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Existence, Uniqueness and the Multi-Stability Results for a $\mathscr{W}$-Hilfer Fractional Differential Equation

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Abstract: In this paper, we apply the well-known aggregation mappings on Mittag-Leffler-type functions to investigating new approximation error estimates of a $\mathscr{W}$-Hilfer fractional differential equation, by a different concept of Ulam-type stability in both bounded and unbounded domains.

Keywords: stability; fractional calculus; special functions; aggregation maps

MSC: 39B62; 46L05; 47B47; 47H10; 46L57

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1. Introduction

The main issue we are studying in the present paper is that of aggregation mapping, which refers to the procedure of merging some inputs into a single output. Probably the oldest instance is the concept of the arithmetic mean, which has been applied throughout the history of empirical sciences. Any map, such as the arithmetic mean, that computes a unique output value from a vector of input values, is named an aggregation map [1].

Aggregation maps play a significant role in various technical tasks scholars that are faced with in current times. They are particularly significant in regard to the diverse problems relevant to the fusion of information. Generally, aggregation maps are widely applied in applied mathematics (e.g., statistics, probability, decision mathematics), pure mathematics (e.g., theory of means and averages, functional equations, measure theory), social sciences (e.g., mathematical psychology), computer and engineering sciences (e.g., operations research, engineering design, artificial intelligence, information theory, data fusion, image analysis), economics and finance (e.g., game theory, decision making, voting theory) and many other natural sciences. Thus, an important characteristic of aggregation maps is that they are applied in different fields [2,3].

Here, we apply n-ary aggregation maps on well-known special functions, including Mittag-Leffler-type functions, to define a class of matrix-valued controller, which helps us to present a new concept of Ulam-type stability. The aggregation maps allow us to obtain the best approximation error estimates by a different concept of perturbation stability, depending on the variant special functions that are initially chosen, and to study minimal errors and optimal stability, which enables us to obtain a single optimal solution.

The history of Ulam-type stability commenced in the 19th century. This stability was first presented by Stanisław Marcin Ulam [4], for an additive function, which was investigated by Donald Hyers [5], for a group of homomorphisms given on a Banach space. Thereafter, the stability notion was extended by Themistocles Rassias [6], and was named Ulam–Hyers–Rassias (UHR) stability. In addition, Ger and Alsina [7] investigated the Ulam-type stability of ODEs, by replacing functional equations. In [8], Jung and Algifiary proposed the Ulam stability of nth-order ODEs, by means of the Laplace transform technique. Jung, Rezaei and Rassias [9] studied the Ulam stability of ODEs by the Laplace transform technique. Applying the UHR technique, Baleanu and Wu [10] demonstrated the Mittag-Leffler (ML)-type stability of fractional equations; Baleanu, Wu and Huang [11] demonstrated the ML-type stability of fractional delay difference equations with impulse; Wu [12] proved the ML-type stability of fractional neural networks through the fixed point (FP) theory see also [13].

Here, we present some novel notions concerning the stability of fractional equations in the Mittag-Leffler–Hyers–Ulam sense, by the FP technique, which is the most popular technique for studying the stability of different types of equations. The FP technique was applied for the first time by Baker [14], who used it to get the UHR stability of a functional equation in a single variable. At present, numerous authors follow Radu's technique [15], and make use of a theorem of Margolis and Diaz.

Here, we study existence, uniqueness and the multi-stability results for the fractional system below:

$$\begin{cases} {}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\Theta^+}\mathscr{J}(\mathcal{S}) = \rho(\mathcal{S},\mathscr{J}(\mathcal{S}), {}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\Theta^+}\mathscr{J}(\mathcal{S})), \\ I^{1-\mathcal{W};\mathscr{W}}_{\Theta^+}\mathscr{J}(\Theta^+) = J_\Theta, \end{cases} \tag{1}$$

where ${}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\Theta^+}(.)$ is a fractional-order derivative in the Hilfer sense of order $\mathcal{X} \in (0,1]$ and type $\mathcal{Z} \in [0,1]$, $I^{1-\mathcal{W};\mathscr{W}}_{\Theta^+}(.)$ is a fractional integral of order $1-\mathcal{W}$, $\mathcal{W} = \mathcal{X} + \mathcal{Z}(1-\mathcal{X})$, in regard to the function $\mathscr{W}$ defined in Definition 2, and $\rho : \Upsilon \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is an arbitrary function. Presume $\mathcal{S} \in \Upsilon = [\Theta, D]$ with $D > \Theta$ and $J_\Theta \in \mathbb{R}$.

Let $[\mathfrak{L}_1,\mathfrak{L}_2](0 \le \mathfrak{L}_1 < \mathfrak{L}_2 < \infty)$ be an interval, and $C[\mathfrak{L}_1,\mathfrak{L}_2]$ be the space of continuous functions $h : [\mathfrak{L}_1,\mathfrak{L}_2] \longrightarrow \mathbb{R}$ with norm

$$\|h\|_{C[\mathfrak{L}_1,\mathfrak{L}_2]} = \max_{\mathfrak{L}_1 \le \zeta \le \mathfrak{L}_2} |h(\zeta)|.$$

The weighted space $C_{1-\mathcal{W};\mathscr{W}}[\mathfrak{L}_1,\mathfrak{L}_2]$ of continuous functions h on $(\mathfrak{L}_1,\mathfrak{L}_2]$ is defined by

$$C_{1-\mathcal{W};\mathscr{W}}[\mathfrak{L}_1,\mathfrak{L}_2] = \left\{ h : (\mathfrak{L}_1,\mathfrak{L}_2] \longrightarrow \mathbb{R}; (\mathscr{W}(\zeta) - \mathscr{W}(\mathfrak{L}_1))^{1-\mathcal{W}} h(\zeta) \in C[\mathfrak{L}_1,\mathfrak{L}_2] \right\}, \quad 0 \le \mathcal{W} < 1,$$

with norm

$$\|h\|_{C_{1-\mathcal{W};\mathscr{W}}[\mathfrak{L}_1,\mathfrak{L}_2]} = \max_{\zeta \in [\mathfrak{L}_1,\mathfrak{L}_2]} \left| (\mathscr{W}(\zeta) - \mathscr{W}(\mathfrak{L}_1))^{1-\mathcal{W}} h(\zeta) \right|.$$

2. Preliminaries

2.1. On Fractional Derivatives

Definition 1 ([1])**.** *Let the interval* $(\mathfrak{L}_1,\mathfrak{L}_2)(\overbrace{\mathfrak{L}_i}^{i=1,2} \in \mathbb{R})$*, and* $\mathcal{X} > 0$*. Presume* $\mathscr{W}(\zeta)$ *is a monotonically increasing and positive function on* $(\mathfrak{L}_1,\mathfrak{L}_2]$ *that has a continuous derivative* $\mathscr{W}'(\zeta)$ *on* $(\mathfrak{L}_1,\mathfrak{L}_2)$*. The fractional integral respecting* $\mathscr{W}$ *on* $[\mathfrak{L}_1,\mathfrak{L}_2]$ *is given by*

$$I^{\mathcal{X};\mathscr{W}}_{\mathfrak{L}_1^+} h(\zeta) = \frac{1}{\Gamma(\mathcal{X})} \int_{\mathfrak{L}_1}^{\mathcal{S}} \mathscr{W}'(\mathcal{S})(\mathscr{W}(\zeta) - \mathscr{W}(\mathcal{S}))^{\mathcal{X}-1} h(\mathcal{S}) d\mathcal{S}.$$

