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Study on the Nonlinear Dynamics of the (3+1)-Dimensional Jimbo-Miwa Equation in Plasma Physics

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Abstract: The Jimbo-Miwa equation (JME) that describes certain interesting (3+1)-dimensional waves in plasma physics is studied in this work. The Hirota bilinear equation is developed via the Cole-Hopf transform. Then, the symbolic computation, together with the ansatz function schemes, are utilized to seek exact solutions. Some new solutions, such as the multi-wave complexiton solution (MWCS), multi-wave solution (MWS) and periodic lump solution (PLS), are successfully constructed. Additionally, different types of travelling wave solutions (TWS), including the dark, bright-dark and singular periodic wave solutions, are disclosed by employing the sub-equation method. Finally, the physical characteristics and interaction behaviors of the extracted solutions are depicted graphically by assigning appropriate parameters. The obtained outcomes in this paper are more general and newer. Additionally, they reveal that the used methods are concise, direct, and can be employed to study other partial differential equations (PDEs) in physics.

Keywords: Hirota bilinear equation; Cole-Hopf transform; multi-wave complexiton solution; multi-wave solution; periodic lump solution; sub-equation method

MSC: 35C07; 35A22

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1. Introduction

Complex phenomena in engineering and physics can usually be reduced to PDEs [1–6]. The study on the properties of these equations such as the explicit analytical solutions, especially the soliton solutions, is of great significance since they can help us to better understand complex phenomena and their inner nature. Up to now, a series of different effective methods have been developed to construct the exact solutions of PDEs such as the Hirota bilinear method [7–10], Wang's Bäcklund transformation-based method [11,12], trial equation method [13,14], Sardar subequation method [15–17], exp-function method [18,19], Riccati equation mapping method [20] and so on [21–28]. In this work, we aim to examine the (3+1)-dimensional JME given by [29]:

$$\Pi_{xxxy} + 3\Pi_x\Pi_{xy} + 3\Pi_y\Pi_{xx} + 2\Pi_{yt} - 3\Pi_{xz} = 0, \tag{1}$$

Equation (1) is derived from the second equation in the well-known KP hierarchy of integrable systems and used widely to describe some interesting (3+1)-dimensional waves in plasma and optics. Up to now, some important research achievements have been developed to deal with Equation (1). In [29], the Kudryashov method is used with the symbolic computation and different solutions are obtained. In [30], four kinds of different wave forms are constructed via the Hirota bilinear method. In [31], the authors employ the direct algebraic method to handle Equation (1) and some different wave forms are constructed. In [32], several closed-form solutions are developed by using the singular manifold method. In [33], the Riccati equation mapping method is adopted. The exp-function method is utilized in [34] and some generalized solutions with parameters are constructed. In [35], the authors carry out the linear superposition principle to seek for multi-resonant solutions of Equation (1). In [36], the authors make use of the generalized Bernoulli equation method to inquire into Equation (1). In this study, we will present th results of a detailed investigation of Equation (1). The rest of the content of this work is given as follows. In Section 2, the Cole-Hopf transform is adopted to establish the Hirota bilinear form, and symbolic computation, combined with the ansatz function schemes, is utilized to search for the MWCS, MWS and PLS. In Section 3, the sub-equation method is used to seek for the TWSs. In Section 4, the physical characteristics and interaction behaviors are presented. Finally, we reach a conclusion in Section 5.

2. The Hirota Bilinear Equation and the Exact Solutions

To obtain the Hirota bilinear form of Equation (1), we adopt the Cole-Hopf transform as:

$$\Pi = 2\ln(\Xi)_x, \tag{2}$$

Taking it into Equation (1), we can obtain the bilinear form as:

$$\left(D_x^3 D_y + 2D_y D_t - 3D_x D_z\right)\Xi \cdot \Xi = 0. \tag{3}$$

Here, the definition of the operators $D_x^m D_\tau^n$ is [37,38]:

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t) g(x',t')|_{x=x',t=t'}. \tag{4}$$

Additionally, there are

$$D_x(f \cdot g) = f_x g - f g_x,$$

$$D_x^2(f \cdot g) = f_{xx} g - 2 f_x g_x + f g_{xx},$$

$$D_x^2(f \cdot f) = 2\left(f_{xx} f - f_x^2\right),$$

$$D_t D_x(f \cdot g) = f_{tx} g - f_t g_x - f_x g_t + g_{tx} f.$$

2.1. The MWCS

In order to find the MWCS, it is assumed that the solution of Equation (3) is:

$$\Xi = u_1 e^p + u_2 e^{-p} + u_3 \sin(q) + u_4 \sin\mathrm{h}(\rho), \tag{5}$$

with

$$\begin{cases} p = x + k_1 y + k_2 z + k_3 t \\ q = x + k_4 y + k_5 z + k_6 t \\ \rho = x + k_7 y + k_8 z + k_9 t \end{cases},$$

where $u_i (i = 1,2,3,4.)$ and $k_i (i = 1,2,3,4,5,6,7,8,9.)$ are constants that can be determined later. Substituting Equation (5) into Equation (3) and setting the coefficients of different terms to zero, an algebraic equation system is attained. Solving it, we derive:

Case 1:

$$k_1 = \frac{3k_2}{2(k_9+2)},\ k_2 = k_2,\ k_3 = k_9,\ k_4 = -\frac{3k_2}{2(k_9+2)},\ k_5 = \frac{k_2 - k_2 k_9}{2+k_9},\ k_6 = k_9 + 1,\ k_7 = \frac{3k_2}{2(2+k_9)},$$
$$k_8 = k_2,\ k_9 = k_9,\ u_1 = u_1,\ u_2 = u_2,\ u_3 = u_3,\ u_4 = u_4.$$

