminaries

In this paper, we always use $\mathcal{I} \triangleq \{1, 2, ..., n\}$, and we define

$$\mathcal{A}^{l} = \min_{s \in [0, +\infty)} \mathcal{A}(s), \quad \mathcal{A}^{u} = \max_{s \in [0, +\infty)} \mathcal{A}(s), \quad \bar{\mathcal{F}} = \frac{1}{\Omega} \int_{0}^{\Omega} \mathcal{F}(s) ds.$$

where $\mathcal{A}(s)$ is any bounded continuous function defined on $[0, +\infty)$ and $\mathcal{F}(s)$ is any Ω -periodic continuous function on $[0, +\infty)$.

Now, we shall give some assumptions for convenience.

Hypothesis 1 (H1). $\epsilon > 0$, $s_i(t) > 0$, $b_{ij}(t) > 0$, $c_{ij}(t) > 0$ $(i, j \in \mathcal{I})$ are all continuous bounded functions on $[0, +\infty)$.

Hypothesis 2 (H2). $\epsilon > 0$, $s_i(t) > 0$, $b_{ij}(t) > 0$, $c_{ij}(t) > 0$ $(i, j \in \mathcal{I})$ are all continuous bounded and Ω -periodic functions on $[0, +\infty)$.

The initial conditions associated with system (6) are given by

$$z_i(t) = \Gamma_i(t), \quad \forall t \in [-\sigma, 0], \quad i \in \mathcal{I},$$
(7)

where $\Gamma_i(t)$ ($i \in \mathcal{I}$) are satisfying $\Gamma_i(0) > 0$ and continuous non-negative functions defined on $[-\sigma, 0]$.

For further study, we need the following lemmas in this paper.

Lemma 1 ([21]). *If* p > 0, q > 0 and $\dot{s}(\mu) \le q - ps(\mu)$, we have

$$s(\mu) \le \frac{q}{p} [1 + (\frac{ps(0)}{q} - 1)e^{-p\mu})],$$

when $\mu \ge 0$ *and* s(0) > 0*.*

Lemma 2 ([22]). If $s(\mu) \ge 0$ and $\lim_{\mu\to\infty} s(\mu) \le M_s$ such that

$$\dot{s}(\mu) \ge s(\mu) \left[\theta - \sum_{l=0}^{m} \lambda^{l} s(\mu - k\epsilon) \right], \quad \lambda = \sum_{l=0}^{m} \lambda^{l},$$

then there is a $0 < m_s < \infty$, such that

$$\lim_{\mu\to\infty}s(\mu)\geq m_s=\frac{\theta}{\lambda}e^{(\theta-\lambda M_s)m\tau}>0.$$

3. Main Results

Theorem 1. If H1 holds and $A_i > 0 (i \in \mathcal{I})$, then system (6) is permanent. Where

$$A_{i} = \min_{t \in R} \{ b_{i0}(t) - \frac{1}{2} \sum_{j=1, j \neq i}^{n} (c_{ij}(t) + c_{ji}(t)) \}.$$

Proof. Suppose that $(z_1(t), z_2(t), \dots, z_n(t))$ is a solution of system (6) with initial conditions (7). We firstly define that

$$W(t) = \sum_{i=1}^{n} z_i(t), \quad s = \min_{i \in \mathcal{I}} s_i(t), \quad S = \max_{i \in \mathcal{I}} s_i(t), \quad t \in \mathbb{R}$$

Then, we have

$$\begin{split} \dot{W}(t) + sW(t) &\leq 2\sum_{i=1}^{n} Sz_{i}(t) - \sum_{i=1}^{n} b_{i0}(t)z_{i}^{2}(t) + z_{i}(t)\sum_{j=1, j \neq i}^{n} c_{ij}(t)z_{j}(t) \\ &= 2\sum_{i=1}^{n} Sz_{i}(t) - \sum_{i=1}^{n} b_{i0}(t)z_{i}^{2}(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j \neq i}^{n} (c_{ij}(t) + c_{ji}(t))z_{i}^{2}(t) \\ &= -\sum_{i=1}^{n} \left[\left(b_{i0}(t) - \frac{1}{2}\sum_{j=1, j \neq i}^{n} (c_{ij}(t) + c_{ji}(t)) \right) z_{i}^{2}(t) - 2Sz_{i}(t) \right] \\ &\leq -\sum_{i=1}^{n} \left[A_{i}z_{i}^{2}(t) - 2Sz_{i}(t) \right]. \end{split}$$
(8)

Since $A_i > 0$, then one can easily obtain

$$A_i z_i^2(t) - 2Sz_i(t) = A_i [z_i(t) - \frac{2S}{A_i} z_i(t) + (\frac{S}{A_i})^2 - (\frac{S}{A_i})^2] = A_i (z_i(t) - \frac{S}{A_i})^2 - \frac{S^2}{A_i} z_i(t) + (\frac{S}{A_i})^2 - \frac{S^2}{A_i} z_i(t) + (\frac{S}{A_i})^2 - (\frac{S}{A_i})^2 - \frac{S^2}{A_i} z_i(t) + (\frac{S}{A_i})^2 - (\frac{$$

Hence, from (8), we derive that

$$\dot{W}(t) + sW(t) \le \sum_{i=1}^{n} \frac{S^2}{A_i} := \theta.$$

Then, by using the Lemma 1, we can find that

$$W(t) \le \frac{\theta}{s} + (W(0) - \frac{\theta}{s})e^{-\theta t}.$$
(9)

