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A Novel Study of Fuzzy Bi-Ideals in Ordered Semirings

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Abstract: In this study, by generalizing the notion of fuzzy bi-ideals of ordered semirings, the notion of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals is established. We prove that $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals are fuzzy bi-ideals but that the converse is not true, and an example is provided to support this proof. A condition is given under which fuzzy bi-ideals of ordered semirings coincide with $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals. An equivalent condition and certain correspondences between bi-ideals and $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals are presented. Moreover, the (κ^*, κ)-lower part of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals is described and depicted in terms of several classes of ordered semirings. Furthermore, it is shown that the ordered semiring is bi-simple if and only if it is $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-simple.

Keywords: $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals; regular ordered semirings; intra-regular ordered semirings

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1. Introduction

The concept of an "ordered semiring" was first used by Gan and Jiang [1] in connection to a semiring with a compatible partial order relation. They also proposed the idea of ideals in ordered semirings. Good et al. [2] developed the concept of bi-ideals in semigroups. Following that, Lajos et al. [3] established bi-ideals in associative rings. Bi-ideals of ordered semirings were described and characterized in terms of regularity, and the relationship between bi-ideals and quasi-ideals was characterized by Palakawong et al. [4]. Senarat et al. [5] developed the terms-ordered k-bi-ideal, strong-prime-ordered k-bi-ideal, and prime-ordered k-bi-ideal in ordered semirings. By expanding on the idea of bi-ideals of ordered semirings, Davvaz et al. [6] introduced the concept of bi-hyperideals in ordered semi-hyperrings. The notions of (m, n)-bi-hyperideals and Prime (m, n)-bi-hyperideals were established and inter-related properties were considered by Omidi and Davvaz [7]. The characterization of ordered h-regular semirings was considered by Anjum et al. [8]. In [9], Patchakhieo and Pibaljommee characterized ordered k-regular semirings using ordered k-ideals. The ordered intra-k-regular semirings have been introduced and defined in different ways by Ayutthaya and Pibaljommee [10]. Omidi and Davvaz [7] considered the concepts of (m, n)-bi-hyperideals and Prime (m, n)-bi-hyperideals and established inter-related features. Anjum et al. [8] proposed characterizing ordered h-regular semirings. By using ordered k-ideals, Patchakhieo and Pibaljommee described ordered k-regular semirings in [9]. The ordered intra-k-regular semirings have been presented and characterized in various ways by Ayutthaya and Pibaljommee [10].

Fuzzy sets to semirings were initially discussed by Ahsan et al. in [11] and Kuroki [12] applied the idea to semigroups. Mandal [13] pioneered the study of ideals and interior ideals in ordered semirings, as well as their characterizations in the sense of regularity.

He developed the concepts of fuzzy bi-ideals and fuzzy quasi-ideals in ordered semirings in [14]. Gao et al. [15] presented semisimple fuzzy ordered semirings and weakly regular fuzzy ordered semirings in terms of different kinds of fuzzy ideals. Saba et al. [16] initiated the study of ordered semirings based on single-valued neutrosophic sets. Several characterizations of regular and intra-regular ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy generalized bi-ideals were presented by Jun et al. [17], who also proposed the idea of (α, β)-fuzzy generalized bi-ideal in ordered semigroups. Similar semiring concepts, such as $(\in, \in \vee q)$-fuzzy bi-ideals on semirings, were investigated by Hedayati [18]. Additionally, other ideas connected to our research in several domains have been examined in [19–25].

In this study, we describe a novel form of fuzzy ideal in ordered semirings. The concept of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal is presented. We show that any fuzzy bi-ideal is the $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal, but the converse assertion is invalid, and an example is shown. A criterion for an $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal to be a fuzzy bi-ideal is given. Furthermore, some correspondences between bi-ideal and $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal are included. Furthermore, regularly ordered semirings are described in terms of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals and their (κ^*, κ)-lower parts. The structure of the paper is as follows: Section 2 highlights some of the ideas and properties of ordered semirings, ideals, fuzzy subsets, and fuzzy subsemirings that are necessary to generate our key results. Section 3 focuses on the concept of the $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal of ordered semirings. Section 4 examines the (κ^*, κ)-lower part of the $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal. Section 5 contains instructions for some potential future research work.

2. Preliminaries

An *ordered semiring* $(\Upsilon, +, \cdot, \leq)$ is a semiring with compatible order relation "$\leq$", i.e., $\wp \leq \varrho \Rightarrow \wp\tau \leq \varrho\tau, \tau\wp \leq \tau\varrho$ and $\wp + \tau \leq \varrho + \tau, \tau + \wp \leq \tau + \varrho$, $\forall\, \wp, \varrho, \tau \in \Upsilon$.

If $\wp + \varrho = \varrho + \wp$, $\forall\, \wp, \varrho \in \Upsilon$, then Υ is said to be additively commutative. An element $0 \in \Upsilon$ is an absorbing zero if $0\wp = 0 = \wp 0$ and $\wp + 0 = \wp = 0 + \wp$, $\forall\, \wp \in \Upsilon$.

For $P \subseteq \Upsilon$, we define $(P] = \{\wp \in \Upsilon \mid \wp \leq \varrho \text{ for some } \wp \in P\}$. For $(\emptyset \neq)P, Q \subseteq \Upsilon$, PQ is defined as $\{\wp\varrho \mid \wp \in P \text{ and } \varrho \in Q\}$.

A subset $(\emptyset \neq)\Sigma$ of Υ is said to be a *sub-semiring* if $\Sigma\Sigma \subseteq \Sigma$ and $\Sigma + \Sigma \subseteq \Sigma$. Additionally, Σ refers to the *left (resp. right)* ideal of Υ if $\Sigma + \Sigma \subseteq \Sigma$ and $\Upsilon\Sigma \subseteq \Sigma$ (resp. $\Sigma\Upsilon \subseteq \Sigma$), and $(\Sigma] \subseteq \Sigma$. If it is both the left and right ideals of Υ, it is referred to as an ideal. A sub-semiring P of Υ is called a *bi-ideal* (in brief, *BI*) of Υ if $P\Upsilon P \subseteq P$ and $(P] \subseteq P$.

A mapping $\widetilde{\lambda}^f : \Upsilon \to [0,1]$ is said to be fuzzy set (in brief, *FS*) of Υ. For the *FSs* $\widetilde{\lambda}^f$ and $\widetilde{\pounds}^f$ of Υ, $\widetilde{\lambda}^f \cap \widetilde{\pounds}^f, \widetilde{\lambda}^f \cup \widetilde{\pounds}^f, \widetilde{\lambda}^f + \widetilde{\pounds}^f$ and $\widetilde{\lambda}^f \circ \widetilde{\pounds}^f$ are described as:

$$(\widetilde{\lambda}^f \cap \widetilde{\pounds}^f)(\wp) = \widetilde{\lambda}^f(\wp) \wedge \widetilde{\pounds}^f(\wp) = \min\{\widetilde{\lambda}^f(\wp), \widetilde{\pounds}^f(\wp)\},$$
$$(\widetilde{\lambda}^f \cup \widetilde{\pounds}^f)(\wp) = \widetilde{\lambda}^f(\wp) \vee \widetilde{\pounds}^f(\wp) = \max\{\widetilde{\lambda}^f(\wp), \widetilde{\pounds}^f(\wp)\},$$

$$(\widetilde{\lambda}^f + \widetilde{\pounds}^f)(\wp) = \begin{cases} \bigvee\limits_{\wp \leq \varrho + \varkappa} \widetilde{\lambda}^f(\varrho) \wedge \widetilde{\pounds}^f(\varkappa), \\ 0, \quad \text{if } \wp \text{ can not be written as } \wp \leq \varrho + \varkappa, \end{cases}$$

and

$$(\widetilde{\lambda}^f \circ \widetilde{\pounds}^f)(\wp) = \begin{cases} \bigvee\limits_{\wp \leq \varrho\varkappa} \widetilde{\lambda}^f(\varrho) \wedge \widetilde{\pounds}^f(\varkappa), & , \\ 0, \quad \text{if } \wp \text{ cannot be written as } \wp \leq \varrho\varkappa. & , \end{cases}$$

For $\Omega \subseteq \Upsilon$, the *characteristic function* χ^f_Ω is defined as:

$$\chi^f_\Omega(\wp) = \begin{cases} 1, & \text{if } \wp \in \Omega; \\ 0, & \text{if } \wp \notin \Omega. \end{cases}$$

Define $\preceq$ on the set $\mathcal{F}(\Upsilon)$ of all FSs of Υ by

$$\widetilde{\lambda}^f \preceq \widetilde{\pounds}^f \Leftrightarrow \widetilde{\lambda}^f(\wp) \leq \widetilde{\pounds}^f(\wp),\ \forall\, \wp \in \Upsilon.$$

If $\widetilde{\lambda}^f$, $\widetilde{\pounds}^f \in \mathcal{F}(\Upsilon)$ such that $\widetilde{\lambda}^f \preceq \widetilde{\pounds}^f$, then $\forall\ \widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$, $\widetilde{\lambda}^f \circ \widetilde{\lambda}^f \preceq \widetilde{\pounds}^f \circ \widetilde{\lambda}^f$ and $\widetilde{\lambda}^f \circ \widetilde{\lambda}^f \preceq \widetilde{\lambda}^f \circ \widetilde{\pounds}^f$. We represent by 1^f the *FS* of Υ given by $1^f : \Upsilon \to [0,1] | r \mapsto 1^f(r) = 1$.