The MWCS is obtained as:

$$\Pi(x,y,z,t)=\frac{2\left[\begin{array}{l} u_1e^{x+\frac{3k_2}{2(k_9+2)}y+k_2z+k_9t}-u_2e^{-(x+\frac{3k_2}{2(k_9+2)}y+k_2z+k_9t)}+u_3\cos\left(x-\frac{3k_2}{2(k_9+2)}y+\frac{k_2-k_2k_9}{2+k_9}z+(k_9+1)t\right)\\ +u_4\cos\mathrm{h}\left(x+\frac{3k_2}{2(2+k_9)}y+k_2z+k_9t\right)\end{array}\right]}{\begin{array}{l} u_1e^{x+\frac{3k_2}{2(k_9+2)}y+k_2z+k_9t}+u_2e^{-(x+\frac{3k_2}{2(k_9+2)}y+k_2z+k_9t)}+u_3\sin\left(x-\frac{3k_2}{2(k_9+2)}y+\frac{k_2-k_2k_9}{2+k_9}z+(k_9+1)t\right)\\ +u_4\sin\mathrm{h}\left(x+\frac{3k_2}{2(2+k_9)}y+k_2z+k_9t\right)\end{array}}. \quad (6)$$

For the special case $u_1=-u_2=2u_4$, Equation (6) becomes:
Case 2:

$$k_1=k_1,\ k_2=k_2,\ k_3=-2+\frac{3k_2}{2k_1},\ k_4=-k_1,\ k_5=2k_1-k_2,\ k_6=-1+\frac{3k_2}{2k_1},\ k_7=k_1,\ k_8=k_2,$$
$$k_9=-2+\frac{3k_2}{2k_1},\ u_1=u_1,\ u_2=u_2,\ u_3=u_3,\ u_4=u_4.$$

Thus, we can obtain the MWCS as:

$$\Pi(x,y,z,t)=\frac{2\left[\begin{array}{l} u_1e^{x+k_1y+k_2z+(-2+\frac{3k_2}{2k_1})t}-u_2e^{-(x+k_1y+k_2z+(-2+\frac{3k_2}{2k_1})t)}+u_3\cos\left(x-k_1y+(2k_1-k_2)z+\left(-1+\frac{3k_2}{2k_1}\right)t\right)\\ +u_4\cos\mathrm{h}\left(x+k_1y+k_2z+\left(-2+\frac{3k_2}{2k_1}\right)t\right)\end{array}\right]}{\begin{array}{l} u_1e^{x+k_1y+k_2z+(-2+\frac{3k_2}{2k_1})t}+u_2e^{-(x+k_1y+k_2z+(-2+\frac{3k_2}{2k_1})t)}+u_3\sin\left(x-k_1y+(2k_1-k_2)z+\left(-1+\frac{3k_2}{2k_1}\right)t\right)\\ +u_4\sin\mathrm{h}\left(x+k_1y+k_2z+\left(-2+\frac{3k_2}{2k_1}\right)t\right)\end{array}}. \quad (7)$$

Case 3:

$$k_1=k_1,\ k_2=\frac{2}{3}(2k_1+k_1k_9),\ k_3=k_9,\ k_4=-k_1,\ k_5=-\frac{2}{3}(-k_1+k_1k_9),\ k_6=-1+k_9,\ k_7=k_1,$$
$$k_8=\frac{2}{3}(2k_1+k_1k_9),\ k_9=k_9,\ u_1=u_1,\ u_2=u_2,\ u_3=u_3,\ u_4=u_4.$$

Thus, we obtain the MWCS solution as:

$$\Pi(x,y,z,t)=\frac{2\left[\begin{array}{l} u_1e^{x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_3t}-u_2e^{-(x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_3t)}+u_3\cos(x+k_4y+k_5z+(k_9-1)t)\\ +u_4\cos\mathrm{h}\left(x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_9t\right)\end{array}\right]}{\begin{array}{l} u_1e^{x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_3t}+u_2e^{-(x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_3t)}+u_3\sin(x+k_4y+k_5z+(k_9-1)t)\\ +u_4\sin\mathrm{h}\left(x+k_1y+\frac{2}{3}(2k_1+k_1k_9)z+k_9t\right)\end{array}}. \quad (8)$$

Case 4:

$$k_1=-\frac{3k_5}{2(k_9-1)},\ k_2=-\frac{2k_5(1+k_9)}{k_9-1},\ k_3=k_9,\ k_4=\frac{3k_5}{2(k_9-1)},\ k_5=k_5,\ k_6=k_9-1,$$
$$k_7=-\frac{3k_5}{2(k_9-1)},\ k_8=-\frac{2k_5(1+k_9)}{k_9-1},\ k_9=k_9,\ u_1=u_1,\ u_2=u_2,\ u_3=u_3,\ u_4=u_4.$$

Accordingly, the MWCS is:

$$\Pi(x,y,z,t)=\frac{2\left[\begin{array}{l} u_1e^{x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t}-u_2e^{-(x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t)}+u_3\cos\left(x+\frac{3k_5}{2(k_9-1)}y+k_5z+(k_9-1)t\right)\\ +u_4\cos\mathrm{h}\left(x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t\right)\end{array}\right]}{\begin{array}{l} u_1e^{x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t}+u_2e^{-(x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t)}+u_3\sin\left(x+\frac{3k_5}{2(k_9-1)}y+k_5z+(k_9-1)t\right)\\ +u_4\sin\mathrm{h}\left(x-\frac{3k_5}{2(k_9-1)}y-\frac{2k_5(1+k_9)}{k_9-1}z+k_9t\right)\end{array}}. \quad (9)$$

Case 5:

$$k_1=-k_4,\ k_2=k_8,\ k_3=-\left(2+\tfrac{3k_8}{2k_4}\right),\ k_4=k_4,\ k_5=-(2k_4+k_8),\ k_6=-\left(1+\tfrac{3k_8}{2k_4}\right),\ k_7=-k_4,$$
$$k_8=k_8,\ k_9=-\left(2+\tfrac{3k_8}{2k_4}\right),\ u_1=u_1,\ u_2=u_2,\ u_3=u_3,\ u_4=u_4.$$

where $k_4\neq 0$. Thus, we can obtain the MWCS as:

$$\Pi(x,y,z,t)=\frac{2\left[\begin{array}{l} u_1e^{x-k_4y+k_8z-(2+\frac{3k_8}{2k_4})t}-u_2e^{-(x-k_4y+k_8z-(2+\frac{3k_8}{2k_4})t)}+u_3\cos\left(x+k_4y-(2k_4+k_8)z-\left(1+\frac{3k_8}{2k_4}\right)t\right)\\ +u_4\cos\mathrm{h}\left(x-k_4y+k_8z-\left(2+\frac{3k_8}{2k_4}\right)t\right)\end{array}\right]}{\begin{array}{l} u_1e^{x-k_4y+k_8z-(2+\frac{3k_8}{2k_4})t}+u_2e^{-(x-k_4y+k_8z-(2+\frac{3k_8}{2k_4})t)}+u_3\sin\left(x+k_4y-(2k_4+k_8)z-\left(1+\frac{3k_8}{2k_4}\right)t\right)\\ +u_4\sin\mathrm{h}\left(x-k_4y+k_8z-\left(2+\frac{3k_8}{2k_4}\right)t\right)\end{array}}. \quad (10)$$

