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An Efficient Class of Estimators in Stratified Random Sampling with an Application to Real Data

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Abstract: This research article addresses an efficient separate and combined class of estimators for the population mean estimation based on stratified random sampling (StRS). The first order approximated expressions of bias and mean square error of the proposed separate and combined class of estimators are obtained. A comparative study is conducted to determine the efficiency conditions in which the suggested class of estimators outperforms the contemporary estimators. These efficiency conditions are examined through an extensive simulation study by employing a hypothetically drawn symmetrical and asymmetrical populations. The simulation results have shown that the suggested class of estimators is more effective than the other available estimators. In addition, an application of the proposed methods is also presented by examining a real data set.

Keywords: bias; separate and combined estimators; efficiency; mean square error

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1. Introduction

In the sampling theory, the main goal of survey researchers is to increase the accuracy of population estimates that depend not only on the sampling percentage and sample size but also on the variability or non-uniformity of the populations. A suitable sampling technique to reduce population variability is StRS, which combines the underlying flexibility of simple random sampling (SRS) with additional characteristics. In sampling theory, the ratio estimators are used in case of simple random sampling (SRS) as well as in case of StRS (See [1]). There are two methods for estimating parameters that are generally used when the sampling design is StRS, namely, the combined estimator and the separate estimator. Unless the relationship is similar across the strata, the separate estimator will be more efficient (have lesser variability) than the combined estimator. The lower efficiency of the combined estimator, however, is often offset by smaller bias and the fact that we do not need to know the separate stratum means. Various estimation procedures have been developed for the estimation of parameters under StRS till date. Following [2–4] investigated few combined ratio-type estimators in StRS, however, motivated by [5,6] envisaged StRS based ratio-type estimator for population mean. Ref. [7] proposed a new population mean estimator by modifying [8] estimator and applying the transformation of [9]. Ref. [10] introduced [11] estimator in StRS; however, ref. [12] enforced combined type of [13] estimator under StRS. Ref. [14] investigated a general family of estimators employing information on auxiliary variable in StRS, however [15] suggested an efficient class of estimators under StRS. Ref. [16] suggested separate estimators of ratio and ratio type using a known coefficient of variation. An improved regression estimator was developed by [17] under StRS by utilizing robust regression techniques and covariance matrices. Ref. [18] investigated the memory type product and ratio estimators using StRS. Motivated by [12,19] examined a general class of estimators under StRS. Ref. [20] constructed some efficient classes of estimators based on StRS, whereas [21] suggested some improved classes of estimators under StRS. For more detailed study about StRS, the researchers are advised to refer [22–28], and the cited references of these articles.

Consider a population $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_N)$ of size N based on research variable y and the supplementary variable x. Let Ω be partitioned into L mutually exclusive and exhaustive strata with stratum h employing N_h units, $h = 1, 2, \ldots, L$, where $\sum_{h=1}^{L} N_h = N$. Let a sample of size n_h be measured using simple random sampling without replacement $SRSWOR$ from the stratum h such that $\sum_{h=1}^{L} n_h = n$ and let (x_{hi}, y_{hi}) are the observed values of the variables (x, y), respectively, on the unit i of the stratum h. The notations considered throughout this article are defined hereunder.

N; size of population,
N_h; size of population in stratum h,
n; size of sample,
n_h; size of sample in stratum h,
$W_h = N_h/N$; weight of stratum h,
$\bar{y}_h = \sum_{i=1}^{n_h} y_{hi}/n_h$; sample mean of variable y in stratum h,
$\bar{y}_{st} = \sum_{h=1}^{L} W_h y_h$; sample mean of variable y,
$\bar{x}_h = \sum_{i=1}^{n_h} x_{hi}/n_h$; sample mean of variable x in stratum h,
$\bar{x}_{st} = \sum_{h=1}^{L} W_h x_h$; sample mean of variable x,
$\bar{Y}_h = \sum_{i=1}^{N_h} y_{hi}/N_h$; population mean of variable y in stratum h,
$\bar{Y} = \bar{Y}_{st} = \sum_{h=1}^{L} W_h Y_h$; population mean of variable y,
$\bar{X}_h = \sum_{i=1}^{N_h} x_{hi}/N_h$; population mean of variable x in stratum h,
$\bar{X} = \bar{X}_{st} = \sum_{h=1}^{L} W_h X_h$; population mean of variable x,
$R = \bar{Y}/\bar{X}$; population ratio,
$R_h = \bar{Y}_h/\bar{X}_h$; population ratio in stratum h,
$S^2_{y_h} = (N_h - 1)^{-1} \sum_{i=1}^{N_h} (y_{hi} - \bar{Y}_h)^2$; population variance of variable y in stratum h,
$S^2_{x_h} = (N_h - 1)^{-1} \sum_{i=1}^{N_h} (x_{hi} - \bar{X}_h)^2$; population variance of variable x in stratum h,
$S_{xy_h} = (N_h - 1)^{-1} \sum_{i=1}^{N_h} (x_{hi} - \bar{X}_h)(y_{hi} - \bar{Y}_h)$; population covariance between variables x and y in stratum h,
$\rho_{xy_h} = S_{xy_h}/S_{x_h} S_{y_h}$; population correlation coefficient between variables x and y in stratum h,
C_{y_h}; population coefficient of variation for variable y in stratum h,
C_{x_h}; population coefficient of variation for variable x in stratum h,
$\beta_1(x_h) = (E(\bar{x}_h - \bar{X}_h)^3)^2/(E(\bar{x}_h - \bar{X}_h)^2)^2$; population coefficient of skewness for variable x in stratum h, and
$\beta_2(x_h) = (E(\bar{x}_h - \bar{X}_h))^4/(E(\bar{x}_h - \bar{X}_h)^2)^2$; population coefficient of kurtosis for variable x in stratum h.

To obtain the characteristics of the separate estimators, we utilize the following notations throughout this article.

$\bar{y}_h = \bar{Y}_h + e_{0h}$, $\bar{x}_h = \bar{X}_h + e_{1h}$, provided that $E(e_{th}) = 0$, $t = 0, 1$, $E({e_{0h}}^2) = \gamma_h S^2_{y_h} = U_0$, $E({e_{1h}}^2) = \gamma_h S^2_{x_h} = U_1$ and $E(e_{0h} e_{1h}) = \gamma_h \rho_{xy_h} S_{x_h} S_{y_h} = U_{10}$.
where $\gamma_h = 1/n_h$.

Also, we utilize the following notations to obtain the characteristics of the combined estimators:

$\bar{y}_{st} = \bar{Y} + \epsilon_0$, $\bar{x}_{st} = \bar{X} + \epsilon_1$, provided that $E(\epsilon_t) = 0$, $t = 0, 1$
and

$$V_{r,s} = \sum_{h=1}^{L} W_h^{r+s} E[(\bar{x}_h - \bar{X}_h)^r (\bar{y}_h - \bar{Y}_h)^s] \tag{1}$$

Using (1), it can be written as

$$E(\epsilon_0{}^2) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{y_h}^2 = V_{0,2}, \qquad E(\epsilon_1{}^2) = \sum_{h=1}^{L} W_h^2 \gamma_h S_{x_h}^2 = V_{2,0}, \qquad \text{and}$$
$$E(\epsilon_0 \epsilon_1) = \sum_{h=1}^{L} W_h^2 \gamma_h \rho_{xy_h} S_{x_h} S_{y_h} = V_{1,1}.$$

The main objective of the article is to provide an efficient class of separate and combined estimators of the population mean $\bar{Y}$ of the research variable y using data on the auxiliary variable x. The remainder of the article is divided into sections. The detailed literature study of the existing separate and combined estimators in StRS is presented in Section 2. The suggested separate and combined class of estimators, together with their characteristics, are provided in Section 3. The efficiency criteria are established in Section 4, which are enhanced through an empirical study utilizing artificial and real data in Section 5. Section 6 draws the conclusion of the article.

2. Existing Estimators

The current section provides a review of the prominent conventional separate and combined class of population mean estimators along with their characteristics.

The commonly used mean per unit estimator in StRS is suggested as

$$t_m = \bar{y}_{st} \tag{2}$$

2.1. Separate Estimators

When study and auxiliary variables are strongly positively correlated and when the regression line of y on x follows a straight line through the origin, then the ratio estimator is more effective. Considering this advantage, ref. [29] suggested the classical StRS based separate ratio estimator as

$$t_r^s = \sum_{h=1}^{L} W_h \bar{y}_h \frac{\bar{X}_h}{\bar{x}_h} \tag{3}$$

Sometimes the study and auxiliary variables are linearly regressed but the regression line passes a location far from the origin. In such situations, the efficiency of the ratio estimator is very poor. Regression estimator is the best option in these situations. The commonly used separate regression estimator under StRS is prescribed as

$$t_\beta^s = \sum_{h=1}^{L} W_h [\bar{y}_h + \beta_h (\bar{X}_h - \bar{x}_h)] \tag{4}$$

where the coefficient of regression of variable y on variable x is β_h in the stratum h.

Since, the use of auxiliary information improves the efficiency of the estimators. Therefore, using auxiliary information, we define the separate type population mean estimators based on [4] as follows

$$t_{kc_1}^s = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h + C_{x_h}}{\bar{x}_h + C_{x_h}} \right) \tag{5}$$

$$t_{kc_2}^s = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h + \beta_2(x_h)}{\bar{x}_h + \beta_2(x_h)} \right) \tag{6}$$

$$t_{kc_3}^s = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h \beta_2(x_h) + C_{x_h}}{\bar{x}_h \beta_2(x_h) + C_{x_h}} \right) \tag{7}$$

$$t_{kc_4}^s = \sum_{h=1}^{L} W_h \bar{y}_h \left(\frac{\bar{X}_h C_{x_h} + \beta_2(x_h)}{\bar{x}_h C_{x_h} + \beta_2(x_h)} \right) \tag{8}$$

Ref. [30] proposed a transformation that involved multiplying a tuning parameter in the estimators improves the efficiency of the estimator. Therefore, taking inspiration from the above work, we define the separate type population mean estimator based on [6] as follows

$$t^s_{kc} = \sum_{h=1}^{L} W_h k_h \bar{y}_h \left(\frac{\bar{X}_h}{\bar{x}_h} \right) \tag{9}$$

where scalar k_h in stratum h is appropriately selected to minimize MSE.

The separate type estimator based on [7] is given by

$$t^s_{sg} = \sum_{h=1}^{L} W_h \lambda_h [\bar{y}_h + \beta_h (\bar{X}_h - \bar{x}_h)] \left(\frac{\bar{z}_h}{\bar{Z}_h} \right) \tag{10}$$

where λ_h is an appropriately selected scalars, $z_h = (\bar{x}_h + X_h)$, and $\bar{Z}_h = (\bar{X}_h + X_h)$, X_h is the h^{th} stratum population total.

The separate type estimator based on [10] is given by

$$t^s_{kk_1} = \sum_{h=1}^{L} W_h \lambda_{k_h} \bar{y}_h \left[\frac{a_h \bar{X}_h + b_h}{\alpha_h (a_h \bar{x}_h + b_h) + (1 - \alpha_h)(a_h \bar{X}_h + b_h)} \right]^g \tag{11}$$

where α_h in stratum h is a constant and g is a prescribed constant that considers values -1 and 1 to produce, respectively, the product and ratio types estimators, while $(a_h \neq 0)$ and b_h are either real numbers or the function of some commonly available parameters of the variable x like the population standard deviation S_{x_h}, the population variation coefficient C_{x_h}, the population kurtosis coefficient $\beta_2(x_h)$, etc., in the stratum h.

The separate type estimator based on [12] is given by

$$t^s_{kk_2} = \sum_{h=1}^{L} W_h [k_{1h} \bar{y}_h + k_{2h} (\bar{X}_h - \bar{x}_h)] \left(\frac{\bar{X}^*_h}{\bar{x}^*_h} \right) \tag{12}$$

where $\bar{X}^*_h = (a_h \bar{X}_h + b_h)$, $\bar{x}^*_h = (a_h \bar{x}_h + b_h)$, and k_{1h} and k_{2h} are appropriately selected scalars for the h^{th} stratum.

The separate type estimator based on [14] is given by

$$t^s_{sv} = \sum_{h=1}^{L} W_h \left[\Lambda_{1h} \bar{y}_h + \Lambda_{2h} \bar{y}_h \left(\frac{\bar{x}^*_h}{\bar{X}^*_h} \right)^{\beta_h} \right] \tag{13}$$

where Λ_{1h}, Λ_{2h} in the stratum h are appropriately selected scalars.

The separate family of population mean estimators based on [15] is suggested as

$$t^s_{ss_1} = \sum_{h=1}^{L} W_h \left[\phi_{1h} \bar{y}_h \left\{ \frac{\alpha_h (a_h \bar{x}_h + b_h) + (1 - \alpha_h)(a_h \bar{X}_h + b_h)}{(a_h \bar{X}_h + b_h)} \right\}^{\delta} + \phi_{2h} \bar{y}_h \left\{ \frac{(a_h \bar{X}_h + b_h)}{\alpha_h (a_h \bar{x}_h + b_h) + (1 - \alpha_h)(a_h \bar{X}_h + b_h)} \right\}^{g} \right] \tag{14}$$

where δ, g, α_h are the prescribed constants in stratum h and ϕ_{1h}, ϕ_{2h} in the stratum h are appropriately selected scalars.

The separate class of ratio exponential estimators based on [31] is suggested as

$$t^s_u = \sum_{h=1}^{L} W_h \bar{y}_h \exp\left(\frac{\bar{X}_h - \bar{x}_h}{\bar{X}_h + (a_h - 1)\bar{x}_h} \right) \tag{15}$$

where a_h in the stratum h is a real constant that is positive.

Ref. [24] introduced the under mentioned separate class of population mean estimators as

$$t^s_{ss_2} = \sum_{h=1}^{L} W_h \left[w_{1h}\bar{y}_h \left(\frac{\bar{X}^*_h}{\bar{x}^*_h}\right)^{\alpha_h} \exp\left(\frac{\beta_h(\bar{X}^*_h - \bar{x}^*_h)}{(\bar{X}^*_h + \bar{x}^*_h)}\right) + w_{2h}\bar{y}_h \left(\frac{\bar{x}^*_h}{\bar{X}^*_h}\right)^{\delta_h} \exp\left(\frac{\lambda_h(\bar{x}^*_h - \bar{X}^*_h)}{(\bar{X}^*_h + \bar{x}^*_h)}\right) \right] \tag{16}$$

where α_h, β_h, δ_h, λ_h are constants in stratum h which take the values -1, 0, 1 for generating various estimators and w_{1h}, w_{2h} in the stratum h are appropriately selected scalars.