Let $P, Q \subseteq \Upsilon$. Then $P \subseteq Q \Leftrightarrow \chi^f_P \preceq \chi^f_Q; \chi^f_P \cap \chi^f_Q = \chi^f_{P\cap Q}; \chi^f_P \circ \chi^f_Q = \chi_{(PQ]}$.

A FS $\widetilde{\lambda}^f$ is called a:

1. *Fuzzy subsemiring* of Υ if $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho)$ and $\widetilde{\lambda}^f(\wp+\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho)$, $\forall\ \wp, \varrho \in \Upsilon$.
2. *Fuzzy left (resp. right) ideal* (in brief, *FL(R)I*) of Υ if $\wp \leq \varrho \Rightarrow \widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho)$, $\widetilde{\lambda}^f(\wp+\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho)$ and $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\varrho)$ (resp. $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp)$), $\forall\ \wp, \varrho \in \Upsilon$.
3. *Fuzzy ideal* of Υ if $\widetilde{\lambda}^f$ is both fuzzy right and left ideals of Υ.
4. *Fuzzy bi-ideal* (in brief, *FBI*) if it is fuzzy subsemiring and $\wp \leq \varrho \Rightarrow \widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho)$ and $\widetilde{\lambda}^f(\wp t\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho)$, $\forall\ \wp, t, \varrho, \in \Upsilon$.

3. $(\in, \in \vee(\kappa^*, q_\kappa))$-Fuzzy Bi-Ideals of Ordered Semirings

In this section, the concept of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals of Υ is introduced.

Let $\wp \in \Upsilon$ and $\iota \in (0,1]$. The *ordered fuzzy point* (OFP) $\wp_\iota$ of Υ is defined by

$$\wp_\iota(\varkappa) = \begin{cases} \iota, & \text{if } \varkappa \in (\wp]; \\ 0, & \text{if } \varkappa \notin (\wp]. \end{cases}$$

For $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$, $\wp_\iota \in \widetilde{\lambda}^f$ represents for $\wp_\iota \subseteq \widetilde{\lambda}^f$. Thus $\wp_\iota \in \widetilde{\lambda}^f \Leftrightarrow \widetilde{\lambda}^f(\wp) \geq \iota$.

Definition 1. *An OFP $\wp_\iota$ of Υ is said to be (κ^*, q)-quasi-coincident with a FS $\widetilde{\lambda}^f$ of Υ for $\kappa^* \in (0,1]$, denoted as $\wp_\iota(\kappa^*, q)\widetilde{\lambda}^f$, and defined as:*

$$\widetilde{\lambda}^f(\wp) + \iota > \kappa^*.$$

For the OFP $\wp_\iota$, we define

(1) $\wp_\iota(\kappa^*, q_\kappa)\widetilde{\lambda}^f$, *if* $\widetilde{\lambda}^f(\wp) + \iota + \kappa > \kappa^*$;

(2) $\wp_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$, *if* $\wp_\iota \in \widetilde{\lambda}^f$ *or* $\wp_\iota(\kappa^*, q_\kappa)\widetilde{\lambda}^f$;

(3) $\wp_\iota \overline{\alpha} \widetilde{\lambda}^f$, *if* $\wp_\iota \alpha \widetilde{\lambda}^f$ *does not hold for* $\alpha \in \{(\kappa^*, q_\kappa), \in \vee(\kappa^*, q_\kappa)\}$;

for $1 \geq \kappa^* > \kappa \geq 0$.

Definition 2. *A FS $\widetilde{\lambda}^f$ of Υ is said to be an $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal (in brief, $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI) of Υ if:*

(1) $\wp \leq \varrho$, $\varrho_\iota \in \widetilde{\lambda}^f \Rightarrow \wp_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$,

(2) $\wp_\iota \in \widetilde{\lambda}^f$ *and* $\varrho_\theta \in \widetilde{\lambda}^f \Rightarrow (\wp+\varrho)_{\iota\wedge\theta} \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$,

(3) $\wp_\iota \in \widetilde{\lambda}^f$ *and* $\varrho_\theta \in \widetilde{\lambda}^f \Rightarrow (\wp\varrho)_{\iota\wedge\theta} \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$, *and*

(4) $t \in \Upsilon$, $\wp_\iota \in \widetilde{\lambda}^f$, $\varrho_\iota \in \lambda \Rightarrow (\wp t\varrho)_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$.

$\forall\ \iota, \theta \in (0,1]$ *and* $\wp, t, \varrho \in \Upsilon$.

Example 1. *On $\Upsilon = \{\wp_1, \wp_2, \wp_3\}$, define the opertaions and order relation as*

$+$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_1$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_2$	$\wp_2$	$\wp_2$	$\wp_2$
$\wp_3$	$\wp_3$	$\wp_2$	$\wp_2$

$\cdot$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$
$\wp_2$	$\wp_1$	$\wp_2$	$\wp_2$
$\wp_3$	$\wp_1$	$\wp_3$	$\wp_3$

$$\leq := \{(\wp_1, \wp_1), (\wp_2, \wp_2), (\wp_3, \wp_3), (\wp_1, \wp_2), (\wp_1, \wp_3)\}.$$

Then $(\Upsilon, +, \cdot, \leq)$ is an ordered semiring. Define an FS $\widetilde{\lambda}^f$ of Υ as

$$\widetilde{\lambda}^f(\varkappa) = \begin{cases} 0.5, & \textit{if } \varkappa = \wp_1; \\ 0.4, & \textit{if } \varkappa = \wp_2; \\ 0.3, & \textit{if } \varkappa = \wp_3. \end{cases}$$

$\widetilde{\lambda}^f$ *is the* $(\in, \in \vee(0.2, q_{0.6}))$*-FBI of* Υ *and can be easily verified.*

Lemma 1. *Each FBI of* Υ *is the* $(\in, \in \vee(\kappa^*, q_\kappa))$*-FBI of* Υ.

Proof. Straightforward. □

Remark 1. *In general, the converse of Lemma 1 does not hold. It is illustrated by the following example:*

Example 2. *Define operations and ordered relations on* $\Upsilon = \{\wp_1, \wp_2, \wp_3\}$ *as follows:*

$+$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_1$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_2$	$\wp_2$	$\wp_2$	$\wp_3$
$\wp_3$	$\wp_3$	$\wp_3$	$\wp_3$

$\cdot$	$\wp_1$	$\wp_2$	$\wp_3$
$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$
$\wp_2$	$\wp_1$	$\wp_2$	$\wp_2$
$\wp_3$	$\wp_1$	$\wp_2$	$\wp_2$

$$\leq := \{(\wp_1, \wp_1), (\wp_2, \wp_2), (b, \wp_3), (\wp_1, \wp_2), (\wp_2, \wp_3)\}.$$

Then, $(\Upsilon, +, \cdot, \leq)$ *is an ordered semiring. Define the FS* $\widetilde{\lambda}^f$ *of* Υ *as*

$$\widetilde{\lambda}^f(\varkappa) = \begin{cases} 0.6, & \text{if } \varkappa = \wp_1; \\ 0.5, & \text{if } \varkappa = \wp_2; \\ 0.7, & \text{if } \varkappa = \wp_3. \end{cases}$$

It can be easily verified that $\widetilde{\lambda}^f$ *is the* $(\in, \in \vee(0.9, q_{0.1}))$*-fuzzy bi-deal of* Υ *but not an FBI of* Υ *as follows:* $\wp_1 \leq \wp_3 \nRightarrow \widetilde{\lambda}^f(\wp_1) \geq \widetilde{\lambda}^f(\wp_3)$.

Theorem 1. *An FS* $\widetilde{\lambda}^f$ *is an* $(\in, \in \vee(\kappa^*, q_\kappa))$*-FBI of* $\Upsilon \Leftrightarrow$

(1) $\wp \leq \varrho \Rightarrow \widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$

(2) $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$,

(3) $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, *and*

(4) $\widetilde{\lambda}^f(\wp t \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$,

$\forall\, \wp, t, \varrho \in \Upsilon$.