It follows from (9) when $t \to \infty$,

$$\lim_{t \to \infty} W(t) \le \lim_{t \to \infty} \left(\frac{\theta}{s} + (W(0) - \frac{\theta}{s})e^{-\theta t}\right) = \frac{\theta}{s} := M.$$
(10)

Hence, there exists a $T_0 > 0$ such that $z_i(t) \le M(i \in \mathcal{I})$ for $t > T_0$. Next, for each $z_i(t)$, we can find that

$$\dot{z}_i(t) \ge z_i(t)[s_i^l - \sum_{k=0}^m b_{ik}^u z_i(t-k\epsilon)].$$

Let $B_i = \sum_{k=0}^{m} b_{ik}^u$, then by Lemma 2, for $t > T_0$, we directly obtain

$$\lim_{t \to \infty} z_i(t) \ge \frac{s_i^l}{B_i} e^{(s_i^l - B_i M)m\epsilon} := m_i > 0, \quad i \in \mathcal{I}.$$
(11)

Then, there is a $T_1 > T_0$ such that $z_i(t) \ge m_i$ for $t > T_1$. This implies that system (6) is permanent. \Box

Theorem 2. If H2 holds and

$$\min_{t \in R} \left\{ b_{i0}(t) + \sum_{k=1}^{m} b_{ik}(t+k\epsilon) - \sum_{j=1, j \neq i}^{n} c_{ji}(t) \right\} := D_i > 0, \quad i \in \mathcal{I},$$

and the system of algebraic equations

$$ar{s}_i - ar{b}_{i0} v_i - \sum_{k=1}^m ar{b}_{ik} v_i + \sum_{j
eq i}^n ar{c}_{ij} v_j = 0, \quad i \in \mathcal{I}_k$$

has a unique positive solution, then system (6) has a positive Ω -periodic solution. Where $v_i > 0$ is a constant.

Proof. Let

$$z_i(t) = e^{u_i(t)}, \quad i \in \mathcal{I}$$

Then, we obtain

$$\dot{u}_i(t) = s_i(t) - b_{i0}(t)e^{u_i(t)} - \sum_{k=1}^m b_{ik}(t)e^{u_i(t-k\epsilon)} + \sum_{j=1, j\neq i}^n c_{ij}(t)e^{u_j(t)}.$$
(12)

Let \mathcal{Z} and \mathcal{X} be the normed vector spaces, and $\mathcal{C}(\mathcal{R}, \mathcal{R}^n)$ the space of all continuous functions $u(t) = (u_1(t), u_2(t), \dots, u_n(t)) : \mathcal{R} \to \mathcal{R}^n$. We take

$$\mathcal{X} = \mathcal{Z} = \{u(t) \in \mathcal{C}(\mathcal{R}, \mathcal{R}^n) : u(t) \text{ is an } \Omega\text{-periodic function}\},\$$

with norm

$$||u|| = \sum_{i=1}^{n} \max_{t \in [0,\Omega]} |u_i(t)|$$

Then, \mathcal{X} and \mathcal{Z} are the Banach spaces. We let \mathcal{L} : Dom $\mathcal{L} \subset \mathcal{X} \to \mathcal{Z}$ be a linear operator and $\mathcal{N} : \mathcal{X} \to \mathcal{Z}$ be a continuous operator satisfying:

$$\mathcal{L}u(t) = \dot{u}(t),$$

$$\mathcal{N}u(t) = (\mathcal{N}u_1(t), \mathcal{N}u_2(t), \dots, \mathcal{N}u_n(t)),$$

where

$$\mathcal{N}u_{i}(t) = s_{i}(t) - b_{i0}(t)e^{u_{i}(t)} - \sum_{k=1}^{m} b_{ik}(t)e^{u_{i}(t-k\epsilon)} + \sum_{j=1, j\neq i}^{n} c_{ij}(t)e^{u_{j}(t)}.$$
(13)

Next, we let $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ and $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ are continuous projectors satisfying:

$$\mathcal{P}u(t) = \frac{1}{\Omega} \int_0^\Omega u(t) dt, \quad \mathcal{Q}v(t) = \frac{1}{\Omega} \int_0^\Omega v(t) dt.$$

Then, Im $\mathcal{L} = \left\{ v \in Z : \int_0^{\Omega} v(t) dt = 0 \right\}$ and Ker $\mathcal{L} = \mathcal{R}^n$. Further, Im \mathcal{L} is closed in \mathcal{Z} and dim Ker $\mathcal{L} = n$. Since for any $v \in \mathcal{Z}$ there are unique $v_1 \in \mathbb{R}^n$ and $v_2 \in \text{Im } \mathcal{L}$ with

$$v_1 = \frac{1}{\Omega} \int_0^{\Omega} v(t) dt, \quad v_2(t) = v(t) - v_1,$$

such that $v(t) = v_1 + v_2(t)$, we obtain codimIm $\mathcal{L} = n$. Thus, \mathcal{L} is a Fredholm mapping of index zero. Moreover, the following form is the generalized inverse (to \mathcal{L}) $\mathcal{K}_{\mathcal{P}}$: Im $\mathcal{L} \to \text{Ker } \mathcal{P} \cap \text{Dom } \mathcal{L}$:

$$\mathcal{K}_{\mathcal{P}}v(t) = \int_0^t v(s)ds - \frac{1}{\Omega}\int_0^\Omega \int_0^t v(s)dsdt.$$