For the special case $u_1=-u_2=2u_4$, Equations (6)–(10) become:

$$\Pi(x,y,z,t)=2\cot\left(x-\frac{3k_2}{2(k_9+2)}y+\frac{k_2-k_2k_9}{2+k_9}z+(k_9+1)t\right).$$

$$\Pi(x,y,z,t)=2\cot\left(x-k_1y+(2k_1-k_2)z+\left(-1+\frac{3k_2}{2k_1}\right)t\right).$$

$$\Pi(x,y,z,t)=2\cot(x+k_4y+k_5z+(k_9-1)t).$$

$$\Pi(x,y,z,t)=2\cot\left(x+\frac{3k_5}{2(k_9-1)}y+k_5z+(k_9-1)t\right).$$

$$\Pi(x,y,z,t)=2\cot\left(x+k_4y-(2k_4+k_8)z-\left(1+\frac{3k_8}{2k_4}\right)t\right).$$

2.2. The MWS

Here, we can use the following ansatz function:

$$\Xi=u_1\cos(p)+u_2\cos\mathrm{h}(q)+u_3\cos\mathrm{h}(\rho), \quad (11)$$

with

$$\begin{cases} p=x+k_1y+k_2z+k_3t\\ q=x+k_4y+k_5z+k_6t\\ \rho=x+k_7y+k_8z+k_9t\end{cases},$$

where $u_i(i=1,2,3.)$ and $k_i(i=1,2,3,4,5,6,7,8,9.)$ are constants that can be determined later. In the same manner, substituting Equation (11) into Equation (3) and making the corresponding adjustments, we derive:

Case 1:

$$k_1=k_1,\ k_2=k_2,\ k_3=2+\frac{3k_2}{2k_1},\ k_7=-k_1,\ k_8=-2k_1-k_2,\ k_9=1+\frac{3k_2}{2k_1},\ u_1=u_1,\ u_2=0,\ u_3=u_3.$$

Then, we obtain the MWS as:

$$\Pi(x,y,z,t)=\frac{2\left[-u_1\sin\left(x+k_1y+k_2z+\left(2+\frac{3k_2}{2k_1}\right)t\right)+u_3\sin\mathrm{h}\left(x-k_1y-(2k_1+k_2)z+\left(1+\frac{3k_2}{2k_1}\right)t\right)\right]}{u_1\cos\left(x+k_1y+k_2z+\left(2+\frac{3k_2}{2k_1}\right)t\right)+u_3\cos\mathrm{h}\left(x-k_1y-(2k_1+k_2)z+\left(1+\frac{3k_2}{2k_1}\right)t\right)}. \quad (12)$$

Case 2:

$$k_1=k_1,\ k_2=-2k_1-k_5,\ k_3=-1-\frac{3k_5}{2k_1},\ k_4=-k_1,\ k_5=k_5,\ k_6=-2-\frac{3k_5}{2k_1},\ u_1=u_1,\ u_2=u_2,\ u_3=0.$$

Thus, we obtain the MWS as:

$$\Pi(x,y,z,t) = \frac{2\left[-u_1 \sin\left(x + k_1 y - (2k_1 + k_5)z - \left(1 + \frac{3k_5}{2k_1}\right)t\right) + u_2 \sin\mathrm{h}\left(x - k_1 y + k_5 z - \left(2 + \frac{3k_5}{2k_1}\right)t\right)\right]}{u_1 \cos\left(x + k_1 y - (2k_1 + k_5)z - \left(1 + \frac{3k_5}{2k_1}\right)t\right) + u_2 \cos\mathrm{h}\left(x - k_1 y + k_5 z - \left(2 + \frac{3k_5}{2k_1}\right)t\right)}. \quad (13)$$

2.3. The PLS

The solution of Equation (3) is assumed as:

$$\Xi = u_1 \sin(p) + u_2 \cos\mathrm{h}(q) + k_7, \quad (14)$$

with

$$\begin{cases} p = x + k_1 y + k_2 z + k_3 t \\ q = x + k_4 y + k_5 z + k_6 t \end{cases},$$

where $u_i (i = 1, 2.)$ and $k_i (i = 1, 2, 3, 4, 5, 6, 7.)$ are constants to be determined later. In the same manner, substituting Equation (14) into Equation (3) and making the corresponding adjustments, we derive:

Case 1:

$k_1 = k_1,\ k_2 = \frac{2}{3}k_1(k_6 - 1),\ k_3 = k_6 + 1,\ k_4 = -k_1,\ k_5 = -\frac{2}{3}k_1(2 + k_6),\ k_6 = k_6,\ k_7 = 0,\ u_1 = u_1,$
$u_2 = u_2.$

The PLS to Equation (1) is:

$$\Pi(x,y,z,t) = \frac{2\left[u_1 \cos\left(x + k_1 y + \frac{2}{3}k_1(k_6 - 1)z + (k_6 + 1)t\right) + u_2 \sin\mathrm{h}\left(x - k_1 y - \frac{2}{3}k_1(2 + k_6)z + k_6 t\right)\right]}{u_1 \sin\left(x + k_1 y + \frac{2}{3}k_1(k_6 - 1)z + (k_6 + 1)t\right) + u_2 \cos\mathrm{h}\left(x - k_1 y - \frac{2}{3}k_1(2 + k_6)z + k_6 t\right)}. \quad (15)$$

Case 2:

$k_1 = \frac{3k_2}{2(k_6-1)},\ k_2 = k_2,\ k_3 = k_6 + 1,\ k_4 = -\frac{3k_2}{2(k_6-1)},\ k_5 = -\frac{2k_2(k_6+1)}{k_6-1},\ k_6 = k_6,\ k_7 = 0,\ u_1 = u_1,$
$u_2 = u_2.$