Proof. $(\Rightarrow)$ Let $\wp, \varrho \in \Upsilon$ such that $\wp \leq \varrho$. If $\widetilde{\lambda}^f(\wp) < \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, then $\exists\, \iota \in (0,1]$ such that $\widetilde{\lambda}^f(\wp) < \iota \leq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. So $s_\iota \in \widetilde{\lambda}^f$, but $(\wp)_\iota \overline{\in \vee(\kappa^*, q_\kappa)} \widetilde{\lambda}^f$, which is a contradiction. Therefore $\widetilde{\lambda}^f(\wp) \geq \min\{\widetilde{\lambda}^f(\varrho), \frac{\kappa^*-\kappa}{2}\}$. Next, if $\widetilde{\lambda}^f(\wp + \varrho) < \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\wp, \varrho \in \Upsilon$, then $\widetilde{\lambda}^f(\wp + \varrho) < \iota \leq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\iota \in (0,1]$. Thus, $\wp_\iota, \varrho_\iota \in \widetilde{\lambda}^f$, but $(\wp + \varrho)_\iota \overline{\in \vee(\kappa^*, q_\kappa)} \widetilde{\lambda}^f$, which is a contradiction. Therefore, $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Similarly, $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, $\forall\, \wp, \varrho \in \Upsilon$. Again, if $\widetilde{\lambda}^f(\wp t \varrho) < \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\wp, t, \varrho \in \Upsilon$, then $\widetilde{\lambda}^f(\wp t \varrho) < \iota \leq \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2}$ for some $\iota \in (0,1]$. Thus, $\wp_\iota, \varrho_\iota \in \widetilde{\lambda}^f$, but $(\wp t \varrho)_\iota \overline{\in \vee(\kappa^*, q_\kappa)} \widetilde{\lambda}^f$, again a contradiction. Consequently, $\widetilde{\lambda}^f(\wp t \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2}$.

$(\Leftarrow)$ Take any $\wp, \varrho \in \Upsilon$ and $\iota, \theta \in (0,1]$ such that $\wp \leq \varrho$ and $\varrho_\theta \in \widetilde{\lambda}^f$. Then, $\widetilde{\lambda}^f(\varrho) \geq \iota$, and it follows that $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \frac{\kappa^*-\kappa}{2}$. If $\iota \leq \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp) \geq \iota$ implies $\wp_\iota \in \widetilde{\lambda}^f$. Again, if $\iota > \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp) \geq \frac{\kappa^*-\kappa}{2}$. Thus, $\widetilde{\lambda}^f(\wp) + \iota > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$, so $\wp_\iota (\kappa^*, q_\kappa) \widetilde{\lambda}^f$. Therefore, $\wp_\iota \in \vee(\kappa^*, q_\kappa) \widetilde{\lambda}^f$. Again, take any $\wp_\theta \in \widetilde{\lambda}^f$ and $\varrho_\theta \in \widetilde{\lambda}^f$. Then, $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. Therefore, $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \theta \wedge \frac{\kappa^*-\kappa}{2}$. Now, if $\iota \wedge \theta \leq \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp + \varrho) \geq \iota \wedge \theta$ implies $(\wp + \varrho)_{\iota \wedge \theta} \in \widetilde{\lambda}^f$. Again, if $\iota \wedge \theta > \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp + \varrho) \geq \frac{\kappa^*-\kappa}{2}$. Therefore, $\widetilde{\lambda}^f(\wp + \varrho) + \iota \wedge \theta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$ implies that $(\wp + \varrho)_{\iota \wedge \theta} (\kappa^*, q_\kappa) \widetilde{\lambda}^f$. Therefore, $(\wp + \varrho)_{\iota \wedge \theta} \in \vee(\kappa^*, q_\kappa) \widetilde{\lambda}^f$. Similarly, $(\wp\varrho)_{\iota \wedge \theta} \in \vee(\kappa^*, q_\kappa) \widetilde{\lambda}^f$ for any $\wp_\theta \in \widetilde{\lambda}^f$ and $\varrho_\theta \in \widetilde{\lambda}^f$. Further, take any $t \in \Upsilon$ and $\wp_\iota, \varrho_\iota \in \widetilde{\lambda}^f$, $\forall\, \iota \in (0,1]$. Then $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. Therefore, $\widetilde{\lambda}^f(\wp t \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \frac{\kappa^*-\kappa}{2}$. Now if $\iota \leq \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp t \varrho) \geq \iota$ implies $(\wp t \varrho)_\iota \in \widetilde{\lambda}^f$. If $\iota > \frac{\kappa^*-\kappa}{2}$, then $\widetilde{\lambda}^f(\wp t \varrho) \geq \frac{\kappa^*-\kappa}{2}$. Thus

$\widetilde{\lambda}^f(\wp t\varrho) + \iota > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$ i.e., $(\wp t\varrho)_\iota(\kappa^*, q_\kappa)\widetilde{\lambda}^f$. Therefore, $(\wp t\varrho)_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$, as required. □

Theorem 2. *If $\widetilde{\lambda}^f(\in \mathcal{F}(\Upsilon))$ is $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI of Υ with $\widetilde{\lambda}^f(\wp) < \frac{\kappa^*-\kappa}{2}, \forall\, \wp \in \Upsilon$. Then $\widetilde{\lambda}^f$ is an FBI of Υ.*

Proof. Suppose that $\wp, \varrho \in \Upsilon$ such that $\wp \leq \varrho$. Since $\widetilde{\lambda}^f$ is $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI*, $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. By hypothesis, $\widetilde{\lambda}^f(\varrho) < \frac{\kappa^*-\kappa}{2}$; thus, it implies $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho)$. Again, for any $\wp, \varrho \in \Upsilon$, we have

$$\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^* - \kappa}{2}$$

and

$$\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^* - \kappa}{2}.$$

Since $\widetilde{\lambda}^f(\varrho) < \frac{\kappa^*-\kappa}{2}$ and $\widetilde{\lambda}^f(\wp) < \frac{\kappa^*-\kappa}{2}$, so

$$\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho)$$

and

$$\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho).$$

Finally, take any $\wp, t, \varrho \in \Upsilon$. Since $\widetilde{\lambda}^f$ is $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI*, by Theorem 1 and the hypothesis

$$\widetilde{\lambda}^f(\wp t\varrho) \geq \widetilde{\lambda}^f(\wp), \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^* - \kappa}{2} = \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho),$$

as required. □

Theorem 3. *Let $(\emptyset \neq)\Omega \subseteq \Upsilon$. Then Ω is a BI of $\Upsilon \Leftrightarrow \chi^f_\Omega$, an $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI.*

Proof. Straightforward. □

Theorem 4. *An FS $\widetilde{\lambda}^f$ is the $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI of $\Upsilon \Leftrightarrow U(\widetilde{\lambda}^f; \iota)(\neq \emptyset)$ $(\iota \in (0, \frac{\kappa^*-\kappa}{2}])$, a BI of Υ.*

Proof. $(\Rightarrow)$ Let $\wp \in \Upsilon$ and $\varrho \in U(\widetilde{\lambda}^f; \iota)$ be such that $\wp \leq \varrho$. Then, $\widetilde{\lambda}^f(\varrho) \geq \iota$. By Theorem 1, $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \frac{\kappa^*-\kappa}{2} = \iota$. Therefore, $\wp \in U(\widetilde{\lambda}^f; \iota)$. Let $\wp, \varrho \in U(\widetilde{\lambda}^f; \iota)$, where $\iota \in (0, \frac{\kappa^*-\kappa}{2}]$. Then $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. By Theorem 1, $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \frac{\kappa^*-\kappa}{2} = \iota$. Therefore, $\wp + \varrho \in U(\widetilde{\lambda}^f; \iota)$. Similarly, $\wp\varrho \in U(\widetilde{\lambda}^f; \iota)$ for $\wp, \varrho \in U(\widetilde{\lambda}^f; \iota)$. Let $\wp, \varrho \in U(\widetilde{\lambda}^f; \iota)$ and $t \in \Upsilon$. Then, $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. So, by Theorem 1, $\widetilde{\lambda}^f(\wp t\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \iota \wedge \iota \wedge \frac{\kappa^*-\kappa}{2} = \iota$. Thus $\widetilde{\lambda}^f(\wp t\varrho) \geq \iota$. Therefore $\wp t\varrho \in U(\widetilde{\lambda}^f; \iota)$. Hence $U(\widetilde{\lambda}^f; \iota)$ is a *BI*.

$(\Leftarrow)$ Take any $\wp, \varrho \in \Upsilon$ with $\wp \leq \varrho$. If $\widetilde{\lambda}^f(\wp) < \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, then for some $\iota \in (0, \frac{\kappa^*-\kappa}{2}]$, $\widetilde{\lambda}^f(\wp) < \iota \leq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. So $\varrho \in U(\widetilde{\lambda}^f; \iota)$, but $\wp \notin U(\widetilde{\lambda}^f; \iota)$, which is a contradiction. Thus $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}, \forall\, \wp, \varrho \in \Upsilon$ with $\wp \leq \varrho$. Again, if $\widetilde{\lambda}^f(\wp + \varrho) < \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\wp, \varrho \in \Upsilon$, then $\widetilde{\lambda}^f(\wp + \varrho) < \iota \leq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\iota \in (0, \frac{\kappa^*-\kappa}{2}]$. Thus, $\wp, \varrho \in U(\widetilde{\lambda}^f; \iota)$, but $\wp + \varrho \notin U(\widetilde{\lambda}^f; \iota)$, a contradiction. Therefore, $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}, \forall\, \wp, \varrho \in \Upsilon$. Similarly, $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}, \forall\, \wp, \varrho \in \Upsilon$. Further, if $\widetilde{\lambda}^f(\wp t\varrho) < \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\wp, t, \varrho \in \Upsilon$. Then, $\exists\, \iota \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\widetilde{\lambda}^f(\wp t\varrho) < \iota \leq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$ implies $\wp_\iota, \varrho_\iota \in U(\widetilde{\lambda}^f; \iota)$, but $(\wp t\varrho)_\iota \notin U(\widetilde{\lambda}^f; \iota)$, again a contradiction. Therefore $\widetilde{\lambda}^f(\wp t\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}, \forall\, \wp, \varrho \in \Upsilon$, as required. □