Let $\mathcal{F}(t) = (\mathcal{F}_1(t), \mathcal{F}_2(t), \dots, \mathcal{F}_n(t))$ as follows:

$$\mathcal{F}_{i}(t) = s_{i}(t) - b_{i0}(t)e^{u_{i}(t)} - \sum_{k=1}^{m} b_{ik}(t)e^{u_{i}(t-k\epsilon)} + \sum_{j=1, j\neq i}^{n} c_{ij}(t)e^{u_{j}(t)}.$$
 (14)

Thus, we have

$$Q\mathcal{N}u(t) = \frac{1}{\Omega} \int_0^\Omega \mathcal{F}(t) dt, \qquad (15)$$

and

$$\mathcal{K}_{\mathcal{P}}(I-\mathcal{Q})\mathcal{N}u(t) = \mathcal{K}_{\mathcal{P}}I\mathcal{N}u(t) - \mathcal{K}_{\mathcal{P}}\mathcal{Q}\mathcal{N}u(t)$$

= $\int_{0}^{t} \mathcal{F}(s)ds - \frac{1}{\Omega}\int_{0}^{\Omega}\int_{0}^{t}\mathcal{F}(s)dsd t$
+ $\left(\frac{1}{2} - \frac{t}{\Omega}\right)\int_{0}^{\Omega}\mathcal{F}(s)ds.$ (16)

So, one can see that \mathcal{QN} and $\mathcal{K}_{\mathcal{P}}(I-\mathcal{Q})\mathcal{N}$ are continuous operators. Further, using the Arzela–Ascoli theorem, $\mathcal{QN}(\overline{\Omega}_0)$ is bounded and $\overline{\mathcal{K}_{\mathcal{P}}(I-\mathcal{Q})\mathcal{N}(\overline{\Omega}_0)}$ is compact for any open bounded set $\Omega_0 \subset \mathcal{X}$. Hence, \mathcal{N} is \mathcal{L} -compact on $\overline{\Omega}_0$ for any open bounded subset $\Omega_0 \subset \mathcal{X}$.

Corresponding to $\mathcal{L}u(t) = \lambda \mathcal{N}u(t)$ with $\lambda \in (0, 1)$ and from (14), we obtain

$$\dot{u}_i(t) = \lambda \mathcal{F}_i(t), \quad i \in \mathcal{I}.$$
(17)

Suppose that $u(t) = (u_1(t), u_2(t), \dots, u_n(t)) \in \mathcal{X}$ is a solution of system (7) for some parameter $\lambda \in (0, 1)$. Then ,integrating system (12) over the interval $[0, \Omega]$, we have

$$\bar{s}_i \Omega - \int_0^\Omega \left(b_{i0}(t) e^{u_i(t)} + \sum_{k=1}^m b_{ik}(t+k\epsilon) e^{u_i(t)} - \sum_{j=1, j \neq i}^n c_{ij}(t) e^{u_j(t)} \right) dt = 0, \quad i \in \mathcal{I}.$$

consequently, we have

$$\int_{0}^{\Omega} \left(b_{i0}(t) e^{u_i(t)} + \sum_{k=1}^{m} b_{ik}(t+k\epsilon) e^{u_i(t)} - \sum_{j=1, j \neq i}^{n} c_{ij}(t) e^{u_j(t)} \right) dt = \bar{s}_i \Omega, \quad i \in \mathcal{I}.$$
(18)

For each $b_{ik}(t)$, we have

$$\int_{0}^{\Omega} \sum_{i=1}^{m} b_{ik}(t) e^{u_i(t-k\varepsilon)} dt = \int_{k\varepsilon}^{\Omega+\varepsilon} \sum_{i=1}^{m} b_{ik}(s+k\varepsilon) e^{u_i(s)} ds$$
$$= \int_{0}^{\Omega} \sum_{i=1}^{m} b_{ik}(s+k\varepsilon) e^{u_i(s)} ds$$
$$= \int_{0}^{\Omega} \sum_{i=1}^{m} b_{ik}(t+k\varepsilon) e^{u_i(t)} dt.$$
(19)

From (17) and (18), one can easily obtain

$$\begin{split} \int_{0}^{\Omega} |\dot{u}_{i}(t)| dt &= \int_{0}^{\Omega} |s_{i}(t) - b_{i0}(t)e^{u_{i}(t)} - \sum_{k=1}^{m} b_{ik}(t+k\epsilon)e^{u_{i}(t)} \\ &+ \sum_{j=1, j \neq i}^{n} c_{ij}(t)e^{u_{j}(t)}| dt \\ &\leq \int_{0}^{\Omega} |s_{i}(t)| dt + \int_{0}^{\Omega} \left[b_{i0}(t)e^{u_{i}(t)} + \sum_{k=1}^{m} b_{ik}(t+k\epsilon)e^{u_{i}(t)} \right. \\ &+ \sum_{j=1, j \neq i}^{n} c_{ij}(t)e^{u_{j}(t)} \right] dt \\ &\leq (\bar{s}_{i} + \bar{S}_{i})w := C_{i}, \quad i \in \mathcal{I}, \end{split}$$
(20)

where $\bar{S}_i = \frac{1}{\Omega} \int_0^{\Omega} |s_i| dt$. From the properties of solutions $u(t) = (u_1(t), u_2(t), \dots, u_n(t))$, we have some constants ξ_i , $\eta_i \in [0, \Omega]$ ($i \in \mathcal{I}$) and satisfying

$$u_i(\xi_i) = \max_{t \in [0,\Omega]} u_i(t), \quad u_i(\eta_i) = \min_{t \in [0,\Omega]} u_i(t), \qquad i \in \mathcal{I}.$$
 (21)