Definition 2 ([1]). *Let $\mathcal{X} \in (\alpha - 1, \alpha)$ with $\alpha \in \mathbb{N}$, and let $h, \mathscr{W} \in C^{\alpha}[\mathfrak{L}_1, \mathfrak{L}_2]$ be two functions, such that $\mathscr{W}$ is increasing and $\mathscr{W}'(\zeta) \neq 0$ for any $\zeta \in [\mathfrak{L}_1, \mathfrak{L}_2]$; therefore, the $\mathscr{W}$-Hilfer fractional derivative ${}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\mathfrak{L}_1^+}(.)$ of order $\mathcal{X}$ and type $\mathcal{Z} \in [0,1]$ is defined as follows:*

$$
{}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\mathfrak{L}_1^+} h(\zeta) = I^{\mathcal{Z}(\alpha-\mathcal{X});\mathscr{W}}_{\mathfrak{L}_1^+} \left(\frac{1}{\mathscr{W}'(\zeta)} \frac{d}{d\zeta} \right)^{\alpha} I^{(1-\mathcal{Z})(\alpha-\mathcal{X});\mathscr{W}}_{\mathfrak{L}_1^+} h(\zeta).
$$

Theorem 1 ([1]). *Let $h \in C^1[\mathfrak{L}_1, \mathfrak{L}_2]$, $\mathcal{X} \in (0,1)$ and $\mathcal{Z} \in [0,1]$. Then,*

$$
{}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\mathfrak{L}_1^+} I^{\mathcal{X};\mathscr{W}}_{\mathfrak{L}_1^+} h(\zeta) = h(\zeta).
$$

Theorem 2 ([1]). *Let $h \in C^1[\mathfrak{L}_1, \mathfrak{L}_2]$, $\mathcal{X} \in (0,1)$ and $\mathcal{Z} \in [0,1]$. Then,*

$$
I^{\mathcal{X};\mathscr{W}}_{\mathfrak{L}_1^+} {}^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\mathfrak{L}_1^+} h(\zeta) = h(\zeta) - \frac{(\mathscr{W}(\zeta) - \mathscr{W}'(\mathfrak{L}_1))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} I^{(1-\mathcal{Z})(1-\mathcal{X});\mathscr{W}}_{\mathfrak{L}_1^+} h(\mathfrak{L}_1).
$$

Lemma 1 ([16]). *Let $\eta_1, \eta_2 > 0$. If $h(\mathcal{S}) = (\mathscr{W}(\mathcal{S}) - \mathscr{W}(\mathfrak{L}_1))^{\eta_2 - 1}$. Then, we obtain*

$$
I^{\eta_1;\mathscr{W}}_{\mathfrak{L}_1^+} h(\mathcal{S}) = \frac{\Gamma(\eta_2)}{\Gamma(\eta_2 + \eta_1)} (\mathscr{W}(\mathcal{S}) - \mathscr{W}(\mathfrak{L}_1))^{\eta_1 + \eta_2 - 1}. \tag{2}
$$

2.2. On the Alternative Theory

Theorem 3 ([17]). *Let $n \in \mathbb{N}$. Consider the set Υ with a complete $[0,\infty]^n$-valued metric χ (see [18]), and consider the self-map $\propto$ on Υ, satisfying the inequality*

$$
\chi(\propto \lambda_1, \propto \lambda_2) \leq \overbrace{(\beta, \cdots, \beta)}^{n} \chi(\lambda_2, \lambda_1), \ \ \beta < 1 \text{ is a Lipschitz constant,}
$$

for every λ_1, $\lambda_2 \in \Upsilon$. Thus, we have two options:

(I) $\chi(\propto^m \lambda_1, \propto^{m+1} \lambda_1) = \overbrace{(+\infty, \cdots, +\infty)}^{n}$, $\forall m \in \mathbb{N}$,
or
(II) we obtain $m_0 \in \mathbb{N}$ s.t.:

(1) $\chi(\propto^m \lambda_1, \propto^{m+1} \lambda_1) < \overbrace{(+\infty, \cdots, +\infty)}^{n}$, $\quad \forall m \geq m_0$;
(2) *the fixed point λ_2^* of $\propto$ is the convergent point of the sequence $\{\propto^m \lambda_1\}$;*
(3) *λ_2^* is the unique fixed point of $\propto$, in the set $V = \{\lambda_2 \in \Upsilon \mid \chi(\propto^{m_0} \lambda_1, \lambda_2) < \overbrace{(+\infty, \cdots, +\infty)}^{n}\}$;*
(4) $\overbrace{\left((1-\beta), \cdots, (1-\beta)\right)}^{n} \chi(\lambda_2, \lambda_2^*) \leq \chi(\lambda_2, \propto \lambda_2)$, *for every $\lambda_2 \in \Upsilon$.*

2.3. On Aggregation Maps and Special Functions

Firstly, we introduce the concept of aggregation maps. Next, we apply a small list of aggregation maps, to study optimal stability, which helps us to obtain a unique optimum solution.

Now, let

$$
\mathrm{diag}[\triangle_1, \cdots, \triangle_n] := \mathrm{diag} \begin{bmatrix} \triangle_1 & & \\ & \ddots & \\ & & \triangle_n \end{bmatrix}, \quad n \in \mathbb{N}.
$$

Note that $\mathrm{diag}[\triangle_1, \cdots, \triangle_n] \leq \mathrm{diag}[\nabla_1, \cdots, \nabla_n]$, iff $\triangle_i \leq \nabla_i$, for any $i = 1, \cdots, n$.

Let $n \in \mathbb{N}$ and $[n] := \{1, \cdots, n\}$. We usually apply bold symbols to demonstrate n-tuples: for instance, $\mathrm{diag}[y_1, \cdots, y_n]_{n \times n}$ will usually be written $\mathbf{Y}$. Let $\mathbb{J} \neq \emptyset$ be a real interval.

Definition 3 ([19]). *A function* $\mathsf{P}^{(n)} : diag[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ *is an aggregation function, if:*

(i) *it is nondecreasing in each variable;*
(ii) *it satisfies the boundary conditions*

$$\sup_{\mathbf{Y}\in\mathbb{J}^n} \mathsf{P}^{(n)}(\mathbf{Y}) = \sup \mathbb{J}, \quad \textit{and} \quad \inf_{\mathbf{Y}\in\mathbb{J}^n} \mathsf{P}^{(n)}(\mathbf{Y}) = \inf \mathbb{J}. \tag{3}$$

$n \in \mathbb{N}$ shows the arty of the aggregation map. Note that the aggregation maps will be written P instead of $\mathsf{P}^{(n)}$.

We present a common list of aggregation maps, as follows:

- The geometric mean function $\mathsf{GM} : \mathrm{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ and the arithmetic mean function $\mathsf{AM} : \mathrm{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ are, respectively, given by

$$\mathsf{AG}_1(\mathbf{Y}) := \mathsf{AM}(\mathbf{Y}) := \frac{1}{n}\sum_{i=1}^{n} y_i, \tag{4}$$

$$\mathsf{AG}_2(\mathbf{Y}) := \mathsf{GM}(\mathbf{Y}) := (\prod_{i=1}^{n} y_i)^{\frac{1}{n}}; \tag{5}$$

- For every $\Bbbk \in [n]$, the projection function $\mathsf{P}_{\Bbbk} : \mathrm{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ and the order statistic function $\mathsf{OS}_{\Bbbk} : \mathrm{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ related to the $\Bbbk^{\text{th}}$ argument, are correspondingly given by

$$\mathsf{AG}_3(\mathbf{Y}) := \mathsf{P}_{\Bbbk}(\mathbf{Y}) := y_{\Bbbk}, \tag{6}$$

$$\mathsf{AG}_4(\mathbf{Y}) := \mathsf{OS}_{\Bbbk}(\mathbf{Y}) := (y)_{\Bbbk}, \tag{7}$$

where $(y)_{\Bbbk}$ is the $\Bbbk^{\text{th}}$ lowest coordinate of y:

$$y_{(1)} \leqslant \cdots \leqslant y_{(k)} \leqslant \cdots y_{(n)}.$$

The projections onto the first and the last coordinates are given by

$$\mathsf{AG}_5(\mathbf{Y}) := \mathsf{P}_F(\mathbf{Y}) := \mathsf{P}_1(\mathbf{Y}) = y_1, \tag{8}$$

$$\mathsf{AG}_6(\mathbf{Y}) := \mathsf{P}_L(\mathbf{Y}) := \mathsf{P}_n(\mathbf{Y}) = y_n. \tag{9}$$