Ref. [20] envisaged the following efficient separate classes of population mean estimators in StRS as

$$t^s_{s_1} = \sum_{h=1}^{L} W_h \alpha_{1h} \bar{y}_h \left[1 + \log\left(\frac{\bar{x}^*_h}{\bar{X}^*_h}\right)\right]^{\beta_{1h}} \tag{17}$$

$$t^s_{s_2} = \sum_{h=1}^{L} W_h \alpha_{2h} \bar{y}_h \left[1 + \beta_{2h} \log\left(\frac{\bar{x}^*_h}{\bar{X}^*_h}\right)\right] \tag{18}$$

$$t^s_{s_3} = \sum_{h=1}^{L} W_h [\alpha_{3h} \bar{y}_h + \beta_{3h}(\bar{x}^*_h - \bar{X}^*_h)] \tag{19}$$

$$t^s_{s_4} = \sum_{h=1}^{L} W_h \alpha_{4h} \bar{y}_h \left(\frac{\bar{X}^*_h}{\bar{x}^*_h}\right)^{\beta_{4h}} \tag{20}$$

$$t^s_{s_5} = \sum_{h=1}^{L} W_h \alpha_{5h} \bar{y}_h \left[\frac{\bar{X}^*_h}{\bar{X}^*_h + \beta_{5h}(\bar{x}^*_h - \bar{X}^*_h)}\right] \tag{21}$$

where α_{ih} and β_{ih}; $i = 1, 2, \ldots, 5$ in the stratum h are appropriately selected scalars.

For easy reference, the mean square error (MSE) of the above reviewed estimators is provided in Appendix A.

2.2. Combined Estimators

Considering positive correlation between study and auxiliary variables, ref. [29] introduced a combined ratio estimator in StRS as

$$t^c_r = \frac{\bar{y}_{st}}{\bar{x}_{st}} \bar{X} \tag{22}$$

The commonly used combined regression estimator in StRS is expressed as

$$t^c_\beta = \bar{y}_{st} + \beta(\bar{X} - \bar{x}_{st}) \tag{23}$$

where the regression coefficient of y on x is β.

On the lines of [2,3,32], the following combined ratio type estimators of population mean were suggested by [4] as

$$t^c_{kc_1} = \bar{y}_{st} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + C_{x_h})}{\sum_{h=1}^{L} W_h(\bar{x}_h + C_{x_h})} \tag{24}$$

$$t^c_{kc_2} = \bar{y}_{st} \frac{\sum_{h=1}^{L} W_h(\bar{X}_h + \beta_2(x_h))}{\sum_{h=1}^{L} W_h(\bar{x}_h + \beta_2(x_h))} \tag{25}$$

$$t^c_{kc_3} = \bar{y}_{st}\frac{\sum\limits_{h=1}^{L} W_h(\bar{X}_h\beta_2(x_h) + C_{x_h})}{\sum\limits_{h=1}^{L} W_h(\bar{x}_h\beta_2(x_h) + C_{x_h})} \tag{26}$$

$$t^c_{kc_4} = \bar{y}_{st}\frac{\sum\limits_{h=1}^{L} W_h(\bar{X}_hC_{x_h} + \beta_2(x_h))}{\sum\limits_{h=1}^{L} W_h(\bar{x}_hC_{x_h} + \beta_2(x_h))} \tag{27}$$

where the kurtosis coefficient of the auxiliary variable x is $\beta_2(x_h)$ in the stratum h.

Following [5], the StRS based combined ratio estimator is examined by [6] as

$$t^c_{kc} = k\bar{y}_{st}\left(\frac{\bar{X}}{\bar{x}_{st}}\right) \tag{28}$$

where k is an appropriately selected scalar.

Using the transformation of [7,9] enforced the regression cum ratio estimator as

$$t^c_{sg} = \lambda[\bar{y}_{st} + \beta(\bar{X} - \bar{x}_{st})]\left(\frac{\bar{z}_{st}}{\bar{Z}}\right) \tag{29}$$

where λ is an appropriately selected scalar, $z_{st} = \sum\limits_{h=1}^{L} W_h(X + \bar{x}_h)$, and $\bar{Z} = \sum\limits_{h=1}^{L} W_h(X + \bar{X}_h)$.

Ref. [10] suggested the following StRS based population mean estimator as

$$t^c_{kk_1} = \lambda_k\bar{y}_{st}\left[\frac{a\bar{X} + b}{\alpha(a\bar{x}_{st} + b) + (1-\alpha)(a\bar{X} + b)}\right]^g \tag{30}$$

where α is a constant that is fixed, g is a scalar that has been carefully chosen, taking amount -1 and 1 to provide, respectively, the product and the ratio types of estimators, and $(a \neq 0)$ and b are the real valuation or functions of the available parameters of the supplementary variable x.

Ref. [12] investigated the StRS based combined type of population mean estimators as

$$t^c_{kk_2} = [k_1\bar{y}_{st} + k_2(\bar{X} - \bar{x}_{st})]\left(\frac{\bar{X}^*}{\bar{x}^*_{st}}\right) \tag{31}$$

$\bar{x}^*_{st} = \sum_{h=1}^{L} W_h(a\bar{x}_h + b)$, $\bar{X}^* = \sum_{h=1}^{L} W_h(a\bar{X}_h + b)$ and k_1 and k_2 are appropriately selected scalars.

Ref. [14] invoked a general estimation method for computing the population mean as

$$t^c_{sv} = \Lambda_1\bar{y}_{st} + \Lambda_2\bar{y}_{st}\left(\frac{\bar{x}^*_{st}}{\bar{X}^*}\right)^\beta \tag{32}$$

where Λ_1 and Λ_2 are appropriately selected scalars.

The following StRS based new family of population mean estimators is proposed by [15] as

$$t^c_{ss_1} = \left[\phi_1\bar{y}_{st}\left\{\frac{\alpha(a\bar{x}_{st}+b)+(1-\alpha)(a\bar{X}+b)}{(a\bar{X}+b)}\right\}^\delta + \phi_2\bar{y}_{st}\left\{\frac{(a\bar{X}+b)}{\alpha(a\bar{x}_{st}+b)+(1-\alpha)(a\bar{X}+b)}\right\}^g\right] \tag{33}$$

where ϕ_1, ϕ_2 are appropriately selected scalars, δ, g and, α are constants.

On the lines of [31,33] developed the following StRS based ratio exponential kind of estimators as

$$t_u^c = \bar{y}_{st} \exp\left(\frac{\bar{X} - \bar{x}_{st}}{\bar{X} + (a-1)\bar{x}_{st}}\right) \tag{34}$$

where the real constant a is positive.

Ref. [24] introduced the undermentioned StRS based population mean estimator as

$$t_{ss_2}^c = \left[\ w_1 \bar{y}_{st} \left(\frac{\bar{X}^*}{\bar{x}_{st}^*}\right)^{\alpha} \exp\left(\frac{\beta(\bar{X}^* - \bar{x}_{st}^*)}{(\bar{X}^* + \bar{x}_{st}^*)}\right) + w_2 \bar{y}_{st} \left(\frac{\bar{x}_{st}^*}{\bar{X}^*}\right)^{\delta} \exp\left(\frac{\lambda(\bar{x}_{st}^* - \bar{X}^*)}{(\bar{X}^* + \bar{x}_{st}^*)}\right)\ \right] \tag{35}$$

where $w_1,\ w_2$ are appropriately selected scalars and $\lambda,\ \delta,\ \beta$ and α are constants that take valuations $0,\ -1$, and 1 for developing various types of estimators.

Ref. [20] investigated the following efficient combined classes of population mean estimators in StRS as

$$t_{s_1}^c = \alpha_1 \bar{y}_{st} \left[1 + \log\left(\frac{\bar{x}_{st}^*}{\bar{X}^*}\right)\right]^{\beta_1} \tag{36}$$

$$t_{s_2}^c = \alpha_2 \bar{y}_{st} \left[1 + \beta_2 \log\left(\frac{\bar{x}_{st}^*}{\bar{X}^*}\right)\right] \tag{37}$$

$$t_{s_3}^c = \alpha_3 \bar{y}_{st} + \beta_3(\bar{x}_{st}^* - \bar{X}^*) \tag{38}$$

$$t_{s_4}^c = \alpha_4 \bar{y}_{st} \left(\frac{\bar{X}^*}{\bar{x}_{st}^*}\right)^{\beta_4} \tag{39}$$

$$t_{s_5}^c = \alpha_5 \bar{y}_{st} \left[\frac{\bar{X}^*}{\bar{X}^* + \beta_5(\bar{x}_{st}^* - \bar{X}^*)}\right] \tag{40}$$

where α_i and β_i; $(i = 1, 2, \ldots, 5)$ are appropriately selected scalars.

For easy reference, the MSE of the above reviewed estimators is provided in Appendix B.

3. Suggested Class of Estimators

The goal of the present article is to suggest an efficient class of separate and combined estimators as an alternative for the conventional estimators covered in the preceding section. On the lines of [20], we have extended the work of [34] in StRS to compute the population mean $\bar{Y}$ of the research variable y utilizing the data from the supplementary variable x.

3.1. Separate Estimators

We suggest an efficient separate class of estimators for estimating the population mean as

$$T_{bk}^s = \sum_{h=1}^{L} W_h \left[\varphi_{1h} \bar{y}_h + \varphi_{2h} \bar{y}_h \left(\frac{\bar{X}_h^*}{\xi_h \bar{x}_h^* + (1 - \xi_h)\bar{X}_h^*}\right)^g\right] \left[1 + \log\left(\frac{\bar{x}_h^*}{\bar{X}_h^*}\right)\right]^{\digamma_h} \tag{41}$$

where φ_{1h}, φ_{2h} and $\digamma_h$ are appropriately selected scalars in the stratum h. Also, ξ_h and g are constants taking real values. In practice, these parameters can be estimated by their consistent estimates.

Theorem 1. *The characteristics including bias and minimum MSE of the proposed separate class of estimators are expressed as*

$$Bias(T_{bk}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 [\varphi_{1h} E_{4h} + \varphi_{2h} E_{5h} - 1] \tag{42}$$

$$minMSE(T_{bk}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1 - \varsigma_h) \tag{43}$$

where $\varsigma_h = (E_{1h}E_{5h}^2 + E_{2h}E_{4h}^2 - 2E_{3h}E_{4h}E_{5h})/(E_{1h}E_{2h} - E_{3h}^2)$.

Proof. Using the notations defined in earlier section, we express the estimator T_{bk}^s as

$$T_{bk}^s - \sum_{h=1}^{L} W_h\bar{Y}_h = \sum_{h=1}^{L} W_h\bar{Y}_h \begin{bmatrix} \varphi_{1h} + \varphi_{2h} - 1 + \varphi_{1h}e_{0h} + \varphi_{2h}e_{0h} + \varphi_{1h}F_h e_{1h} \\ +\varphi_{2h}F_h e_{1h} - \varphi_{2h}g\xi_h e_{1h} - \varphi_{2h}g\xi_h e_{0h}e_{1h} \\ +\varphi_{2h}\frac{g(g+1)}{2}\xi_h^2 e_{1h}^2 + \varphi_{1h}F_h e_{0h}e_{1h} + \varphi_{2h}F_h e_{0h}e_{1h} \\ -\varphi_{2h}F_h g\xi_h e_{1h}^2 - \varphi_{1h}F_h e_{1h}^2 - \varphi_{2h}F_h e_{1h}^2 \\ +\varphi_{1h}\frac{F_h^2}{2}e_{1h}^2 + \varphi_{2h}\frac{F_h^2}{2}e_{1h}^2 \end{bmatrix} \tag{44}$$

Taking expectation both the sides of (44), we get bias to the first order of approximation as follows:

$$Bias(T_{bk}^s) = \sum_{h=1}^{L} W_h^2\bar{Y}_h^2[\varphi_{1h}E_{4h} + \varphi_{2h}E_{5h} - 1] \tag{45}$$

Again, squaring and taking expectation both sides of (44), we get MSE to the first order of approximation as follows:

$$MSE(T_{bk}^s) = \sum_{h=1}^{L} W_h^2\bar{Y}_h^2\left[1 + \varphi_{1h}^2E_{1h} + \varphi_{2h}^2E_{2h} + 2\varphi_{1h}\varphi_{2h}E_{3h} - 2\varphi_{1h}E_{4h} - 2\varphi_{2h}E_{5h}\right] \tag{46}$$

where

$$E_{1h} = 1 + U_0 + (2F_h^2\nu^2 - 2F_h\nu^2)U_1 + 4F_h\nu U_{10};$$
$$E_{2h} = \begin{bmatrix} 1 + U_0 + \{2F_h^2\nu^2 - 2F_h\nu^2 + g^2\xi_h^2\nu^2 - 4F_hg\xi_h\nu^2 + g(g+1)\xi_h^2\nu^2\}U_1 \\ +4\nu(F_h - g\xi_h)U_{10} \end{bmatrix};$$
$$E_{3h} = 1 + U_0 + \left\{2F_h^2\nu^2 - 2F_hg\xi_h\nu^2 - 2F_h\nu^2 + \frac{g(g+1)}{2}\xi_h^2\nu^2\right\}U_1 + 2\nu(2F_h - g\xi_h)U_{10};$$
$$E_{4h} = 1 + \left\{\frac{F_h^2}{2}\nu^2 - F_h\nu^2\right\}U_1 + F_h\nu U_{10};$$
and
$$E_{5h} = 1 + \left\{\frac{F_h^2}{2}\nu^2 - F_h\nu^2 - F_hg\xi_h\nu^2 + \frac{g(g+1)}{2}\xi_h^2\nu^2\right\}U_1 + (F_h\nu - g\xi_h\nu)U_{10}.$$

We obtain the optimum values of φ_{1h} and φ_{2h} by differentiating (44) by differentiating it with respect to the scalars φ_{1h} and φ_{2h} and equating to zero.

$$\varphi_{1h} = \frac{(E_{2h}E_{4h} - E_{3h}E_{5h})}{(E_{1h}E_{2h} - E_{3h}^2)} \tag{47}$$

$$\varphi_{2h} = \frac{(E_{1h}E_{5h} - E_{3h}E_{4h})}{(E_{1h}E_{2h} - E_{3h}^2)} \tag{48}$$

The optimum MSE is obtained by using the optimum values of φ_{1h} and φ_{2h} in the $MSE(T_{bk}^s)$.

$$\begin{aligned} minMSE(T_{bk}^s) &= \sum_{h=1}^{L} W_h^2\bar{Y}_h^2\left[1 - \frac{(E_{1h}E_{5h}^2 + E_{2h}E_{4h}^2 - 2E_{3h}E_{4h}E_{5h})}{(E_{1h}E_{2h} - E_{3h}^2)}\right] \\ &= \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) \end{aligned} \tag{49}$$

□

3.2. Combined Estimators

We suggest an efficient combined class of estimators for estimating the population mean as

$$T_{bk}^{c} = \left[\varphi_1 \bar{y}_{st} + \varphi_2 \bar{y}_{st} \left(\frac{\bar{X}^*}{\xi \bar{x}_{st}^* + (1-\xi)\bar{X}^*} \right)^{g} \right] \left[1 + \log\left(\frac{\bar{x}_{st}^*}{\bar{X}^*} \right) \right]^{F} \tag{50}$$

where φ_1, φ_2 and F are the appropriately selected scalars. Furthermore, ξ and g are constants taking real values. In practice, these parameters can be estimated by their consistent estimates.