Considering the integration of periodic functions, we have

$$\int_{0}^{\Omega} b_{ik}(t+k\varepsilon)dt = \int_{k\varepsilon}^{\Omega+k\varepsilon} b_{ik}(s)ds = \int_{0}^{\Omega} b_{ik}(t)dt, \quad i \in \mathcal{I}.$$
 (22)

Then from (18) and (22), we have

$$\bar{s}_{i}\Omega \leq \int_{0}^{\Omega} b_{i0}(t)e^{u_{i}(\xi_{i})}dt + \sum_{k=1}^{m} \int_{0}^{\Omega} b_{ik}(t+k\epsilon)e^{u_{i}(\xi_{i})}dt
= \int_{0}^{\Omega} b_{i0}(t)e^{u_{i}(\xi_{i})}dt + \sum_{k=1}^{m} \int_{0}^{\Omega} b_{ik}(t)e^{u_{i}(\xi_{i})}dt
= e^{u_{i}(\xi_{i})} \int_{0}^{\Omega} \left[b_{i0}(t) + \sum_{k=1}^{m} b_{ik}(t)\right]dt, \quad i \in \mathcal{I}.$$
(23)

Therefore, we find that

$$u_i(\xi_i) \ge \ln(\frac{\bar{s}_i}{\bar{b}_{i0} + \sum_{k=1}^m \bar{b}_{ik}}), \qquad i \in \mathcal{I}.$$
(24)

On the other hand from (18) and (19), we have

$$\int_{0}^{\Omega} [b_{i0}(t)e^{u_{i}(t)} + \sum_{k=1}^{m} b_{ik}(t+k\epsilon)e^{u_{i}(t)} - \sum_{j=1, j\neq i}^{n} c_{ij}(t)e^{u_{j}(t)}]dt = \bar{s}_{i}\Omega, \ i \in \mathcal{I}.$$
 (25)

From (25), we obtain

$$\int_{0}^{\Omega} [b_{10}(t)e^{u_{1}(t)} + \sum_{k=1}^{m} b_{1k}(t+k\epsilon)e^{u_{1}(t)} - \sum_{j=1,j\neq i}^{n} c_{1j}(t)e^{u_{j}(t)}]dt + \int_{0}^{\Omega} [b_{20}(t)e^{u_{2}(t)} + \sum_{k=1}^{m} b_{2k}(t+k\epsilon)e^{u_{2}(t)} - \sum_{j=1,j\neq i}^{n} c_{2j}(t)e^{u_{j}(t)}]dt + \dots + \int_{0}^{\Omega} [b_{n0}(t)e^{u_{2}(t)} + \sum_{k=1}^{m} b_{nk}(t+k\epsilon)e^{u_{n}(t)} - \sum_{j=1,j\neq i}^{n} c_{nj}(t)e^{u_{j}(t)}]dt = \int_{0}^{\Omega} [b_{10}(t) + \sum_{k=1}^{m} b_{1k}(t+k\epsilon) - \sum_{j=1,j\neq i}^{n} c_{j1}(t)]e^{u_{1}(t)}dt$$
(26)
$$+ \int_{0}^{\Omega} [b_{20}(t) + \sum_{k=1}^{m} b_{2k}(t+k\epsilon) - \sum_{j=1,j\neq i}^{n} c_{j2}(t)]e^{u_{2}(t)}dt + \dots + \int_{0}^{\Omega} [b_{n0}(t) + \sum_{k=1}^{m} b_{nk}(t+k\epsilon) - \sum_{j=1,j\neq i}^{n} c_{jn}(t)]e^{u_{n}(t)}dt = \sum_{i=1}^{n} \bar{s}_{i}\Omega.$$

Thus, we have

$$\int_0^{\Omega} \left[b_{i0}(t) + \sum_{k=1}^m b_{ik}(t+k\epsilon) - \sum_{j=1, j\neq i}^n c_{ji}(t) \right] e^{u_i(t)} dt \leq \sum_{i=1}^n \bar{s}_i \Omega, \quad i \in \mathcal{I},$$

and

$$\int_{0}^{\Omega} e^{u_i(t)} dt \le \frac{\sum_{i=1}^{n} \bar{s}_i \Omega}{D_i}, \quad i \in \mathcal{I}.$$
(27)

From (21) and (27), we further obtain

$$u(\eta_i) \le \ln \frac{\sum_{i=1}^n \bar{s}_i}{D_i} := B_i, \quad i \in \mathcal{I}.$$
(28)

From (20) and (28), we have

$$u_{i}(t) \leq u_{i}(\eta_{i}) + \int_{0}^{\Omega} |\dot{u}_{i}(t)| dt \leq B_{i} + C_{i} := M_{i},$$

$$u_{i}(t) \geq u_{i}(\varepsilon_{i}) - \int_{0}^{\Omega} |\dot{u}_{i}(t)| dt := N_{i},$$
(29)

and

$$\max |u_i(t)| \le \max(M_i, N_i) := H_i.$$
(30)

It is obvious that the constants H_i are independent of parameter $\lambda \in (0, 1)$. For any $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$, from (13) we obtain

$$\mathcal{QN}u = (\mathcal{QN}u_1, \mathcal{QN}u_2, \dots, \mathcal{QN}u_n),$$
$$\mathcal{QN}u_i = \bar{s}_i - \bar{b}_{i0}e^{u_i} - \sum_{k=1}^m \bar{b}_{ik}e^{u_i} + \sum_{j \neq i}^n \bar{c}_{ij}e^{u_j}, \quad i \in \mathcal{I}.$$