Likewise, the extreme order statistics y_1 and y_n are correspondingly the minimum and maximum functions

$$\mathsf{AG}_7(\mathbf{Y}) := \mathsf{Min}(\mathbf{Y}) := \mathsf{OS}_1(\mathbf{Y}) = \min\{y_1, \cdots, y_n\}, \tag{10}$$

$$\mathsf{AG}_8(\mathbf{Y}) := \mathsf{Max}(\mathbf{Y}) := \mathsf{OS}_n(\mathbf{Y}) = \max\{y_1, \cdots, y_n\}, \tag{11}$$

which will be written through the operations $\vee$ and $\wedge$, respectively:

$$\mathsf{Max}(\mathbf{Y}) = \bigvee_{i=1}^{n} y_i, \quad \text{and} \quad \mathsf{Min}(\mathbf{Y}) = \bigwedge_{i=1}^{n} y_i.$$

Similarly, the median of odd numbers of values $\mathrm{diag}[y_1, \cdots, y_{2\Bbbk-1}]_{(-1+2\Bbbk)\times(-1+2\Bbbk)}$ is simply given by

$$\mathsf{Med}\Big(\mathrm{diag}[y_1, \cdots, y_{-1+2\Bbbk}]_{(-1+2\Bbbk)\times(2\Bbbk-1)}\Big) = y_{(\Bbbk)}.$$

For an even number of values $\mathrm{diag}[y_1, \cdots, y_{2\Bbbk}]$, the median is given by

$$\mathsf{Med}\Big(\mathrm{diag}[y_1, \cdots, y_{2\Bbbk}]_{2\Bbbk\times 2\Bbbk}\Big) := \mathsf{AM}\Big(\mathrm{diag}[y_{(\Bbbk+1)}, y_{(\Bbbk)}]_{2\times 2}\Big) = \frac{y_{(\Bbbk+1)} + y_{(\Bbbk)}}{2}.$$

For every $\varphi \in \mathbb{J}$, we also define the φ-median, $\mathsf{Med}_\varphi : \text{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$, by

$$\begin{aligned} \mathsf{AG}_9(\mathbf{Y}) := \mathsf{Med}_\varphi(\mathbf{Y}) &= \mathsf{Med}\Big(\text{diag}[y_1, \cdots, y_n, \underbrace{\varphi, \cdots, \varphi}_{n-1}]_{(2n-1)\times(2n-1)}\Big) \\ &= \mathsf{Med}(\mathsf{Min}(\mathbf{Y}), \varphi, \mathsf{Max}(\mathbf{Y})); \end{aligned} \tag{12}$$

- For every $\emptyset \neq K \subseteq [n]$, the partial minimum $\mathsf{Min}_{\Bbbk} : \text{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$ and the partial maximum $\mathsf{Max}_{\Bbbk} : \text{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$, associated with K, are, respectively, given by

$$\mathsf{AG}_{10}(\mathbf{Y}) := \mathsf{Min}_{\Bbbk}(\mathbf{Y}) := \bigwedge_{i\in K} y_i, \tag{13}$$

$$\mathsf{AG}_{11}(\mathbf{Y}) := \mathsf{Max}_{\Bbbk}(\mathbf{Y}) := \bigvee_{i\in K} y_i; \tag{14}$$

- For every weight vector $\mathbf{V} = \text{diag}[v_1, \cdots, v_n]_{n\times n} \in \text{diag}[[0,1], \cdots, [0,1]]_{n\times n}$ s.t. $\sum_{i=1}^n v_i = 1$, the weighted arithmetic mean function

$$\mathsf{WAM}_{\mathbf{V}} : \text{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$$

and the ordered weighted averaging function $\mathsf{OWA}_{\mathbf{V}} : \text{diag}[\mathbb{J}, \cdots, \mathbb{J}]_{n\times n} \longrightarrow \mathbb{J}$, associated with $\mathbf{V}$, are, respectively, given by

$$\mathsf{AG}_{12}(\mathbf{Y}) := \mathsf{WAM}_{\mathbf{V}}(\mathbf{Y}) := \sum_{i=1}^n v_i y_i, \qquad \mathsf{AG}_{13}(\mathbf{Y}) := \mathsf{OWA}_{\mathbf{V}}(\mathbf{Y}) := \sum_{i=1}^n v_i y_{(i)}; \tag{15}$$

- The sum and product functions $\sum, \Pi : \text{diag}[\overline{\mathbb{R}}, \cdots, \overline{\mathbb{R}}]_{n\times n} \longrightarrow \mathbb{R}$ are correspondingly given by

$$\mathsf{AG}_{14}(\mathbf{Y}) := \sum(\mathbf{Y}) := \sum_{i=1}^n y_i, \qquad \mathsf{AG}_{15}(\mathbf{Y}) := \Pi(\mathbf{Y}) := \prod_{i=1}^n y_i. \tag{16}$$

The main issue we are studying in this section is that of aggregation mapping, which refers to the process of combining various input values into a single output. We will apply the above aggregation mappings on Mittag-Leffler functions, to study the stability results for the governing model.

Consider the following special functions:

- The one-parameter Mittag-Leffler function [20],

$$\ni_1(\Lambda) := \nabla_{\mathcal{X}}(\Lambda) = \sum_{i=0}^{\infty} \frac{\Lambda^i}{\Gamma(i\mathcal{X}+1)}, \tag{17}$$

in which $\Lambda, \mathcal{X} \in \mathbb{C}, i \in \mathbb{N}$, and $\Re(\wp) > 0$;

- The pre-superhyperbolic supercosine through (17) [20],

$$\begin{aligned} \ni_2(\Lambda) &:= \mathit{precosh}_{\mathcal{X}}(\Lambda) \\ &= 0.5\Big(\nabla_{\mathcal{X}}(\Lambda) + \nabla_{\mathcal{X}}(-\Lambda)\Big) \\ &= \sum_{i=0}^{\infty} \frac{\Lambda^{2i}}{\Gamma((2i)\mathcal{X}+1)}, \end{aligned}$$

where $\Lambda, \mathcal{X} \in \mathbb{C}$, and $\Re(\mathcal{X}) > 0$;

- The pre-supercosine function through (17) [20],

$$\begin{aligned}\backepsilon_3\,(\Lambda) &:= precos_{\mathcal{X}}(\Lambda)\\ &= \frac{1}{2}\Big(\nabla_{\mathcal{X}}(i\Lambda)+\nabla_{\mathcal{X}}(-i\Lambda)\Big)\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\Lambda^{2i}}{\Gamma((2i)\mathcal{X}+1)},\end{aligned}$$

where $\Lambda, \mathcal{X} \in \mathbb{C}$, and $\Re(\mathcal{X}) > 0$;
- The pre-superhyperbolic supersine through (17) [20],

$$\begin{aligned}\backepsilon_4\,(\Lambda) &:= presinh_{\mathcal{X}}(\Lambda)\\ &= \frac{1}{2}\Big(\nabla_{\mathcal{X}}(\Lambda)-\nabla_{\mathcal{X}}(-\Lambda)\Big)\\ &= \sum_{i=0}^{\infty}\frac{\Lambda^{2i+1}}{\Gamma((2i+1)\mathcal{X}+1)},\end{aligned}$$

in which $\Lambda, \mathcal{X} \in \mathbb{C}$, and $\Re(\mathcal{X}) > 0$;
- The pre-supersine function through (17) [20],

$$\begin{aligned}\backepsilon_5\,(\Lambda) &:= presin_{\mathcal{X}}(\Lambda)\\ &= \frac{1}{2i}\Big(\nabla_{\mathcal{X}}(i\Lambda)-\nabla_{\mathcal{X}}(-i\Lambda)\Big)\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\Lambda^{2i+1}}{\Gamma((2i+1)\mathcal{X}+1)},\end{aligned}$$

where $\Lambda, \mathcal{X} \in \mathbb{C}$, and $\Re(\mathcal{X}) > 0$.