Theorem 2. *The characteristics including bias and minimum MSE of the proposed combined class of estimators are expressed as*

$$Bias(T_{bk}^{c}) = \bar{Y}[\varphi_1 E_4 + \varphi_2 E_5 - 1] \tag{51}$$

$$minMSE(T_{bk}^{c}) = \bar{Y}^2(1-\varsigma) \tag{52}$$

where $\varsigma = (E_1 E_5^2 + E_2 E_4^2 - 2E_3 E_4 E_5)/(E_1 E_2 - E_3^2)$.

Proof. Using the notations defined in earlier section, we express the estimator T_{bk}^{c} as

$$T_{bk}^{c} - \bar{Y} = \bar{Y} \left[\begin{array}{l} \varphi_1 + \varphi_2 - 1 + \varphi_1 \epsilon_0 + \varphi_2 \epsilon_0 + \varphi_1 F \epsilon_1 + \varphi_2 F \epsilon_1 - \varphi_2 g \xi \epsilon_1 \\ -\varphi_2 g \xi \epsilon_0 \epsilon_1 + \varphi_2 \frac{g(g+1)}{2} \xi^2 \epsilon_1^2 + \varphi_1 F \epsilon_0 \epsilon_1 + \varphi_2 F \epsilon_0 \epsilon_1 \\ -\varphi_2 F g \xi \epsilon_1^2 - \varphi_1 F \epsilon_1^2 - \varphi_2 F \epsilon_1^2 + \varphi_1 \frac{F^2}{2} \epsilon_1^2 + \varphi_2 \frac{F^2}{2} \epsilon_1^2 \end{array} \right] \tag{53}$$

Taking expectation both the sides of (53), we get bias to the first order of approximation as follows:

$$Bias(T_{bk}^{c}) = \bar{Y}^2[\varphi_1 E_4 + \varphi_2 E_5 - 1] \tag{54}$$

Again, squaring and taking expectation both sides of (53), we get MSE to the first order of approximation as follows:

$$MSE(T_{bk}^{c}) = \bar{Y}^2 \left[1 + \varphi_1^2 E_1 + \varphi_2^2 E_2 + 2\varphi_1 \varphi_2 E_3 - 2\varphi_1 E_4 - 2\varphi_2 E_5 \right] \tag{55}$$

where

$$\begin{aligned} E_1 &= 1 + V_{0,2} + (2F^2 \nu^2 - 2F \nu^2) V_{2,0} + 4F \nu V_{1,1}; \\ E_2 &= \left[\begin{array}{l} 1 + V_{0,2} + \{ 2F^2 \nu^2 - 2F \nu^2 + g^2 \xi^2 \nu^2 - 4F g \xi \nu^2 + g(g+1) \xi^2 \nu^2 \} V_{2,0} \\ +4\nu(F - g\xi) V_{1,1} \end{array} \right]; \\ E_3 &= \ 1 + V_{0,2} + \left\{ 2F^2 \nu^2 - 2F g \xi \nu^2 - 2F \nu^2 + \tfrac{g(g+1)}{2} \xi^2 \nu^2 \right\} V_{2,0} + 2\nu(2F - g\xi) V_{1,1} \ ; \\ E_4 &= \ 1 + \left(\tfrac{F^2}{2} \nu^2 - F \nu^2 \right) V_{2,0} + F \nu V_{1,1} \ ; \\ \text{and } E_5 &= \ 1 + \left\{ \tfrac{F^2}{2} \nu^2 - F \nu^2 - F g \xi \nu^2 + \tfrac{g(g+1)}{2} \xi^2 \nu^2 \right\} V_{2,0} + (F \nu - g \xi \nu) V_{1,1}. \end{aligned}$$

We obtain the optimum values of φ_1 and φ_2 by differentiating (53) with respect to the scalars φ_1 and φ_2 and equating to zero.

$$\varphi_{1(opt)} = \frac{(E_2 E_4 - E_3 E_5)}{(E_1 E_2 - E_3^2)} \tag{56}$$

$$\varphi_{2(opt)} = \frac{(E_1 E_5 - E_3 E_4)}{(E_1 E_2 - E_3^2)} \tag{57}$$

The optimum MSE is obtained by using the optimum values of φ_1 and φ_2 in the $MSE(T_{bk}^c)$.

$$minMSE(T_{bk}^c) = \bar{Y}^2\left[1 - \frac{(E_1E_5^2 + E_2E_4^2 - 2E_3E_4E_5)}{(E_1E_2 - E_3^2)}\right] \tag{58}$$

$$= \bar{Y}^2(1 - \varsigma) \tag{59}$$

□

It is important to note that the efficiency criteria must be derived using the MSE expressions of the suggested separate and combined class of estimators provided in (43) and (52).

4. Efficiency Conditions

The efficiency conditions for separate and combined estimators are discussed in this section.

4.1. Separate Estimators

By comparing the minimum MSE of the suggested separate estimators T_{bk}^s with the minimum MSE of existing separate estimators, the following efficiency criteria are listed here.

1. The proposed separate estimators dominate the conventional mean estimator if

$$MSE(T_{bk}^s) < MSE(t_m) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2U_0$$

2. The proposed separate estimators dominate the separate ratio estimator if

$$MSE(T_{bk}^s) < MSE(t_r^s) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(U_0 + U_1 - 2U_{10})$$

3. The proposed separate estimators dominate the separate regression estimator if

$$MSE(T_{bk}^s) < MSE(t_\beta^s) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2\left[U_0 - \frac{U_{10}^2}{U_1}\right]$$

4. The proposed separate estimators dominate the separate form of [4] estimator if

$$MSE(T_{bk}^s) < MSE(t_{kc_i}^s) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2\left[U_0 + \lambda_{ih}^2U_1 - 2\lambda_{ih}U_{10}\right]$$

5. The proposed separate estimators dominate the separate form of [6] estimator if

$$MSE(T_{bk}^s) < MSE(t_{kc}^s) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2[1 - \varsigma_h] < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2\left[\begin{array}{c}(k_h^* - 1)^2 \\ +\left\{\begin{array}{c} U_0 + k_h^{*2}U_1 \\ -2k_h^*U_{10}\end{array}\right\}\end{array}\right]$$

6. The proposed separate estimators dominate the separate form of [7] estimator if

$$MSE(T_{bk}^s) < MSE(t_{sg}^s) \implies \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \varsigma_h) < \sum_{h=1}^{L} W_h^2\bar{Y}_h^2(1 - \lambda_{s_h(opt)})$$

7. The proposed separate estimators dominate the separate form of [10] estimator if

$$MSE(T_{bk}^{s}) < MSE(t_{kk_1}^{s}) \implies \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{A_h^2}{4B_h}\right)$$

8. The proposed separate estimators dominate the separate form of [12] estimator if

$$MSE(T_{bk}^{s}) < MSE(t_{kk_2}^{s}) \implies \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \left[\frac{\bar{Y}_h^2 (v_h^2 U_1 - 1)(U_1 U_0 - U_{10}^2)}{v_h^2 U_1^2 + U_{10}^2 - U_1(1+U_0)}\right]$$

9. The proposed separate estimators dominate the separate form of [14] estimator if

$$MSE(T_{bk}^{s}) < MSE(t_{sv}^{s}) \implies \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \Lambda_{1_h(opt)} - \Lambda_{2_h(opt)} B_{4h}\right]$$

10. The proposed separate estimators dominate the separate form of [15] estimator if

$$MSE(T_{bk}^{s}) < MSE(t_{ss_1}^{s}) \implies \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \begin{bmatrix} 1 - \phi_{1_h(opt)} C_{4h} \\ -\phi_{2_h(opt)} C_{5h} \end{bmatrix}$$

11. The proposed separate estimators dominate the separate form of [24] estimator if

$$\begin{aligned} & MSE(T_{bk}^{s}) < MSE(t_{ss_2}^{s}) \\ \implies & \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \frac{\left(D_{1h} D_{5h}^2 + D_{2h} D_{4h}^2 - 2D_{3h} D_{4h} D_{5h}\right)}{(D_{1h} D_{2h} - D_{3h}^2)}\right] \end{aligned} \quad (60)$$

12. The proposed separate estimators dominate the separate form of [31] estimator if

$$MSE(T_{bk}^{s}) < MSE(t_{u}^{s}) \implies \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 (1-\varsigma_h) < \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_0 - \frac{U_{10}^2}{U_1}\right]$$

The suggested separate class of estimators represses the conventional separate estimators under the aforementioned efficiency criteria.

4.2. Combined Estimators

By comparing the minimum MSE of the suggested combined estimators T_{bk}^{c} with the minimum MSE of existing combined estimators, the following efficiency criteria are listed here.

1. The proposed combined estimators dominate the mean per unit estimator if

$$MSE(t_m) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - V_{0,2}$$

2. The proposed combined estimators dominate the combined form of ratio estimator if

$$MSE(t_r^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - V_{0,2} - V_{2,0} + 2V_{1,1}$$

3. The proposed combined estimators dominate the combined form of regression estimator if

$$MSE(t_\beta^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - V_{0,2} + \frac{V_{1,1}^2}{V_{2,0}}$$

4. The proposed combined estimators dominate the combined form [4] estimator if

$$MSE(t_{kc_i}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - \lambda_i^2 V_{2,0} - V_{0,2} + 2\lambda_i V_{1,1}$$

5. The proposed combined estimators dominate the combined form [6] estimator if

$$MSE(t_{kc}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - (k^{*} - 1)^{2} - V_{0,2} - k^{*^{2}} V_{2,0} + 2k^{*} V_{1,1}$$

6. The proposed combined estimators dominate the combined form [7] estimator if

$$MSE(t_{sg}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > \lambda_{s(opt)}$$

7. The proposed combined estimators dominate the combined form [10] estimator if

$$MSE(t_{kk_1}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > \frac{A^2}{B}$$

8. The proposed combined estimators dominate the combined form [12] estimator if

$$MSE(t_{kk_2}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - \frac{(\upsilon^2 V_{2,0} - 1)(V_{2,0}V_{0,2} - V_{1,1}^2)}{\upsilon^2 V_{2,0}^2 + V_{1,1}^2 - V_{2,0}(1 + V_{0,2})}$$

9. The proposed combined estimators dominate the combined form [14] estimator if

$$MSE(t_{sv}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > \Lambda_{1(opt)} + \Lambda_{2(opt)} B_4$$

10. The proposed combined estimators dominate the combined form [15] estimator if

$$MSE(t_{ss_1}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > \phi_{1(opt)} C_4 + \phi_{2(opt)} C_5$$

11. The proposed combined estimators dominate the combined form [31] estimator if

$$MSE(t_{u}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > 1 - V_{0,2} + \frac{V_{1,1}^2}{V_{2,0}}$$

12. The proposed combined estimators dominate the combined form [24] estimator if

$$MSE(t_{ss_2}^{c}) > MSE(T_{bk}^{c}) \implies \varsigma > \left[\frac{D_4^2 D_2 - 2D_3 D_4 D_5 + D_1 D_5^2}{D_1 D_2 - D_3^2} \right]$$

The suggested combined class of estimators outperforms the conventional combined estimators under the aforementioned efficiency conditions.

4.3. Comparison of Suggested Separate and Combined Estimators

By comparing the minimized MSE of the suggested separate and combined classes of estimators T_{bk}^{c} and T_{bk}^{s} from (52) and (43), we get

$$minMSE(T_{bk}^{c}) - minMSE(T_{bk}^{s}) = \sum_{h=1}^{L} \left\{ (\bar{Y}^2 - W_h^2 \bar{Y}_h^2) - \left(\bar{Y}^2 \varsigma - W_h^2 \bar{Y}_h^2 \varsigma_h \right) \right\} \quad (61)$$

In general, the final term on the right side is least, and it decreases if and only if the correlation between the research and supplementary variables is a straight line within each stratum that goes through the origin.

In addition, unless R_h is invariant from one stratum to another, separate estimators probably become more precise in each stratum if the sample in each stratum is large enough so that the approximate expression for $MSE(T_{bk}^{s})$ is valid and cumulative bias that may change the suggested estimators is miniscule, while the suggested combined estimators are recommended preferably with a small sample only in each stratum, refer [1].

5. Empirical Study

In this part, an empirical investigation is completed under two headings: a simulation study employing hypothetically constructed populations and a numerical study employing real population.