Thus, we obtain the PLS of Equation (1) as:

$$\Pi(x,y,z,t) = \frac{2\left[u_1 \cos\left(x + \frac{3k_2}{2(k_6-1)}y + k_2 z + (k_6 + 1)t\right) + u_2 \sin\mathrm{h}\left(x - \frac{3k_2}{2(k_6-1)}y - \frac{2k_2(k_6+1)}{k_6-1}z + k_6 t\right)\right]}{u_1 \sin\left(x + \frac{3k_2}{2(k_6-1)}y + k_2 z + (k_6 + 1)t\right) + u_2 \cos\mathrm{h}\left(x - \frac{3k_2}{2(k_6-1)}y - \frac{2k_2(k_6+1)}{k_6-1}z + k_6 t\right)}. \quad (16)$$

Case 3:

$k_1 = -\frac{3k_5}{2(k_3+1)},\ k_2 = \frac{k_5(2-k_3)}{k_3+1},\ k_3 = k_3,\ k_4 = \frac{3k_5}{2(k_3+1)},\ k_5 = k_5,\ k_6 = k_3 - 1,\ k_7 = 0,\ u_1 = u_1,$
$u_2 = u_2$

The PLS of Equation (1) is obtained as:

$$\Pi(x,y,z,t) = \frac{2\left[u_1 \cos\left(x - \frac{3k_5}{2(k_3+1)}y + \frac{k_5(2-k_3)}{k_3+1}z + k_3 t\right) + u_2 \sin\mathrm{h}\left(x + \frac{3k_5}{2(k_3+1)}y + k_5 z + (k_3 - 1)t\right)\right]}{u_1 \sin\left(x - \frac{3k_5}{2(k_3+1)}y + \frac{k_5(2-k_3)}{k_3+1}z + k_3 t\right) + u_2 \cos\mathrm{h}\left(x + \frac{3k_5}{2(k_3+1)}y + k_5 z + (k_3 - 1)t\right)}. \quad (17)$$

Case 4:

$k_1 = -k_4,\ k_2 = -\frac{2}{3}k_4(k_3 - 2),\ k_3 = k_3,\ k_4 = k_4,\ k_5 = \frac{2}{3}k_4(k_3 + 1),\ k_6 = k_3 - 1,\ k_7 = 0,\ u_1 = u_1,$
$u_2 = u_2.$

Thus, the PLS of Equation (1) is attained as:

$$\Pi(x,y,z,t)=\frac{2\left[u_1\cos\left(x-k_4y-\frac{2}{3}k_4(k_3-2)z+k_3t\right)+u_2\sin\mathrm{h}\left(x+k_4y+\frac{2}{3}k_4(k_3+1)z+(k_3-1)t\right)\right]}{u_1\sin\left(x-k_4y-\frac{2}{3}k_4(k_3-2)z+k_3t\right)+u_2\cos\mathrm{h}\left(x+k_4y+\frac{2}{3}k_4(k_3+1)z+(k_3-1)t\right)}. \tag{18}$$

Case 5:

$$k_1=k_1,\ k_2=k_2,\ k_3=2+\frac{3k_2}{2k_1},\ k_4=-k_1,\ k_5=-2k_1-k_2,\ k_6=1+\frac{3k_2}{2k_1},\ k_7=0,\ u_1=u_1,\ u_2=u_2.$$

We obtain the PLS of Equation (1) as:

$$\Pi(x,y,z,t)=\frac{2\left[u_1\cos\left(x+k_1y+k_2z+\left(2+\frac{3k_2}{2k_1}\right)t\right)+u_2\sin\mathrm{h}\left(x-k_1y-(2k_1+k_2)z+\left(1+\frac{3k_2}{2k_1}\right)t\right)\right]}{u_1\sin\left(x+k_1y+k_2z+\left(2+\frac{3k_2}{2k_1}\right)t\right)+u_2\cos\mathrm{h}\left(x-k_1y-(2k_1+k_2)z+\left(1+\frac{3k_2}{2k_1}\right)t\right)}. \tag{19}$$

3. The TWS

This section aims to study the TWS using the sub-equation method [39,40]. For this end, we apply the following variable transformation to Equation (1):

$$\Pi(x,y,z,t)=\Im(\chi),\ \chi=mx+ny+kz+st, \tag{20}$$

where m, n, k, and s are non-zero constants. Equation (1) can be converted as:

$$m^3n\Im^{(4)}+6m^2n\Im'\Im''+(2ns-3mk)\Im''=0, \tag{21}$$

where $\Im^{(4)}=\frac{d^4\Im}{d\chi^4}$, $\Im''=\frac{d^2\Im}{d\chi^2}$, $\Im'=\frac{d\Im}{d\chi}$. Integrating Equation (21) with respect to χ once and setting the integral constant to zero, we derive:

$$m^3n\Im'''+3m^2n(\Im')^2+(2ns-3mk)\Im'=0. \tag{22}$$

Based on the sub-equation method, the solution of Equation (22) can be assumed as:

$$\Im(\chi)=\sum_{i=0}^{c}\varepsilon_i\aleph^i(\chi). \tag{23}$$

where $\varepsilon_i(i=0,1,2,\ldots,c.)$ are constants that can be determined later. Additionally, there is:

$$\aleph'(\chi)=\sigma+\aleph^2(\chi). \tag{24}$$

Here, σ is a constant. Equation (24) has the following different solutions:

$$\aleph(\chi)=\begin{cases}-\sqrt{-\sigma}\tan\mathrm{h}\left(\sqrt{-\sigma}\chi\right), & \sigma<0\\ -\sqrt{-\sigma}\cot\mathrm{h}\left(\sqrt{-\sigma}\chi\right), & \sigma<0\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\chi\right), & \sigma>0\\ -\sqrt{\sigma}\cot\left(\sqrt{\sigma}\chi\right), & \sigma>0\\ -\frac{1}{\zeta+\Lambda},\ \Lambda\ \text{is a constant}, & \sigma=0\end{cases}. \tag{25}$$

We can determine the value of c in Equation (23) via balancing $\Im'''$ and $(\Im')^2$ in Equation (22) as:

$$c=1. \tag{26}$$

Then, Equation (23) becomes:

$$\Im(\chi)=\varepsilon_0+\varepsilon_1\Im(\chi). \tag{27}$$

Substituting Equation (27) with Equation (24) into Equation (22) and setting their coefficients of the different powers of $\Im(\chi)$ to zero, it yields:

Solving them, we derive:

$$\Im^0(\chi): -3km\sigma\varepsilon_1 + 2ns\sigma\varepsilon_1 + 2m^3n\sigma^2\varepsilon_1 + 3m^2n\sigma^2\varepsilon_1^2 = 0,$$
$$\Im^2(\chi): -3km\varepsilon_1 - ns\varepsilon_1 + 8m^3n\sigma\varepsilon_1 + 6m^2n\sigma\varepsilon_1^2 = 0,$$
$$\Im^4(\chi): \ 6m^3n\varepsilon_1 + 3m^2n\varepsilon_1^2 = 0.$$

Case 1:

$$\varepsilon_0 = \varepsilon_0,\ \varepsilon_1 = \varepsilon_1,\ m = -\frac{\varepsilon_1}{2},\ n = n,\ k = k,\ s = -\frac{\varepsilon_1\left(3k + n\sigma\varepsilon_1^2\right)}{4n},\ \sigma = \sigma.$$

Thus, the TWS of Equation (1) can be obtained as:

$$\Pi(x,y,z,t) = \varepsilon_0 - \varepsilon_1\sqrt{-\sigma}\tan\text{h}\left[\sqrt{-\sigma}\left(-\frac{\varepsilon_1}{2}x + ny + kz - \frac{\varepsilon_1\left(3k + n\sigma\varepsilon_1^2\right)}{4n}t\right)\right], \sigma < 0. \tag{28}$$

$$\Pi(x,y,z,t) = \varepsilon_0 - \varepsilon_1\sqrt{-\sigma}\cot\text{h}\left[\sqrt{-\sigma}\left(-\frac{\varepsilon_1}{2}x + ny + kz - \frac{\varepsilon_1\left(3k + n\sigma\varepsilon_1^2\right)}{4n}t\right)\right], \sigma < 0. \tag{29}$$

$$\Pi(x,y,z,t) = \varepsilon_0 + \varepsilon_1\sqrt{\sigma}\tan\left[\sqrt{\sigma}\left(-\frac{\varepsilon_1}{2}x + ny + kz - \frac{\varepsilon_1\left(3k + n\sigma\varepsilon_1^2\right)}{4n}t\right)\right], \sigma > 0. \tag{30}$$

$$\Pi(x,y,z,t) = \varepsilon_0 - \varepsilon_1\sqrt{\sigma}\cot\left[\sqrt{\sigma}\left(-\frac{\alpha_1}{2}x + ny + kz - \frac{\varepsilon_1\left(3k + n\sigma\varepsilon_1^2\right)}{4n}t\right)\right],\ \sigma > 0. \tag{31}$$

Case 2:

$$\varepsilon_0 = \varepsilon_0,\ \varepsilon_1 = -2m,\ m = m,\ n = n,\ k = -\frac{2\left(2m^3n\sigma - ns\right)}{3m},\ s = s,\ \sigma = \sigma.$$

Thus, the TWS of Equation (1) can be obtained as:

$$\Pi(x,y,z,t) = \varepsilon_0 + 2m\sqrt{-\sigma}\tan\text{h}\left[\sqrt{-\sigma}\left(mx + ny - \frac{2\left(2m^3n\sigma - ns\right)}{3m}z + st\right)\right], \sigma < 0. \tag{32}$$

$$\Pi(x,y,z,t) = \alpha_0 + 2m\sqrt{-\sigma}\cot\text{h}\left[\sqrt{-\sigma}\left(mx + ny - \frac{2\left(2m^3n\sigma - ns\right)}{3m}z + st\right)\right], \sigma < 0. \tag{33}$$

$$\Pi(x,y,z,t) = \varepsilon_0 - 2m\sqrt{\sigma}\tan\left[\sqrt{\sigma}\left(mx + ny - \frac{2\left(2m^3n\sigma - ns\right)}{3m}z + st\right)\right], \sigma > 0. \tag{34}$$

$$\Pi(x,y,z,t) = \varepsilon_0 + 2m\sqrt{\sigma}\cot\left[\sqrt{\sigma}\left(mx + ny - \frac{2\left(2m^3n\sigma - ns\right)}{3m}z + st\right)\right],\ \sigma > 0. \tag{35}$$

4. The Physical Interpretations

The obtained solutions will be presented by the 3D plot and 2D contour in this section by taking the reasonable parameters.

By assigning the parameters as $k_2 = 1$, $k_9 = 2$, $u_1 = 1$, $u_2 = 1$, $u_3 = 1$, the multi-wave complexiton solution given by Equation (6) for the different time is illustrated in Figure 1 in the form of the 3D plot and 2D contour. Obviously, we can find there is a collision phenomenon between the singular periodic wave and the lump in the outline. As t increases, the waveform propagates in the negative direction of the x axis and y axis.