Let $0 < \mathcal{B} < +\infty$ be big enough, satisfying $|u_1^*| + |u_2^*| + \cdots + |u_n^*| < \mathcal{B}$ and $\mathcal{B} > B_1 + B_2 + \cdots + B_n$. Let $\Omega_0 \subset \mathcal{X}$ be a bounded open set, satisfying

$$\Omega_0 = \{ u \in X : \|u\| < \mathcal{B} \}.$$

One can see that Ω_0 fulfills the first two conditions of coincidence degree theory [23]. Moreover, we can obtain

$$\deg\{\mathcal{JQN},\Omega_0 \cap \operatorname{Ker} \mathcal{L}, (0,0,\ldots,0)\} = \operatorname{sgn} \begin{vmatrix} f_{u_1}^1 & f_{u_2}^1 & \cdots & f_{u_n}^1 \\ f_{u_1}^2 & f_{u_2}^2 & \cdots & f_{u_n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ f_{u_1}^n & f_{u_2}^n & \cdots & f_{u_n}^n \end{vmatrix} \neq 0.$$

It can be seen that the Ω_0 satisfies the last condition of coincidence degree theory [23]. Therefore, system (12) has an Ω -periodic solution $u^*(t) = (u_1^*(t), u_2^*(t), \dots, u_n^*(t)) \in \overline{\Omega}_0$. Finally, we find that system (6) has a positive Ω -periodic solution. \Box

Theorem 3. If H1 and $\Theta > 0$ hold, the system (6) is globally attractive. Where

$$\Theta = \min\{\Theta_1, \Theta_2, \cdots, \Theta_n\},\$$

and

$$\Theta_i = \min_{t \in \mathbb{R}} \left\{ b_{i0}(t) - \sum_{k=1}^m b_{ik}(t+k\varepsilon) - \sum_{j=1, j \neq i}^n c_{ji}(t) \right\}, \ i \in \mathcal{I}.$$

Proof. Suppose that $(x_1(t), x_2(t), ..., x_n(t))$ and $(z_1(t), z_2(t), ..., z_n(t))$ are any two positive solutions of system (6), then from the conditions of Theorem 3, there exist real numbers T^* and $M > m_0 > 0$ such that

$$m_0 \le x_i(t), z_i(t) \le M,\tag{31}$$

as $t \ge T^*$. Construct the following Lyapunov functional:

$$\mathcal{V}(t) = \sum_{i=1}^{n} \left[\left| \ln x_i(t) - \ln z_i(t) \right| + \sum_{k=1}^{m} \int_{t-k\varepsilon}^{t} b_{ik}(s+k\varepsilon) |x_i(s) - z_i(s)| ds \right],$$

then we obtain

$$D^{+}\mathcal{V}(t) = \sum_{i=1}^{n} \operatorname{sign}(x_{i}(t) - z_{i}(t)) \left[-b_{i0}(t)(x_{i}(t) - z_{i}(t)) + \sum_{j=1, j \neq i}^{n} c_{ij}(t)(x_{j}(t) - z_{j}(t)) - \sum_{k=1}^{m} b_{ik}(t)(x_{i}(t - k\varepsilon) - z_{i}(t - k\varepsilon)) \right] + \sum_{i=1}^{n} \sum_{k=1}^{m} \left[b_{ik}(t + k\varepsilon) |x_{i}(t) - z_{i}(t)| - b_{ik}(t)|x_{i}(t - k\varepsilon) - z_{i}(t - k\varepsilon)| \right] \\ = b_{ik}(t) |x_{i}(t - k\varepsilon) - z_{i}(t - k\varepsilon)| \right] \\ \leq \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{i}(t) - z_{i}(t)| + \sum_{k=1}^{m} b_{ik}(t) |x_{i}(t - k\varepsilon) - z_{i}(t - k\varepsilon)| + \sum_{j=1, j \neq i}^{n} c_{ij}(t) |x_{j}(t) - z_{j}(t)| \right] - \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik}(t) |x_{i}(t - k\varepsilon) - z_{i}(t - k\varepsilon)| + \sum_{i=1}^{n} \sum_{k=1}^{m} b_{ik}(t + k\varepsilon) |x_{i}(t) - z_{i}(t)| \\ = \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{i}(t) - z_{i}(t)| + \sum_{k=1}^{m} b_{ik}(t + k\varepsilon) |x_{i}(t) - z_{i}(t)| + \sum_{j=1, j \neq i}^{n} c_{ij}(t) |x_{j}(t) - z_{j}(t)| \right] \\ \leq \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{i}(t) - z_{i}(t)| \right] \\ \leq \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{j}(t) - z_{j}(t)| \right] \\ \leq \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{i}(t) - z_{i}(t)| \right] \\ \leq \sum_{i=1}^{n} \left[-b_{i0}(t) |x_{i}(t) - z_{i}(t)| \right] \\ \leq \sum_{i=1}^{n} \left[b_{i0}(t) - \sum_{k=1}^{m} b_{ik}(t + k\varepsilon) - \sum_{j=1, j \neq i}^{n} c_{ji}(t) \right] |x_{i}(t) - z_{i}(t)| \\ \leq -\Theta |x_{i}(t) - z_{i}(t)|.$$

Integrating from T_* to *t* on both sides of (32), we obtain

$$\mathcal{V}(t) + \int_{T^*}^t \Theta |x_i(t) - z_i(t)| \le \mathcal{V}(T^*).$$
(33)