5.1. Simulation Study

To gain a better understanding, following [35], we accomplished a simulation analysis over some arbitrarily drawn symmetrical (Normal) and asymmetrical (Chi-square) populations of size N = 1500 units by utilizing the models as follows:

$$y_i = 4.6 + \sqrt{(1-\rho_{xy}^2)}\, y_i^* + \rho_{xy}\left(\frac{S_y}{S_x}\right)x_i^*$$
$$x_i = 4.2 + x_i^*$$

where y_i^* and x_i^* are two independent variables of a certain distribution. In particular, for generating normal population, we consider $y^* \sim N(8, 13)$ and $x^* \sim N(10, 15)$, while for generating chi-square population, we consider $y^* \sim \chi^2(23)$ and $x^* \sim \chi^2(25)$. We take into account different correlation coefficient values, such as ρ_{xy} = 0.1, 0.5, and 0.9, to examine how the separate and combined estimators behave. The stratification of the populations has been done under two cases. In case 1, both populations have been divided into $h = 3$ strata having sizes (N_1, N_2, N_3) = (300, 700, 500). The proportional allocation has been considered to draw samples of sizes $n = 300$ such that (n_1, n_2, n_3)=(60, 140, 100) and $n = 400$ such that (n_1, n_2, n_3) = (80, 187, 133) from the stratum (N_1, N_2, N_3), respectively, using $SRSWOR$. Further, in case 2, both populations have been divided into $h = 6$ strata having sizes (N_1, N_2, N_3, N_4, N_5, N_6) = (150, 250, 300, 500, 180, 120). The proportional allocation has been considered to draw samples of sizes $n = 300$ such that (n_1, n_2, n_3, n_4, n_5, n_6) = (30, 50, 60, 100, 36, 24) and $n = 400$ such that (n_1, n_2, n_3, n_4, n_5, n_6) = (40, 67, 80, 133, 48, 32) from the stratum (N_1, N_2, N_3, N_4, N_5, N_6), respectively, through $SRSWOR$. Using 10,000 repeated samples, the MSE and percent relative efficiency (PRE) of the suggested separate and combined class of estimators relative to the traditional mean estimator are determined as follows.

$$MSE(T_i) = \frac{1}{10,000}\sum_{i=1}^{10,000}(T_i - \bar{Y})^2$$
$$PRE = \frac{MSE(t_m)}{MSE(T_i)} \times 100$$

where T_i is the existing and proposed separate and combined estimators.

The simulation findings are presented in Tables 1–4 by MSE and PRE for $N_h = 3$, $N_h = 6$, $n = 300,\ 400$ and various amount of correlation coefficient $\rho_{xy} = 0.1, 0.5, 0.9$.

Table 1. MSE and PRE of different separate estimators for artificially generated normal population.

	Stratum	$N_h = 3$						$N_h = 6$					
	ρ_{xy}	**0.1**		**0.5**		**0.9**		**0.1**		**0.5**		**0.9**	
n	**Estimators**	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
300	t_m	0.3367	100.00	0.3367	100.00	0.3367	100.00	0.0562	100.00	0.0562	100.00	0.0562	100.00
	t_r^s	0.4584	73.44	0.2508	134.25	0.1015	331.61	0.0873	64.37	0.0489	114.82	0.0107	521.08
	t_β^s/t_u^s	0.3232	104.16	0.2456	137.06	0.0627	536.49	0.0521	107.86	0.0392	143.36	0.0104	540.38
	$t_{kc_1}^s$	0.3539	95.14	0.2516	133.79	0.1422	236.80	0.0558	100.57	0.0403	139.17	0.0215	260.68
	$t_{kc_2}^s$	0.3522	95.61	0.2516	133.79	0.1503	224.00	0.0736	76.28	0.0436	128.65	0.0124	452.17
	$t_{kc_3}^s$	0.3979	84.62	0.2492	135.09	0.1177	285.87	0.0635	88.42	0.0411	136.47	0.0145	386.01
	$t_{kc_4}^s$	0.4014	83.87	0.2474	136.07	0.1248	269.72	0.0860	65.31	0.0483	116.20	0.0108	519.15
	t_{kc}^s	0.4552	73.96	0.2611	128.93	0.1128	298.45	0.0854	65.78	0.0487	115.28	0.0110	507.88
	t_{sg}^s	0.3004	112.07	0.2321	145.07	0.0621	541.76	0.0492	114.20	0.0375	149.57	0.0103	545.63
	$t_{kk_1}^s$	0.3888	86.61	0.2354	143.02	0.0996	338.04	0.0722	77.83	0.0444	126.56	0.0106	525.81
	$t_{kk_2}^s$	0.2984	112.83	0.2314	145.50	0.0616	546.05	0.0488	114.97	0.0373	150.36	0.0103	545.07
	t_{sv}^s	0.3005	112.05	0.2317	145.33	0.0614	547.83	0.0498	112.69	0.0379	148.12	0.0101	554.14
	$t_{ss_1}^s$	0.2939	114.56	0.2315	145.43	0.0622	541.05	0.0477	117.71	0.0364	154.00	0.0103	542.12
	$t_{ss_2}^s$	0.2894	116.32	0.2312	145.63	0.0616	545.87	0.0510	110.19	0.0382	147.12	0.0099	567.67
	$t_{s_1}^s$	0.2944	114.36	0.2265	148.63	0.0612	550.22	0.0476	117.87	0.0356	157.72	0.0098	573.46
	$t_{s_2}^s$	0.2987	112.73	0.2356	142.90	0.0698	482.22	0.0488	115.17	0.0377	148.91	0.0103	544.29
	$t_{s_3}^s$	0.3003	112.12	0.2318	145.23	0.0617	545.73	0.0491	114.29	0.0375	149.85	0.0103	544.52
	$t_{s_4}^s$	0.2978	113.04	0.2316	145.38	0.0626	537.52	0.0485	115.70	0.0370	151.72	0.0103	545.93
	$t_{s_5}^s$	0.3003	112.12	0.2318	145.23	0.0617	545.73	0.0491	114.29	0.0375	149.85	0.0103	544.52
	T_{bk}^s	**0.2801**	**120.20**	**0.2240**	**150.31**	**0.0601**	**560.23**	**0.0462**	**121.64**	**0.0350**	**160.57**	**0.0094**	**592.05**
400	t_m	0.2559	100.00	0.2559	100.00	0.2559	100.00	0.0420	100.00	0.0420	100.00	0.0420	100.00
	t_r^s	0.2992	85.53	0.1789	143.04	0.0755	338.90	0.0585	71.76	0.0348	120.64	0.0076	550.14
	t_β^s/t_u^s	0.2373	107.86	0.1771	144.51	0.0472	541.73	0.0374	112.22	0.0283	148.12	0.0075	558.14
	$t_{kc_1}^s$	0.2503	102.24	0.1826	140.15	0.1063	240.70	0.0392	107.13	0.0291	144.18	0.0155	269.93
	$t_{kc_2}^s$	0.2475	103.38	0.1836	139.35	0.1137	225.00	0.0500	84.07	0.0311	134.87	0.0089	471.77
	$t_{kc_3}^s$	0.2724	93.94	0.1791	142.91	0.0872	293.22	0.0439	95.60	0.0295	142.33	0.0102	411.11
	$t_{kc_4}^s$	0.2691	95.10	0.1787	143.17	0.0939	272.59	0.0577	72.78	0.0344	122.08	0.0076	547.57
	t_{kc}^s	0.3004	85.19	0.1846	138.59	0.0814	314.30	0.0578	72.72	0.0347	121.03	0.0077	541.38
	t_{sg}^s	0.2261	113.16	0.1709	149.76	0.0471	543.38	0.0360	116.49	0.0276	152.21	0.0075	556.03
	$t_{kk_1}^s$	0.2726	93.89	0.1722	148.63	0.0750	341.23	0.0518	81.10	0.0326	128.92	0.0076	553.21
	$t_{kk_2}^s$	0.2256	113.45	0.1705	150.07	0.0467	547.57	0.0359	116.87	0.0275	152.75	0.0074	562.81
	t_{sv}^s	0.2261	113.19	0.1705	150.06	0.0466	549.23	0.0364	115.34	0.0275	152.56	0.0072	583.33
	$t_{ss_1}^s$	0.2230	114.77	0.1708	149.84	0.0468	546.61	0.0352	119.18	0.0270	155.50	0.0074	561.05
	$t_{ss_2}^s$	0.2210	115.82	0.1707	149.92	0.0474	539.00	0.0370	113.42	0.0283	148.16	0.0073	575.34
	$t_{s_1}^s$	0.2223	115.14	0.1699	150.54	0.0463	552.58	0.0351	119.70	0.0265	158.45	0.0070	600.01
	$t_{s_2}^s$	0.2255	113.50	0.1730	147.92	0.0464	551.08	0.0358	117.16	0.0277	151.74	0.0074	567.94
	$t_{s_3}^s$	0.2260	113.22	0.1707	149.94	0.0467	547.45	0.0360	116.61	0.0275	152.53	0.0074	562.60
	$t_{s_4}^s$	0.2246	113.93	0.1707	149.94	0.0472	542.07	0.0357	117.74	0.0273	153.89	0.0074	561.68
	$t_{s_5}^s$	0.2260	113.22	0.1707	149.94	0.0467	547.45	0.0360	116.61	0.0275	152.53	0.0074	562.60
	T_{bk}^s	**0.2092**	**122.32**	**0.1676**	**152.71**	**0.0453**	**564.90**	**0.0341**	**123.16**	**0.0259**	**162.16**	**0.0069**	**608.69**

Bold numerical values indicate minimum MSE and maximum PRE.

Table 2. MSE and PRE of different combined estimators for artificially generated normal population.

	Stratum	$N_h = 3$						$N_h = 6$					
	ρ_{xy}	**0.1**		**0.5**		**0.9**		**0.1**		**0.5**		**0.9**	
n	**Estimators**	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
300	t_m	0.3367	100.00	0.3367	100.00	0.3367	100.00	0.0562	100.00	0.0562	100.00	0.0562	100.00
	t_r^c	0.6019	55.93	0.2595	129.74	0.1024	328.64	0.0955	58.85	0.0513	109.39	0.0108	516.77
	t_β^c/t_u^c	0.3332	101.01	0.2525	133.33	0.0639	526.31	0.0534	105.08	0.0404	139.06	0.0106	526.31
	$t_{kc_1}^c$	0.3882	86.73	0.2575	130.73	0.1428	235.76	0.0585	95.95	0.0415	135.43	0.0216	260.16
	$t_{kc_2}^c$	0.3962	84.99	0.2571	130.96	0.1509	223.12	0.0796	70.55	0.0456	123.09	0.0125	449.57
	$t_{kc_3}^c$	0.4668	72.13	0.2565	131.24	0.1185	283.94	0.0677	82.95	0.0427	131.53	0.0146	384.24
	$t_{kc_4}^c$	0.5044	66.75	0.2543	132.40	0.1256	268.05	0.0940	59.74	0.0507	110.73	0.0109	514.95
	t_{kc}^c	0.5829	57.76	0.2695	124.91	0.1137	295.97	0.0930	60.38	0.0511	109.96	0.0111	503.76
	t_{sg}^c	0.3091	108.91	0.2382	141.33	0.0633	531.72	0.0504	111.43	0.0386	145.28	0.0106	527.20
	$t_{kk_1}^c$	0.4671	72.08	0.2427	138.69	0.1005	335.05	0.0773	72.64	0.0463	121.27	0.0107	521.55
	$t_{kk_2}^c$	0.3069	109.71	0.2375	141.76	0.0628	535.85	0.0500	112.29	0.0384	146.06	0.0105	533.54
	t_{sv}^c	0.3093	108.86	0.2378	141.57	0.0626	537.59	0.0509	110.34	0.0387	145.01	0.0102	548.10
	$t_{ss_1}^c$	0.2972	113.28	0.2376	141.69	0.0640	525.65	0.0490	114.62	0.0375	149.69	0.0105	530.64
	$t_{ss_2}^c$	0.2945	114.34	0.2374	141.79	0.0984	342.19	0.0522	107.64	0.0398	141.06	0.0106	530.11
	$t_{s_1}^c$	0.3050	110.37	0.2380	141.48	0.0623	539.92	0.0498	112.87	0.0369	152.30	0.0098	573.42
	$t_{s_2}^c$	0.3074	109.53	0.2414	139.48	0.0625	538.73	0.0500	112.39	0.0388	144.85	0.0104	539.20
	$t_{s_3}^c$	0.3091	108.94	0.2380	141.48	0.0628	535.54	0.0504	111.51	0.0386	145.54	0.0105	532.99
	$t_{s_4}^c$	0.3072	109.60	0.2375	141.76	0.0638	527.12	0.0498	112.73	0.0381	147.41	0.0106	531.44
	$t_{s_5}^c$	0.3091	108.94	0.2380	141.48	0.0628	535.54	0.0504	111.51	0.0386	145.54	0.0105	532.99
	T_{bk}^c	**0.2851**	**118.09**	**0.2318**	**145.22**	**0.0617**	**545.70**	**0.0486**	**115.45**	**0.0360**	**156.12**	**0.0095**	**586.82**
400	t_m	0.2559	100.00	0.2559	100.00	0.2559	100.00	0.0420	100.00	0.0420	100.00	0.0420	100.00
	t_r^c	0.3332	76.81	0.1805	141.79	0.0758	337.32	0.0631	66.62	0.0357	117.62	0.0077	539.59
	t_β^c/t_u^c	0.2435	105.08	0.1785	143.36	0.0477	536.49	0.0385	109.18	0.0288	145.68	0.0076	547.09
	$t_{kc_1}^c$	0.2633	97.21	0.1837	139.30	0.1065	240.16	0.0409	102.79	0.0295	142.12	0.0156	268.66
	$t_{kc_2}^c$	0.2611	98.03	0.1847	138.58	0.1139	224.57	0.0534	78.68	0.0319	131.78	0.0090	465.57
	$t_{kc_3}^c$	0.2943	86.95	0.1804	141.84	0.0876	292.18	0.0464	90.45	0.0301	139.57	0.0103	406.58
	$t_{kc_4}^c$	0.2934	87.23	0.1800	142.18	0.0941	271.73	0.0622	67.59	0.0353	119.04	0.0078	537.31
	t_{kc}^c	0.3324	77.00	0.1862	137.44	0.0818	312.90	0.0621	67.63	0.0356	118.05	0.0079	531.15
	t_{sg}^c	0.2318	110.39	0.1722	148.61	0.0475	538.22	0.0370	113.47	0.0280	149.79	0.0077	545.25
	$t_{kk_1}^c$	0.2980	85.87	0.1736	147.43	0.0753	339.66	0.0552	76.11	0.0333	125.95	0.0077	542.75
	$t_{kk_2}^c$	0.2311	110.72	0.1718	148.92	0.0471	542.32	0.0369	113.84	0.0279	150.31	0.0076	551.76
	t_{sv}^c	0.2318	110.39	0.1718	148.91	0.0470	543.96	0.0373	112.50	0.0281	149.38	0.0073	572.57
	$t_{ss_1}^c$	0.2285	112.01	0.1721	148.72	0.0473	540.14	0.0362	115.98	0.0274	153.07	0.0076	550.09
	$t_{ss_2}^c$	0.2257	113.36	0.1719	148.82	0.0490	521.59	0.0380	110.57	0.0288	145.74	0.0077	545.87
	$t_{s_1}^c$	0.2282	112.15	0.1712	149.47	0.0467	547.27	0.0361	116.29	0.0269	155.85	0.0073	575.33
	$t_{s_2}^c$	0.2309	110.85	0.1743	146.83	0.0468	546.78	0.0368	114.18	0.0281	149.40	0.0075	556.02
	$t_{s_3}^c$	0.2317	110.44	0.1720	148.80	0.0472	542.20	0.0370	113.57	0.0280	150.09	0.0076	551.54
	$t_{s_4}^c$	0.2303	111.13	0.1719	148.82	0.0476	536.82	0.0366	114.62	0.0277	151.45	0.0076	550.68
	$t_{s_5}^c$	0.2317	110.44	0.1720	148.80	0.0472	542.20	0.0370	113.57	0.0280	150.09	0.0076	551.54
	T_{bk}^c	**0.2241**	**114.19**	**0.1688**	**151.55**	**0.0465**	**550.32**	**0.0352**	**119.61**	**0.0261**	**160.92**	**0.0071**	**590.66**

Bold numerical values indicate minimum MSE and maximum PRE.

