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Left and Right Operator Rings of a Γ Ring in Terms of Rough Fuzzy Ideals

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Abstract: The relationship between Rough Set (RS) and algebraic systems has been long studied by mathematicians. RS is a growing research area that encourages studies into both real-world applications and the theory itself. In RS, a universe subset is characterized by a pair of ordinary sets called lower and upper approximations. In this study, we look attentively at the use of rough sets when the universe set has a ring structure. The main contribution of the paper is to concentrate on the study of rough fuzzy ideals concerning the gamma ring and to describe some properties of its lower and upper approximations. This paper deals with the connection between Rough Fuzzy Sets (RFS) and ring theory. The goal of this paper is to present the notion of Left Operator Rings (LOR) and Right Operator Rings (ROR) in the gamma ring structure. We introduce some basic concepts of rough fuzzy left and right operator rings. Furthermore, we investigate some characterizations of left and right operator rings and prove some theorems based on these results.

Keywords: Γ Rings; Rough set; Rough fuzzy set; Rough fuzzy ideal; Left operator ring; Right operator ring

MSC: 08A72; 41A65

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Citation: Pushpanathan, D.; Devarasan, E. Left and Right Operator Rings of a Γ Ring in Terms of Rough Fuzzy Ideals. *Axioms* **2023**, *12*, 808. https://doi.org/10.3390/axioms12090808

Academic Editor: Boldizsár Tüű-Szabó

Received: 7 July 2023
Revised: 8 August 2023
Accepted: 10 August 2023
Published: 22 August 2023

1. Introduction

Researchers have studied abstract algebra in fuzzy settings since the introduction of Zadeh's fuzzy sets in 1965 [1]. The use of fuzzy sets with algebraic structures plays a major role in mathematics, with numerous applications. As a result, academics have plenty of motivation to explore concepts and conclusions from the area of abstract algebras and apply them to fuzzy settings more broadly. The literature on numerous fuzzy algebraic ideas is rapidly expanding. Many researchers have also addressed various algebraic structures in fuzzy versions [2]. The gamma ring is a type of algebraic structure. Nobusawa proposed the gamma ring concept in 1964 [3]. Compared to ring structures, this is more common. Barnes lowered the requirements of Nobusawa's gamma ring [4]. A fuzzy set was applied in the theory of gamma rings by Jun et al. [5,6]. In ring theory, gamma rings have been used to extend several fundamental conclusions. Dutta et al. discussed several compositions of fuzzy ideals in gamma rings [7]. Kyuno and Luh investigated the structure of gamma rings and discovered several generalizations that are similar to equivalent portions in ring theory [8–10]. Muhiuddin et al. studied fuzzy bi ideals in semirings [11]. Murray et al. focused on the operator ring and discussed some of its related results [12]. Alam studied the concept of fuzzy rings with operators and proved some of its properties [13].

IFS is an extension of a fuzzy set. In 1986 the idea of IFS was stated by Atanassov to address the issue of non determinacy caused by a single membership function. IFS can be useful in explaining decision-making uncertainty and ambiguity. A study of the intuitionistic fuzzy ideals of gamma rings was conducted by Palaniappan et al. [14,15]. Ezhilmaran et al. explored the characteristic properties of gamma near rings in 2017 [16].

Yamin et al. examined some new ideas such as an intuitionistic fuzzy ring with operators, an intuitionistic fuzzy ideal with operators, and an intuitionistic fuzzy quotient ring with operators [17].

Pawlak introduced RS theory [18,19]. Pawlak's RS theory and Zadeh's fuzzy set theory are complementary generalizations of classical set theory. A fuzzy set deals with possibility uncertainty, which is associated with the imprecision of states, perceptions, and preferences, whereas RS deals with the uncertainty caused by the ambiguity of information. Fuzzy set theory combined with RS theory led to a wide range of models. Many researchers discussed fuzzy sets with algebraic structure. RS is a growing field of study that encourages investigation into both practical applications and the theory itself. Applications in science were more effective when the algebraic structure of a mathematical theory was studied. This is the main motivation for our research into the algebraic structures of these generalized rough sets. In addition to providing additional insight into RS theory, such research may result in the development of new application methods. Davvaz et al. proved the connection between rough sets with ring theory and also discussed rough subrings [20]. Some authors discussed the results and methods of rough algebraic structures [21,22]. Agusfrianto et al. discussed rough rings and proved some results [23].

Fuzzy set theory and RS theory are two widely used approaches for dealing with the ambiguity and imprecision of the data. These theories can combine in a very helpful way even if they are different from one another. In 1990, Dubois et al. investigated RFS and fuzzy rough sets [24]. Using a crisp approximation space, RFS is a pair of fuzzy sets derived from a fuzzy set and a fuzzy rough set is an approximation of a crisp set in a fuzzy approximation space. RFS can be used in analyzing improbability in classification, especially vagueness. Subha et al. focused on rough semiprime and rough fuzzy ideals in semigroups [25]. Few researchers have discussed rough fuzzy ideals in rings [26–28]. Recently, many authors have discussed RFI in gamma ring structures [29,30]. Researchers discussed the gamma ring using fuzzy and intuitionistic fuzzy sets. An extension of this is proposed in a new work using rough fuzzy ideals in gamma rings.

The present work aims at giving the RFI in left and right operators of the gamma ring structure. The arrangement of the article is as follows. In Section 2, we include the prerequisites of the concept. We discuss some properties of ROR and prove some results in Section 3. Section 4 investigates the notion of a rough fuzzy ideal with a left operator ring and presents the relevant results. Finally, Section 5 concludes the study.

2. Preliminaries

Definition 1 ([3])**.** *If* $N = \{p, q, r \ldots .\}$ *and* $\Gamma = \{\alpha, \beta, \gamma \ldots .\}$ *be two additive abelian groups and for all* $p, q, r \in N$ *and* $\alpha, \beta \in \Gamma$ *the following axioms are satisfied*

(1) $p\alpha q \in N$
(2) $(p+q)\alpha\, r = p\alpha\, r + q\alpha\, r,\ p(\alpha+\beta)q = p\alpha q + p\beta q,\ p\alpha(q+r) = p\alpha q + p\alpha r,$
(3) $(p\alpha q)\beta\, r = p\alpha(q\beta r).$

Then N is called a Γ *Ring. If these axioms are enriched by Barnes ([4)]*

(1′) $p\alpha q \in N, \alpha p\beta \in \Gamma$
(2′) $(p+q)\alpha\, r = p\alpha\, r + q\alpha\, r,\ p(\alpha+\beta)q = p\alpha q + p\beta q,\ p\alpha(q+r) = p\alpha q + p\alpha r,$
(3′) $(p\alpha q)\beta\, r = p(\alpha q\beta)r = p\alpha(q\beta r),$
(4′) $p\alpha q = 0$ *for all* $p, q \in N$ *implies* $\alpha = 0$.