Here, we define the matrix-valued controller $\mathfrak{S}$ as follows:

$$\mathfrak{S}(\Lambda) := \mathrm{diag}\Big[\,\backepsilon_1\,(\Lambda),\cdots,\backepsilon_5\,(\Lambda)\Big].$$

Note that we have the following inequalities:

$$\begin{aligned}&\frac{\theta}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(s)(\mathscr{W}(\mathcal{S})-\mathscr{W}(s))^{\mathcal{X}-1}\nabla_{\mathcal{X}}\Big((\mathscr{W}(s)-\mathscr{W}(0))^{\mathcal{X}}\Big)ds\\ &\le \frac{\theta}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(s)(\mathscr{W}(\mathcal{S})-\mathscr{W}(s))^{\mathcal{X}-1}\sum_{k=0}^{\infty}\frac{(\mathscr{W}(s)-\mathscr{W}(0))^{k\mathcal{X}}}{\Gamma(k\mathcal{X}+1)}ds\\ &= \frac{\theta}{\Gamma(\mathcal{X})}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathcal{X}+1)}\int_0^{\mathcal{S}}(\mathscr{W}(\mathcal{S})-\mathscr{W}(s))^{\mathcal{X}-1}(\mathscr{W}(s)-\mathscr{W}(0))^{k\mathcal{X}}d\mathscr{W}(s)\\ &= \frac{\theta}{\Gamma(\mathcal{X})}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathcal{X}+1)}\int_0^{\mathscr{W}(\mathcal{S})-\mathscr{W}(0)}(\mathscr{W}(\mathcal{S})-\mathscr{W}(0)-u)^{\mathcal{X}-1}(u)^{k\mathcal{X}}du\\ &(u=\mathscr{W}(s)-\mathscr{W}(0))\\ &\le \frac{\theta}{\Gamma(\mathcal{X})}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathcal{X}+1)}(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}-1}\int_0^{\mathscr{W}(\mathcal{S})-\mathscr{W}(0)}(1-\frac{u}{\mathscr{W}(\mathcal{S})-\mathscr{W}(0)})^{\mathcal{X}-1}u^{k\mathcal{X}}du\\ &= \frac{\theta}{\Gamma(\mathcal{X})}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\mathcal{X}+1)}(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{(k+1)\mathcal{X}}\int_0^1(1-v)^{\mathcal{X}-1}v^{k\mathcal{X}}dv\end{aligned}$$

$$\left(v = \frac{u}{\mathscr{W}(\mathcal{S}) - \mathscr{W}(0)}\right)$$
$$= \frac{\theta}{\Gamma(\mathcal{X})} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\mathcal{X}+1)} (\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{(k+1)\mathcal{X}} \frac{\Gamma(k\mathcal{X}+1)\Gamma(\mathcal{X})}{\Gamma((k+1)\mathcal{X}+1)}$$
$$\leq \theta \sum_{n=0}^{\infty} \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{n\mathcal{X}}}{\Gamma(n\mathcal{X}+1)}$$
$$= \theta \nabla_{\mathcal{X}}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big).$$

Now, we obtain

$$\begin{aligned} &\tfrac{\theta}{\Gamma(\mathcal{X})} \int_0^{\mathcal{S}} \mathscr{W}'(s)(\mathscr{W}(\mathcal{S}) - \mathscr{W}(s))^{\mathcal{X}-1} \mathfrak{S}\Big((\mathscr{W}(s) - \mathscr{W}(0))^{\mathcal{X}}\Big) ds \\ &\leq \theta \mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big). \end{aligned} \tag{18}$$

3. Existence, Uniqueness and Multi-Stability

Making use of Theorem 3, we study existence, uniqueness and the multi-stability results of the system (19) and (20).

Now, for $\rho : \Upsilon \times \mathbb{R}^2 \to \mathbb{R}$ and $0 < \theta$, we consider the following equations:

$$^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{J}(\mathcal{S}) = \rho(\mathcal{S}, \mathscr{J}(\mathcal{S}), ^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{J}(\mathcal{S})); \tag{19}$$

$$I_{\Theta^+}^{\mathcal{W};\mathscr{W}} \mathscr{J}(\Theta^+) = J_\Theta, \quad J_\Theta \in \mathbb{R}, \tag{20}$$

and the following inequality:

$$\begin{aligned} &\mathrm{diag}\Bigg[\Big|^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{L}(\mathcal{S}) - \rho(\mathcal{S}, \mathscr{L}(\mathcal{S}), ^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{L}(\mathcal{S}))\Big|, \cdots, \\ &\qquad \Big|^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{L}(\mathcal{S}) - \rho(\mathcal{S}, \mathscr{L}(\mathcal{S}), ^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{L}(\mathcal{S}))\Big|\Bigg]_{15\times 15} \\ &\leq \mathrm{diag}\Bigg[\theta_1 \mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \cdots, \theta_{15} \mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg], \end{aligned} \tag{21}$$

where $\mathcal{S} \in \Upsilon$.

Now, we define the notion of multi-stability.

Definition 4. *Equations (19) and (20) have multi-stability with respect to*

$$diag\Bigg[\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \cdots, \mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg],$$

if there exists $\underbrace{c_i}_{i=1,\cdots,15} > 0$*, such that for every* $\underbrace{\theta_i}_{i=1,\cdots,15} > 0$ *and every solution* $\mathscr{L} \in C_{1-\mathcal{W};\mathscr{W}}(\Upsilon, \mathbb{R})$ *to (45) and* $I_{\Theta^+}^{1-\mathcal{W};\mathscr{W}} \mathscr{L}(\Theta^+) = J_\Theta$*, there exists a solution* $\mathscr{J} \in C_{1-\mathcal{W};\mathscr{W}}(\Upsilon, \mathbb{R})$ *to (19) and (20) with*

$$\begin{aligned} &diag\Big[|\mathscr{L}(\mathcal{S}) - \mathscr{J}(\mathcal{S})|, \cdots, |\mathscr{L}(\mathcal{S}) - \mathscr{J}(\mathcal{S})|\Big]_{15\times 15} \\ &\leq diag\Bigg[c_1\theta_1 \mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \cdots, c_{15}\theta_{15} \mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg], \end{aligned}$$

for every $\mathcal{S} \in \delta$.

Lemma 2 ([21]). *Let a continuous function $\rho(\mathcal{S},\kappa,v) : \Upsilon\times\mathbb{R}^2 \to \mathbb{R}$. Then, Equation (1) is equivalent to*

$$\mathscr{J}(\mathcal{S}) = \frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})}J_\Theta + I_{\Theta^+}^{\mathcal{X};\mathscr{W}}\curlywedge(\mathcal{S}), \tag{22}$$

where $\curlywedge\in C(\Upsilon,\mathbb{R})$ satisfies the equality below:

$$\curlywedge_{\mathscr{J}}(\mathcal{S}) = \rho\Big(\mathcal{S},\mathscr{J}(\mathcal{S}),\curlywedge_{\mathscr{J}}(\mathcal{S})\Big).$$