Example 3. *Define the operations $(+,\ \cdot)$ and order relation $\leq$ on $\Upsilon = \{\wp_1, \wp_2, \wp_3, \tau\}$ in the following ways:*

$+$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_1$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_2$	$\wp_2$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_3$	$\wp_3$	$\wp_3$	$\wp_3$	$\wp_4$
$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$	$\wp_4$

$\cdot$	$\wp_1$	$\wp_2$	$\wp_3$	$\wp_4$
$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$	$\wp_1$
$\wp_2$	$\wp_1$	$\wp_2$	$\wp_2$	$\wp_2$
$\wp_3$	$\wp_1$	$\wp_2$	$\wp_2$	$\wp_2$
$\wp_4$	$\wp_1$	$\wp_2$	$\wp_2$	$\wp_2$

$$\leq := \{(\wp_1,\wp_1),(\wp_2,\wp_2),(\wp_3,\wp_3),(\wp_1,\wp_2),(\wp_2,\wp_3),(\wp_3,c)\}$$

Then, $(\Upsilon,+,\cdot,\leq)$ is an ordered semiring. Now define an FS $\widetilde{\lambda}^f$ of Υ as $\widetilde{\lambda}^f(\wp_1) = 0.5$, $\widetilde{\lambda}^f(\wp_2) = 0.4, \widetilde{\lambda}^f(\wp_3) = 0.1$ and $\widetilde{\lambda}^f(\tau) = 0.3$. Therefore,

$$U(\widetilde{\lambda}^f;\iota) = \begin{cases} \Upsilon, & \text{if } \wp_1 < \iota \leq 0.1; \\ \{\wp_1,\wp_2,\tau\}, & \text{if } 0.1 < \iota \leq 0.3; \\ \{\wp_1,\wp_2\}, & \text{if } 0.3 < \iota \leq 0.4; \\ \{\wp_1\}, & \text{if } 0.4 < \iota \leq 0.5; \\ \emptyset, & \text{if } 0.5 < \iota \leq 1. \end{cases}$$

By Theorem 4, $\widetilde{\lambda}^f$ is an $(\in,\in\vee(\kappa^*,q_\kappa))$-FBI of Υ as $U(\widetilde{\lambda}^f;\iota)$ is a BI of Υ, $\forall\, \iota \in (0, \frac{\kappa^*-\kappa}{2}]$, with $\kappa^* = 1$ and $\kappa = 0$.

Definition 3. *Let $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$. The set*

$$[\widetilde{\lambda}^f]_\iota \;=\; \{\wp \in \Upsilon \mid \wp_\iota \in \vee(\kappa^*,q_\kappa)\widetilde{\lambda}^f\},$$

where $\iota \in (0,1]$, is said to be an $(\in\vee(\kappa^*,q_\kappa))$-level subset of $\widetilde{\lambda}^f$.

Theorem 5. *Let $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$ such that $\wp \leq \varrho$ implies $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho)$. Then. $\widetilde{\lambda}^f$ is an $(\in,\in\vee(\kappa^*,q_\kappa))$-FBI of $\Upsilon \Leftrightarrow \forall\, \iota \in (0,1]$, the $(\in\vee(\kappa^*,q_\kappa))$-level subset $[\widetilde{\lambda}^f]_\iota$ of $\widetilde{\lambda}^f$ is a bi-deal of Υ.*

Proof. $(\Rightarrow)$ Take any $\wp \in \Upsilon$ and $\varrho \in [\widetilde{\lambda}^f]_\iota$ such that $\wp \leq \varrho$. As $\varrho \in [\widetilde{\lambda}^f]_\iota$, we have $\varrho_\iota \in \vee(\kappa^*,q_\kappa)\widetilde{\lambda}^f$ implies $\widetilde{\lambda}^f(\varrho) \geq \iota$ or $\widetilde{\lambda}^f(\varrho)+\iota+\kappa > \kappa^*$. By hypothesis, we have $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \geq \iota$ or $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \geq \kappa^*-\iota-\kappa$. Thus, $\wp_\iota \in \vee(\kappa^*,q_\kappa)\widetilde{\lambda}^f$. Therefore, $\wp \in [\widetilde{\lambda}^f]_\iota$. Next, take any $\wp,\varrho \in [\widetilde{\lambda}^f]_\iota$. Then, $\wp_\iota,\varrho_\iota \in \vee(\kappa^*,q_\kappa)\widetilde{\lambda}^f$; that is, $\widetilde{\lambda}^f(\wp) \geq \iota$ or $\widetilde{\lambda}^f(\wp)+\iota+\kappa > \kappa^*$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$ or $\widetilde{\lambda}^f(\varrho)+\iota+\kappa > \kappa^*$.
Case (i). Let $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. If $\iota > \frac{\kappa^*-\kappa}{2}$; then,

$$\begin{aligned} \widetilde{\lambda}^f(\wp+\varrho) &\geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \\ &\geq \iota \wedge \iota \wedge \frac{\kappa^*-\kappa}{2} \\ &= \frac{\kappa^*-\kappa}{2}, \end{aligned}$$

and, so, $(\wp+\varrho)_\iota(\kappa^*,q_\kappa)\widetilde{\lambda}^f$. If $\iota \leq \frac{\kappa^*-\kappa}{2}$, then

$$\begin{aligned} \widetilde{\lambda}^f(\wp+\varrho) &\geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \\ &\geq \iota \wedge \iota \wedge \frac{\kappa^*-\kappa}{2} = \iota, \end{aligned}$$

and so $(\wp+\varrho)_\iota \in \widetilde{\lambda}^f$. Hence, $(\wp+\varrho)_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$.
Case (ii). Let $\widetilde{\lambda}^f(\wp) \geq \iota$ and $\widetilde{\lambda}^f(\varrho)+\iota+\kappa > \kappa^*$. If $\iota > \frac{\kappa^*-\kappa}{2}$, then

$$\begin{aligned}\widetilde{\lambda}^f(\wp+\varrho) &\geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \\ &= \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \\ &> (\kappa^*-\iota-\kappa) \wedge \frac{\kappa^*-\kappa}{2} \\ &= \kappa^*-\iota-\kappa,\end{aligned}$$

that is, $\widetilde{\lambda}^f(\wp+\varrho)+\iota+\kappa > \kappa^*$, and thus $(\wp+\varrho)_\iota(\kappa^*, q_\kappa)\widetilde{\lambda}^f$. If $\iota \leq \frac{\kappa^*-\kappa}{2}$, then

$$\begin{aligned}\widetilde{\lambda}^f(\wp+\varrho) &\geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \\ &\geq \iota \wedge (\kappa^*-\iota-\kappa) \wedge \frac{\kappa^*-\kappa}{2} \\ &= \iota,\end{aligned}$$

and so $(\wp+\varrho)_\iota \in \widetilde{\lambda}^f$. Hence, $(\wp+\varrho)_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$.
Case (iii). Let $\widetilde{\lambda}^f(\wp)+\iota+\kappa > \kappa^*$ and $\widetilde{\lambda}^f(\varrho) \geq \iota$. Proof is analogous to case proof (ii).
Case (iv). Let $\widetilde{\lambda}^f(\wp)+\iota+\kappa > \kappa^*$ and $\widetilde{\lambda}^f(\varrho)+\iota+\kappa > \kappa^*$. Proof is analogous to previous two cases.

Thus for all cases, we have $(\wp+\varrho)_\iota \in \vee(\kappa^*, q_\kappa)\widetilde{\lambda}^f$, and thus $\wp+\varrho \in [\widetilde{\lambda}^f]_\iota$. Similarly, for any $t \in \mathrm{Y}$ and $\wp, \varrho \in [\widetilde{\lambda}^f]_\iota$, we have $\wp\varrho \in [\widetilde{\lambda}^f]_\iota$ and $\wp t \varrho \in [\widetilde{\lambda}^f]_\iota$. Hence, $[\widetilde{\lambda}^f]_\iota$ is a *BI* of Y.