Definition 2 ([19])**.** *Suppose the knowledge base K = (U, R) with each subset* $P \subseteq U$ *and an equivalence relation* $R \in IND(K)$ *we associated two subsets* $\underline{apr}(P) = \bigcup\{Y \in U/R : Y \subseteq P\}$ *and* $\overline{apr}(P) = \bigcup\{Y \in U/R : Y \cap P \neq \phi\}$*, called apr-lower and apr-upper approximations of P respectively.*

Definition 3 ([24])**.** *Let* $X \subseteq U$ *be a set, R be an equivalence relation on U and P be a fuzzy subset in U. Then upper and lower approximation of* $\overline{apr}(X)$ *and* $\underline{apr}(X)$ *be the fuzzy subsets P by R are* *the fuzzy subset of* U/R *with membership function is* $\mu_{\overline{apr}(P)}(X_i) = sup\{\mu_P(x)/\omega(X_i) = [x]_R\}$ *and* $\mu_{\underline{apr}(P)}(P_i) = inf\{\mu_P(x)/\omega(X_i) = [x]_R\}$. *Where* $\mu_{\overline{apr}(P)}(X_i)$ *(resp.*$\mu_{\underline{apr}(P)}(X_i)$*) is the membership of* X_i in $\overline{apr}(P)$ *(resp.* $\underline{apr}(P)$*).* $(\overline{apr}(P), \underline{apr}(P))$ *is called a RFS.*

Definition 4 ([30])**.** *An upper (resp. Lower) RFS* $P =< \overline{apr}_P, \underline{apr}_P >$ *in N is called a RFLI (resp. RFRI) of a Γ Ring N.*

$(1)\overline{apr}_P(a-b) \geq \{\overline{apr}_P(a) \wedge \overline{apr}_P(b)\}$,
$\overline{apr}_P(a\lambda b) \geq \overline{apr}_P(b)[resp.\overline{apr}_P(a\lambda b) \geq \overline{apr}_P(a)]$
$(2)\ \underline{apr}_P(a-b) \leq \left\{\underline{apr}_P(a) \vee \underline{apr}_P(b)\right\}$,
$\underline{apr}_P(a\lambda b) \leq \underline{apr}_P(b)\,[resp.\underline{apr}_P(a\lambda b) \leq \underline{apr}_P(a)\,]$*, for all* $a, b \in N$ *and* $\lambda \in \Gamma$.

Example 1 ([30])**.** *Let* $N = \{a,\ b,\ c,\ d\}$ *and* λ = {*e*, *f*, *g*, *h*}*. Define N and α as follows*

-	a	b	c	d
a	a	b	c	d
b	b	b	d	c
c	c	d	d	c
d	d	c	c	c

λ	e	f	g	h
e	e	f	g	h
f	f	f	h	g
g	g	h	h	g
h	h	g	g	g

$$\overline{apr}_p(x) = \begin{cases} 0.5 & if\ x = a, e \\ 0.6 & if\ x = b, f \\ 0.6 & if\ x = c, d, g, h \end{cases},\ \underline{apr}_P(x) = \begin{cases} 0.7 & if\ x = a, e \\ 0.5 & if\ x = b, f \\ 0.4 & if\ x = c, d, g, h \end{cases}$$

By routine calculation, clearly N is a RFI.

3. Right Operator Ring

In this section, we establish some of the properties of ROR of a Γ Ring and proved some related theorems. Throughout the paper, we assume that for any rough fuzzy left [resp. right, two sided] ideal P of N, $\overline{apr}_P(0_N) = 1$, $\underline{apr}_P(0_N) = 0$, and $\overline{apr}_P(0_L) = 1$, $\underline{apr}_P(0_L) = 0$.

Definition 5 ([9])**.** *Let N be a Γ Ring and F be the free abelian group generated by* $\Gamma \times N$*, the set of all ordered pairs* (λ, a) *with* $a \in N$, $\lambda \in \Gamma$*. Let A be the subgroup of elements* $\sum_i n_i(\lambda_i, a_i) \in F$*, where* n_i *are integers such that* $\sum_i n_i(a\lambda_i a_i) = 0$ *for all* $a \in N$*. Let* $R = F/A$*, the factor group of F by A and the coset* $(\lambda,\ a) + A$ *by* $[\lambda,\ a]$*. Clearly every element in R can be expressed as a finite sum* $\sum_i [\lambda_i,\ a_i]$*. Also, for all* $a, b \in N$ *and* $\lambda, \mu \in \Gamma$, $[\lambda,\ a] + [\mu,\ a] = [\lambda + \mu,\ a]$ *and* $[\lambda,\ a] + [\lambda,\ b] = [\lambda,\ a + b]$*. We define a multiplication in R by* $\sum_i [\lambda_i,\ a_i] \sum_j [\mu_j,\ b_j] = \sum_{i,j} [\lambda_i,\ a_i\mu_j b_j]$*. Then R forms the Ring. Furthermore, N is a right R-module, with the definition* $a. \sum_i [\lambda_i, a_i] = \sum_i a\lambda_i a_i$ *for* $a \in N$, $\sum_i [\lambda_i, a_i] \in R$*. We call the ring R is the ROR of the Γ Ring N.*

Definition 6 ([9])**.** *A right unity of Γ Ring N is an element* $\sum_i [\delta_i,\ e_i] \in R$ *such that* $\sum_i a\delta_i e_i = a$ *for every element* $a \in N$. $\sum_i [\delta_i,\ e_i]$ *is the unity of R.*

3.1. Rough Fuzzy Sets in Right Operator Rings of a Γ Ring

Definition 7 ([7])**.** *For a fuzzy subset* φ *of R, define a fuzzy subset* φ^* *of N by*

$$\varphi^*(a) = \bigwedge_{\vartheta \in \Gamma} \varphi([\vartheta, a])\ where\ a \in N.$$

For a fuzzy subset ρ of N define a fuzzy subset $\rho^{*\prime}$ of R by

$$\rho^{*\prime}\left(\sum_i [\lambda_i, a_i]\right) = \bigwedge_{n\in N} \rho\left(\sum_i n\lambda_i a_i\right), \textit{where } \sum_i [\lambda_i, a_i] \in R.$$

Definition 8. *For a rough fuzzy subset $P = <\overline{apr}_P,\ \underline{apr}_P>$ of R, define a rough fuzzy subset $P^* = <\overline{apr}_{P^*},\ \underline{apr}_{P^*}>$ of N by*

$$\overline{apr}_{P^*}(a) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta, a]) \textit{ and}$$
$$\underline{apr}_{P^*}(a) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta, a]), \textit{ where } a \in N.$$