Remark 1. *Let $\mathscr{L}\in C_{1-\mathcal{W};\mathscr{W}}(\Upsilon,\mathbb{R})$ be a solution of (45), and $I_{\Theta^+}^{1-\mathcal{W};\mathscr{W}}\mathscr{L}(\Theta^+) = J_\Theta$. Then, $\mathscr{L}$ is a solution of the inequality below:*

$$\begin{aligned}
&diag\Bigg[\Bigg|\mathscr{L}(\mathcal{S}) - \frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})}J_\Theta \\
&-\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(s))^{\mathcal{X}-1}\curlywedge_{\mathscr{L}}(s)ds\Bigg|,\cdots, \\
&\qquad\Bigg|\mathscr{L}(\mathcal{S}) \\
&-\frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})}J_\Theta - \frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(s))^{\mathcal{X}-1}\curlywedge_{\mathscr{L}}(s)ds\Bigg|\Bigg]_{15\times15} \\
&\leq diag\Bigg[\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots,\theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg],
\end{aligned}$$

where $\curlywedge_{\mathscr{L}}\in C(\Upsilon,\mathbb{R})$ satisfies the equality below:

$$\curlywedge_{\mathscr{L}}(\mathcal{S}) = \rho\Big(\mathcal{S},\mathscr{L}(\mathcal{S}),\curlywedge_{\mathscr{L}}(\mathcal{S})\Big). \tag{23}$$

Let us suppose the following axioms are satisfied:
$(\mathcal{F}_1)$ $\rho : \Upsilon\times\mathbb{R}^2\to\mathbb{R}$ is continuous;
$(\mathcal{F}_2)$ There is $0<\tau_1,\tau_2$, with $0<\dfrac{\tau_1}{(1-\tau_2)}<1$ s.t.

$$|\rho(\mathcal{S},\Omega,\varpi)-\rho(\mathcal{S},\overline{\Omega},\overline{\varpi})| \leq \tau_1|\Omega-\overline{\Omega}|+\tau_2|\varpi-\overline{\varpi}| \ \text{ for each }\ \Omega,\varpi,\overline{\Omega},\overline{\varpi}\in\mathbb{R}\ \text{and}\ \mathcal{S}\in\Upsilon.$$

Theorem 4. *Let $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ be satisfied. If $\mathscr{L}\in C_{1-\mathcal{W};\mathscr{W}}(\Upsilon,\mathbb{R})$ satisfies (45) and $I_{\Theta^+}^{1-\mathcal{W};\mathscr{W}}\mathscr{L}(\Theta^+) = J_\Theta$, then there is a single function $\mathscr{J}$ satisfying (19) and (20), s.t.,*

$$\begin{aligned}
&diag\Big[|\mathscr{J}(\mathcal{S})-\mathscr{L}(\mathcal{S})|,\cdots,|\mathscr{J}(\mathcal{S})-\mathscr{L}(\mathcal{S})|\Big]_{15\times15} \\
&\leq diag\Bigg[\frac{\theta_1}{1-\frac{\tau_1}{(1-\tau_2)}}\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots, \\
&\qquad\frac{\theta_n}{1-\frac{\tau_1}{(1-\tau_2)}}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg],
\end{aligned} \tag{24}$$

for every $\mathcal{S}\in\Upsilon$.

Proof. Set $\varrho = C_{1-\mathcal{W};\mathscr{W}}(\Upsilon,\mathbb{R})$, and define a mapping, $\chi:\varrho\times\varrho\to[0,\infty]^n$, by

$$
\begin{aligned}
&\chi(\Phi_1,\Phi_2) \\
= \ & \inf\Big\{(\mathtt{C}_1,\cdots,\mathtt{C}_{15}) \geq 0 : \mathrm{diag}\Big[|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|,\cdots,|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|\Big]_{15\times 15} \\
&\quad \leq \mathrm{diag}\Big[\mathtt{C}_1\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots, \\
&\qquad\qquad \mathtt{C}_{15}\theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big]\Big\}.
\end{aligned} \tag{25}
$$

We show (ϱ,χ) is a complete metric space. Let $\chi(\Phi_1,\Phi_2) > \chi(\Phi_1,\nu)+\chi(\nu,\Phi_2)$, for some Φ_1,Φ_2 and $\nu \in \varrho$. Thus, there exists $\mathcal{S}_\circ \in \delta$ with

$$
\begin{aligned}
&\mathrm{diag}\Big[|\Phi_1(\mathcal{S}_\circ)-\Phi_2(\mathcal{S}_\circ)|,\cdots,|\Phi_1(\mathcal{S}_\circ)-\Phi_2(\mathcal{S}_\circ)|\Big]_{15\times 15} \\
&> (\chi(\Phi_1,\nu)+\chi(\nu,\Phi_2))\mathrm{diag}\Big[\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}_\circ)-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots, \\
&\qquad\qquad \theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}_\circ)-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big].
\end{aligned}
$$

Thus, from the definition of χ, we obtain

$$
\begin{aligned}
&|\Phi_1(\mathcal{S}_\circ)-\Phi_2(\mathcal{S}_\circ)| \\
> \ & |\Phi_1(\mathcal{S}_\circ)-\nu(\mathcal{S}_\circ)|+|(\nu(\mathcal{S}_\circ)-\Phi_2(\mathcal{S}_\circ)|,
\end{aligned}
$$

which is contradictory. We now show (ϱ,χ) is complete. Presume ω_k is a Cauchy sequence in (ϱ,χ). Thus, for every $\underbrace{\epsilon_i}_{i=1,\cdots,15} > 0$, there exists a $\underbrace{\aleph_{\epsilon_i}}_{i=1,\cdots,15} \in \mathbb{N}$, s.t. $\chi(\omega_m,\omega_k) \leq (\epsilon_1,\cdots,\epsilon_{15})$, for every $m,k \geq \underbrace{\aleph_{\epsilon_i}}_{i=1,\cdots,15}$. According to (25), we obtain

$$
\begin{aligned}
&\mathrm{diag}\Big[|\omega_m(\mathcal{S})-\omega_k(\mathcal{S})|,\cdots,|\omega_m(\mathcal{S})-\omega_k(\mathcal{S})|\Big]_{15\times 15} \\
&\leq \mathrm{diag}\Big[\epsilon_1\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots,\epsilon_{15}\theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big],
\end{aligned} \tag{26}
$$

for each $\mathcal{S} \in \mathrm{Y}$. If $\mathcal{S}$ is fixed, (26) concludes that $\{\omega_k(\mathcal{S})\}$ is a Cauchy sequence in $\mathbb{R}$. As the set of real numbers $\mathbb{R}$ is complete, $\{\omega_k(\mathcal{S})\}$ converges for any $\mathcal{S} \in \mathrm{Y}$. Then, we obtain a function ω, defined by

$$
\omega(\mathcal{S}) := \lim_{k\to\infty}\omega_k(\mathcal{S}), \quad (\mathcal{S} \in \mathrm{Y}), \tag{27}
$$

which gives us $\omega \in \varrho$, (because $\{\omega_k(\mathcal{S})\}$ is Cauchy in complete space $\mathbb{R}$, so they are uniformly convergent on the mapping ω defined in (27). The uniform convergence leads us to the fact that ω is continuous, and is an element of ϱ). If we set $m \to \infty$, it follows from (26) that

$$
\begin{aligned}
&\mathrm{diag}\Big[|\omega(\mathcal{S})-\omega_k(\mathcal{S})|,\cdots,|\omega(\mathcal{S})-\omega_k(\mathcal{S})|\Big]_{15\times 15} \\
&\leq \mathrm{diag}\Big[\epsilon_1\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots,\epsilon_{15}\theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big].
\end{aligned} \tag{28}
$$

Considering (25), we obtain

$$
\chi(\omega,\omega_k) \leq (\epsilon_1,\cdots,\epsilon_{15}).
$$

This confirms that the Cauchy sequence $\{\omega_k\}$ converges to ω in (ϱ, χ). Thus, (ϱ, χ) is complete. In view of Lemma 2, Equations (19) and (20) are equivalent to the system below:

$$\mathscr{J}(\mathcal{S}) = \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + I_{\Theta^+}^{\mathcal{X};\mathscr{W}} \curlywedge(\mathcal{S}), \tag{29}$$

in which $\curlywedge \in C(\Upsilon, \mathbb{R})$ satisfies the equality,

$$\curlywedge(\mathcal{S}) = \rho\left(\mathcal{S}, \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + I_{\Theta^+}^{\mathcal{X};\mathscr{W}} \curlywedge(\mathcal{S}), \curlywedge(\mathcal{S})\right),$$

for all $\mathcal{S} \in \Upsilon$. To prove this note, by using $I_{\Theta^+}^{\mathcal{X};\mathscr{W}}(.)$ on both sides of (1), and utilizing Theorem 2, we obtain

$$\mathscr{J}(\mathcal{S}) - \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} I_{\Theta^+}^{(1-\mathcal{X})(1-\mathcal{Z});\mathscr{W}} J_\Theta = I_{\Theta^+}^{\mathcal{X};\mathscr{W}} \curlywedge(\mathcal{S}).$$