Then, from (31) and the similar method in [22], we can find that $(x_i(t) - z_i(t))$ and $(\dot{x}_i(t) - \dot{z}_i(t))$ are bounded on $[T_*, \infty)$ and $|x_i(t) - z_i(t)| \in L^1[T^*, +\infty)$. Then, by Barbalat's lemma, we find that for $i \in \mathcal{I}$,

$$\lim_{t \to +\infty} |x_i(t) - z_i(t)| = 0.$$

From the conditions and proofs of the above theorems, we can derive the following result for system (6). \Box

Corollary 1. If H2 holds and

$$\min_{t\in R}\left\{b_{i0}(t)-\sum_{k=1}^m b_{ik}(t+k\epsilon)-\sum_{j=1,j\neq i}^n c_{ji}(t)\right\}:=\Pi_i>0,\quad i\in\mathcal{I},$$

and the system of algebraic equations

$$ar{s}_i-ar{b}_{i0}v_i-\sum_{k=1}^mar{b}_{ik}v_i+\sum_{j
eq i}^nar{c}_{ij}v_j=0,\quad i\in\mathcal{I},$$

a unique positive solution, then system (6) is permanent and has a global attractive positive ω -periodic solution.

Remark 1. Because model (6) can be specialized as systems in [7–9,11,14–18], and compared with the results in [7–9,11,14–18], in the present paper we derived the aforementioned results. Therefore, system (6) and the results in this paper can be seen as the supplements and extensions of the previously known results and models in [7–9,11,14–18].

4. Examples

In this section, we provide two examples to validate the practicability and feasibility of our results.

Example 1.

$$\begin{split} \dot{z}_1(t) &= z_1(t) \left[3 + \sin(t) - (4 + 0.5\sin(t))z_1(t) - (0.2 + 0.1\sin(t))z_1(t - 0.4) \right. \\ &\quad - (0.2 + 0.1\cos(t))z_1(t - 0.8) + (0.4 + 0.2\sin(t))z_2(t) \\ &\quad + (0.4 + 0.2\cos(t))z_3(t) \right], \\ \dot{z}_2(t) &= z_2(t) \left[2 + \sin(t) - (5 + 1.5\sin(t))z_2(t) - (0.3 + 0.1\sin(t))z_2(t - 0.4) \right. \\ &\quad - (0.3 + 0.1\cos(t))z_2(t - 0.8) + (0.5 + 0.2\sin(t))z_1(t) \\ &\quad + (0.5 + 0.2\cos(t))z_3(t) \right], \end{split}$$
(34)
$$&\quad + (0.5 + 0.2\cos(t))z_3(t) \right], \\ \dot{z}_3(t) &= z_3(t) \left[2 + \sin(t) - (6 + 2\sin(t))z_3(t) - (0.2 + 0.1\cos(t))z_3(t - 0.4) \right. \\ &\quad - (0.3 + 0.1\sin(t))z_3(t - 0.8) + (0.3 + 0.1\sin(t))z_1(t) \\ &\quad + (0.3 + 0.1\cos(t))z_2(t) \right]. \end{split}$$

From direct calculation, we obtain

$$\begin{split} \min_{t \in R} \left\{ b_{10}(t) - \sum_{k=1}^{2} b_{1k}(t+k\epsilon) - \sum_{j=2, j \neq 1}^{3} c_{j1}(t) \right\} &\approx 1.7 > 0, \\ \min_{t \in R} \left\{ b_{20}(t) - \sum_{k=1}^{2} b_{2k}(t+k\epsilon) - \sum_{j=1, j \neq 2}^{3} c_{j2}(t) \right\} &\approx 1.3 > 0, \\ \min_{t \in R} \left\{ b_{30}(t) - \sum_{k=1}^{2} b_{3k}(t+k\epsilon) - \sum_{j=1, j \neq 3}^{3} c_{j3}(t) \right\} &\approx 2.5 > 0, \end{split}$$

and

$$4.4v_1 - 0.4v_2 - 0.4v_3 = 3,$$

$$5.6v_2 - 0.5v_1 - 0.5v_3 = 2,$$

$$6.5v_3 - 0.3v_1 - 0.3v_2 = 2,$$

has a unique positive solution $v_1 = 0.7564$, $v_2 = 0.4572$, $v_3 = 0.3537$. Obviously, the assumptions of Corollary 1 are all satisfied. Therefore, system (34) is permanent and has a globally attractive 2π -periodic solution. The corresponding simulations demonstrated in Figure 1.



Figure 1. The global attractivity, periodic solution and permanence of system (34). Here, we take different initial values. (a) Permanence, periodicity and global attractivity of $z_1(t)$, $z_2(t)$ and $z_3(t)$. (b) Permanence, periodicity and global attractivity of $z_1(t)$ and $z_2(t)$. (c) Permanence, periodicity and global attractivity of $z_1(t)$ and $z_2(t)$. (c) Permanence, periodicity and global attractivity of $z_1(t)$ and $z_2(t)$. (d) Permanence, periodicity and global attractivity of $z_1(t)$ and $z_2(t)$. (e) Permanence, periodicity and global attractivity of $z_1(t)$, $z_2(t)$ and $z_3(t)$.

As shown in Figure 1, numerical simulations suggest that system (34) is permanent and has a global attractive 2π -periodic solution.