For a rough fuzzy subset $Q = <\overline{apr}_Q,\ \underline{apr}_Q>$ of N, define a rough fuzzy subset $Q^{*\prime} = <\overline{apr}_{Q^{*\prime}},\ \underline{apr}_{Q^{*\prime}}>$ of R by

$$\overline{apr}_{Q^{*\prime}}\left(\sum_i [\lambda_i, a_i]\right) = \bigwedge_{n\in N} \overline{apr}_Q\left(\sum_i n\lambda_i a_i\right) \textit{ and}$$
$$\underline{apr}_{Q^{*\prime}}\left(\sum_i [\lambda_i, a_i]\right) = \bigvee_{n\in N} \underline{apr}_Q\left(\sum_i n\lambda_i a_i\right), \textit{ where} \sum_i [\lambda_i, a_i] \in R.$$

3.2. Characterizations of Rough Fuzzy Ideals in Right Operator Rings of a Γ Ring

Theorem 1. *If $\{P_i / i \in I\}$ is a family of rough fuzzy subsets of R, then*

$$\left(\bigcap_{i\in I} \overline{apr}_{P_i^*}\right) = \left(\bigcap_{i\in I} \overline{apr}_{P_i}\right)^* \textit{and} \left(\bigcup_{i\in I} \underline{apr}_{P_i^*}\right) = \left(\bigcup_{i\in I} \underline{apr}_{P_i}\right)^*.$$

Proof. Let a ∈ N.

$$\left(\bigcap_{i\in I} \overline{apr}_{P_i}\right)^*(a) = \bigwedge_{\vartheta\in\Gamma}\left[\left(\bigcap_{i\in I} \overline{apr}_{P_i}\right)([\vartheta, a])\right] = \bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{i\in I}\left(\overline{apr}_{P_i}[\vartheta, a]\right)\right]$$
$$= \bigwedge_{i\in I}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_{P_i}([\vartheta, a])\right]\right] = \bigwedge_{i\in I}\left[\overline{apr}_{P_i^*}(a)\right] = \left(\bigcap_{i\in I} \overline{apr}_{P_i^*}\right)(a).$$

So $\left(\bigcap_{i\in I} \overline{apr}_{P_i^*}\right) = \left(\bigcap_{i\in I} \overline{apr}_{P_i}\right)^*$.

Also

$$\left(\bigcup_{i\in I} \underline{apr}_{P_i}\right)^*(a) = \bigvee_{\vartheta\in\Gamma}\left[\left(\bigcup_{i\in I} \underline{apr}_{P_i^*}\right)([\vartheta, a])\right] = \bigvee_{\vartheta\in\Gamma}\left[\bigvee_{i\in I}\left(\underline{apr}_{P_i}[\vartheta, a]\right)\right]$$
$$= \bigvee_{i\in I}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P_i}([\vartheta, a])\right]\right] = \bigvee_{i\in I}\left[\underline{apr}_{P_i^*}(a)\right] = \left(\bigcup_{i\in I} \underline{apr}_{P_i^*}\right)(a).$$

So $\left(\bigcup_{i\in I} \underline{apr}_{P_i^*}\right) = \left(\bigcup_{i\in I} \underline{apr}_{P_i}\right)^*$. □

Theorem 2. *If $P = <\overline{apr}_P,\ \underline{apr}_P> \in$ RFI (R) [resp. RFRI (R), RFLI (R)], then the RFS $P^* = <\overline{apr}_{P^*},\ \underline{apr}_{P^*}> \in$ RFI (N) [resp. RFRI (N), RFLI (N)].*

Proof. Consider P be the RFI of R. Then $\overline{apr}_P(0_R) = 1$ and $\underline{apr}_P(0_R) = 0$. Now

$$\overline{apr}_{P^*}(0_N) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ 0_N]) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P(0_R) = 1 \text{ and}$$
$$\underline{apr}_{P^*}(0_N) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,\ 0_N]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P(0_R) = 0.$$

So P^* is nonempty. If $a,\ b\ \in N\ \ and\ \lambda \in \Gamma$.

$$\overline{apr}_{P^*}(a-b) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ a-b]) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,a]-[\vartheta,b])$$
$$\geq \left\{\bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ a])\right\} \bigwedge \left\{\bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ b])\right\} = \overline{apr}_{P^*}(a) \bigwedge \overline{apr}_{P^*}(b).$$
$$\underline{apr}_{P^*}(a-b) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a-b]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a]-[\vartheta,b])$$
$$\leq \left\{\bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a])\right\} \bigvee \left\{\bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,b])\right\} = \underline{apr}_{P^*}(a) \bigvee \underline{apr}_{P^*}(b).$$

Also

$$\overline{apr}_{P^*}(a\lambda b) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ a\lambda b]) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,a][\lambda,b])$$
$$\geq \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ a]) = \overline{apr}_{P^*}(a).$$

For right ideals

$$\overline{apr}_{P^*}(a\lambda b) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,\ a\lambda b]) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,a][\lambda,b])$$
$$\geq \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\lambda,b]) = \overline{apr}_P([\lambda,b]) \geq \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([\vartheta,b]) = \overline{apr}_{P^*}(b)$$

Similarly,

$$\underline{apr}_{P^*}(a\lambda b) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a\lambda b]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a][\lambda,b])$$
$$\leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a]) = \underline{apr}_{P^*}(a).$$

For right ideals

$$\underline{apr}_{P^*}(a\lambda b) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a\lambda b]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,a][\lambda,b])$$

$$\leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\lambda,b]) = \underline{apr}_P([\lambda,b]) \leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([\vartheta,b]) = \underline{apr}_{P^*}(b).$$

So, P^* is a RFI of N. □

Theorem 3. *If* $P = <\overline{apr}_P, \underline{apr}_P> \in$ *RFI (N) [resp. RFLI (N), RFRI (N)], then the RFS* $P^{*\prime} = <\overline{apr}_{P^{*\prime}},\ \underline{apr}_{P^{*\prime}}> \in$*RFI (R) [resp. RFLI (R), RFRI (R)].*

Proof. Let P be a RFI of N. Then $\overline{apr}_P(0_N) = 1$ and $\underline{apr}_P(0_N) = 0$.

$$\overline{apr}_{P^{*\prime}}([\vartheta,\ 0_N]) = \bigwedge_{n\in N} \overline{apr}_P(n\vartheta 0_N) = \overline{apr}_P(0_N) = 1 \ and$$
$$\underline{apr}_{P^{*\prime}}([\vartheta,\ 0_N]) = \bigvee_{m\in M} \underline{apr}_P(n\vartheta 0_N) = \underline{apr}_P(0_N) = 0.$$

So, $P^{*'}$ is nonempty.