Thus,

$$\mathscr{J}(\mathcal{S}) = \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + I_{\Theta^+}^{\mathcal{X};\mathscr{W}} \curlywedge(\mathcal{S}). \tag{30}$$

In addition, if $\mathscr{J}$ satisfies (30), then it satisfies (1). To see this, apply ${}^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}}(.)$ on both sides of Equation (30). Then, according to Theorem 1, we obtain

$${}^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \mathscr{J}(\mathcal{S}) = {}^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + {}^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}} I_{\Theta^+}^{\mathcal{X};\mathscr{W}} \curlywedge(\mathcal{S}) = \curlywedge(\mathcal{S}),$$

where, for $\mathcal{W} \in (0,1)$, we apply ${}^{\mathscr{H}}D_{\Theta^+}^{\mathcal{X},\mathcal{Z};\mathscr{W}}(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\Theta))^{\mathcal{W}-1} = 0$. We deduce that $\mathscr{J}(\mathcal{S})$ satisfies (1) if $\mathscr{J}(\mathcal{S})$ satisfies (29).

Let $\propto$: $\varrho \longrightarrow \varrho$, such that $\Phi_1 \in \varrho$

$$\begin{aligned} &\propto(\Phi_1(\mathcal{S})) \\ = \; & \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + \frac{1}{\Gamma(\mathcal{X})} \int_0^{\mathcal{S}} \mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\Phi_1}(\ni) d\ni, \end{aligned} \tag{31}$$

where $\curlywedge_{\Phi_1} \in C(\Upsilon, \mathbb{R})$ satisfies the following equality:

$$\curlywedge_{\Phi_1}(\mathcal{S}) = \rho\left(\mathcal{S}, \Phi_1(\mathcal{S}), \curlywedge_{\Phi_1}(\mathcal{S})\right). \tag{32}$$

For $\Phi_1 \in \varrho$, we obtain

$$\begin{aligned} &|\propto(\Phi_1(\mathcal{S})) - \propto(\Phi_1(\mathcal{S}_0))| \\ &= \left| \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + \frac{1}{\Gamma(\mathcal{X})} \int_0^{\mathcal{S}} \mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\Phi_1}(\ni) d\ni \right. \\ &\quad \left. - \frac{(\mathscr{W}(\mathcal{S}_0) - \mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_\Theta + \frac{1}{\Gamma(\mathcal{X})} \int_0^{\mathcal{S}} \mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S}_0) - \mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\Phi_1}(\ni) d\ni \right| \\ &\longrightarrow 0, \quad \text{as } \mathcal{S} \longrightarrow \mathcal{S}_0, \end{aligned}$$

so $\propto$: $\varrho \longrightarrow \varrho$ is continuous.

We now prove $\propto$ is contractive on ϱ. Let $\propto$: $\varrho \longrightarrow \varrho$ defined in (31). Let $\Phi_1, \Phi_2 \in C_{1-\mathcal{W};\mathscr{W}}(\Upsilon, \mathbb{R})$, and $\chi(\Phi_1(\mathcal{S}), \Phi_2(\mathcal{S})) \leq (\Bbbk_1, \cdots, \Bbbk_{15})$, and $\Bbbk_1, \cdots, \Bbbk_{15} \in [0, +\infty]$. Then, for all $\mathcal{S} \in \Upsilon$, we obtain

$$\operatorname{diag}\Big[|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|,\cdots,|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|\Big]_{15\times 15}$$
$$\leq \operatorname{diag}\Big[\Bbbk_1\theta_1\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots,\Bbbk_{15}\theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big].$$

For all $\mathcal{S}\in \Upsilon$, we obtain

$$\begin{aligned}&\Big|\propto(\Phi_1(\mathcal{S}))-\propto(\Phi_2(\mathcal{S}))\Big|\\ \leq\ &\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1}|\curlywedge_{\Phi_1}(\ni)-\curlywedge_{\Phi_2}(\ni)|d\ni,\end{aligned}\tag{33}$$

where

$$\curlywedge_{\Phi_1}(\mathcal{S})=\rho(\mathcal{S},\Phi_1(\mathcal{S}),\curlywedge_{\Phi_1}(\mathcal{S})),$$

and

$$\curlywedge_{\Phi_2}(\mathcal{S})=\rho(\mathcal{S},\Phi_2(\mathcal{S}),\curlywedge_{\Phi_2}(\mathcal{S})).$$

Through hypothesis $(\mathcal{F}_2)$, we obtain, for any $\mathcal{S}\in\Upsilon$,

$$|\curlywedge_{\Phi_1}(\mathcal{S})-\curlywedge_{\Phi_2}(\mathcal{S})|\leq\tau_1|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|+\tau_2|\curlywedge_{\Phi_1}(\mathcal{S})-\curlywedge_{\Phi_2}(\mathcal{S})|,$$

which can be written as

$$|\curlywedge_{\Phi_1}(\mathcal{S})-\curlywedge_{\Phi_2}(\mathcal{S})|\leq\frac{\tau_1}{1-\tau_2}|\Phi_1(\mathcal{S})-\Phi_2(\mathcal{S})|.\tag{34}$$

Next, using Remark 1, (33) and (34), we obtain

$$\begin{aligned}&\operatorname{diag}\Big[\Big|\propto(\Phi_1(\mathcal{S}))-\propto(\Phi_2(\mathcal{S}))\Big|,\cdots,\Big|\propto(\Phi_1(\mathcal{S}))-\propto(\Phi_2(\mathcal{S}))\Big|\Big]_{15\times15}\\
\leq\ &\operatorname{diag}\Big[\frac{\tau_1}{(1-\tau_2)\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1}|\Phi_1(\ni)-\Phi_2(\ni)|d\ni,\cdots,\\
&\frac{\tau_1}{(1-\tau_2)\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1}|\Phi_1(\ni)-\Phi_2(\ni)|d\ni\Big]_{15\times15}\\
\leq\ &\operatorname{diag}\Big[\frac{\Bbbk_1\theta_1\tau_1}{(1-\tau_2)}\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1}\\
&\times\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\ni)-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)d\ni,\cdots,\\
&\frac{\Bbbk_{15}\theta_{15}\tau_1}{(1-\tau_2)}\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1}\\
&\times\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\ni)-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)d\ni\Big]\\
\leq\ &\operatorname{diag}\Big[\frac{\Bbbk_1\theta_1\tau_1}{(1-\tau_2)}\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big),\cdots,\\
&\frac{\Bbbk_{15}\theta_{15}\tau_1}{(1-\tau_2)}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big].\end{aligned}$$

Then, we obtain

$$\chi(\propto(\Phi_1),\propto(\Phi_2))\leq\underbrace{(\frac{\tau_1}{(1-\tau_2)},\cdots,\frac{\tau_1}{(1-\tau_2)})}_{15}\ \chi(\Phi_1,\Phi_2).$$

As $0 < \dfrac{\tau_1}{(1-\tau_2)} < 1$, we deduce the contractive property of $\propto$.