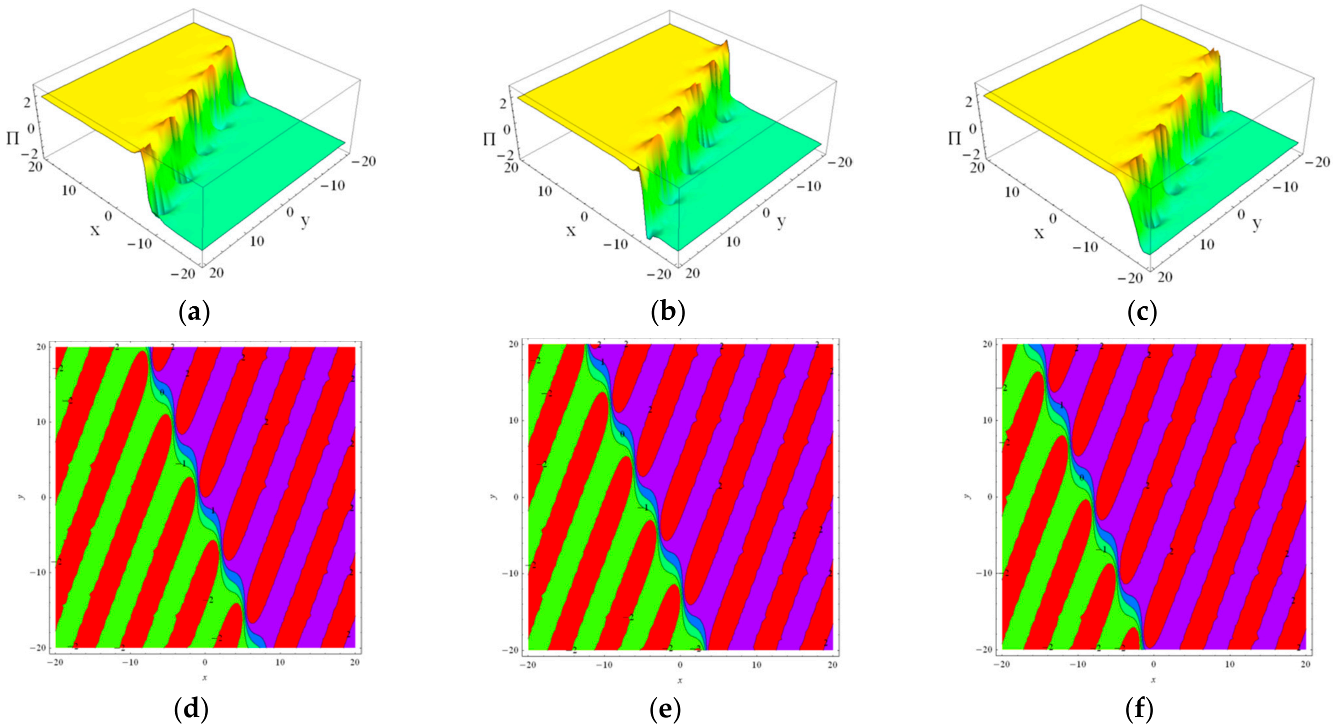


Figure 1. The graphical description of Equation (6) with $k_2 = 1$, $k_9 = 2$, $u_1 = 1$, $u_2 = 1$, $u_3 = 1$ at $z = 0$, (**a**,**d**) for $t = 0$, (**b**,**e**) for $t = 2$, (**c**,**f**) for $t = 4$.

We illustrate the dynamic behavior of Equation (12) by selecting $k_1 = 1$, $k_2 = 1$, $u_1 = 0.6$, $u_2 = 0.4$ in Figure 2. From this, collision phenomena between the breather waves and singular periodic waves are revealed. We can observe that the waveform travels along the negative direction of the x axis and positive direction of y axis.

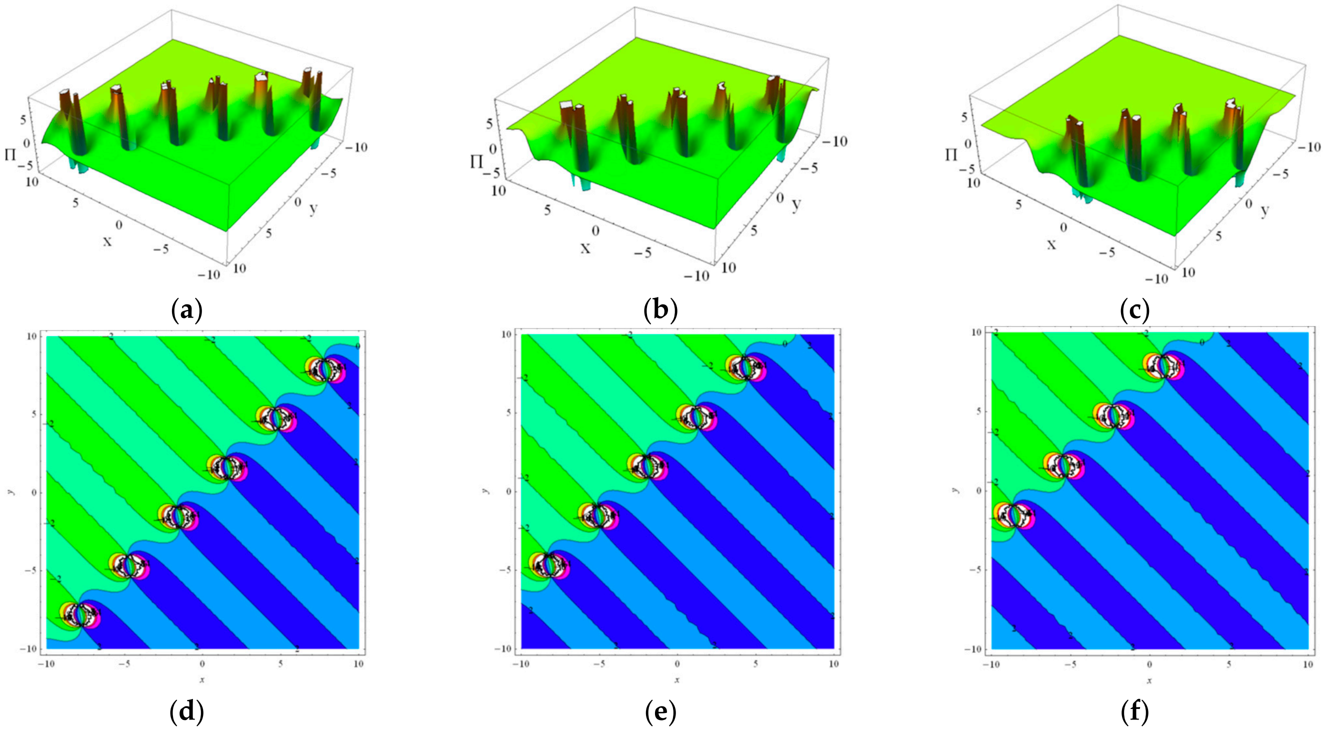


Figure 2. The graphical description of Equation (12) with $k_1 = 1$, $k_2 = 1$,$u_1 = 0.6$, $u_2 = 0.4$ at $z = 0$. (**a**,**d**) for $t = 0$, (**b**,**e**) for $t = 1$, (**c**,**f**) for $t = 2$.

Selecting $k_1 = 1, k_2 = 2, u_1 = 1, u_2 = 1, u_3 = 1, u_4 = 1$, we present the performance of Equation (15) in Figure 3. Here, it can be found the waveform propagates along the negative direction of the x axis and positive direction of y axis. Additionally, the outline of the wave can be explained as the interaction between lump solution and trigonometric function solution.