Let $\sum_i [\lambda_i, a_i], \sum_j [\mu_j, b_j] \in R$. Then

$$\begin{aligned}
\overline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i] - \sum_j [\mu_j, b_j]\right) &= \bigwedge_{n\in N} \overline{apr}_P\left(\sum_i n\lambda_i a_i - \sum_j n\mu_j b_j\right) \\
&\geq \bigwedge_{n\in N}\left[\overline{apr}_P\left(\sum_i n\lambda_i a_i\right) \wedge \overline{apr}_P\left(\sum_j n\mu_j b_j\right)\right] \\
&= \left\{\bigwedge_{n\in N} \overline{apr}_P\left(\sum_i n\lambda_i a_i\right)\right\} \wedge \left\{\bigwedge_{n\in N} \overline{apr}_P\left(\sum_j n\mu_j b_j\right)\right\} \\
&= \overline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i]\right) \wedge \overline{apr}_{P^{*'}}\left(\sum_j [\mu_j, b_j]\right) \text{and}
\end{aligned}$$

$$\begin{aligned}
\underline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i] - \sum_j [\mu_j, b_j]\right) &= \bigvee_{n\in N} \underline{apr}_P\left(\sum_i n\lambda_i a_i - \sum_j n\mu_j b_j\right) \\
&\leq \bigvee_{n\in N}\left[\underline{apr}_P\left(\sum_i n\lambda_i a_i\right) \vee \underline{apr}_P\left(\sum_j n\mu_j b_j)\right)\right] \\
&= \left\{\bigvee_{n\in N} \underline{apr}_P\left(\sum_i n\lambda_i a_i\right)\right\} \vee \left\{\bigvee_{n\in N} \underline{apr}_P\left(\sum_j n\mu_j b_j\right)\right\} \\
&= \underline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i]\right) \vee \underline{apr}_{P^{*'}}\left(\sum_j [\mu_j, b_j]\right).
\end{aligned}$$

Again,

$$\begin{aligned}
\overline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i] \sum_j [\mu_j, b_j]\right) &= \overline{apr}_{P^{*'}}\left(\sum_{i,j}\left[\lambda_i,\ a_i\mu_j b_j\right]\right) = \bigwedge_{n\in N} \overline{apr}_P\left(\sum_{i,j} n\lambda_i a_i \mu_j b_j\right) \\
&\geq \bigwedge_{n\in N}\left[\bigwedge_i\left[\overline{apr}_P\left(n\lambda_1\left(\sum_j\left(a_1\mu_j b_j\right)\right)\right), \overline{apr}_P\left(n\lambda_2\left(\sum_j\left(a_2\mu_j b_j\right)\right)\right)\ldots\right]\right] \\
&\geq \bigwedge_{n\in N}\left[\bigwedge_i\left[\overline{apr}_P\left(\sum_j\left(a_1\mu_j b_j\right)\right), \overline{apr}_P\left(\sum_j\left(a_2\mu_j b_j\right)\right)\ldots\right]\right] \\
&= \bigwedge\left[\overline{apr}_P\left(\sum_j a_1\mu_j b_j\right), \overline{apr}_P\left(\sum_j a_2\mu_j b_j\right)\ldots\right] \\
&\geq \bigwedge_{n\in N}\left[\overline{apr}_P\left(\sum_j n\mu_j b_j\right)\right] = \overline{apr}_{P^{*'}}\left(\sum_j [\mu_j, b_j]\right)
\end{aligned}$$

Similarly, we can prove that

$$\overline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i] \sum_j [\mu_j, b_j]\right) \geq \overline{apr}_{P^{*'}}\left(\sum_i [\lambda_i, a_i]\right)$$

and

$$\underline{apr}_{P^{*\prime}}\left(\sum_i[\lambda_i,a_i]\sum_j[\mu_j,b_j]\right)=\underline{apr}_{P^{*\prime}}\left(\sum_{i,j}[\lambda_i,\ a_i\mu_j b_j]\right)=\bigvee_{n\in N}\underline{apr}_P\left(\sum_{i,j}n\lambda_i a_i\mu_j b_j\right)$$
$$\le\bigvee_{n\in N}\left[\bigvee_i\left[\underline{apr}_P\left(n\lambda_1\left(\sum_j a_1\mu_j b_j\right)\right),\underline{apr}_P\left(n\lambda_2\left(\sum_j a_2\mu_j b_j\right)\right)\ldots\right]\right]$$
$$\le\bigvee_{n\in N}\left[\bigvee_i\left[\underline{apr}_P\left(\sum_j a_1\mu_j b_j\right),\ \underline{apr}_P\left(\sum_j a_2\mu_j b_j\right)\ldots\right]\right]$$
$$=\bigvee\left[\underline{apr}_P\left(\sum_j a_1\mu_j b_j\right),\underline{apr}_P\left(\sum_j a_2\mu_j b_j\right)\ldots\ldots\ldots\right]$$
$$\le\bigvee_{n\in N}\left[\underline{apr}_P\left(\sum_j n\mu_j b_j\right)\right]=\underline{apr}_{P^{*\prime}}\left(\sum_j[\mu_j,b_j]\right).$$

Similarly, we can prove that,

$$\underline{apr}_{P^{*\prime}}\left[\sum_i[\lambda_i,a_i]\sum_j[\mu_j,b_j]\right]\le\underline{apr}_{P^{*\prime}}\left(\sum_i[\lambda_i,a_i]\right).$$

So $P^{*\prime}$ is a RFI of R. □

Theorem 4. *The lattices of all RFI (resp. RFLI) of N and R are bijective mapping with respect to the inclusion and if preserves the isomorphic condition $P\to P^{*\prime}$ where $P\in$ RFI(N) [resp. RFLI(N)] and $P^{*\prime}\in RFI(R)$ [resp.RFLI(R)].*

Proof. First, we show that $\left(P^{*\prime}\right)^*=P$, where $P\in$ RFI(N). Let $a\in N$. Then

$$\begin{aligned}\left(\overline{apr}_{P^{*\prime}}\right)^*(a)&=\bigwedge_{\vartheta\in\Gamma}[\overline{apr}_{P^{*\prime}}([\vartheta,a])]=\bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{n\in N}[\overline{apr}_P\ (n\vartheta a)]\right]\\&\ge\bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{n\in N}[\ \overline{apr}_P\ (a)]\right]=\overline{apr}_P(a).\\\left(\underline{apr}_{P^{*\prime}}\right)^*(a)&=\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P^{*\prime}}([\vartheta,a])\right]=\bigvee_{\vartheta\in\Gamma}\left[\bigvee_{n\in N}\left[\underline{apr}_A(n\vartheta a)\right]\right]\\&\le\bigvee_{\vartheta\in\Gamma}\left[\bigvee_{n\in N}\left[\underline{apr}_P(a)\right]\right]=\underline{apr}_P(a).\end{aligned}$$

So, $P\subseteq\left(P^{*\prime}\right)^*$.