Let $\mathscr{L} \in \varrho$. As $\propto(\mathscr{L}) \in \varrho$, we obtain

$$\begin{aligned}
& \operatorname{diag}\left[\left|\propto(\mathscr{L}(\mathcal{S}))-\mathscr{L}(\mathcal{S})\right|, \cdots, \left|\propto(\mathscr{L}(\mathcal{S}))-\mathscr{L}(\mathcal{S})\right|\right]_{15\times 15} \\
\leq \; & \operatorname{diag}\left[\left|\mathscr{L}(\mathcal{S})-\frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_{\Theta}\right.\right. \\
& \qquad \left. -\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}} \mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\mathscr{L}}(\ni) d\ni\right|, \cdots, \\
& \left|\mathscr{L}(\mathcal{S})-\frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_{\Theta}-\frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}} \mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S})\right. \\
& \left.\left. -\mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\mathscr{L}}(\ni) d\ni\right|\right]_{15\times 15} \\
\leq \; & \operatorname{diag}\left[\theta_1 \mathsf{AG}_1\left(\mathfrak{S}\left((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\right)\right), \cdots, \theta_{15}\mathsf{AG}_{15}\left(\mathfrak{S}\left((\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{X}}\right)\right)\right],
\end{aligned}$$

for $\mathcal{S} \in \mathrm{Y}$, which implies that

$$\chi(\propto(\mathscr{L}), \mathscr{L}) \leq \underbrace{(1, \cdots, 1)}_{15}; \tag{35}$$

hence, for all $k \in \mathbb{N}$, we obtain $\chi(\propto^k(\mathscr{L}), \propto^{k+1}(\mathscr{L})) < \underbrace{(+\infty, \cdots, +\infty)}_{15}$. We now use Theorem 3, and so we obtain a single map $\mathscr{J} \in \{\tilde{\sigma} \in \varrho : \chi(\propto \mathscr{L}, \tilde{\sigma}) < \underbrace{(+\infty, \cdots, +\infty)}_{15}\}$, such that $\propto \mathscr{J} = \mathscr{J}$. Thus,

$$\mathscr{J}(\mathcal{S}) = \frac{(\mathscr{W}(\mathcal{S})-\mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})} J_{\Theta} + \frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}} \mathscr{W}'(\ni)(\mathscr{W}(\mathcal{S})-\mathscr{W}(\ni))^{\mathcal{X}-1} \curlywedge_{\mathscr{J}}(\ni) d\ni, \tag{36}$$

for every $\mathcal{S} \in \mathrm{Y}$, where $\curlywedge_{\mathscr{J}} \in C(\mathrm{Y}, \mathbb{R})$ satisfies the equality (32), and $I_{\Theta^+}^{1-\mathcal{W};\mathscr{W}} \mathscr{J}(\Theta^+) = J_{\Theta} \in \mathbb{R}$.

Based on Theorem 3 and (35), we obtain

$$\begin{aligned}
\chi(\mathscr{J}, \mathscr{L}) \; & \leq \; \underbrace{\left(\frac{1}{1-\frac{\tau_1}{(1-\tau_2)}}, \cdots, \frac{1}{1-\frac{\tau_1}{(1-\tau_2)}}\right)}_{15} \chi(\propto(\mathscr{L}), \mathscr{L}) \\
& \leq \; \underbrace{\left(\frac{1}{1-\frac{\tau_1}{(1-\tau_2)}}, \cdots, \left(\frac{1}{1-\frac{\tau_1}{(1-\tau_2)}}\right)\right.}_{15},
\end{aligned}$$

which concludes (24). □

We study the next theorem for the set of real numbers. By a similar method, we can investigate the theorem for $[0, +\infty)$ and $(-\infty, 0]$. Let $\pitchfork = C_{1-\mathcal{W};\mathscr{W}}(\mathbb{R})$.

Theorem 5. *Let $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ be satisfied. If $\mathscr{L}$ in $\pitchfork$ satisfies (45), and*

$$I_{\Theta^+}^{1-\mathcal{W};\mathscr{W}} \mathscr{L}(\Theta^+) = J_{\Theta},$$

then there exists a single function $\mathscr{J}$ satisfying (19) and (20), and s.t. (24) is satisfied for all $\mathcal{S} \in \mathbb{R}$.

Proof. For all $k \in \mathbb{N}$, we consider $\mathsf{P}_k = [\Im - k, \Im + k]$. Based on Theorem 4, there exists a single function $\mathscr{J}_k \in \mathsf{C}_{1-\mathcal{W};\mathscr{W}}(\mathsf{P}_k)$, s.t.,

$$^{\mathscr{H}}D^{\mathcal{X},\mathcal{Z};\mathscr{W}}_{\Theta^+}\mathscr{J}_n(\mathcal{S}) = \mathcal{W}(\mathcal{S}, \mathscr{J}_k(\mathcal{S}), \mathscr{J}_k(\eta(\mathcal{S}))), \qquad \mathcal{S} \in \mathsf{P}_k, \tag{37}$$

$$I^{1-\mathcal{W};\mathscr{W}}_{\Theta^+}\mathscr{J}_k(\Theta^+) = J_\Theta, \qquad J_\Theta \in \mathbb{R}, \tag{38}$$

and

$$\begin{aligned}
&\operatorname{diag}\Big[\,|\mathscr{J}_k(\mathcal{S}) - \mathscr{L}(\mathcal{S})|, \cdots, |\mathscr{J}_k(\mathcal{S}) - \mathscr{L}(\mathcal{S})|\,\Big]_{15\times 15}\\
&\le \operatorname{diag}\Bigg[\frac{\theta_1}{1 - \dfrac{\tau_1}{(1-\tau_2)}}\mathsf{AG}_1\Big(\mathfrak{S}\Big(\big(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0)\big)^{\mathcal{X}}\Big)\Big), \cdots,\\
&\qquad\qquad \frac{\theta_{15}}{1 - \dfrac{\tau_1}{(1-\tau_2)}}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big(\big(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0)\big)^{\mathcal{X}}\Big)\Big)\Bigg],
\end{aligned} \tag{39}$$

for all $\mathcal{S} \in \mathsf{P}_k$. The uniqueness of $\mathscr{J}_k$ implies that if $\mathcal{S} \in \mathsf{P}_k$, then

$$\mathscr{J}_k(\mathcal{S}) = \mathscr{J}_{k+1}(\mathcal{S}) = \mathscr{J}_{k+2}(\mathcal{S}) = \cdots. \tag{40}$$

Consider $k(\mathcal{S}) \in \mathbb{N}$ as

$$k(\mathcal{S}) = \min\{k \in \mathbb{N} \mid \mathcal{S} \in \mathsf{P}_k\}.$$

In addition, consider a function $\mathscr{J}$ given by

$$\mathscr{J}(\mathcal{S}) = \mathscr{J}_{k(\mathcal{S})}(\mathcal{S}), \quad \mathcal{S} \in \mathbb{R},$$

and we claim $\mathscr{J} \in \pitchfork$. For $\mathcal{S}_1 \in \mathbb{R}$, we let the integer $k_1 = k(\mathcal{S}_1)$. Thus, $\mathcal{S}_1$ belongs to the interior of P_{k_1+1}, and there is an $\epsilon > 0$, such that $\mathscr{J}(\mathcal{S}) = \mathscr{J}_{k_1+1}(\mathcal{S})$ for all $\mathcal{S}$ with $\mathcal{S}_1 - \epsilon < \mathcal{S} < \mathcal{S}_1 + \epsilon$. Then, we prove that $\mathscr{J}$ satisfies (19), (20) and (24) for all $\mathcal{S} \in \mathbb{R}$. For all $\mathcal{S} \in \mathbb{R}$, allow the integer $k(\mathcal{S})$. Thus, $\mathcal{S} \in \mathsf{P}_{k(\mathcal{S})}$, and we infer from (37) and (38) that