Example 2.

$$\begin{aligned} \dot{z}_1(t) &= z_1(t) \Big[3.5 + 0.5\cos(t) - (3 + 0.5\cos(t))z_1(t) - (5.2 + 0.1\cos(t))z_1(t - 0.65) \\ &- (5.3 + 0.1\cos(t))z_1(t - 1.3) + (0.89 + 0.02\cos(t))z_2(t) \\ &+ (0.85 + 0.02\cos(t))z_3(t) \Big], \\ \dot{z}_2(t) &= z_2(t) \Big[4 + 0.45\cos(t) - (3 + 0.55\cos(t))z_2(t) - (5.8 + 0.1\cos(t))z_2(t - 0.4) \\ &- (5.7 + 0.1\cos(t))z_2(t - 0.8) + (0.9 + 0.02\cos(t))z_1(t) \\ &+ (0.88 + 0.02\cos(t))z_3(t) \Big], \end{aligned}$$
(35)
$$\begin{aligned} &+ (0.88 + 0.02\cos(t))z_3(t) \Big], \\ \dot{z}_3(t) &= z_3(t) \Big[5 + 0.55\cos(t) - (3 + 0.45\cos(t))z_3(t) - (5.5 + 0.1\cos(t))z_3(t - 0.4) \\ &- (5.6 + 0.1\cos(t))z_3(t - 0.8) + (0.93 + 0.01\cos(t))z_1(t) \\ &+ (0.91 + 0.01\cos(t))z_2(t) \Big]. \end{aligned}$$

From direct calculation, we obtain

$$\begin{split} \min_{t \in R} \{ b_{10}(t) - \frac{1}{2} \sum_{j=2, j \neq 1}^{3} (c_{1j}(t) + c_{j1}(t)) \} &\approx 0.68 > 0, \\ \min_{t \in R} \{ b_{20}(t) - \frac{1}{2} \sum_{j=1, j \neq 2}^{3} (c_{2j}(t) + c_{j2}(t)) \} &\approx 0.645 > 0, \\ \min_{t \in R} \{ b_{30}(t) - \frac{1}{2} \sum_{j=1, j \neq 3}^{3} (c_{3j}(t) + c_{j3}(t)) \} &\approx 0.735 > 0, \\ \end{split}$$

$$\begin{split} \min_{t \in R} \left\{ b_{10}(t) + \sum_{k=1}^{2} b_{1k}(t+k\epsilon) - \sum_{j=2, j \neq 1}^{3} c_{j1}(t) \right\} &\approx 10.94 > 0, \\ \min_{t \in R} \left\{ b_{20}(t) + \sum_{k=1}^{2} b_{2k}(t+k\epsilon) - \sum_{j=1, j \neq 2}^{3} c_{j2}(t) \right\} &\approx 11.92 > 0, \\ \min_{t \in R} \left\{ b_{30}(t) + \sum_{k=1}^{2} b_{3k}(t+k\epsilon) - \sum_{j=1, j \neq 3}^{3} c_{j3}(t) \right\} &\approx 11.68 > 0, \end{split}$$

and

$$\begin{aligned} &13.5v_1 - 0.89v_2 - 0.87v_3 = 3.5, \\ &14.5v_2 - 0.9v_1 - 0.88v_3 = 4, \\ &14.1v_3 - 0.93v_1 - 0.91v_2 = 5, \end{aligned}$$

has a unique positive solution $v_1 = 0.3058, v_2 = 0.3188, v_3 = 0.3954.$ However, we find that

$$\begin{split} \min_{t \in R} \left\{ b_{10}(t) - \sum_{k=1}^{2} b_{1k}(t+k\epsilon) - \sum_{j=2, j \neq 1}^{3} c_{j1}(t) \right\} &\approx -10.06 < 0, \\ \min_{t \in R} \left\{ b_{20}(t) - \sum_{k=1}^{2} b_{2k}(t+k\epsilon) - \sum_{j=1, j \neq 2}^{3} c_{j2}(t) \right\} &\approx -11.08 < 0, \\ \min_{t \in R} \left\{ b_{30}(t) - \sum_{k=1}^{2} b_{3k}(t+k\epsilon) - \sum_{j=1, j \neq 3}^{3} c_{j3}(t) \right\} &\approx -10.52 < 0. \end{split}$$

Obviously, the assumptions of Theorems 1 and 2 are all satisfied. However, the assumptions of Theorem 3 do not hold. The corresponding simulation is demonstrated in Figure 2.



Figure 2. Dynamical behaviors of system (35). Here, we take different initial values. (a) Permanence and non-global attractivity of $z_1(t)$. (b) Permanence and non-global attractivity of $z_2(t)$. (c) Permanence and non-global attractivity of $z_1(t)$ and $z_2(t)$. (e) Permanence and non-global attractivity of $z_1(t)$ and $z_2(t)$. (e) Permanence and non-global attractivity of $z_1(t)$ and $z_2(t)$. (f) Permanence and non-global attractivity of $z_2(t)$ and $z_3(t)$. (g) Permanence and non-global attractivity of $z_1(t)$ and $z_2(t)$.

As shown in Figure 2, numerical simulations suggest that system (35) is permanent, but has no globally attractive positive periodic solution.

5. Conclusions

By considering the non-negligible nature of time delay and cooperative relationships between populations, and by extending and generalizing previous works, in this paper, we established and investigated an n-species Lotka–Volterra cooperative system with time delays (6). First, we obtained new conditions for the boundedness, permanence, and periodic solution by using inequality techniques, the comparison principle, and the coincidence degree theory. Second, by means of the Lyapunov method, we obtained conditions for the global attractiveness of the system. Moreover, we derived the conditions for the existence and global attractivity of the periodic solution. Finally, numerical simulations were employed to illustrate the effectiveness and feasibility of our results. Recently, the dynamic properties of fractional-order delay differential equations have been extensively investigated both in theory and applications [24,25]. Therefore, we have interesting future work such as the dynamical behavior on the fractional-order n-species Lotka–Volterra cooperative population systems with delays.