Table 3. MSE and PRE of different separate estimators for artificially generated χ^2 population.

	Stratum	$N_h = 3$						$N_h = 6$					
	ρ_{xy}	**0.1**		**0.5**		**0.9**		**0.1**		**0.5**		**0.9**	
n	**Estimators**	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
300	t_m	0.1963	100.00	0.1963	100.00	0.1963	100.00	0.0121	100.00	0.0121	100.00	0.0121	100.00
	t_r^s	0.2226	88.15	0.1448	135.48	0.0543	361.01	0.0133	91.49	0.0086	141.37	0.0032	376.50
	t_β^s/t_u^s	0.1786	109.91	0.1390	141.16	0.0516	380.42	0.0108	112.22	0.0082	147.13	0.0030	394.47
	$t_{kc_1}^s$	0.2213	109.89	0.1445	135.83	0.0546	358.96	0.0127	95.60	0.0084	144.02	0.0033	358.56
	$t_{kc_2}^s$	0.2111	88.69	0.1418	138.40	0.0576	340.81	0.0126	96.19	0.0084	144.29	0.0034	357.07
	$t_{kc_3}^s$	0.2222	92.97	0.1447	135.60	0.0544	360.29	0.0131	93.025	0.0085	142.50	0.0032	369.92
	$t_{kc_4}^s$	0.1964	88.33	0.1392	140.99	0.0661	296.55	0.0130	93.46	0.0085	142.77	0.0033	369.16
	t_{kc}^s	0.2226	99.92	0.1449	135.45	0.0544	360.83	0.0133	91.48	0.0086	141.24	0.0032	375.72
	t_{sg}^s	0.1786	88.15	0.1392	141.02	0.0514	381.63	0.0108	112.44	0.0082	147.20	0.0031	392.09
	$t_{kk_1}^s$	0.2223	109.87	0.1448	135.57	0.0543	361.04	0.0132	92.03	0.0085	141.79	0.0032	376.65
	$t_{kk_2}^s$	0.1785	88.27	0.1390	141.22	0.0511	384.08	0.0108	112.53	0.0082	147.44	0.0030	394.78
	t_{sv}^s	0.1785	109.95	0.1390	141.22	0.0511	384.10	0.0113	107.68	0.0082	146.98	0.0030	394.91
	$t_{ss_1}^s$	0.1785	109.94	0.1389	141.24	0.0511	384.04	0.0108	112.64	0.0082	147.53	0.0031	394.59
	$t_{ss_2}^s$	0.1784	109.97	0.1389	141.25	0.0511	384.03	0.0108	112.34	0.0083	147.14	0.0030	397.61
	$t_{s_1}^s$	0.1784	109.99	0.1390	141.27	0.0510	384.27	0.0108	112.68	0.0082	147.72	0.0030	395.84
	$t_{s_2}^s$	0.1785	109.98	0.1390	141.21	0.0511	384.05	0.0108	112.54	0.0083	147.35	0.0030	394.60
	$t_{s_3}^s$	0.1785	109.95	0.1390	141.22	0.0511	384.08	0.0108	112.53	0.0082	147.44	0.0030	394.78
	$t_{s_4}^s$	0.1785	109.96	0.1389	141.23	0.0511	384.06	0.0108	112.58	0.0082	147.49	0.0030	394.67
	$t_{s_5}^s$	0.1785	109.95	0.1390	141.22	0.0511	384.08	0.0108	112.53	0.0082	147.44	0.0030	394.78
	T_{bk}^s	**0.1681**	**116.77**	**0.1301**	**150.88**	**0.0499**	**393.38**	**0.0019**	**119.80**	**0.0077**	**157.11**	**0.0029**	**404.68**
400	t_m	0.0409	100.00	0.0409	100.00	0.0409	100.00	0.0119	100.00	0.0119	100.00	0.0119	100.00
	t_r^s	0.0434	94.23	0.0281	145.45	0.0112	363.29	0.0125	95.51	0.0078	152.34	0.0031	385.22
	t_β^s/t_u^s	0.0359	113.96	0.0271	150.69	0.0101	402.72	0.0104	114.88	0.0076	155.95	0.0029	411.35
	$t_{kc_1}^s$	0.0432	94.63	0.0281	145.73	0.0113	361.55	0.0120	99.41	0.0077	154.32	0.0032	366.70
	$t_{kc_2}^s$	0.0415	98.72	0.0276	148.29	0.0120	338.72	0.0119	100.41	0.0077	154.70	0.0032	364.23
	$t_{kc_3}^s$	0.0434	94.39	0.0281	145.57	0.0112	362.61	0.0123	96.86	0.0078	153.15	0.0031	378.20
	$t_{kc_4}^s$	0.0376	108.79	0.0273	149.97	0.0160	255.98	0.0122	97.65	0.0078	153.57	0.0031	376.52
	t_{kc}^s	0.0434	94.23	0.0281	145.45	0.0112	363.23	0.0125	95.50	0.0078	152.24	0.0031	384.63
	t_{sg}^s	0.0359	113.88	0.0272	150.45	0.0102	400.12	0.0104	115.00	0.0076	155.88	0.0029	408.66
	$t_{kk_1}^s$	0.0434	94.27	0.0281	150.45	0.0112	363.30	0.0124	95.87	0.0078	152.61	0.0031	385.31
	$t_{kk_2}^s$	0.0359	113.97	0.0271	145.48	0.0102	402.74	0.0104	115.09	0.0076	156.16	0.0029	411.56
	t_{sv}^s	0.0359	113.97	0.0271	150.71	0.0101	403.31	0.0108	110.77	0.0077	155.61	0.0029	408.91
	$t_{ss_1}^s$	0.0359	113.98	0.0271	150.70	0.0102	402.72	0.0104	115.17	0.0076	156.21	0.0029	411.41
	$t_{s_1}^s$	0.0359	113.98	0.0271	150.73	0.0101	402.72	0.0104	114.96	0.0076	155.95	0.0028	413.72
	$t_{s_2}^s$	0.0359	113.97	0.0271	150.70	0.0102	402.79	0.0103	115.21	0.0076	156.36	0.0029	412.27
	$t_{s_3}^s$	0.0359	113.97	0.0271	150.71	0.0102	402.73	0.0104	115.10	0.0076	156.08	0.0029	411.48
	$t_{s_4}^s$	0.0359	113.98	0.0271	150.71	0.0102	402.74	0.0104	115.09	0.0076	156.16	0.0029	411.56
	$t_{s_5}^s$	0.0359	113.97	0.0271	150.71	0.0102	402.73	0.0104	115.10	0.0076	156.08	0.0029	411.48
	T_{bk}^s	**0.0339**	**120.64**	**0.0264**	**154.92**	**0.0099**	**413.11**	**0.0098**	**121.42**	**0.0072**	**165.22**	**0.0028**	**425.07**

Bold numerical values indicate minimum MSE and maximum PRE.

Table 4. MSE and PRE of different combined estimators for artificially generated χ^2 population.

	Stratum	$N_h = 3$						$N_h = 6$					
	ρ_{xy}	**0.1**		**0.5**		**0.9**		**0.1**		**0.5**		**0.9**	
n	**Estimators**	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
300	t_m	0.1963	100.00	0.1963	100.00	0.1963	100.00	0.0121	100.00	0.0121	100.00	0.0121	100.00
	t_r^c	0.2631	74.60	0.1566	125.33	0.0574	341.85	0.0158	76.58	0.0094	128.70	0.0034	361.38
	t_β^c/t_u^c	0.1884	104.16	0.1472	133.33	0.0544	360.36	0.0115	105.21	0.0089	137.06	0.0032	376.57
	$t_{kc_1}^c$	0.2610	75.18	0.1561	125.73	0.0577	340.12	0.0150	80.83	0.0092	131.78	0.0035	345.50
	$t_{kc_2}^c$	0.2460	79.78	0.1524	128.73	0.0604	324.55	0.0149	81.41	0.0092	132.14	0.0035	344.15
	$t_{kc_3}^c$	0.2624	74.78	0.1564	125.47	0.0575	341.25	0.0156	77.93	0.0094	129.97	0.0034	355.57
	$t_{kc_4}^c$	0.2252	87.16	0.1481	132.54	0.0688	285.29	0.0156	78.33	0.0093	130.29	0.0034	354.89
	t_{kc}^c	0.2631	74.60	0.1566	125.31	0.0574	341.69	0.0160	76.34	0.0095	128.61	0.0034	360.64
	t_{sg}^c	0.1884	104.18	0.1473	133.22	0.0547	358.29	0.0116	105.34	0.0089	137.18	0.0032	374.46
	$t_{kk_1}^c$	0.2627	74.72	0.1565	125.42	0.0574	341.88	0.0158	76.90	0.0094	129.15	0.0034	361.54
	$t_{kk_2}^c$	0.1883	104.23	0.1471	133.39	0.0545	360.42	0.0116	105.39	0.0089	137.37	0.0032	376.88
	t_{sv}^c	0.1883	104.22	0.1471	133.39	0.0545	360.43	0.0122	100.22	0.0152	80.05	0.0032	376.90
	$t_{ss_1}^c$	0.1883	104.24	0.1471	133.41	0.0545	360.38	0.0116	105.48	0.0089	137.47	0.0032	376.71
	$t_{ss_2}^c$	0.1882	104.26	0.1471	133.42	0.0545	360.37	0.0116	105.25	0.0089	137.47	0.0032	379.24
	$t_{s_1}^c$	0.1883	104.55	0.1471	133.44	0.0544	360.59	0.0116	105.50	0.0088	137.09	0.0032	377.87
	$t_{s_2}^c$	0.1883	104.53	0.1471	133.38	0.0545	360.37	0.0116	105.42	0.0089	137.61	0.0032	376.67
	$t_{s_3}^c$	0.1883	104.53	0.1471	133.39	0.0545	360.42	0.0116	105.39	0.0089	137.30	0.0032	376.88
	$t_{s_4}^c$	0.1882	104.53	0.1471	133.40	0.0544	360.40	0.0116	105.44	0.0088	137.37	0.0032	376.79
	$t_{s_5}^c$	0.1883	104.53	0.1471	133.39	0.0545	360.42	0.0116	105.39	0.0089	137.30	0.0032	376.88
	T_{bk}^c	**0.1776**	**110.52**	**0.1358**	**144.55**	**0.0530**	**370.37**	**0.0109**	**111.00**	**0.0083**	**145.78**	**0.0031**	**384.12**
400	t_m	0.0409	100.00	0.0409	100.00	0.0409	100.00	0.0119	100.00	0.0119	100.00	0.0119	100.00
	t_r^c	0.0506	80.89	0.0318	128.82	0.0119	341.57	0.0139	85.87	0.0087	137.06	0.0032	363.14
	t_β^c/t_u^c	0.0382	107.25	0.0298	137.45	0.0110	371.78	0.0109	109.89	0.0083	143.36	0.0031	384.02
	$t_{kc_1}^c$	0.0503	81.31	0.0317	129.14	0.0120	340.13	0.0133	90.06	0.0086	139.63	0.0034	347.51
	$t_{kc_2}^c$	0.0478	85.62	0.0309	132.33	0.0127	321.01	0.0131	91.06	0.0085	140.21	0.0035	345.31
	$t_{kc_3}^c$	0.0505	81.06	0.0317	128.95	0.0120	341.01	0.0137	87.29	0.0087	138.05	0.0033	357.25
	$t_{kc_4}^c$	0.0422	96.90	0.0298	137.29	0.0165	248.02	0.0136	88.06	0.0086	138.62	0.0034	355.76
	t_{kc}^c	0.0506	80.89	0.0318	128.82	0.0119	341.51	0.0139	85.88	0.0087	136.98	0.0033	362.59
	t_{sg}^c	0.0382	107.20	0.0298	137.27	0.0110	369.60	0.0109	110.02	0.0083	143.35	0.0031	381.75
	$t_{kk_1}^c$	0.0506	80.92	0.0318	128.85	0.0119	341.58	0.0108	110.15	0.0087	137.36	0.0033	363.24
	$t_{kk_2}^c$	0.0382	107.26	0.0298	137.47	0.0110	371.80	0.0109	110.09	0.0083	143.57	0.0031	384.23
	t_{sv}^c	0.0383	106.76	0.0298	136.88	0.0110	372.27	0.0109	109.16	0.0089	134.61	0.0047	254.06
	$t_{ss_1}^c$	0.0382	107.06	0.0299	136.83	0.0109	375.22	0.0108	110.17	0.0083	143.64	0.0031	384.09
	$t_{ss_2}^c$	0.0383	106.83	0.0296	138.17	0.0110	371.78	0.0108	109.98	0.0083	143.38	0.0031	385.89
	$t_{s_1}^c$	0.0381	107.36	0.0294	139.11	0.0108	376.26	0.0108	110.19	0.0083	143.75	0.0031	384.87
	$t_{s_2}^c$	0.0382	107.12	0.0295	138.64	0.0110	371.48	0.0108	110.11	0.0083	143.52	0.0031	384.10
	$t_{s_3}^c$	0.0383	107.23	0.0297	137.71	0.0110	371.81	0.0108	110.09	0.0083	143.57	0.0031	384.23
	$t_{s_4}^c$	0.0382	107.44	0.0296	138.13	0.0109	374.88	0.0108	110.13	0.0083	143.60	0.0031	384.15
	$t_{s_5}^c$	0.0383	107.23	0.0297	137.71	0.0110	371.81	0.0108	110.09	0.0083	143.57	0.0031	384.23
	T_{bk}^c	**0.0358**	**114.29**	**0.0278**	**147.23**	**0.0657**	**389.47**	**0.0101**	**117.82**	**0.0078**	**152.56**	**0.0029**	**400.66**

Bold numerical values indicate minimum MSE and maximum PRE.