$$\begin{aligned}
\mathscr{J}(\mathcal{S}) &= \mathscr{J}_{k(\mathcal{S})}(\mathcal{S})\\
&= \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})}J_\Theta + \frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\ni))^{\mathcal{X}-1}\curlywedge_{\mathscr{J}_{n(\mathcal{S})}}(\ni)d\ni\\
&= \frac{(\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{W}-1}}{\Gamma(\mathcal{W})}J_\Theta + \frac{1}{\Gamma(\mathcal{X})}\int_0^{\mathcal{S}}\mathscr{W}'(\mathcal{X})(\mathscr{W}(\mathcal{S}) - \mathscr{W}(\ni))^{\mathcal{X}-1}\curlywedge_{\mathscr{J}}(\ni)d\ni,
\end{aligned} \tag{41}$$

where

$$\curlywedge_{\mathscr{J}}(\mathcal{S}) = \rho(\mathcal{S}, \mathscr{J}(\mathcal{S}), \curlywedge_{\mathscr{J}}(\mathcal{S}))$$

and

$$\curlywedge_{\mathscr{J}_{k(\mathcal{S})}}(\mathcal{S}) = \rho(\mathcal{S}, \mathscr{J}_{k(\mathcal{S})}(\mathcal{S}), \curlywedge_{\mathscr{J}_{k(\mathcal{S})}}(\mathcal{S}))$$

are in $C(\mathrm{Y}, \mathbb{R})$. The above, (41), is true because $k(\ni) \le k(\mathcal{S})$ for all $\ni \in \mathsf{P}_{k(\mathcal{S})}$, and we deduce from (40) that

$$\mathscr{J}(\ni) = \mathscr{J}_{k(\ni)}(\ni) = \mathscr{J}_{k(\mathcal{S})}(\ni). \tag{42}$$

As $\mathscr{J}(\mathcal{S}) = \mathscr{J}_{k(\mathcal{S})}(\mathcal{S})$ and $\mathcal{S} \in \mathsf{P}_{k(\mathcal{S})}$ for all $\mathcal{S} \in \mathbb{R}$, (39) concludes that

$$\begin{aligned}
&\operatorname{diag}\Big[|\mathscr{J}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|, \cdots, |\mathscr{J}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|\Big]_{15\times 15} \\
&= \operatorname{diag}\Big[|\mathscr{J}_{k(\mathcal{S})}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|, \cdots, |\mathscr{J}_{k(\mathcal{S})}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|\Big]_{15\times 15} \\
&\leq \operatorname{diag}\Bigg[\frac{\theta_1}{1 - \frac{\tau_1}{(1-\tau_2)}} \mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \cdots, \\
&\qquad \frac{\theta_{15}}{1 - \frac{\tau_1}{(1-\tau_2)}} \mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg].
\end{aligned} \tag{43}$$

Eventually, we claim $\mathscr{J}$ is single. Let $\mathscr{J}' \in ⫛$ be another function satisfying (19), (20) and (24), for any $\mathcal{S} \in \mathbb{R}$. As $\mathscr{J}|_{\mathsf{P}_{k(\mathcal{S})}} (= \mathscr{J}_{k(\mathcal{S})})$ and $\mathscr{J}'|_{\mathsf{P}_{k(\mathcal{S})}}$ both satisfy (19), (20) and (24) for any $\mathcal{S} \in \mathsf{P}_{k(\mathcal{S})}$, and the uniqueness of $\mathscr{J}_{k(\mathcal{S})} = \mathscr{J}|_{\mathsf{P}_{k(\mathcal{S})}}$, we conclude that

$$\mathscr{J}(\mathcal{S}) = \mathscr{J}|_{\mathsf{P}_{k(\mathcal{S})}}(\mathcal{S}) = \mathscr{J}'|_{\mathsf{P}_{k(\mathcal{S})}}(\mathcal{S}) = \mathscr{J}'(\mathcal{S}),$$

as required. $\square$

4. Example

Example 1. *Consider the system (1) for* $\mathscr{W}(\mathcal{S}) = \mathcal{S}^2$, $\Theta = 0$, $D = 1$, $\mathcal{X} = \mathcal{Z} = \frac{1}{2}$, $\mathscr{J} : [0,1] \to \mathbb{R}$ *and* $\rho : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\rho(\mathcal{S}, \phi, \psi) = \frac{1}{5} {}_2\mathbb{F}_1(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S})\phi + \frac{1}{8}\psi,$$

and we obtain

$$\begin{cases}
{}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{J}(\mathcal{S}) = \frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)\mathscr{J}(\mathcal{S}) + \frac{1}{8} {}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{J}(\mathcal{S}), \\
I_{0^+}^{\frac{1}{4};\mathcal{S}^2} \mathscr{J}(0^+) = J_0 \in \mathbb{R},
\end{cases} \tag{44}$$

in which $\mathsf{P} \in \mathbb{R}^+$ *and* $\mathsf{Q} \in \mathbb{R}$, *and* ${}_2\mathbb{F}_1(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; .)$ *is the hypergeometric function.*

In addition, consider the inequality below:

$$\begin{aligned}
&diag\Bigg[\Big|{}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{L}(\mathcal{S}) - \frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)\mathscr{L}(\mathcal{S}) - \frac{1}{8} {}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{L}(\mathcal{S})\Big|, \cdots, \\
&\quad \Big|{}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{L}(\mathcal{S}) - \frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)\mathscr{L}(\mathcal{S}) - \frac{1}{8} {}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2} \mathscr{L}(\mathcal{S})\Big|\Bigg]_{15\times 15} \\
&\leq diag\Big[\theta_1 \mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \cdots, \theta_{15}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Big],
\end{aligned} \tag{45}$$

for every $\mathcal{S} \in [0,1]$.

For any $\kappa, \upsilon, \overline{\kappa}, \overline{\upsilon} \in \mathbb{R}$ *and* $\mathcal{S} \in [0,1]$, *we obtain*

$$\begin{aligned}
&|\rho(\mathcal{S}, \Omega, \varpi) - \rho(\mathcal{S}, \overline{\Omega}, \overline{\varpi})| \\
=\ & \Big|\frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)\Omega + \frac{1}{8} {}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2}\varpi - \frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)\overline{\Omega} - \frac{1}{8} {}^{\mathscr{H}}D_{0^+}^{\frac{1}{2},\frac{1}{2};\mathcal{S}^2}\overline{\varpi}\Big| \\
\leq\ & \frac{1}{5} {}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)|\Omega - \overline{\Omega}| + \frac{1}{8}|\varpi - \overline{\varpi}|.
\end{aligned}$$

Thus, condition $(\mathcal{F}_2)$ *is satisfied with* $\tau_1 = \frac{1}{5}{}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)$ *and* $\tau_2 = \frac{1}{8}$, *and the condition*

$$\frac{8}{35}{}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big) < 1$$

is satisfied. Theorem 4 implies that (44) has a single solution, and is stable with

$$\begin{aligned} &diag\Big[|\mathscr{J}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|, \cdots, |\mathscr{J}(\mathcal{S}) - \mathscr{L}(\mathcal{S})|\Big]_{15\times 15} \\ &\leq diag\Bigg[\frac{\theta_1}{1 - \frac{8}{35}{}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)}\mathsf{AG}_1\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big), \\ &\qquad \cdots, \frac{\theta_{15}}{1 - \frac{8}{35}{}_2\mathbb{F}_1\Big(\mathsf{P}, \mathsf{Q}, \mathsf{Q}; \mathcal{S}\Big)}\mathsf{AG}_{15}\Big(\mathfrak{S}\Big((\mathscr{W}(\mathcal{S}) - \mathscr{W}(0))^{\mathcal{X}}\Big)\Big)\Bigg]_{15\times 15}, \end{aligned}$$

where $\mathcal{S} \in [0, 1]$.

5. Conclusions

We used the aggregation maps on diverse special functions such as the Mittag-Leffler function, supertrigonometric and superhyperbolic functions, to propose a novel controller that helps us study a different notion of stability: namely, multi-stability. Multi-stability enables us to obtain various approximations, depending on various special functions, and to obtain optimal stability, which, in turn, enables us to obtain a unique optimum solution.

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