($\Leftarrow$) Let $\widetilde{\lambda}^f(\wp) < \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, for some $\wp, \varrho \in \mathrm{Y}$. Then, $\iota \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\widetilde{\lambda}^f(\wp) < \iota \leq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Thus, it follows that $\varrho \in [\widetilde{\lambda}^f]_\iota$ but $\wp \notin [\widetilde{\lambda}^f]_\iota$, which is a contradiction, and hence $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Let $\widetilde{\lambda}^f(\wp+\varrho) < \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$ for some $\wp, \varrho \in \mathrm{Y}$. Then $\exists\, \iota \in (0, \frac{\kappa^*-\kappa}{2}]$ such that $\widetilde{\lambda}^f(\wp+\varrho) < \iota \leq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Thus, it follows that $\wp, \varrho \in [\widetilde{\lambda}^f]_\iota$ but $\wp+\varrho \notin [\widetilde{\lambda}^f]_\iota$, which is a contradiction. Therefore, $\widetilde{\lambda}^f(\wp+\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, $\forall\, \wp, \varrho \in \mathrm{Y}$. Similarly, $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, $\forall\, \wp, \varrho \in \mathrm{Y}$. Next, suppose that $\widetilde{\lambda}^f(\wp t \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$ for some $\wp, t, \varrho \in \mathrm{Y}$. It follows that $\wp, \varrho \in [\widetilde{\lambda}^f]_\iota$ but $\wp t \varrho \notin [\widetilde{\lambda}^f]_\iota$ which is again a contradiction. Thus $\widetilde{\lambda}^f(\wp t \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, as required. □

4. Lower Part of $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI*

The concept of the lower part of the $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y is defined and characterized.

Definition 4. *The (κ^*, κ)-lower part $\underline{\lambda^{f}{}^{\kappa^*}_{\kappa}}$ of $\widetilde{\lambda}^f$ is defined as*

$$\underline{\lambda^{f}{}^{\kappa^*}_{\kappa}}(\wp) \;=\; \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2},$$

$\forall\, \wp \in \mathrm{Y}$ *and* $1 \geq \kappa^* > \kappa \geq 0$.

The (κ^*, κ)-lower part $(\underline{\chi^{f}{}^{\kappa^*}_{\kappa}})_\Omega$ of the characteristic function χ^f_Ω is defined for $\Omega \subseteq R$ as

$$(\underline{\chi^{f}{}^{\kappa^*}_{\kappa}})_\Omega(\wp) = \begin{cases} \frac{\kappa^*-\kappa}{2}, & \text{if } \wp \in \Omega; \\ 0, & \text{if } \wp \notin \Omega. \end{cases}$$

Definition 5. *Let $\widetilde{\pounds}^f, \widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$. Define $\widetilde{\pounds}^f(\cap)_\kappa^{\kappa^*}\widetilde{\lambda}^f$, $\widetilde{\pounds}^f(\cup)_\kappa^{\kappa^*}\widetilde{\lambda}^f$, and $\widetilde{\pounds}^f(\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f$ as follows:*

$$(\widetilde{\pounds}^f(\cap)_\kappa^{\kappa^*}\widetilde{\lambda}^f)(\wp) = (\widetilde{\pounds}^f \cap \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2}$$
$$(\widetilde{\pounds}^f(\cup)_\kappa^{\kappa^*}\widetilde{\lambda}^f)(\wp) = (\widetilde{\pounds}^f \cup \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2}$$
$$(\widetilde{\pounds}^f(\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f)(\wp) = (\widetilde{\pounds}^f \circ \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2}$$
$$(\widetilde{\pounds}^f(+)_\kappa^{\kappa^*}\widetilde{\lambda}^f)(\wp) = (\widetilde{\pounds}^f + \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2}$$

$\forall\ \wp \in \Upsilon$ *and* $1 \geq \kappa^* > \kappa \geq 0$.

Lemma 2. $\widetilde{\pounds}^f, \widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$. *Then,*

(1) $(\underline{\pounds f_\kappa^{\kappa^*}})_\kappa^{\kappa^*} = \underline{\pounds f_\kappa^{\kappa^*}}$ *and* $\underline{\pounds f_\kappa^{\kappa^*}} \subseteq \widetilde{\pounds}^f$;
(2) *If* $\widetilde{\pounds}^f \subseteq \widetilde{\lambda}^f$, *and* $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$, *then* $\widetilde{\pounds}^f(\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f \subseteq \widetilde{\lambda}^f(\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f$ *and* $\widetilde{\lambda}^f(\circ)_\kappa^{\kappa^*}\widetilde{\pounds}^f \subseteq \widetilde{\lambda}^f\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f$;
(3) *If* $\widetilde{\pounds}^f \subseteq \widetilde{\lambda}^f$, *and* $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$, *then* $\widetilde{\pounds}^f(+)_\kappa^{\kappa^*}\widetilde{\lambda}^f \subseteq \widetilde{\lambda}^f(+)_\kappa^{\kappa^*}\widetilde{\lambda}^f$ *and* $\lambda(+)_\kappa^{\kappa^*}\widetilde{\pounds}^f \subseteq \lambda(+)_\kappa^{\kappa^*}\widetilde{\lambda}^f$;
(4) $\widetilde{\pounds}^f(\cap)_\kappa^{\kappa^*}\widetilde{\lambda}^f = \underline{\pounds f_\kappa^{\kappa^*}} \cap \underline{\lambda f_\kappa^{\kappa^*}}$;
(5) $\widetilde{\pounds}^f(\cup)_\kappa^{\kappa^*}\widetilde{\lambda}^f = \underline{\pounds f_\kappa^{\kappa^*}} \cup \underline{\lambda f_\kappa^{\kappa^*}}$;
(6) $\widetilde{\pounds}^f(\circ)_\kappa^{\kappa^*}\widetilde{\lambda}^f = \underline{\pounds f_\kappa^{\kappa^*}} \circ \underline{\lambda f_\kappa^{\kappa^*}}$;
(7) $\widetilde{\pounds}^f(+)_\kappa^{\kappa^*}\widetilde{\lambda}^f = \underline{\pounds f_\kappa^{\kappa^*}} + \underline{\lambda f_\kappa^{\kappa^*}}$.

Proof. Straightforward. □

Lemma 3. *Let* $\Sigma, \Omega \subseteq \Upsilon$. *Then,*

(1) $\chi_\Sigma(+)_\kappa^{\kappa^*}\chi_\Omega = (\underline{\chi_\kappa^{\kappa^*}})_{\Sigma+\Omega}$;
(2) $\chi_\Sigma(\cap)_\kappa^{\kappa^*}\chi_\Omega = (\underline{\chi_\kappa^{\kappa^*}})_{\Sigma\cap\Omega}$;
(3) $\chi_\Sigma(\cup)_\kappa^{\kappa^*}\chi_\Omega = (\underline{\chi_\kappa^{\kappa^*}})_{\Sigma\cup\Omega}$;
(4) $\chi_\Sigma(\circ)_\kappa^{\kappa^*}\chi_\Omega = (\underline{\chi_\kappa^{\kappa^*}})_{(\Sigma\Omega]}$.

Proof. Straightforward. □

Lemma 4. *If $\widetilde{\lambda}^f$ is the $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI of* Υ, *then* $\underline{\lambda f_\kappa^{\kappa^*}}$ *is an FBI of* Υ.

Proof. Let $\wp, \varrho \in \Upsilon$ be such that $\wp \leq \varrho$. Then, $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Thus, it implies $\widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2} \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, and, so, $(\underline{\lambda f_\kappa^{\kappa^*}})(\wp) \geq (\underline{\lambda f_\kappa^{\kappa^*}})(\varrho)$. Next suppose that $\wp, \varrho \in \Upsilon$. Since $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Υ $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. It follows that $\widetilde{\lambda}^f(\wp + \varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \wedge \frac{\kappa^*-\kappa}{2}$, and hence, $(\underline{\lambda f_\kappa^{\kappa^*}})(\wp + \varrho) \geq (\underline{\lambda f_\kappa^{\kappa^*}})(\wp) \wedge (\underline{\lambda f_\kappa^{\kappa^*}})(\varrho)$. Similarly, $(\underline{\lambda f_\kappa^{\kappa^*}})(\wp\varrho) \geq (\underline{\lambda f_\kappa^{\kappa^*}})(\wp) \wedge (\underline{\lambda f_\kappa^{\kappa^*}})(\varrho)$, $\forall\ \wp, \varrho \in \Upsilon$. Let $\wp, t, \varrho \in \Upsilon$; we have $\widetilde{\lambda}^f(\wp t\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Then $\widetilde{\lambda}^f(\wp\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$, and so $(\underline{\lambda f_\kappa^{\kappa^*}})(\wp t\varrho) \geq (\underline{\lambda f_\kappa^{\kappa^*}})(\wp) \wedge (\underline{\lambda f_\kappa^{\kappa^*}})(\varrho)$. Therefore, $\underline{\lambda f_\kappa^{\kappa^*}}$ is an *FBI* of Υ. □

Lemma 5. *Let* $(\emptyset \neq)\Omega \subseteq S$. *Then,* Ω *is a BI of* $\Upsilon \Leftrightarrow (\underline{\chi f_\kappa^{\kappa^*}})_\Omega$, *the* $(\in, \in \vee(\kappa^*, q_\kappa))$*-FBI of* Υ.