After carefully reading the simulation results shown in Tables 1–4, we have made the following findings:

- Table 1 consisting of the simulation outcomes of the separate estimators for normal population demonstrates the dominance of the suggested separate estimator T_{bk}^s over the usual mean per unit estimator t_m, separate classical ratio and regression estimators t_r^s, t_β^s, ref. [4,6] estimators $t_{kc_i}^s$, $i = 1,2,3,4$, t_{kc}^s, ref. [7] estimator t_{sg}^s, ref. [10,12] estimators $t_{kk_1}^s$, $t_{kk_2}^s$, ref. [14] estimator t_{sv}^s, ref. [15] estimator $t_{ss_1}^s$, ref. [31] t_u^s, ref. [24] estimator $t_{ss_2}^s$ and [20] estimators $t_{s_i}^s$, $i = 1,2,\ldots,5$ by lesser MSE and greater PRE for several values of correlation coefficient.

- Table 2 based on the simulation outcomes of the separate estimators for χ^2 population shows the similar inclination.
- The findings of the simulation study presented in Table 3 show the dominance of the suggested combined class of estimators T^c_{bk} over the conventional mean estimator t_m, the combined form of conventional ratio and regression estimators t^c_r, t^c_β, refs. [4,6] estimators $t^c_{kc_i}$, $i = 1,2,3,4$, t^c_{kc}, ref. [7] estimator t^c_{sg}, refs. [10,12] estimators $t^c_{kk_1}$, $t^c_{kk_2}$, ref. [14] estimator t^c_{sv}, ref. [15] estimators $t^c_{ss_1}$, ref. [31] estimator t^c_u, ref. [24] estimator $t^c_{ss_2}$, and [20] estimators $t^c_{s_i}$, $i = 1,2,\ldots,5$ by minimum MSE and maximum PRE for different values of correlation coefficients.
- Table 4 based on the simulation outcomes of the combined estimators exhibits the similar proclivity when the nature of population is χ^2.
- The findings of Tables 1–4 show that when the correlation coefficient varies from 0.1 to 0.9 with increment 0.4, the MSE and PRE of the proposed separate and combined estimators decreases and increases, respectively. Apart from this, the gain in the efficiency of the proposed separate and combined estimators is more in most of the cases in asymmetric (χ^2) population as compare to symmetric (normal) population.
- From the findings of Tables 1–4, it can be observed that when number of strata increases from 3 to 6, the MSE and PRE of the proposed separate and combined estimators decreases and increases, respectively.
- Further, from the findings of Tables 1–4, it can also be observed that when sample size increases from 300 to 400, the MSE and PRE of the proposed separate and combined estimators decreases and increases, respectively.

5.2. Real Data Application

To demonstrate the execution of the suggested estimators concerning to the conventional estimators, a real data is taken from [36], page number 228. In this data, let the output of $N = 80$ factories be the study variable y and the fixed capital of these factories be auxiliary variable x which are recorded from 4 regions (strata) of these 80 factories. A total sample of size $n = 45$ is drawn by using Neyman allocation from $h = 4$ strata. For easy reference, the descriptive statistics are provided in Table 5.

Table 5. Summary of real population.

	Total	**Symbol for Stratum *h***	**1**	**2**	**3**	**4**
Population size	N = 80	N_h	19	32	14	15
Sample size	n = 45	n_h	11	18	8	8
Population mean	$\bar{X}$ = 1126.46	$\bar{X}_h$	349.68	706.59	1539.57	2620.53
Population mean	$\bar{Y}$ = 5182.64	$\bar{Y}_h$	2967.95	4657.63	6537.21	7843.67
Kurtosis coefficient	$\beta_2(x)$ = 12.18	$\beta_2(x_h)$	4.59	18.54	15.44	10.16
Correlation coefficient	ρ_{xy} = 0.94	ρ_{xy_h}	0.93	0.92	0.98	0.96
Standard deviation	S_x = 845.61	S_{x_h}	109.44	109.22	277.18	370.97
Standard deviation	S_y = 1835.66	S_{y_h}	757.08	669.12	416.11	645.68

Now, utilising the parameters reported in Table 5, we have calculated the MSE and PRE of several separate and combined estimators T_i with regard to the mean per unit estimator t_m. The following formula is used to tabulate the PRE:

$$PRE = \frac{MSE(t_m)}{MSE(T_i)} \times 100 \tag{62}$$

where MSE(T_i) is the MSE of the conventional and proposed separate and combined estimators.

The results obtained using real data are shown in Table 6 , which show the outperformance of the suggested classes of estimators over the currently used estimators.

Table 6. *MSE* and *PRE* of different separate and combined estimators utilizing real populations.

Combined Estimators	MSE	PRE	**Separate Estimators**	MSE	PRE
t_m	46,819.71	100.00	t_m	85,499.73	100.00
t_r^c	29,209.75	160.28	t_r^s	53,341.34	160.28
t_β^c/t_u^c	2839.67	1648.76	t_β^s/t_u^s	5185.67	1648.76
$t_{kc_1}^c$	29,203.22	160.32	$t_{kc_1}^s$	53,329.42	160.32
$t_{kc_2}^c$	28,744.99	162.87	$t_{kc_2}^s$	52,492.61	162.87
$t_{kc_3}^c$	29,209.11	160.29	$t_{kc_3}^s$	53,340.17	160.29
$t_{kc_4}^c$	26,086.00	179.48	$t_{kc_4}^s$	47,636.91	179.48
t_{kc}^c	29,350.51	159.51	t_{kc}^s	53,414.51	160.06
t_{sg}^c	21,359.16	256.00	t_{sg}^s	41,791.27	204.58
$t_{kk_1}^c$	24,646.96	189.96	$t_{kk_1}^s$	53,217.62	160.66
$t_{kk_2}^c$	2839.43	1648.90	$t_{kk_2}^s$	5185.55	1648.80
t_{sv}^c	2814.58	1663.46	t_{sv}^s	5159.55	1657.11
$t_{ss_1}^c$	2817.14	1661.36	$t_{ss_1}^s$	5164.69	1655.46
$t_{ss_2}^c$	2824.60	1657.56	$t_{ss_2}^s$	5144.01	1662.12
$t_{s_1}^c$	2737.10	1710.55	$t_{s_1}^s$	5132.26	1665.92
$t_{s_2}^c$	2839.54	1648.84	$t_{s_2}^s$	5185.60	1648.79
$t_{s_3}^c$	2839.43	1648.90	$t_{s_3}^s$	5185.55	1648.80
$t_{s_4}^c$	2827.97	1655.59	$t_{s_4}^s$	5179.59	1650.70
$t_{s_5}^c$	2839.43	1648.90	$t_{s_5}^s$	5185.55	1648.80
T_{bk}^c	2735.18	1711.75	T_{bk}^s	5118.07	1670.54

The results obtained from real data analysis are presented in Table 6, show that the suggested combined estimator T_{bk}^c performs superior than the conventional mean estimator t_m, combined conventional ratio and regression estimators t_r^c, t_β^c, refs. [4,6] estimators $t_{kc_i}^c$, $i = 1,2,3,4$, t_{kc}^c, ref. [7] estimator t_{sg}^c, refs. [10,12] estimators $t_{kk_1}^c$, $t_{kk_2}^c$, ref. [14] estimator t_{sv}^c, ref. [15] estimators $t_{ss_1}^c$, ref. [31] estimator t_u^c, ref. [24] estimator $t_{ss_2}^c$, and ref. [20] estimators $t_{s_i}^c$, $i = 1,2,\ldots,5$ by minimum MSE and maximum PRE. The same tendency may be observed from the findings of the separate estimators presented in Table 6.

6. Conclusions

In this article, we have suggested an efficient separate and combined class of estimators under stratified simple random sampling for estimating the population mean of the research variable utilizing auxiliary information. The properties like bias and meas square error of the suggested separate and combined class of estimators are obtained approximately to the order one. The suggested separate and combined class of estimators outperform the currently available separate and combined estimators under efficiency conditions that are deduced. Furthermore, an empirical study is carried out in favour of the theoretical results utilizing some artificially generated symmetric and asymmetric populations and a real population. It has been seen from the empirical findings that the suggested class of separate and combined estimators are most efficient in comparison to the existing separate and combined estimators by lesser MSE and greater PRE. It is also noticed from the simulation results that when the number of strata, correlation coefficient as well as sample size increase, the PRE of the proposed estimators increases, while MSE decreases. Moreover, from empirical results, it has been also seen that the suggested separate class of estimators becomes superior than the suggested combined class of estimators for artificially generated and real populations. This indicates that the sampling ratio is rather high for both artificial and real data. Thus, we enthusiastically recommend our suggested class of estimators over all prominent estimators discussed in the present paper for experimental surveys.

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Appendix A

The *MSEs* of the conventional separate estimators are listed below for ready reference.

$$MSE(t_m) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 U_0 \tag{A1}$$

$$MSE(t_r^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[U_0 + U_1 - 2U_{10} \Big] \tag{A2}$$

$$MSE(t_\beta^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_0 + \frac{\beta_h^2}{R_h^2} U_1 - 2\frac{\beta_h}{R_h} U_{10} \right] \tag{A3}$$

$$minMSE(t_\beta^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_0 - \frac{U_{10}^2}{U_1} \right] \tag{A4}$$

$$MSE(t_{kc_i}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[U_0 + \eta_{ih}^2 U_1 - 2\eta_{ih} U_{10} \Big]; \;\; i = 1,2,3,4 \tag{A5}$$

$$MSE(t_{kc}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[(k_h - 1)^2 + \Big\{ \; U_0 + k_h^2 U_1 - 2k_h U_{10} \; \Big\} \Big] \tag{A6}$$

$$minMSE(t_{kc}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[(k_h^* - 1)^2 + \Big\{ \; U_0 + k_h^{*^2} U_1 - 2k_h^* U_{10} \; \Big\} \Big] \tag{A7}$$

$$MSE(t_{sg}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[(\lambda_{s_h} - 1)^2 + \lambda_{s_h}^2 \left(U_0 - \frac{U_{10}^2}{U_1} + \frac{U_1}{(N+1)^2} \right) \right] \tag{A8}$$

$$minMSE(t_{sg}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 - \lambda_{s_h(opt)} \Big] \tag{A9}$$

$$MSE(t_{kk_1}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[\begin{array}{l} \lambda_{k_h}^2 U_0 + \alpha_h^2 v_h^2 \Big\{ \lambda_{k_h}^2 (2g^2 + g) - \lambda_{k_h} (g^2 + g) \Big\} U_1 \\ -2g\alpha_h v_h (2\lambda_{k_h}^2 - \lambda_{k_h}) U_{10} + (\lambda_{k_h} - 1)^2 \end{array} \right] \tag{A10}$$

$$minMSE(t_{kk_1}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(1 - \frac{A_h^2}{4B_h} \right) \tag{A11}$$

$$MSE(t_{kk_2}^s) = \sum_{h=1}^{L} W_h^2 \left[\begin{array}{l} \bar{Y}_h^2 + k_{2h}^2 \bar{X}_h^2 U_1 + k_{1h}^2 \bar{Y}_h^2 (1 + U_0 + 3v_h^2 U_1 - 4v_h U_{10}) \\ -2k_{2h} \bar{X}_h \bar{Y}_h v_h U_1 - 2k_{1h} \bar{Y}_h^2 (1 + v_h^2 U_1 - v_h U_{10}) \\ +2k_{1h} k_{2h} \bar{X}_h \bar{Y}_h (2v_h U_1 - U_{10}) \end{array} \right] \tag{A12}$$

$$minMSE(t_{kk_2}^s) = \sum_{h=1}^{L} W_h^2 \left[\frac{\bar{Y}_h^2 (v_h^2 U_1 - 1)(U_1 U_0 - U_{10}^2)}{v_h^2 U_1^2 + U_{10}^2 - U_1 (1 + U_0)} \right] \tag{A13}$$

$$minMSE(t_{sv}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 - \Lambda_{1_h(opt)} - \Lambda_{2_h(opt)} B_{4h} \Big] \tag{A14}$$

$$MSE(t_{ss_1}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 + \phi_{1h}^2 C_{1h} + \phi_{2h}^2 C_{2h} + 2\phi_{1h}\phi_{2h} C_{3h} - 2\phi_{1h} C_{4h} - 2\phi_{2h} C_{5h} \Big] \tag{A15}$$

$$minMSE(t_{ss_1}^s) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 - \phi_{1_h(opt)} C_{4h} - \phi_{2_h(opt)} C_{5h} \Big] \tag{A16}$$

$$MSE(t^s_{ss_2}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 + w_{1h}^2 D_{1h} + w_{2h}^2 D_{2h} + 2w_{1h}w_{2h}D_{3h} - 2w_{1h}D_4 - 2w_{2h}D_{5h}\Big] \tag{A17}$$

$$minMSE(t^s_{ss_2}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \frac{(D_{1h}D_{5h}^2 + D_{2h}D_{4h}^2 - 2D_{3h}D_{4h}D_{5h})}{D_{1h}D_{2h} - D_{3h}^2}\right] \tag{A18}$$

$$MSE(t^s_u) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_0 + \frac{U_1}{a_h^2} - 2\frac{U_{10}}{a_h}\right] \tag{A19}$$

$$minMSE(t^s_u) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[U_0 - \frac{U_{10}^2}{U_1}\right] \tag{A20}$$

$$MSE(t^s_{s_i}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \Big[1 + \alpha_{ih}^2 P_{ih} - 2\alpha_{ih} Q_{ih}\Big],\ i = 1,2,4,5 \tag{A21}$$

$$minMSE(t^s_{s_i}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left[1 - \frac{Q_{ih}^2}{P_{ih}}\right] \tag{A22}$$

$$MSE(t^s_{s_3}) = \bar{Y}^2 \left[(\alpha_{3h} - 1)^2 + \alpha_{3h}^2 U_0 + \frac{\bar{X}_h^2}{\bar{Y}_h^2}\beta_{3h}^2 v^2 U_1 + 2\frac{\bar{X}_h}{\bar{Y}_h}\alpha_{3h}\beta_{3h} v U_{10}\right] \tag{A23}$$