Assume $\sum_j[e_j,\delta_j]$ be the element of N with left unity.

We have $\sum_j e_j\delta_j\, a = a$ for every $a\in N$.

Now,

$$\overline{apr}_P(a)=\overline{apr}_P\left(\sum_j e_j\delta_j a\right)\ge\bigwedge_i[\overline{apr}_P(e_1\delta_1 a),\ \overline{apr}_P(e_2\delta_2 a),\ldots]$$
$$\ge\bigwedge_{\vartheta\in\Gamma}[\bigwedge_{n\in N}[\overline{apr}_P(n\vartheta a)]]=(\underline{apr}_{P^{*\prime}})^*(a).$$
$$\underline{apr}_P=\underline{apr}_P\left(\sum_j e_j\delta_j a\right)\le\bigvee_i\left[\underline{apr}_P(e_1\delta_1 a),\ \underline{apr}_P(e_2\delta_2 a),\ldots\right]$$
$$\le\bigvee_{\vartheta\in\Gamma}[\bigvee_{n\in N}[\underline{apr}_P(n\vartheta a)]]=(\underline{apr}_{P^{*\prime}})^*(a).$$

So, $\left(\mathrm{P}^{*\prime}\right)^{*} \subseteq \mathrm{P}$. Hence $\mathrm{P} = \left(\mathrm{P}^{*\prime}\right)^{*}$. Again, let P be a RFI of R. Now,

$$\begin{aligned}
(\overline{apr}_{P^*})^{*\prime}\left(\sum_k[\lambda_k,\ a_k]\right) &= \bigwedge_{n\in N}\left[\overline{apr}_{P^*}\left(\sum_k (n\lambda_k a_k)\right)\right] \\
&= \bigwedge_{n\in N}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_P\left(\vartheta,\ \sum_k n\lambda_k a_k\right)\right]\right] \\
&= \bigwedge_{n\in N}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_P\left([\vartheta, n]\sum_k\ [\lambda_k,\ a_k]\right)\ \right]\right] \\
&\geq \overline{apr}_P\left(\sum_k[\lambda_k,\ a_k]\right).
\end{aligned}$$

$$\begin{aligned}
\left(\underline{apr}_{P^*}\right)^{*\prime}\left(\sum_k[\lambda_k,\ a_k]\right) &= \bigvee_{n\in N}\left[\underline{apr}_{P^*}\left(\sum_k (n\lambda_k a_k)]\right)\right] \\
&= \bigvee_{n\in N}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_P\left(\left[\vartheta,\ \sum_k n\lambda_k a_k\right)\right]\right]\right] \\
&= \bigvee_{n\in N}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_P\left([\vartheta,\ n]\sum_k\ [\lambda_k a_k]\right)\right]\right] \\
&\leq \underline{apr}_P\left(\sum_k\ [\lambda_k,\ a_k]\ \right)
\end{aligned}$$

So, $\mathrm{P} \subseteq \left(\mathrm{P}^{*}\right)^{*\prime}$.

Let $\sum_j[\delta_j{}',\ e_j{}']$ be the element of N with right unity.

$$\begin{aligned}
&\overline{apr}_P\left(\sum_k[\lambda_k,,\ a_k]\right) = \overline{apr}_P\left(\sum_j[\delta_j{}',\ e_j{}']\sum_k[\lambda_k,\ a_k]\right) \\
&\geq \bigwedge_j\left[\overline{apr}_P\left([\delta_1{}', e_1{}']\sum_k[\lambda_k,\ a_k]\right), \overline{apr}_P\left([\delta_2{}', e_2{}']\sum_k[\lambda_k,\ a_k]\right)\ldots\right] \\
&\geq \bigwedge_{n\in N}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_P\left([\vartheta, n]\sum_k[\lambda_k,\ a_k]\right)\right]\right] = (\overline{apr}_{P^*})^{*\prime}\left(\sum_k[\lambda_k,\ a_k]\right). \\
&\underline{apr}_P\left(\sum_k[\lambda_k,,\ a_k]\right) = \underline{apr}_P\left(\sum_j[\delta_j{}',\ e_j{}']\sum_k[\lambda_k,\ a_k]\right) \\
&\leq \bigvee_j\left[\underline{apr}\left([\delta_1{}', e_1{}']\sum_k[\lambda_k,\ a_k]\right),\ \underline{apr}\left([\delta_2{}', e_2{}']\sum_k[\lambda_k,\ a_k]\right)\ldots\right] \\
&\leq \bigvee_{n\in N}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_P\left([\vartheta, n]\sum_k[\lambda_k,\ a_k]\right)\right]\right] = \left(\underline{apr}_{P^*}\right)^{*\prime}\left(\sum_k\ [\lambda_k,\ a_k]\right).
\end{aligned}$$

So, $P \supseteq (P^*)^{*\prime}$. Thus $P = (P^*)^{*\prime}$. Thus, the correspondence $P \to P^{*\prime}$ is a bijection. Now, $P_1, P_2 \in RFI(N)$ such that $P_1 \subseteq P_2$

$$\begin{aligned}
\overline{apr}_{P_1{}^{*\prime}}\left(\sum_i[\lambda_i,\ a_i]\right) &= \bigwedge_{n\in N}\ \overline{apr}_{P_1}\left(\sum_i n\lambda_i a_i\right) \\
&\leq \bigwedge_{n\in N}\ \overline{apr}_{P_2}\left(\sum_i n\lambda_i a_i\right) \\
&= \overline{apr}_{P_2{}^{*\prime}}\left(\sum_i[\lambda_i,\ a_i]\right),\ for\ all\ \sum_i[\lambda_i,\ a_i] \in R. \\
\underline{apr}_{P_1{}^{*\prime}}\left(\sum_i[\lambda_i,\ a_i]\right) &= \bigvee_{n\in N}\underline{apr}_{P_1}\left(\sum_i n\lambda_i a_i\right) \\
&\geq \bigvee_{n\in N}\underline{apr}_{P_2}\left(\sum_i n\lambda_i a_i\right) \\
&= \underline{apr}_{P_2{}^{*\prime}}\left(\sum_i[\lambda_i,\ a_i]\right),\ for\ all\ \sum_i[\lambda_i,\ a_i] \in R.
\end{aligned}$$

So, ${P_1}^{*\prime} \subseteq {P_2}^{*\prime}$.