Proof. Let $\wp, \varrho \in \Upsilon$ and $\iota, \theta \in (0, 1]$ be such that $\wp_\iota \in (\underline{\chi f_\kappa^{\kappa^*}})_\Omega$ and $\varrho_\theta \in (\underline{\chi f_\kappa^{\kappa^*}})_\Omega$. Then, $(\underline{\chi f_\kappa^{\kappa^*}})_\Omega(\wp) \geq \iota > 0$ and $(\underline{\chi f_\kappa^{\kappa^*}})_\Omega(\varrho) \geq \theta > 0$. Therefore, $\wp, \varrho \in \Omega$. As Ω is a *BI* of Υ, $\wp + \varrho \in \Omega$. Thus $(\underline{\chi f_\kappa^{\kappa^*}})_\Omega(\wp + \varrho) = \frac{\kappa^*-\kappa}{2}$. If $\omega \wedge \theta \leq \frac{\kappa^*-\kappa}{2}$, then $(\underline{\chi f_\kappa^{\kappa^*}})_\Omega(\wp + \varrho) \geq \omega$, so we have $(\wp + \varrho)_{\iota\wedge\theta} \in (\underline{\chi f_\kappa^{\kappa^*}})_\Omega$. If $\iota \wedge \theta > \frac{\kappa^*-\kappa}{2}$, then $(\underline{\chi f_\kappa^{\kappa^*}})_\Omega(\wp + \varrho) + \iota \wedge \theta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} =$

$\kappa^* - \kappa$. So $(\wp + \varrho)_{\iota\wedge\theta}(\kappa^*, q_\kappa)(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Similarly, $\wp_\iota \in (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$ and $\varrho_\theta \in (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$ imply $(\wp\varrho)_{\iota\wedge\theta}(\kappa^*, q_\kappa)(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Therefore, $(\wp + \varrho)_{\iota\wedge\theta} \in \vee(\kappa^*, q_\kappa)(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Let $\wp, \varrho, t \in$ Y and $\iota \in (0, 1]$ be such that $\wp_\iota, \varrho_\theta \in (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Then, $\wp, \varrho \in \Omega$, $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) \geq \iota$, $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) \geq \theta$. Since Ω is a *BI* of Y, we have $\wp t\varrho \in \Omega$. Thus, $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp t\varrho) \geq \frac{\kappa^*-\kappa}{2}$. If $\iota \wedge \theta \leq \frac{\kappa^*-\kappa}{2}$, then $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp t\varrho) \geq \iota \wedge \theta$. Therefore $(\wp t\varrho)_{\iota\wedge\theta} \in (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Again, if $\iota \wedge \theta > \frac{\kappa^*-\kappa}{2}$, then $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp t\varrho) + \iota \wedge \theta > \frac{\kappa^*-\kappa}{2} + \frac{\kappa^*-\kappa}{2} = \kappa^* - \kappa$. So $(\wp t\varrho)_{\iota\wedge\theta}(\kappa^*, q_\kappa)(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$. Thus, $(\wp t\varrho)_{\iota\wedge\theta} \in \vee(\kappa^*, q_\kappa)(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$, as required.

Let $\wp \in$ Y and $\varrho \in \Omega$ such that $\wp \leq \varrho$. Then $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) = \frac{\kappa^*-\kappa}{2}$. Since $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y, and $\wp \leq \varrho$, we have $(\underline{f^{\kappa^*}_{\kappa}})_\Omega(\wp) \geq (\underline{f^{\kappa^*}_{\kappa}})_\Omega(\varrho) \wedge \frac{\kappa^*-\kappa}{2} = \frac{\kappa^*-\kappa}{2}$. Thus, $(\underline{f^{\kappa^*}_{\kappa}})_\Omega(\wp) = \frac{\kappa^*-\kappa}{2}$ and so $\wp \in \Omega$. Let $\wp, \varrho \in \Omega$. Then, $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) = \frac{\kappa^*-\kappa}{2}$ and $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) = \frac{\kappa^*-\kappa}{2}$. Since $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y, we have $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp + \varrho) \geq (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) \wedge (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) \wedge \frac{\kappa^*-\kappa}{2} = \frac{\kappa^*-\kappa}{2}$ and also $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp\varrho) \geq (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) \wedge (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) \wedge \frac{\kappa^*-\kappa}{2} = \frac{\kappa^*-\kappa}{2}$. Thus, it implies $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp + \varrho) = \frac{\kappa^*-\kappa}{2}$ and $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp\varrho) = \frac{\kappa^*-\kappa}{2}$. Therefore, $\wp + \varrho, \wp\varrho \in \Omega$. Let $\wp, \varrho \in \Omega$ and $t \in$ Y. Then $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) = \frac{\kappa^*-\kappa}{2}$ and $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) = \frac{\kappa^*-\kappa}{2}$. Now, $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp t\varrho) \geq (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp) \wedge (\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\varrho) \wedge \frac{\kappa^*-\kappa}{2} = \frac{\kappa^*-\kappa}{2}$. Hence $(\underline{\chi^{f^{\kappa^*}_{\kappa}}})_\Omega(\wp t\varrho) = \frac{\kappa^*-\kappa}{2}$. Therefore $\wp t\varrho \in \Omega$. Hence, Ω is a *BI* of Y. $\square$

Theorem 6. *Let $\widetilde{\lambda}^f \in \mathcal{F}(\mathrm{Y})$. Then $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI of* Y $\Leftrightarrow$

(1) $\widetilde{\lambda}^f\ (+)^{k^*}_k \widetilde{\lambda}^f \preceq \underline{\lambda^{f^{\kappa^*}_{\kappa}}}$,

(2) $\widetilde{\lambda}^f\ (\circ)^{k^*}_k \widetilde{\lambda}^f \preceq \underline{\lambda^{f^{\kappa^*}_{\kappa}}}$,

(3) $\widetilde{\lambda}^f (\circ)^{\kappa^*}_\kappa 1^f\ (\circ)^{k^*}_k \widetilde{\lambda}^f \preceq \underline{\lambda^{f^{\kappa^*}_{\kappa}}}$, *and*

(4) $(\forall\ \wp, \varrho \in \mathrm{Y})\ \wp \leq \varrho \Rightarrow \widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$.

Proof. ($\Rightarrow$) Suppose that $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y. If $\widetilde{\lambda}^f\ (+)^{k^*}_k \widetilde{\lambda}^f = 0$, then $\widetilde{\lambda}^f\ (+)^{k^*}_k \widetilde{\lambda}^f \preceq \widetilde{\lambda}^f$. Suppose that $\widetilde{\lambda}^f\ (+)^{k^*}_k \widetilde{\lambda}^f \neq 0$. Then, we have

$$\begin{aligned}
(\widetilde{\lambda}^f\ (+)^{k^*}_k \widetilde{\lambda}^f)(\wp) &= (\widetilde{\lambda}^f\ + \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \bigvee_{\wp \leq \nu + \tau} \{\widetilde{\lambda}^f(\nu) \wedge \widetilde{\lambda}^f(\tau)\} \wedge \frac{\kappa^* - \kappa}{2} \\
&\leq \bigvee_{\wp \leq \nu + \tau} \widetilde{\lambda}^f(\nu + \tau) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \underline{\lambda^{f^{\kappa^*}_{\kappa}}}(\wp).
\end{aligned}$$

Thus, $\widetilde{\lambda}^f\,(+)^{\kappa^*}_{k}\widetilde{\lambda}^f \preceq \underline{\lambda f^{\kappa^*}_{\kappa}}$. Similarly, $\widetilde{\lambda}^f\,(\circ)^{\kappa^*}_{k}\widetilde{\lambda}^f \preceq \underline{\lambda f^{\kappa^*}_{\kappa}}$. Again, $(\widetilde{\lambda}^f(\circ)^{\kappa^*}_{\kappa}1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(\wp) = 0$, then $\widetilde{\lambda}^f(\circ)^{\kappa^*}_{\kappa}1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f \preceq \underline{\lambda f^{\kappa^*}_{\kappa}}$. Suppose that $(\widetilde{\lambda}^f(\circ)^{\kappa^*}_{\kappa}1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(\wp) \neq 0$. Then, we have

$$
\begin{aligned}
&(\widetilde{\lambda}^f(\circ)^{\kappa^*}_{\kappa}1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(\wp)\\
&= (\widetilde{\lambda}^f \circ 1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^*-\kappa}{2}\\
&= \bigvee_{\wp\leq yz}\left\{\widetilde{\lambda}^f(y) \wedge \left\{\left\{\bigvee_{z\leq \nu\tau}\{1^f(\nu)\wedge\widetilde{\lambda}^f(\tau)\right\} \wedge \frac{\kappa^*-\kappa}{2}\right\}\right\} \wedge \frac{\kappa^*-\kappa}{2}\\
&= \bigvee_{\wp\leq yz}\bigvee_{z\leq \nu\tau}\left\{\widetilde{\lambda}^f(y)\wedge\widetilde{\lambda}^f(\tau)\wedge\frac{\kappa^*-\kappa}{2}\right\}\wedge\frac{\kappa^*-\kappa}{2}\\
&\leq \bigvee_{\wp\leq yz}\bigvee_{z\leq \nu\tau}\widetilde{\lambda}^f(y\nu\tau)\wedge\frac{\kappa^*-\kappa}{2}\\
&\leq \widetilde{\lambda}^f(\wp)\wedge\frac{\kappa^*-\kappa}{2}\\
&= \underline{\lambda f^{\kappa^*}_{\kappa}}(\wp).
\end{aligned}
$$

Therefore, $\widetilde{\lambda}^f(\circ)^{\kappa^*}_{\kappa}1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f \preceq \underline{\lambda f^{\kappa^*}_{\kappa}}$.