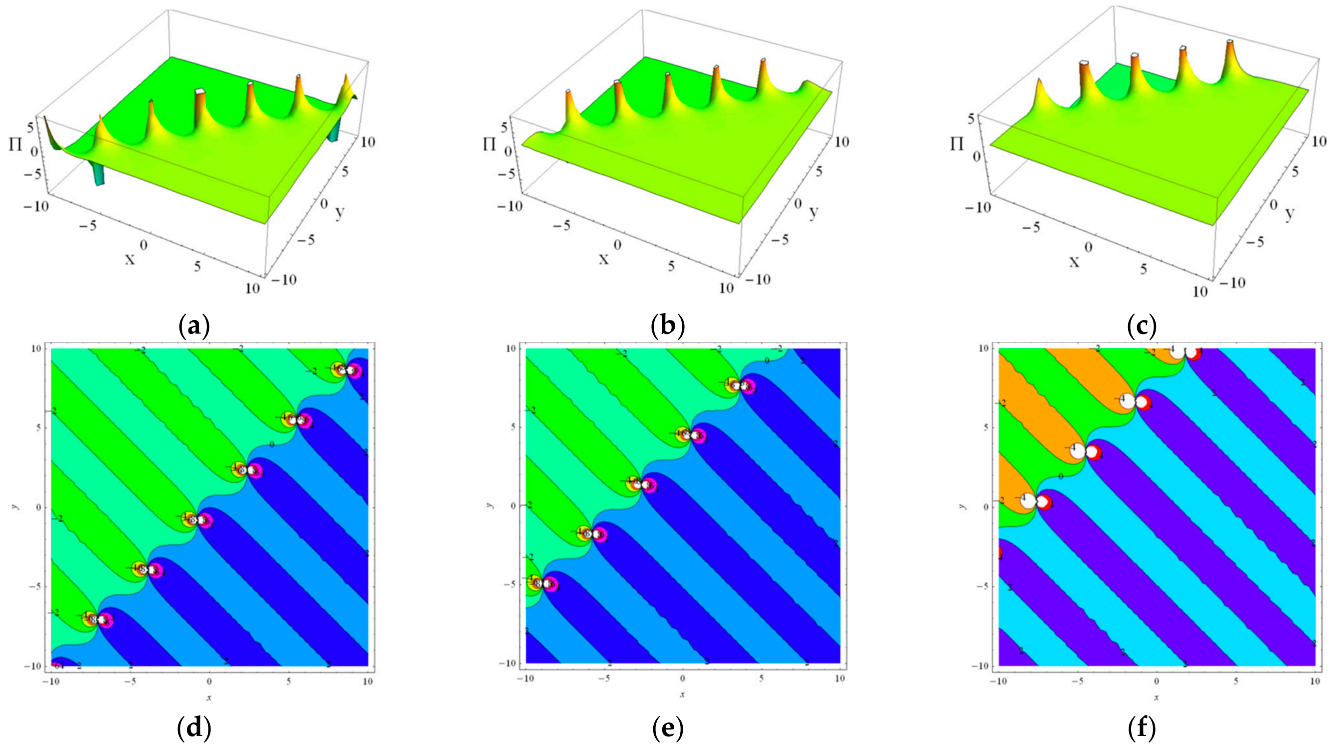


Figure 3. The graphical description of Equation (15) with $k_1 = 1, k_2 = 2, u_1 = 1, u_2 = 1, u_3 = 1$ at $z = 0$. (**a**,**d**) for $t = 0$, (**b**,**e**) for $t = 2$, (**c**,**f**) for $t = 4$.

By using the parameters as $\varepsilon_0 = 1, \varepsilon_1 = 1, n = 1, k = 1, \sigma = -1$, the dynamic characteristics of Equation (28) are revealed in Figure 4, where Figure 4a is the 3D plot, Figure 4b is the 2D contour and Figure 4c represents the 2D curve. In our observation, it is a dark wave. With the same parameters, Figure 5 illustrates the behaviors of Equations (3) and (10), which is a bright-dark wave.

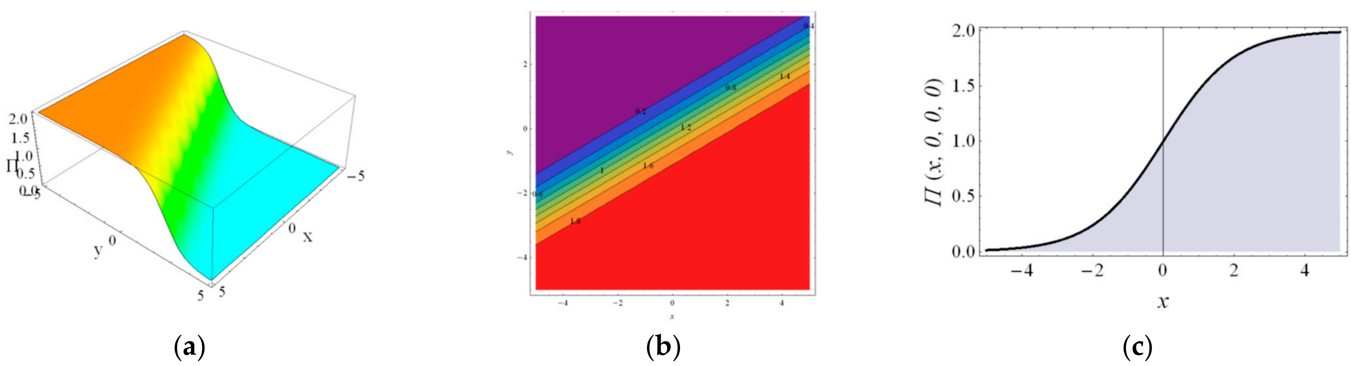
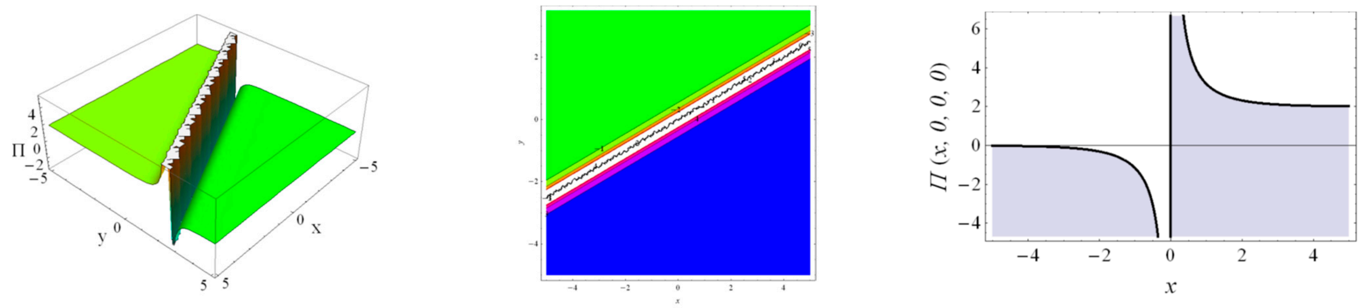