Similarly, we can show that if $P_1 \subseteq P_2$, where $P_1,\ P_2 \in RFI(R)$, then ${P_1}^{*\prime} \subseteq {P_2}^{*\prime}$. So, $P \rightarrow P^{*\prime}$ is a lattice isomorphism. □

4. Left Operator Ring

The following section includes the characterization of RFI in left operator ring and discussed some related theorems.

Definition 9 ([9]). *Let N be a Γ Ring and F be the free abelian group generated by $N \times \Gamma$, the set of all ordered pairs $(a,\ \lambda)$ with $a \in N,\ \lambda \in \Gamma$. Let A be the subgroup of elements $\sum_i n_i(a_i, \lambda_i) \in F$, where n_i are integers such that $\sum_i n_i(a_i\lambda_i a) = 0$ for all $a \in N$. Let $L = F/A$, the factor group of F by A and let us denote the coset $A + (a, \lambda)$ by $[a,\ \lambda]$. Clearly every element in L can be expressed as a finite sum of $\sum_i [a_i,\ \lambda_i]$. Also for all $a, b \in N$ and $\lambda, \mu \in \Gamma$, $[a,\ \lambda] + [a,\ \mu] = [a,\ \lambda + \mu]$ and $[a, \lambda] + [b, \lambda] = [\,a + b,\ \lambda]$. we define a multiplication in L by $\sum_i [a_i,\ \lambda_i] \sum_j \left[b_j,\ \mu_j\right] = \sum_{i,j} \left[a_i\lambda_i b_j,\ \mu_j\right]$. Then L forms the Ring. Furthermore N is a right L-module, with the definition $\sum_i [a_i,\ \lambda_i].\ a = \sum_i a_i\lambda_i a$ for $a \in N$, $\sum_i [a_i,\ \lambda_i] \in L$. We call the ring L the LOR of the Γ Ring N.*

Definition 10 ([9]). *A left unity of a Γ Ring N is an element $\sum_j [f_j,\ \vartheta_j] \in L$ such that $\sum_i f_j\vartheta_j a = a$ for every element $a \in N$. $\sum_j [f_j,\ \vartheta_j]$ is the unity of L.*

4.1. Rough Fuzzy Sets in Left Operator Rings of a Γ Ring

Definition 11 ([7]). *For a fuzzy subset δ of L, define a fuzzy subset δ^+ of N by*

$$\delta^+(a) = \bigwedge_{\vartheta \in \Gamma} \delta([a,\ \vartheta]\Big) \text{ where } a \in N.$$

For a fuzzy subset η of N, define a fuzzy subset $\eta^{+\prime}$ of L by

$$\eta^{+\prime}\left(\sum_i [a_i, \lambda_i]\right) = \bigwedge_{n \in N} \eta\left(\sum_i a_i\lambda_i n\right), \text{ where } \sum_i [a_i, \lambda_i] \in L.$$

Definition 12. *For a rough fuzzy subset $P = <\overline{apr}_P,\ \underline{apr}_P>$ of L, define a rough fuzzy subset $P^+ = <\overline{apr}_{P+},\ \underline{apr}_{P+}>$ of N by*

$$\overline{apr}_{P+}(a) = \bigwedge_{\vartheta \in \Gamma} \overline{apr}_P([a,\ \vartheta]) \text{ and } \underline{apr}_{P+} = \bigvee_{\vartheta \in \Gamma} \underline{apr}_P([a,\ \vartheta]), \text{ where } a \in N.$$

For a rough fuzzy subset $Q = <\overline{apr}_Q,\ \underline{apr}_Q>$ of R, define a RFS$Q^{+\prime} = <\overline{apr}_{Q^{+\prime}},\ \underline{apr}_{Q^{+\prime}}>$ of L by

$$\overline{apr}_{Q^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\right) = \bigwedge_{n \in N} \overline{apr}_Q\left(\sum_i a_i\lambda_i n\right) \text{ and}$$
$$\underline{apr}_{Q^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\right) = \bigvee_{n \in N} \underline{apr}_Q\left(\sum_i a_i\lambda_i n\right) \text{ where } \sum_i [a_i, \lambda_i] \in L.$$

4.2. Characterizations of Rough Fuzzy Ideals in Left Operator Rings of a Γ Ring

Theorem 5. *If $\{P_i / i \in I\}$ is a family of rough fuzzy subsets of L, then*

$$\left(\bigcap_{i\in I}\overline{apr}_{P_i^+}\right)=\left(\bigcap_{i\in I}\overline{apr}_{P_i}\right)^+ \text{ and } \left(\bigcup_{i\in I}\underline{apr}_{P_i^+}\right)=\left(\bigcup_{i\in I}\underline{apr}_{P_i}\right)^+.$$

Proof. Let $a \in N$. Now,

$$\left(\bigcap_{i\in I}\overline{apr}_{P_i}\right)^+(a)=\bigwedge_{\vartheta\in\Gamma}\left[\left(\bigcap_{i\in I}\overline{apr}_{P_i}\right)([a,\ \vartheta])\right]=\bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{i\in I}(\overline{apr}_{P_i}[a,\ \vartheta])\right]$$
$$=\bigwedge_{i\in I}\left[\bigwedge_{\vartheta\in\Gamma}(\overline{apr}_{P_i}[a,\ \vartheta])\right]=\bigwedge_{i\in I}\left[\overline{apr}_{P_i^+}(a)\right]=\left(\bigcap_{i\in I}\overline{apr}_{P_i^+}\right)(a).$$

So $\left(\bigcap_{i\in I}\overline{apr}_{P_i^+}\right)=\left(\bigcap_{i\in I}\overline{apr}_{P_i}\right)^+$ and

$$\left(\bigcup_{i\in I}\underline{apr}_{P_i}\right)^+(a)=\bigvee_{\vartheta\in\Gamma}\left[\bigcup_{i\in I}\underline{apr}_{P_i}([a,\vartheta])\right]=\bigvee_{\vartheta\in\Gamma}\left[\bigvee_{i\in I}\left(\underline{apr}_{P_i}[a,\vartheta]\right)\right]$$
$$=\bigvee_{i\in I}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P_i}([a,\vartheta])\right]\right]=\bigvee_{i\in I}\left[\underline{apr}_{P_i^+}(a)\right]=\left(\bigcup_{i\in I}\underline{apr}_{P_i^+}\right)(a).$$

So, $\left(\bigcup_{i\in I}\underline{apr}_{P_i}\right)^+=\left(\bigcup_{i\in I}\underline{apr}_{P_i^+}\right)$. □

Theorem 6. *If $P = <\overline{apr}_P,\ \underline{apr}_P> \in$ RFI(L) [resp. RFRI (L), RFLI (L)], then the RFS $P^+ = <\overline{apr}_{P+},\ \underline{apr}_{P+}> \in$ RFI (N) [resp. RFRI (N), RFLI (N)].*

Proof. Assume P be the RFI of L. Then $\overline{apr}_P(0_L) = 1$ and $\underline{apr}_P(0_L) = 0$.