($\Leftarrow$) Let $\wp, \varrho \in \Upsilon$. Then, by hypothesis, we have

$$
\begin{aligned}
\widetilde{\lambda}^f(\wp+\varrho) &\geq \underline{\lambda f^{\kappa^*}_{\kappa}}(\wp+\varrho)\\
&\geq (\widetilde{\lambda}^f\,(+)^{\kappa^*}_{k}\widetilde{\lambda}^f)(\wp+\varrho)\\
&= \left\{\bigvee_{\wp+\varrho\leq \nu+\tau}\{\widetilde{\lambda}^f(\nu)\wedge\widetilde{\lambda}^f(\tau)\right\}\wedge\frac{\kappa^*-\kappa}{2}\\
&\geq \widetilde{\lambda}^f(\wp)\wedge\widetilde{\lambda}^f(\varrho)\wedge\frac{\kappa^*-\kappa}{2}.
\end{aligned}
$$

Similarly, by hypothesis, $\widetilde{\lambda}^f(\wp\varrho) \geq \widetilde{\lambda}^f(\wp)\wedge\widetilde{\lambda}^f(\varrho)\wedge\frac{\kappa^*-\kappa}{2}$.

We also have

$$
\begin{aligned}
\widetilde{\lambda}^f(\wp ts) &\geq \underline{\lambda f^{\kappa^*}_{\kappa}}(\wp ts)\\
&= (\widetilde{\lambda}^f \circ 1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(\wp ts)\wedge\frac{\kappa^*-\kappa}{2}\\
&= \left\{\bigvee_{\wp t\varrho\leq pq}\widetilde{\lambda}^f(p)\wedge(1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(q)\right\}\wedge\frac{\kappa^*-\kappa}{2}\\
&\geq \widetilde{\lambda}^f(\wp)\wedge(1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f)(t\varrho)\\
&= \widetilde{\lambda}^f(\wp)\wedge\left\{\bigvee_{(u,v)\in A_{rs}}1^f(u)\wedge\widetilde{\lambda}^f(v)\right\}\wedge\frac{\kappa^*-\kappa}{2}\\
&\geq \widetilde{\lambda}^f(\wp)\wedge 1^f(t)\wedge\widetilde{\lambda}^f(\wp)\wedge\frac{\kappa^*-\kappa}{2}\\
&= \widetilde{\lambda}^f(\wp)\wedge\widetilde{\lambda}^f(\varrho)\wedge\frac{\kappa^*-\kappa}{2},
\end{aligned}
$$

as required. □

Theorem 7. *The following statements are equivalent in* Υ*:*

(1) Υ *is regular.*

(2) $\lambda f^{\kappa^*}_{\kappa} \preceq \widetilde{\lambda}^f \circ 1^f(\circ)^{\kappa^*}_{\kappa}\widetilde{\lambda}^f$ *for any* $(\in, \in\vee(\kappa^*, q_\kappa))$*-FBI of* Υ*.*

Proof. Assume that $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y. If $\wp \in$ Y, then, as Y is regular, $\exists$ $t \in$ Y such that $\wp \leq \wp t \wp$. Now, we have

$$\begin{aligned}
(\widetilde{\lambda}^f \circ 1^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f)(\wp) &= (\widetilde{\lambda}^f \circ 1^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \left\{ \bigvee_{\wp \leq pq} \widetilde{\lambda}^f(p) \wedge (1^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f)(q) \right\} \wedge \frac{\kappa^* - \kappa}{2} \\
&\geq \widetilde{\lambda}^f(\wp) \wedge (1^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f)(t\wp) \\
&= \widetilde{\lambda}^f(\wp) \wedge \left\{ \bigvee_{t\wp \leq uv} 1^f(u) \wedge \widetilde{\lambda}^f(v) \right\} \wedge \frac{\kappa^* - \kappa}{2} \\
&\geq \widetilde{\lambda}^f(\wp) \wedge 1^f(t) \wedge \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^* - \kappa}{2}.
\end{aligned}$$

Thus, $\lambda {f_\kappa^{\kappa^*}} \preceq \lambda^f \circ 1^f (\circ)_\kappa^{\kappa^*} \lambda^f$.

(2) $\Rightarrow$ (1). Let B be a *BI* of Y. Then, by Lemma 5, $(\underline{\chi {f_\kappa^{\kappa^*}}})_B$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y. Thus, by hypothesis, we have

$$(\underline{\chi {f_\kappa^{\kappa^*}}})_B \subseteq \chi_B (\circ)_k^{k^*} \chi_I (\circ)_k^{k^*} \chi_B = (\underline{\chi {f_\kappa^{\kappa^*}}})_{(BIB]} \subseteq (\underline{\chi {f_\kappa^{\kappa^*}}})_{(\sum BIB]}.$$

So $B \subseteq (\sum BRB]$. Since B is *BI*, so $(\sum BRB] \subseteq B$. Thus $B = (\sum BRB]$. Hence, by ([9] Lemma 2.2), Y is regular. $\square$

Theorem 8. *The following statements are equivalent in* Y*:*

(1) Y *is regular and intra-regular.*

(2) $\lambda {f_\kappa^{\kappa^*}} = \widetilde{\lambda}^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f$ *for any* $(\in, \in \vee(\kappa^*, q_\kappa))$*-FBI of* Y.

Proof. ($\Rightarrow$) Suppose that $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y. As Y is regular and intra-regular, $a \leq axa$ and $a \leq ya^2z$. Therefore, $a \leq (axya)(ayxa)$. We have

$$\begin{aligned}
(\widetilde{\lambda}^f\ (\circ)_k^{k^*} \widetilde{\lambda}^f)(\wp) &= (\widetilde{\lambda}^f\ + \widetilde{\lambda}^f)(\wp) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \bigvee_{\wp \leq pq} \{\widetilde{\lambda}^f(p) \wedge \widetilde{\lambda}^f(q)\} \wedge \frac{\kappa^* - \kappa}{2} \\
&\geq \widetilde{\lambda}^f(axya) \wedge \widetilde{\lambda}^f(ayxa) \wedge \frac{\kappa^* - \kappa}{2} \\
&\geq \lambda^f(a) \wedge \frac{\kappa^* - \kappa}{2} \\
&= \underline{\lambda {f_\kappa^{\kappa^*}}}(a).
\end{aligned}$$

Thus $\underline{\lambda {f_\kappa^{\kappa^*}}} \preceq \widetilde{\lambda}^f\ (\circ)_k^{k^*} \widetilde{\lambda}^f$. Since $\widetilde{\lambda}^f$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI*, so $\widetilde{\lambda}^f\ (\circ)_k^{k^*} \widetilde{\lambda}^f \preceq \underline{\lambda {f_\kappa^{\kappa^*}}}$. Hence $\lambda {f_\kappa^{\kappa^*}} = \widetilde{\lambda}^f (\circ)_\kappa^{\kappa^*} \widetilde{\lambda}^f$.

(2) $\Rightarrow$ (1). Let B be a *BI* of Y. Then, by Lemma 5, $(\underline{\chi {f_\kappa^{\kappa^*}}})_B$ is an $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Y. Thus by hypothesis, we have

$$(\underline{\chi {f_k^{k^*}}})_B \subseteq \chi^f{}_B (\circ)_k^{k^*} \chi^f{}_B = (\underline{\chi {f_k^{k^*}}})_{(BB]} \subseteq (\underline{\chi {f_k^{k^*}}})_{(\sum BB]}.$$

Therefore, $B \subseteq (\sum BB]$. Since B is *BI*, so $(\sum BB] \subseteq B$. Thus $B = (\sum BB]$. Hence, by ([9]Theorem 3.12), Y is regular. $\square$

Definition 6. *Let $t \in \Upsilon$ and $\widetilde{\lambda}^f \in \mathcal{F}(\Upsilon)$. Define the following $\mathcal{I}_t$ of Υ as*

$$\mathcal{I}_t = \{\wp \in \Upsilon | \widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^* - \kappa}{2}\}.$$

Lemma 6. *Let $\widetilde{\lambda}^f$ be the $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI of Υ. Then $\mathcal{I}_t$ ($\forall\ t \in \Upsilon$) is the BI of Υ.*

Proof. Let $t \in \Upsilon$. As $t \in \mathcal{I}_t$, we have $\mathcal{I}_t \neq \emptyset$. Take any $\wp, \varrho \in \mathcal{I}_t$. Then, $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(t) \wedge \frac{\kappa^*-\kappa}{2}$ and $\widetilde{\lambda}^f(\varrho) \geq \widetilde{\lambda}^f(t) \wedge \frac{\kappa^*-\kappa}{2}$. Since $\widetilde{\lambda}^f$ is the $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Υ, $\widetilde{\lambda}^f(\wp + \varrho) \geq \widetilde{\lambda}^f(r) \wedge \widetilde{\lambda}^f(y) \wedge \frac{\kappa^*-\kappa}{2} \geq \frac{\kappa^*-\kappa}{2}$, so $\wp + \varrho \in \mathcal{I}_t$. By a similar argument, $\wp\varrho \in \mathcal{I}_t$.