$$minMSE(t^s_{s_3}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 [1 - \alpha_{3_h(opt)}] \tag{A24}$$

where

$$\eta_{1h} = \frac{\bar{X}_h}{(\bar{X}_h + C_{x_h})}, \qquad \eta_{2h} = \frac{\bar{X}_h}{(\bar{X}_h + \beta_2(x_h))},$$

$$\eta_{3h} = \frac{\bar{X}_h \beta_2(x_h)}{(\bar{X}_h \beta_2(x_h) + C_{x_h})}, \qquad \eta_{4h} = \frac{\bar{X}_h C_{x_h}}{(\bar{X}_h C_{x_h} + \beta_2(x_h))}$$

$$A_h = 2 + (g^2 + g)\alpha^2 v_h^2 U_1 - 2g\alpha v_h U_{10}$$

$$B_h = 1 + U_0 + (2g^2 + g)\alpha^2 v_h^2 U_1 - 4g\alpha v_h U_{10}$$

$$B_{1h} = 1 + U_{10}, \qquad B_{2h} = A_{1h} + \beta_h(2\beta_h - 1)U_1 + 4\beta_h U_{10}$$

$$B_{3h} = A_{1h} + \frac{\beta_h(\beta_h - 1)}{2}U_1 + 2\beta_h U_{10}, \qquad B_{4h} = 1 + \frac{\beta_h(\beta_h - 1)}{2}U_1 + \beta_h U_{10}$$

$$C_{1h} = 1 + U_0 + 4\alpha_h \delta v_h U_{10} + \delta(2\delta - 1)\alpha_h^2 v_h^2 U_1$$

$$C_{2h} = 1 + U_0 - 4g\alpha_h v_h U_{10} + g(2g + 1)\alpha_h^2 v_h^2 U_1$$

$$C_{3h} = 1 + U_0 + 2\alpha_h(\delta - g)v_h U_{10} + \frac{\alpha_h^2 v_h^2}{2}(\delta - g)(\delta - g - 1)U_1$$

$$C_{4h} = 1 + \alpha_h \delta v_h U_{10} + \frac{\delta(\delta + 1)}{2}\alpha_h^2 v_h^2 U_1$$

$$C_{5h} = 1 - \alpha_h g v_h U_{10} + \frac{g(g + 1)}{2}\alpha_h^2 v_h^2 U_1$$

$$D_{1h} = 1 + U_0 - 2\Theta_{1h} a_h U_{10} + \frac{\Theta_{1h}(\Theta_{1h} + 1)}{2} a_h^2 U_1$$

$$D_{2h} = 1 + U_0 + 2\Theta_{2h} a_h U_{10} + \frac{\Theta_{2h}(\Theta_{2h} - 1)}{2} a_h^2 U_1$$

$$D_{3h} = 1 + U_0 + (\Theta_{2h} - \Theta_{1h}) a_h U_{10} + \frac{(\Theta_{2h} - \Theta_{1h})(\Theta_{2h} - \Theta_{1h} - 2)}{8} a_h^2 U_1$$

$$D_{4h} = 1 - \frac{\Theta_{1h}}{2}\left(a_h U_{10} - \frac{(\Theta_{1h} + 2)}{4} a_h^2 U_1\right)$$

$$D_{5h} = 1 - \frac{\Theta_{2h}}{2}\left(a_h U_{10} + \frac{(\Theta_{2h} - 2)}{4} a_h^2 U_1\right)$$

$$\Theta_{1h} = 2\alpha_h + \beta_h, \qquad \Theta_{2h} = 2\delta + \lambda$$

$$P_{1h} = 1 + U_0 + 2\beta_{1h}(\beta_{1h} - 1)v^2 U_1 + 4\beta_{1h} v U_{10}$$

$$Q_{1h} = \beta_{1h}\left(\frac{\beta_{1h}}{2} - 1\right)v^2 U_1 - \beta_{1h} v U_{10}$$

$$
\begin{aligned}
P_{2h} &= 1 + U_0 + \beta_{2h}(\beta_{2h} - 1)v^2 U_1 + 4\beta_{2h} v U_{10} \\
Q_{2h} &= 1 + \beta_{2h} v U_{10} - \tfrac{\beta_{2h}}{2} v^2 U_1 \\
P_{4h} &= 1 + U_0 - 4\beta_{4h} v U_{10} + \beta_{4h}(2\beta_{4h} + 1)v^2 U_1 \\
Q_{4h} &= 1 + \frac{\beta_{4h}(\beta_{4h} + 1)}{2} v^2 U_1 - \beta_{4h} v U_{10} \\
P_{5h} &= 1 + U_0 + 3\beta_{5h}^2 v^2 U_1 - 4\beta_{5h} v U_{10} \\
Q_{5h} &= 1 + \beta_{5h}^2 v^2 U_1 - \beta_{5h} v U_{10}
\end{aligned}
$$

The scalars' optimal values in the aforementioned MSE expressions are listed as

$$
\hat{\beta}_{h(opt)} = \hat{R}_h \frac{U_{10}}{U_1}, \qquad k_{h(opt)} = \frac{1 + U_{10}}{1 + U_0} = k_h^*(\text{say}), \qquad \lambda_{s_h(opt)} = \frac{1}{\left(1 + U_0 - \frac{U_{10}^2}{U_1} + \frac{U_1}{(N_h+1)^2}\right)},
$$

$$
\begin{aligned}
\lambda_{k_h(opt)} &= \frac{1 + \alpha_h g v_h U_{10} + \frac{g(g+1)}{2}\alpha_h^2 v_h^2 U_1}{1 + U_0 + g(2g+1)\alpha_h^2 v_h^2 U_1 - 4g\alpha_h v_h U_{10}} = \frac{A_h}{B_h} \ (\text{say}) \\
k_{1_{h(opt)}} &= \frac{U_1(1 - v_h^2 U_1)}{U_0 U_1 + U_1 - v_h^2 U_1^2 - U_{10}^2}, \qquad k_{2_{h(opt)}} = R_h \left[v_h + \frac{(1 - v_h^2 U_1)(U_{10} - 2v_h U_1)}{U_1 + U_0 U_1 - U_{10}^2 - v_h^2 U_1^2} \right] \\
\lambda_{1_{h(opt)}} &= \frac{(A_{2h}A_{4h} - A_{3h}A_{5h})}{(A_{1h}A_{2h} - A_{3h}^2)} \ \text{and} \ \lambda_{2_{h(opt)}} = \frac{(A_{3h}A_{4h} - A_{1h}A_{5h})}{(A_{1h}A_{2h} - A_{3h}^2)} \\
\Lambda_{1_{h(opt)}} &= \frac{(B_{2h} - B_{3h}B_{4h})}{(B_{1h}B_{2h} - B_{3h}^2)} \ \text{and} \ \Lambda_{2_{h(opt)}} = \frac{(B_{1h}B_{4h} - B_{3h})}{(B_{1h}B_{2h} - B_{3h}^2)} \\
\phi_{1_{h(opt)}} &= \frac{(C_{2h}C_{4h} - C_{3h}C_{5h})}{(C_{1h}C_{2h} - C_{3h}^2)} \ \text{and} \ \phi_{2_{h(opt)}} = \frac{(C_{1h}C_{5h} - C_{3h}C_{4h})}{(C_{1h}C_{2h} - C_{3h}^2)} \\
w_{1_{h(opt)}} &= \frac{(D_{2h}D_{4h} - D_{3h}D_{5h})}{(D_{1h}D_{2h} - D_{3h}^2)} \ \text{and} \ w_{2_{h(opt)}} = \frac{(D_{1h}D_{5h} - D_{3h}D_{4h})}{(D_{1h}D_{2h} - D_{3h}^2)} \\
a_{h(opt)} &= \frac{U_1}{U_{10}}, \quad \alpha_{i_h(opt)} = \frac{Q_{ih}}{P_{ih}}, \ i = 1,2,4,5, \quad \alpha_{3_h(opt)} = \frac{1}{\left(1 + U_0 - \frac{U_{10}^2}{U_1}\right)}
\end{aligned}
$$

$$
\beta_{1_h(opt)} = -\frac{U_{10}}{vU_1} = \beta_{2_h(opt)}, \quad \beta_{3_h(opt)} = -\alpha_{3_h(opt)} \frac{U_{10}}{vU_1}, \quad \beta_{4_h(opt)} = \frac{U_{10}}{vU_1} = \beta_{5_h(opt)}
$$

Appendix B

The $MSEs$ of the conventional combined estimators are listed below for ready reference.

$$MSE(t_m) = \bar{Y}^2 V_{0,2} \tag{A25}$$

$$MSE(t_r^c) = \bar{Y}^2 \left[V_{0,2} + V_{2,0} - 2V_{1,1} \right] \tag{A26}$$

$$MSE(t_\beta^c) = \bar{Y}^2 \left[V_{0,2} + \frac{\beta^2}{R^2} V_{2,0} - 2\frac{\beta}{R} V_{1,1} \right] \tag{A27}$$

$$minMSE(t_\beta^c) = \bar{Y}^2 \left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}} \right] \tag{A28}$$

$$MSE(t_{kc_i}^c) = \bar{Y}^2 \left[V_{0,2} + \eta_i^2 V_{2,0} - 2\eta_i V_{1,1} \right]; \ i = 1,2,3,4 \tag{A29}$$

$$MSE(t_{kc}^{c}) = \bar{Y}^2\left[(k-1)^2 + \left\{\ V_{0,2} + k^2V_{2,0} - 2kV_{1,1}\ \right\}\right] \tag{A30}$$

$$minMSE(t_{kc}^{c}) = \bar{Y}^2\left[(k^*-1)^2 + \left\{\ V_{0,2} + k^{*^2}V_{2,0} - 2k^*V_{1,1}\ \right\}\right] \tag{A31}$$

$$MSE(t_{sg}^{c}) = \bar{Y}^2\left[(\lambda_s-1)^2 + \lambda_s^2\left(V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}} + \frac{V_{2,0}}{(N+1)^2}\right)\right] \tag{A32}$$

$$minMSE(t_{sg}^{c}) = \bar{Y}^2\left[1 - \lambda_{s(opt)}\right] \tag{A33}$$

$$MSE(t_{kk_1}^{c}) = \bar{Y}^2\left[\ 1 + \lambda_k^2A - 2\lambda_kB\ \right] \tag{A34}$$

$$minMSE(t_{kk_1}^{c}) = \bar{Y}^2\left(1 - \frac{B^2}{A}\right) \tag{A35}$$

$$MSE(t_{kk_2}^{c}) = \left[\begin{array}{l} \bar{Y}^2 + k_2^2\bar{X}^2V_{2,0} + k_1^2\bar{Y}^2(1 + V_{0,2} + 3v^2V_{2,0} - 4vV_{1,1}) \\ -2k_2\bar{X}\bar{Y}vV_{2,0} - 2k_1\bar{Y}^2(1 + v^2V_{2,0} - vV_{1,1}) \\ +2k_1k_2\bar{X}\bar{Y}(2vV_{2,0} - V_{1,1}) \end{array}\right] \tag{A36}$$

$$minMSE(t_{kk_2}^{c}) = \frac{\bar{Y}^2(v^2V_{2,0}-1)(V_{2,0}V_{0,2} - V_{1,1}^2)}{v^2V_{2,0}^2 + V_{1,1}^2 - V_{2,0}(1+V_{0,2})} \tag{A37}$$

$$MSE(t_{sv}^{c}) = \bar{Y}^2\left[1 + B_1\Lambda_1^2 + B_2\Lambda_2^2 + 2\Lambda_1\Lambda_2B_3 - 2\Lambda_1 - 2\Lambda_2B_4\right] \tag{A38}$$

$$minMSE(t_{sv}^{c}) = \bar{Y}^2\left[1 - \Lambda_{1(opt)} - \Lambda_{2(opt)}B_4\right] \tag{A39}$$

$$MSE(t_{ss_1}^{c}) = \bar{Y}^2\left[1 + \phi_1^2C_1 + \phi_2^2C_2 + 2\phi_1\phi_2C_3 - 2\phi_1C_4 - 2\phi_2C_5\right] \tag{A40}$$

$$minMSE(t_{ss_1}^{c}) = \bar{Y}^2\left[1 - \phi_{1(opt)}C_4 - \phi_{2(opt)}C_5\right] \tag{A41}$$

$$MSE(t_{ss_2}^{c}) = \bar{Y}^2\left[1 + w_1^2D_1 + w_2^2D_2 + 2w_1w_2D_3 - 2w_1D_4 - 2w_2D_5\right] \tag{A42}$$

$$minMSE(t_{ss_2}^{c}) = \bar{Y}^2\left[1 - \frac{(D_1D_5^2 + D_2D_4^2 - 2D_3D_4D_5)}{D_1D_2 - D_3^2}\right] \tag{A43}$$

$$MSE(t_{u}^{c}) = \bar{Y}^2\left[V_{0,2} + \frac{V_{2,0}}{a^2} - 2\frac{V_{1,1}}{a}\right] \tag{A44}$$

$$minMSE(t_{u}^{c}) = \bar{Y}^2\left[V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right] \tag{A45}$$

$$MSE(t_{s_i}^{c}) = \bar{Y}^2\left[1 + \alpha_i^2P_i - 2\alpha_iQ_i\right],\ i = 1,2,4,5 \tag{A46}$$

$$minMSE(t_{s_i}^{c}) = \bar{Y}^2\left[1 - \frac{Q_i^2}{P_i}\right] \tag{A47}$$

$$MSE(t_{s_3}^{c}) = \bar{Y}^2\left[(\alpha_3-1)^2 + \alpha_3^2V_{0,2} + \frac{\bar{X}^2}{\bar{Y}^2}\beta_3^2v^2V_{2,0} + 2\frac{\bar{X}}{\bar{Y}}\alpha_3\beta_3vV_{1,1}\right] \tag{A48}$$

$$minMSE(t_{s_3}^{c}) = \bar{Y}^2(1 - \alpha_{3(opt)}) \tag{A49}$$

where

$$\eta_1 = \frac{\sum_{h=1}^{L} W_h\bar{X}_h}{\sum_{h=1}^{L} W_h(\bar{X}_h + C_xx_h)}, \qquad \eta_2 = \frac{\sum_{h=1}^{L} W_h\bar{X}_h}{\sum_{h=1}^{L} W_h(\bar{X}_h + \beta_2(x_h))},$$