Now,

$$\overline{apr}_{P+}(0_N)=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([0_N,\ \vartheta])=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P(0_L)=1 \text{ and}$$
$$\underline{apr}_{P+}(0_N)=\bigvee_{\vartheta\in\Gamma}\underline{apr}_P([0_N,\ \vartheta])=\bigvee_{\vartheta\in\Gamma}\underline{apr}_P(0_L)=0$$

So, P^+ is nonempty. Let $a, b \in N$ and $\alpha \in \Gamma$.

$$\overline{apr}_{P+}(a-b)=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a-b,\ \vartheta])=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a,\ \vartheta]-[b,\ \vartheta])$$
$$\geq\left\{\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a,\ \vartheta])\right\}\bigwedge\left\{\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([b,\ \vartheta])\right\}=\overline{apr}_{P+}(a)\bigwedge\overline{apr}_{P+}(b).$$
$$\underline{apr}_{P+}(a-b)=\bigvee_{\vartheta\in\Gamma}\underline{apr}_P([a-b,\ \vartheta])=\bigvee_{\vartheta\in\Gamma}\underline{apr}_P([a,\ \vartheta]-[b,\ \vartheta])$$
$$\leq\left\{\bigvee_{\vartheta\in\Gamma}\underline{apr}_P([a,\ \vartheta])\right\}\bigvee\left\{\bigvee_{\vartheta\in\Gamma}\underline{apr}_P([b,\ \vartheta])\right\}=\underline{apr}_{P+}(a)\bigvee\underline{apr}_{P+}(b).$$

Again,

$$\overline{apr}_{P+}(a\lambda b)=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a\lambda b,\ \vartheta])=\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a,\lambda][b,\ \vartheta])$$
$$\geq\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a,\lambda])=\overline{apr}_P([a,\lambda])\geq\bigwedge_{\vartheta\in\Gamma}\overline{apr}_P([a,\vartheta])=\overline{apr}_{P+}(a).$$

For right ideals,

$$\overline{apr}_{P^+}(a\lambda b) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([a\lambda b,\ \vartheta]) = \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P[a,\lambda][b,\ \vartheta])$$
$$\geq \bigwedge_{\vartheta\in\Gamma} \overline{apr}_P([b,\ \vartheta]) = \overline{apr}_{P^+}(b).$$

Similarly,

$$\underline{apr}_{P^+}(a\lambda b) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a\lambda b,\ \vartheta]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a,\lambda][b,\ \vartheta])$$
$$\leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a,\lambda]) = \underline{apr}_P([a,\lambda]) \leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a,\vartheta]) = \underline{apr}_{P^+}(a).$$

For right ideals,

$$\underline{apr}_{P^+}(a\lambda b) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a\lambda b,\vartheta]) = \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([a,\lambda][b,\ \vartheta])$$
$$\leq \bigvee_{\vartheta\in\Gamma} \underline{apr}_P([b,\ \vartheta]) = \underline{apr}_{P^+}(b).$$

So, P^+ is a RFI of N. □

Theorem 7. *If P = <$\overline{apr}_P$, $\underline{apr}_P$ >∈ RFI (N) [resp. RFLI (N), RFRI (N)], then the RFS $P^{+'}$ = <$\overline{apr}_{P^{+'}}$, $\underline{apr}_{P^{+'}}$ > ∈ RFI (L) [resp. RFLI (L), RFRI (L)].*

Proof. Let P be a RFI of N. Then $\overline{apr}_P(0_N) = 1$ and $\underline{apr}_P(0_N) = 0$. Now,

$$\overline{apr}_{P^{+'}}([0_N,\ \vartheta]) = \bigwedge_{n\in N} \overline{apr}_P(0_N\vartheta n) = \overline{apr}_P(0_N) = 1 \text{ and}$$
$$\underline{apr}_{P^{+'}}([0_N,\vartheta]) = \bigvee_{n\in N} \underline{apr}_P(0_N\vartheta n) = \underline{apr}_P(0_N) = 0.$$

So, $P^{+'}$ is nonempty.

Let $\sum_i [a_i,\ \lambda_i]$, $\sum_j [b_j, \mu_j]$ ϵ L. Then

$$\overline{apr}_{P^{+'}}\left(\sum_i [a_i,\ \lambda_i] - \sum_j [b_j,\ \mu_j]\right) = \bigwedge_{n\in N} \overline{apr}_P\left(\sum_i a_i\lambda_i n - \sum_j b_j\mu_j n\right)$$
$$\geq \bigwedge_{n\in N}\left\{\overline{apr}_P\left(\sum_i a_i\lambda_i n\right) \wedge \overline{apr}_P\left(\sum_j b_j\mu_j n\right)\right\}$$
$$= \left\{\bigwedge_{n\in N} \overline{apr}_P\left(\sum_i a_i\lambda_i n\right)\right\} \wedge \left\{\bigwedge_{n\in N} \overline{apr}_P\left(\sum_j b_j\mu_j n\right)\right\}$$
$$= \overline{apr}_{P^{+'}}\left(\sum_i [a_i,\ \lambda_i])\right) \wedge \overline{apr}_{P^{+'}}\left(\sum_j \left[b_j,\ \mu_j\right]\right) \text{and}$$

$$\underline{apr}_{P^{*'}}\left(\sum_i [a_i,\ \lambda_i] - \sum_j [b_j,\ \mu_j]\right) = \bigvee_{n\in N} \underline{apr}_P\left(\sum_i a_i\lambda_i n - \sum_j b_j\mu_j n\right)$$
$$\leq \bigvee_{n\in N}\left\{\underline{apr}_P\left(\sum_i a_i\lambda_i n\right) \vee \underline{apr}_P\left(\sum_j b_j\mu_j n\right)\right\}$$
$$= \left\{\bigvee_{n\in N} \underline{apr}_P\left(\sum_i a_i\lambda_i n)\right)\right\} \vee \left\{\bigvee_{n\in N} \underline{apr}_P\left(\sum_j b_j\mu_j n\right)\right\}$$
$$= \underline{apr}_{P^{+'}}\left(\sum_i [a_i,\ \lambda_i]\right) \vee \underline{apr}_{P^{+'}}\left(\sum_j \left[b_j,\ \mu_j\right]\right).$$