Next, take any $\tau \in \Upsilon$ and $\wp, \varrho \in \mathcal{I}_t$. Then $\widetilde{\lambda}^f(\wp) \geq \widetilde{\lambda}^f(t) \wedge \frac{\kappa^*-\kappa}{2}$ and $\widetilde{\lambda}^f(\varrho) \geq \widetilde{\lambda}^f(t) \wedge \frac{\kappa^*-\kappa}{2}$. By hypothesis, $\widetilde{\lambda}^f(\wp\tau\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2}$. Therefore, $\widetilde{\lambda}^f(\wp\tau\varrho) \geq \widetilde{\lambda}^f(a) \wedge \frac{\kappa^*-\kappa}{2}$. Thus $\wp\tau\varrho \in \mathcal{I}_t$. Additionally, for any $\wp \in \Upsilon$ and $\varrho \in \mathcal{I}_t$ such that $\wp \leq \varrho$, we have $\wp \in \mathcal{I}_t$. Hence, $\mathcal{I}_t$ is a *BI* of Υ. □

Definition 7. *An ordered semiring Υ is called $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-simple if every $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI is constant. That is, $\forall\ \wp, \varrho \in \Upsilon$; we have $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\wp) = \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\varrho)$, for each $(\in, \in \vee(\kappa^*, q_\kappa))$-FBI $\widetilde{\lambda}^f$ of Υ.*

Theorem 9. *The ordered semiring Υ is bi-simple $\Leftrightarrow$ it is $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-simple.*

Proof. ($\Rightarrow$) Let $\widetilde{\lambda}^f$ be the $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Υ and $\wp, \varrho \in \Upsilon$. By Lemma 6, $\mathcal{I}_\wp$ is an left ideal of Υ. As Υ is bi-simple, $\mathcal{I}_\wp = R$. So $\varrho \in \Upsilon$. Thus, $\widetilde{\lambda}^f(\varrho) \geq \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2}$. Therefore, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\varrho) = \widetilde{\lambda}^f(\varrho) \wedge \frac{\kappa^*-\kappa}{2} \geq \widetilde{\lambda}^f(\wp) \wedge \frac{\kappa^*-\kappa}{2} = \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\wp)$. Similarly, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\varrho) \leq \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\wp)$. Thus, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\wp) = \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\varrho)$, as required.

($\Leftarrow$) Assume that I is the proper *BI* of Υ. By Lemma 5, $(\underline{\lambda^{f^{\kappa^*}}_{\kappa}})_I$ is the $(\in, \in \vee(\kappa^*, q_\kappa))$-*FBI* of Υ. As Υ is $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-simple, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\wp) = \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(\varrho)$, $\forall\ \wp, \varrho \in \Upsilon$. Let $p \in I$ and $q \in \Upsilon$. Then, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(p) = \underline{\lambda^{f^{\kappa^*}}_{\kappa}}(q)$. As $p \in I$, we have $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(p) = \frac{\kappa^*-\kappa}{2}$. Therefore, $\underline{\lambda^{f^{\kappa^*}}_{\kappa}}(q) = \frac{\kappa^*-\kappa}{2}$, which implies that $q \in I$. Thus, $I = \Upsilon$, and hence Υ is bi-simple. □

5. Conclusions

The notion of the $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideal, which is broader than the existing terminology, was introduced in this work. A condition is provided under which fuzzy bi-ideals and $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals coincide. Bi-ideals and $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals connections were taken into consideration. Regular and intra-regular ordered semirings were described in terms of $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-ideals and their $(\kappa*, \kappa)$-lower parts. Moreover, $(\in, \in \vee(\kappa^*, q_\kappa))$-fuzzy bi-simple ordered semirings were defined and characterized.

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References

1. Gan, A.P.; Jiang, Y.L. On ordered ideals in ordered semirings. *J. Math. Res. Exp.* **2011**, *31*, 989–996.
2. Good, R.A.; Hughes, D.R. Associated groups for a semigroup. *Bull. Am. Math. Soc.* **1952**, *58*, 624–625.
3. Lajos, S.; Szász, A. On the bi-ideals in associative rings. *Proc. Jpn. Acad.* **1970**, *46*, 505–507. [CrossRef]
4. Palakawong na Ayutthaya, P.; Pibaljommee, B. Characterizations of intra-regular ordered semirings by ordered quasi-ideals. *Int. J. Math. Math. Sci.* **2016**, *2016*, 4272451. [CrossRef]
5. Senarat, P.; Pibaljommee, B. Prime ordered k-bi-ideals in ordered semirings. *Quasigroups Relat. Syst.* **2017**, *25*, 121–132.
6. Davvaz, B.; Omidi, S. Basic notions and properties of ordered semihyperrings. *Categ. Gen. Algebr. Struct. Appl.* **2016**, *4*, 43–62.
7. Omidi, S.; Davvaz, B. Contribution to study special kinds of hyperideals in ordered semihyperrings. *J. Taibah Univ. Sci.* **2017**, *11*, 1083–1094. [CrossRef]
8. Anjum, R.; Ullah, S.; Chu, Y.M.; Munir, M.; Kausar, N.; Kadry, S. Characterizations of ordered h-regular semirings by ordered h-ideals. *AIMS Math.* **2020**, *5*, 5768–5790. [CrossRef]
9. Patchakhieo, S.; Pibaljommee, B. Characterizations of ordered k-regular semirings by ordered k-ideals. *Asian-Eur. J. Math.* **2017**, *10*, 1750020. [CrossRef]
10. Palakawong na Ayutthaya, P.; Pibaljommee, B. Characterizationsof ordered intra-k-regular semirings by ordered k-ideals. *Commun. Korean Math. Soc.* **2018**, *3*, 1–12.
11. Ahsan, J.; Saifullah, K.; Khan, F. Fuzzy semirings. *Fuzzy Sets Syst.* **1993**, *60*, 309–320. [CrossRef]
12. Kuroki, N. Fuzzy bi-ideals in semigroups. *Comment. Math. Univ. St. Pauli* **1979**, *28*, 17–21.
13. Mandal, D. Fuzzy ideals and fuzzy interior ideals in ordered semirings. *Fuzzy Inf. Eng.* **2014**, *6*, 101–114. [CrossRef]
14. Mandal, D. Fuzzy bi-ideals and fuzzy quasi-ideals in ordered semirings. *Gulf J. Math.* **2014**, *2*, 60–67. [CrossRef]
15. Gao, N.; Li, Q.; Huang, X.; Jiang, H. Fuzzy Orders and Pseudo-fuzzy Orders on Semirings. *J. Intell. Fuzzy Syst.* **2019**, *36*, 6443–6454. [CrossRef]
16. Al-Kaseasbeh, S.; Al Tahan, M.; Davvaz, B.; Hariri, M. Single valued neutrosophic (m,n)-ideals of ordered semirings. *AIMS Math.* **2022**, *7*, 1211–1223. [CrossRef]
17. Jun, Y.B.; Khan, A.; Shabir, M. Ordered semigroup characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals. *Bull. Malays. Math. Soc.* **2009**, *32*, 391–408.
18. Hedayati, H. Fuzzy ideals of semirings. *Neural Comput. Appl.* **2011**, *20*, 1219–1228. [CrossRef]
19. Cristea, I.; Mahboob, A.; Khan, N.M. A new type fuzzy quasi-ideals of Ordered Semigrouops. *J. Mult.-Valued Log. Soft Comput.* **2020**, *34*, 283–304.
20. Muhiuddin, G.; Mahboob, A.; Khan, N.M. A new type of fuzzy semiprime subsets in ordered semigroups. *J. Intell. Fuzzy Syst.* **2019**, *37*, 4195–4205. [CrossRef]
21. Muhiuddin, G.; Mahboob, A. Int-soft Ideals over the soft sets in ordered semigroups. *AIMS Math.* **2020**, *5*, 2412–2423. [CrossRef]
22. Muhiuddin, G.; Al-Kadi, D. Interval valued m-polar fuzzy BCK/BCI-algebras. *Int. J. Comput. Intell. Syst.* 2021, 14, 1014–1021. [CrossRef]
23. Muhiuddin, G.; Al-Kadi, D.; Mahboob, A.; Aljohani, A. Generalized fuzzy ideals of BCI-algebras based on interval valued m-polar fuzzy structures. *Int. J. Comput. Intell. Syst.* **2021**, *14*, 169. [CrossRef]
24. Qahtan, S.; Alsattar, H.A.; Zaidan, A.A.; Deveci, M.; Pamucar, D.; Martinez, L. A comparative study of evaluating and benchmarking sign language recognition system-based wearable sensory devices using a single fuzzy set. *Knowl.-Based Syst.* **2023**, *269*, 110519. [CrossRef]
25. Qahtan, S.; Alsattar, H.A.; Zaidan, A.A.; Deveci, M.; Pamucar, D.; Delen, D. Performance assessment of sustainable transportation in the shipping industry using a q-rung orthopair fuzzy rough sets-based decision-making methodology. *Expert Syst. Appl.* **2023**, *223*, 119958. [CrossRef]