$$\eta_3 = \frac{\sum_{h=1}^{L} W_h \bar{X}_h \beta_2(x_h)}{\sum_{h=1}^{L} W_h (\bar{X}_h \beta_2(x_h) + C_{x_h})}, \qquad \eta_4 = \frac{\sum_{h=1}^{L} W_h \bar{X}_h C_{x_h}}{\sum_{h=1}^{L} W_h (\bar{X}_h C_{x_h} + \beta_2(x_h))}$$

$$A = 2 + (g^2 + g)\alpha^2 v^2 V_{2,0} - 2g\alpha v V_{1,1}$$

$$B = 1 + V_{0,2} + (2g^2 + g)\alpha^2 v^2 V_{2,0} - 4g\alpha v V_{1,1}$$

$$B_1 = 1 + V_{0,2}, \qquad B_2 = B_1 + \beta(2\beta - 1)V_{2,0} + 4\beta V_{1,1}$$

$$B_3 = B_1 + \frac{\beta(\beta-1)}{2} V_{2,0} + 2\beta V_{1,1}, \qquad B_4 = 1 + \frac{\beta(\beta-1)}{2} V_{2,0} + \beta V_{1,1}$$

$$C_1 = 1 + V_{0,2} + 4\alpha\delta v V_{1,1} + \delta(2\delta - 1)\alpha^2 v^2 V_{2,0}$$

$$C_2 = 1 + V_{0,2} - 4g\alpha v V_{1,1} + g(2g + 1)\alpha^2 v^2 V_{2,0}$$

$$C_3 = 1 + V_{0,2} + 2\alpha(\delta - g) v V_{1,1} + \frac{\alpha^2 v^2}{2}(\delta - g)(\delta - g - 1)V_{2,0}$$

$$C_4 = 1 + \alpha\delta v V_{1,1} + \frac{\delta(\delta+1)}{2}\alpha^2 v^2 V_{2,0}$$

$$C_5 = 1 - \alpha g v V_{1,1} + \frac{g(g+1)}{2}\alpha^2 v^2 V_{2,0}$$

$$D_1 = 1 + V_{0,2} - 2\Theta_1 a V_{1,1} + \frac{\Theta_1(\Theta_1 + 1)}{2} a^2 V_{2,0}$$

$$D_2 = 1 + V_{0,2} + 2\Theta_2 a V_{1,1} + \frac{\Theta_2(\Theta_2 - 1)}{2} a^2 V_{2,0}$$

$$D_3 = 1 + V_{0,2} + (\Theta_2 - \Theta_1) a V_{1,1} + \frac{(\Theta_2 - \Theta_1)(\Theta_2 - \Theta_1 - 2)}{8} a^2 V_{2,0}$$

$$D_4 = 1 - \frac{\Theta_1}{2}\left(a V_{1,1} - \frac{(\Theta_1 + 2)}{4} a^2 V_{2,0} \right)$$

$$D_5 = 1 - \frac{\Theta_2}{2}\left(a V_{1,1} + \frac{(\Theta_2 - 2)}{4} a^2 V_{2,0} \right)$$

$$\Theta_1 = 2\alpha + \beta, \qquad \Theta_2 = 2\delta + \lambda$$

$$P_1 = 1 + V_{0,2} + 2\beta_1(\beta_1 - 1)v^2 V_{2,0} + 4\beta_1 v V_{1,1}$$

$$Q_1 = \beta_1\left(\tfrac{\beta_1}{2} - 1\right)v^2 V_{2,0} - \beta_1 v V_{1,1}$$

$$P_2 = 1 + V_{0,2} + \beta_2(\beta_2 - 1)v^2 V_{2,0} + 4\beta_2 v V_{1,1}$$

$$Q_2 = 1 + \beta_2 v V_{1,1} - \tfrac{\beta_2}{2} v^2 V_{2,0}$$

$$P_4 = 1 + V_{0,2} - 4\beta_4 v V_{1,1} + \beta_4(2\beta_4 + 1)v^2 V_{2,0}$$

$$Q_4 = 1 + \frac{\beta_4(\beta_4 + 1)}{2} v^2 V_{2,0} - \beta_4 v V_{1,1}$$

$$P_5 = 1 + V_{0,2} + 3\beta_5^2 v^2 V_{2,0} - 4\beta_5 v V_{1,1}$$

$$Q_5 = 1 + \beta_5^2 v^2 V_{2,0} - \beta_5 v V_{1,1}$$

The scalars' optimal values in the aforementioned MSE expressions are listed as

$$\hat{\beta}_{(opt)} = \hat{R}\frac{V_{1,1}}{V_{2,0}}, \qquad k_{(opt)} = \frac{1 + V_{1,1}}{1 + V_{0,2}} = k^*(\text{say}), \qquad \lambda_{s(opt)} = \frac{1}{\left(1 + V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}} + \frac{V_{2,0}}{(N+1)^2}\right)},$$

$$\lambda_{k(opt)} = \frac{1 + \alpha g v V_{1,1} + \frac{g(g+1)}{2}\alpha^2 v^2 V_{2,0}}{1 + V_{0,2} + g(2g+1)\alpha^2 v^2 V_{2,0} - 4g\alpha v V_{1,1}} = \frac{B}{A}\ (\text{say})$$

$$k_{1_{(opt)}} = \frac{V_{2,0}(1 - v^2 V_{2,0})}{V_{0,2}V_{2,0} + V_{2,0} - v^2 V_{2,0}^2 - V_{1,1}^2}, \qquad k_{2_{(opt)}} = R\left[v + \frac{(1 - v^2 V_{2,0})(V_{1,1} - 2v V_{2,0})}{V_{2,0} + V_{0,2}V_{2,0} - V_{1,1}^2 - v^2 V_{2,0}^2} \right]$$

$$\lambda_{1(opt)} = \frac{(A_2A_4 - A_3A_5)}{(A_1A_2 - A_3^2)} \text{ and } \lambda_{2(opt)} = \frac{(A_3A_4 - A_1A_5)}{(A_1A_2 - A_3^2)}$$

$$\Lambda_{1(opt)} = \frac{(B_2 - B_3B_4)}{(B_1B_2 - B_3^2)} \text{ and } \Lambda_{2(opt)} = \frac{(B_1B_4 - B_3)}{(B_1B_2 - B_3^2)}$$

$$\phi_{1(opt)} = \frac{(C_2C_4 - C_3C_5)}{(C_1C_2 - C_3^2)} \text{ and } \phi_{2(opt)} = \frac{(C_1C_5 - C_3C_4)}{(C_1C_2 - C_3^2)}$$

$$w_{1(opt)} = \frac{(D_2D_4 - D_3D_5)}{(D_1D_2 - D_3^2)} \text{ and } w_{2(opt)} = \frac{(D_1D_5 - D_3D_4)}{(D_1D_2 - D_3^2)}$$

$$a_{(opt)} = \frac{V_{2,0}}{V_{1,1}}, \quad \alpha_{i(opt)} = \frac{Q_i}{P_i}, \ i = 1,2,4,5, \quad \alpha_{3(opt)} = \frac{1}{\left(1 + V_{0,2} - \frac{V_{1,1}^2}{V_{2,0}}\right)}$$

$$\beta_{1(opt)} = -\frac{V_{1,1}}{vV_{2,0}} = \beta_{2(opt)}, \qquad \beta_{3(opt)} = -\alpha_{3(opt)}\frac{V_{1,1}}{vV_{2,0}}, \quad \beta_{4(opt)} = \frac{V_{1,1}}{vV_{2,0}} = \beta_{5(opt)}$$

References

1. Cochran, W.G. *Sampling Techniques*; John Wiley and Sons: New York, NY, USA, 1977.
2. Sisodia, B.V.S.; Dwivedi, V.K. A modified ratio estimator using coefficient of variation of auxiliary variable. *J. Ind. Soc. Agric. Stat.* **1981**, *33*, 13–18.
3. Upadhyaya, L.N.; Singh, H.P. Use of transformed auxiliary variable in estimating the finite population mean. *Biom. J.* **1999**, *41*, 627–636. [CrossRef]
4. Kadilar, C.; Cingi, H. Ratio estimator in stratified sampling. *Biom. J.* **2003**, *45*, 218–225. [CrossRef]
5. Prasad, B. Some improved ratio type estimators of population mean and ratio in finite population sample surveys. *Commun. Stat. Theory Methods* **1989**, *18*, 379–392. [CrossRef]
6. Kadilar, C.; Cingi, H. A new estimator in stratified random sampling. *Commun. Stat. Theory Methods* **2005**, *34*, 597–602. [CrossRef]
7. Shabbir, J.; Gupta, S. A new estimator of population mean in stratified sampling. *Commun. Stat. Theory Methods* **2006**, *35*, 1201–1209. [CrossRef]
8. Ray, S.K.; Singh, R.K. Difference cum product type estimators. *J. Ind. Stat. Asso.* **1981**, *19*, 147–151.
9. Bedi, P.K. Efficient utilization of auxiliary information at estimation stage. *Biom. J.* **1996**, *38*, 973–976. [CrossRef]
10. Koyuncu, N.; Kadilar, C. Ratio and product estimators in stratified random sampling. *J. Stat. Plan. Inference* **2009**, *139*, 2552–2558. [CrossRef]
11. Khoshnevisan, M.; Sing, R.; Chauhan, P.; Sawan, N.; Smarandache, F. A general family of estimators for estimating population mean using known value of some population parameters. *Far East J. Theor. Stat.* **2007**, *22*, 181—191.
12. Koyuncu, N.; Kadilar, C. On improvement in estimating population mean in stratified random sampling. *J. Appl. Stat.* **2010**, *37*, 999–1013. [CrossRef]
13. Gupta, S.; Shabbir, J. On improvement in estimating the population mean in simple random sampling. *J. Appl. Stat.* **2008** *35*, 559–566. [CrossRef]
14. Singh, H.P.; Vishwakarma, G.K. A general procedure for estimating the population mean in stratified sampling using auxiliary information. *Metron* **2010**, *68*, 47–65. [CrossRef]
15. Singh, H.P.; Solanki, R.S. Efficient ratio and product estimators in stratified random sampling. *Commun. Stat. Theory Methods* **2013**, *42*, 1008–1023. [CrossRef]
16. Tailor, R.; Mishra, A. Ratio and ratio type separate estimator for population mean when coefficient of variation is known. *J. Reliab. Stat. Stud.* **2018**, *11*, 45–50.
17. Zaman, T.; Bulut, H. Modified regression estimators using robust regression methods and covariance matrices in stratified random sampling. *Commun. Stat. Theory Methods* **2020**, *49*, 3407–3420. [CrossRef]
18. Aslam, I.; Noor-ul-Amin, M.; Yasmeen, U.; Hanif, M. Memory type ratio and product estimators in stratified sampling. *J. Reliab. Stat. Stud.* **2020**, *13*, 1–20. [CrossRef]
19. Tiwari, K.K.; Bhougal, S.; Kumar, S. A general class of estimators in stratified random sampling. *Commun. Stat. B Simul. Comput.* **2020**, *52*, 442–452 . [CrossRef]
20. Bhushan, S.; Kumar, A.; Singh, S. Some efficient classes of estimators under stratified sampling. *Commun. Stat. Theory Methods* **2023**, *52*, 1767–1796. [CrossRef]
21. Bhushan, S.; Kumar, A.; Tyagi, D.; Singh, S. On some improved classes of estimators under stratified sampling using attribute. *J. Reliab. Stat. Stud.* **2022**, *15*, 187–210. [CrossRef]
22. Kaur, P. On the estimation of population mean in stratified sampling. *Biom. J.* **1985**, *27*, 101–105. [CrossRef]
23. Diana, G. A study of k^{th} order approximation of some ratio type strategies. *Metron* **1992**, *1*, 19–32.
24. Solanki, R.S.; Singh, H.P. An improved estimation in stratified random sampling. *Commun. Stat. Theory Methods* **2016**, *45*, 2056–2070. [CrossRef]

25. Kumar, S.; Trehan, M.; Joorel, J.S. A simulation study: Estimation of population mean using two auxiliary variables in stratified random sampling. *J. Stat. Comput. Simul.* **2018**, *88*, 3694–3707. [CrossRef]
26. Saini, M.; Kumar, A. Ratio estimators using stratified random sampling and stratified ranked set sampling. *Life Cycle Reliab. Saf. Eng.* **2019**, *8*, 85–89. [CrossRef]
27. Sharma, V.; Kumar, S. Simulation study of ratio type estimators in stratified random sampling using multi-auxiliary information. *Thail. Stat.* **2020**, *18*, 281–289.
28. Muneer, S.; Khalil, A.; Shabbir, J. A parent-generalized family of chain ratio exponential estimators in stratified random sampling using supplementary variables. *Commun. Stat. B Simul. Comput.* **2020**, *51*, 4727–4748. [CrossRef]
29. Hansen, M.H.; Hurwitz, W.N.; Gurney, M. Problems and methods of the sample survey of business. *J. Am. Stat. Assoc.* **1946**, *41*, 173–189. [CrossRef]
30. Searls, D.T. The utilization of a known coefficient of variation in the estimation procedure. *J. Am. Stat. Assoc.* **1964**, *59*, 1225–1226. [CrossRef]
31. Yadav, R.; Upadhyaya, L.N.; Singh, H.P.; Chatterjee, S. Improved ratio and product exponential type estimators for finite population mean in stratified random sampling. *Commun. Stat. Theory Meth.* **2014**, *43*, 3269–3285. [CrossRef]
32. Singh, H.P.; Tailor, R.; Tailor, R.; Kakran, M.S. An improved estimator of population mean using power transformation. *J. Ind. Soc. Agric. Stat.* **2004**, *58*, 223–230.
33. Upadhyaya, L.N.; Singh, H.P.; Chatterjee, S.; Yadav, R. Improved ratio and product exponential type estimators. *J. Stat. Theor. Pract.* **2011**, *5*, 285–302. [CrossRef]
34. Bhushan, S.; Kumar, A.; Kumar, S.; Singh, S. An improved class of estimators of population mean under simple random sampling. *Philipp. Stat.* **2021**, *70*, 33–47.
35. Singh, H.P.; Horn, S. An alternative estimator for multi-character surveys. *Metrika* **1998**, *48*, 99–107. [CrossRef]
36. Murthy, M.N. *Sampling Theory and Methods*; Statistical Publishing Society: Calcutta, India, 1967.