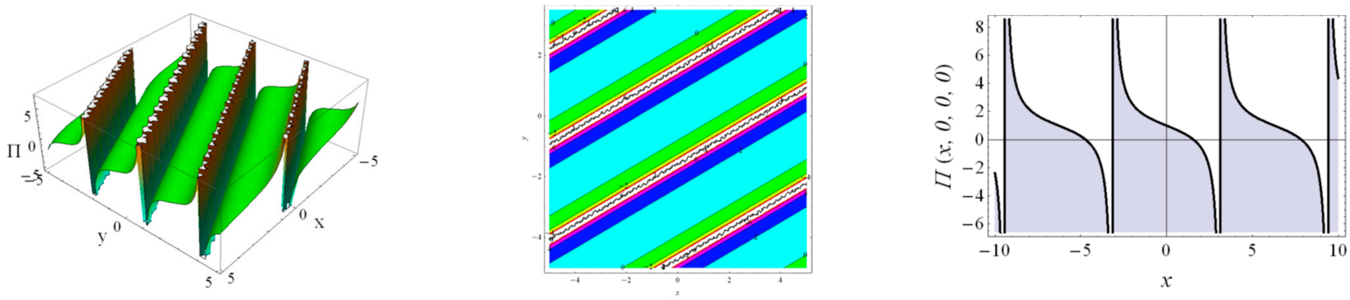
Figure 4. The graphical description of Equation (28) with the parameters as $\varepsilon_0 = 1, \varepsilon_1 = 1, n = 1, k = 1, \sigma = -1$. (**a**) for $z = 0, t = 0$, (**b**) for $z = 0, t = 0$, (**c**) for $y = 0, z = 0, t = 0$.



(**a**) (**b**) (**c**)

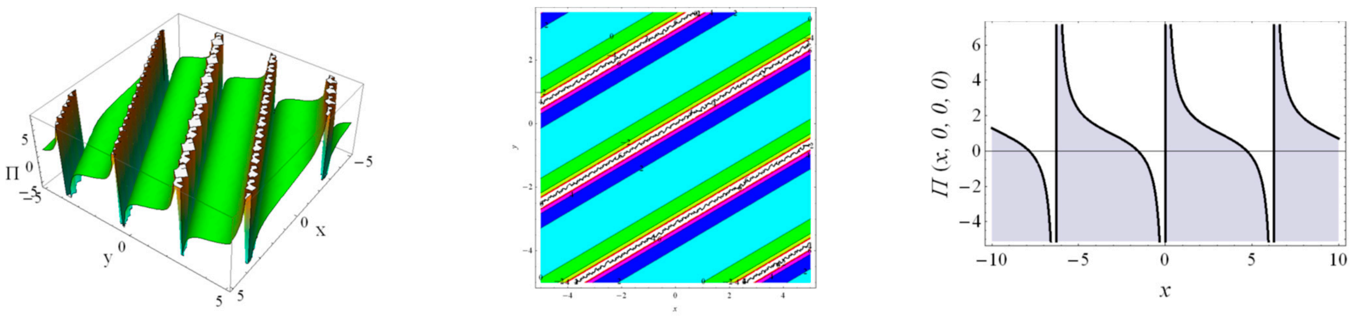
Figure 5. The graphical description of Equation (29) with the parameters as $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $n = 1$, $k = 1$, $\sigma = -1$. (**a**) for $z = 0$, $t = 0$, (**b**) for $z = 0$, $t = 0$, (**c**) for $y = 0$, $z = 0$, $t = 0$.

The performances of Equations (29) and (30) are presented in Figures 6 and 7, respectively with $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $n = 1$, $k = 1$, $\sigma = 1$. We find that the profiles are both singular periodic waves.



(**a**) (**b**) (**c**)

Figure 6. The graphical description of Equation (29) with the parameters as $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $n = 1$, $k = 1$, $\sigma = 1$. (**a**) for $z = 0$, $t = 0$, (**b**) for $z = 0$, $t = 0$, (**c**) for $y = 0$, $z = 0$, $t = 0$.



(**a**) (**b**) (**c**)

Figure 7. The graphical description of Equation (30) with the parameters as $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $n = 1$, $k = 1$, $\sigma = 1$. (**a**) for $z = 0$, $t = 0$, (**b**) for $z = 0$, $t = 0$, (**c**) for $y = 0$, $z = 0$, $t = 0$.

5. Conclusions and Future Recommendation

In this article, we obtained multi-wave complexiton solutions, multi-wave solutions and periodic lump solutions of the (3+1)-dimensional Jimbo-Miwa equation with the help of the Hirota bilinear method. Besides, we also construct its diverse travelling wave solutions like the dark, bright-dark and singular periodic wave solutions by applying the sub-equation method. The evolution phenomenon of these different solutions are described graphically. From these descriptions, the physical behavior and the interaction are presented. The obtained results in this work are all new and have not been reported elsewhere. Additionally, they show that the methods adopted are effective and direct, and can moreover be used to study the other PDEs arising in physics.

In recent years, the interest in fractal and fractional calculus [41–49] has intensified in different fields due to their strong ability to describe complex phenomena. Applying the fractal and fractional calculus to Equation (1) and obtaining the exact solutions will animate our future research.

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Conflicts of Interest: This research does not have any conflicts of interest.

Nomenclature

Multi-wave complexiton solutions	MWCS
Multi-wave solutions	MWS
Periodic lump solutions	PLS
Travelling wave solutions	TWS
The Jimbo-Miwa equation	JME
Travelling wave solutions	TWS

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