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References

- 1. Tang, S.; Xiao, Y.; Liang, J.; Wang, X. Matheynatical Biology; Science Press: Beijing, China, 2019. (In Chinese)
- Zhang, Z. Mutualism or cooperation among competitors promotes coexistence and competitive ability. *Ecol. Model.* 2003, 164, 271–282. [CrossRef]
- 3. Chen, F.; Xie, X. A Study on the Dynamics of Cooperative Population Models; Science Press: Beijing, China, 2014. (In Chinese)
- 4. Zouache, D.; Moussaoui, A.; Abdelaziz, F.B. A cooperative swarm intelligence algorithm for multi-objective discrete optimization with application to the knapsack problem. *Eur. J. Oper. Res.* **2008**, *264*, 74–88. [CrossRef]
- Heimerl, A.; Baur, T.; Lingenfelser, F.; Wagner, J.; Andre, E. Nova—A tool for explainable cooperative machine learning. In Proceedings of the 8th International Conference on Affective Computing and Intelligent Interaction (ACII), Cambridge, UK, 3–6 September 2019; pp. 109–115. [CrossRef]
- Vidhate, D.; Kulkarni, P. Cooperative machine learning with information fusion for dynamic decision making in diagnostic applications. In Proceedings of the 2012 International Conference on Advances in Mobile Network, Communication and Its Applications, Bangalore, India, 1–2 August 2012; pp. 70–74. [CrossRef]
- 7. Abdurahman, X.; Teng, Z. On the persistence of a nonautonomous n-species Lotka–Volterra cooperative system. *Appl. Math. Comput.* **2004**, *152*, 885–895. [CrossRef]
- Enatsu, Y. Permanence for multi-species nonautonomous Lotka–Volterra cooperative systems. *Aip Conf. Proc.* 2009, 1124, 109–118.
 Lu, G.; Lu, Z.; Enatsu, Y. Permanence for Lotka–Volterra systems with multiple delays. *Nonlinear Anal. Real World Appl.* 2011, 12, 2552–2560. [CrossRef]
- 10. Chen, L.; Xie, X. Permanence of an N-species cooperation system with continuous time delays and feedback controls. *Nonlinear Anal. Real World Appl.* **2011**, *12*, 34–38. [CrossRef]
- 11. Abdurahman, X.; Teng, Z. Persistence and Extinction for General Non-autonomous n-Species Lotka–Volterra Cooperative Systems with Delays. *Stud. Appl. Math.* 2007, 118, 17–43. [CrossRef]
- 12. Lu, S. On the existence of positive periodic solutions to a Lotka Volterra cooperative population model with multiple delays. *Nonlinear Anal.* **2008**, *68*, 1746–1753. [CrossRef]
- 13. Liao, X.; Yang, X.; Huang, T. Persistence of delayed cooperative models: Impulsive control method. *Appl. Math. Comput.* **2019**, 342, 130–146. [CrossRef]
- 14. Mukherjee, D. Permanence and global attractivity for facultative mutualism system with delay. *Math. Methods Appl. Sci.* 2003, 26, 1–9. [CrossRef]

- Lin, S.; Lu, Z. Permanence for two-species Lotka–Volterra systems with delays. *Math. Biosci. Eng.* 2006, *3*, 137–144. [CrossRef] [PubMed]
- Lu, G.; Lu, Z. Permanence for two-species Lotka–Volterra cooperative systems with delays. *Math. Biosci. Eng.* 2008, *5*, 477–484. [PubMed]
- 17. Nakata, Y. Permanence for the Lotka–Volterra cooperative system with several delays. Int. J. Biomath. 2009, 2, 267–285. [CrossRef]
- Lu, G.; Lu, Z.; Lian, X. Delay effect on the permanence for Lotka–Volterra cooperative systems. *Nonlinear Anal. Real World Appl.* 2010, 11, 2810–2816. [CrossRef]
- 19. Chen, L.; Lu, Z.; Wang, W. The effect of delays on the permanence for Lotka–Volterra systems. *Appl. Math. Lett.* **1995**, *8*, 71–73. [CrossRef]
- 20. Zhao, L.; Qin, B.; Chen, F. Permanence and global stability of a May cooperative system with strong and weak cooperative partners. *Adv. Differ. Equ.* **2018**, 2018, 172. [CrossRef]
- 21. Jiang, Z.; Muhammadhaji, A.; Hu, C.; Teng, Z. Dynamics of N-Species cooperation models with feedback controls and continuous delays. *Qual. Theory Dyn. Syst.* 2023, 22, 41. [CrossRef]
- Muhammadhaji, A. Dynamics of a predator-prey-competition system with pured elays. *Differ. Equ. Dyn. Syst.* 2022, 30, 35–49. [CrossRef]
- 23. Gaines, R.E.; Mawhin, J.L. Coincidence Degree and Nonlinear Differential Equations; Springer: Berlin/Heidelberg, Germany, 2006.
- 24. Agarwal, R.P.; Hristova, S. Ulam-type stability for a boundary-value problem for multi-term delay fractional differential equations of Caputo type. *Axioms* 2022, *11*, 742. [CrossRef]
- 25. Wattanakejorn, V.; Karthikeyann, P.; Poornima, S.; Karthikeyan, K.; Sitthiwirattham, T. Existence solutions for implicit fractional relaxation differential equations with impulsive delay boundary conditions. *Axioms* **2022**, *11*, 611. [CrossRef]

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