Again,

$$
\begin{aligned}
\overline{apr}_{P^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\sum_j [b_j,\ \mu_j]\right) &= \overline{apr}_{P^{+\prime}}\left(\sum_{i,j}[a_i\lambda_i b_j,\ \mu_j]\right) = \bigwedge_{n\in N}\overline{apr}_P\left(\sum_{i,j} a_i\lambda_i b_j\mu_j n\right) \\
&\geq \bigwedge_{n\in N}\left[\bigwedge_i\left[\overline{apr}_P\left(\sum_i [a_i\lambda_i b_i]\mu_1 n\right),\ \overline{apr}_P\left(\sum_i [a_i\lambda_i b_i]\mu_2 n\right)\ldots\right]\right] \\
&\geq \bigwedge_{n\in N}\left[\bigwedge_i\left[\overline{apr}_P\left(\sum_i a_i\lambda_i b_1\right),\overline{apr}_P\left(\sum_i a_i\lambda_i b_2\right)\ldots\right]\right] \\
&= \bigwedge\left[\overline{apr}_P\left(\sum_j a_i\lambda_i b_1\right),\overline{apr}_P\left(\sum_j a_i\lambda_i b_2\right)\ldots\right] \\
&\geq \bigwedge_{n\in N}\left[[\ \overline{apr}_P\left(\sum_j a_i\lambda_i n\right)\right] = \overline{apr}_{P^{+\prime}}\left(\sum_i [a_i,\lambda_i]\right).
\end{aligned}
$$

Similarly, we can show that

$$
\overline{apr}_{P^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\sum_j [b_j,\ \mu_j]\right) \geq \overline{apr}_{P^{+\prime}}\left(\sum_i [b_j,\ \mu_j]\right)
$$

$$
\begin{aligned}
\underline{apr}_{P^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\sum_j \left[b_j,\mu_j\right]\right) &= \underline{apr}_{P^{+\prime}}\left(\sum_{i,j}[a_i\lambda_i b_j,\ \mu_j]\right) = \bigvee_{n\in N}\underline{apr}_P\left(\sum_{i,j} a_i\lambda_i b_j\mu_j n\right) \\
&\leq \bigvee_{n\in N}\left[\bigvee_i\left[\underline{apr}_P\left(\sum_i [a_i\lambda_i b_i]\mu_1 n\right),\ \underline{apr}_P\left(\sum_i [a_i\lambda_i b_2]\mu_2 n\right)\ldots\right]\right] \\
&\leq \bigvee_{n\in N}\left[\bigvee_i\left[\underline{apr}_P\left(\sum_i a_i\lambda_i b_1\right),\ \underline{apr}_P\left(\sum_i a_i\lambda_i b_2\right)\ldots\right]\right] \\
&= \bigvee\left[\underline{apr}_P\left(\sum_i a_i\lambda_i b_1\right),\underline{apr}_P\left(\sum_i a_i\lambda_i b_2\right)\ldots\right] \\
&\leq \bigvee_{n\in N}\left[\underline{apr}_P\left(\sum_i a_i\lambda_i n\right)\right] = \underline{apr}_{P^{+\prime}}\left(\sum_j [a_i,\lambda_i]\right).
\end{aligned}
$$

Similarly, we can prove that

$$
\underline{apr}_{P^{+\prime}}\left(\sum_i [a_i,\ \lambda_i]\sum_j [b_j,\ \mu_j]\right) \leq \underline{apr}_{P^{+\prime}}\left(\sum_j [b_j,\mu_j]\right).
$$

So, $P^{+'}$ is a RFI of L. □

Theorem 8. *The lattices of all RFI (resp. RFRI) of N and L are bijective mapping with respect to the inclusion and if preserves the isomorphic condition $P \to P^{+'}$ where $P \in$ RFI(N) [resp. RFLI(N)] and $P^{+'} \in RFI(L)$ [resp.RFLI(L)].*

Proof. First, we show that $\left(P^{+'}\right)^{+} = P$. where $P \in RFI(N)$. Let $a \in N$.

Then

$$\begin{aligned}(\overline{apr}_{P^{+\prime}})^{+}(a) &= \bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_{P^{+\prime}}([a,\ \vartheta])\right] = \bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{n\in N}[\overline{apr}_{P}(a\vartheta n)]\right]\\ &\geq \bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{n\in N}[\overline{apr}_{P}\ (a)]\right] = \overline{apr}_{P}(a).\\ \left(\underline{apr}_{P^{+\prime}}\right)^{+}(a) &= \bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P^{+\prime}}([a,\ \vartheta])\right] = \bigvee_{\vartheta\in\Gamma}\left[\bigvee_{n\in N}\left[\underline{apr}_{A}(a\vartheta n)\right]\right]\\ &\leq \bigvee_{\vartheta\in\Gamma}\left[\bigvee_{n\in N}\left[\underline{apr}_{P}(a)\right]\right] = \underline{apr}_{P}(a).\end{aligned}$$

So, $\mathrm{P} \subseteq \left(\mathrm{P}^{+\prime}\right)^{+}$.

Let $\sum_i[\delta_i,\ e_i]$ be the element of N with right unity.

We have $\sum_i a\delta_i e_i = a$ for all $a \in N$.

Now,

$$\begin{aligned}\overline{apr}_{P}(a) = \overline{apr}_{P}\left(\sum_i a\delta_i e_i\right) &\geq \bigwedge_i[\overline{apr}_{P}(a\delta_1 e_1),\ \overline{apr}_{P}(a\delta_2 e_2),\ldots]\\ &\geq \bigwedge_{\vartheta\in\Gamma}\left[\bigwedge_{n\in N}[\overline{apr}_{A}(a\vartheta n)]\right] = (\overline{apr}_{P^{+\prime}})^{+}(a)\\ \underline{apr}_{P}(a) = \underline{apr}_{P}\left(\sum_j a\delta_i e_i\right) &\leq \bigvee_i\left[\underline{apr}_{P}(a\delta_1 e_1),\ \overline{apr}_{P}(a\delta_2 e_2),\ldots\right]\\ &\leq \bigvee_{\vartheta\in\Gamma}\left[\bigvee_{n\in N}\left[\underline{apr}_{P}(a\vartheta n)\right]\right] = \left(\underline{apr}_{P^{+\prime}}\right)^{+}(a).\end{aligned}$$

So, $\left(P^{+\prime}\right)^{+} \subseteq P$. Hence $P = \left(P^{+\prime}\right)^{+}$. Again, let P be a RFI of L. Now

$$\begin{aligned}(\overline{apr}_{P^{+}})^{+\prime}\left(\sum_k[a_k,\lambda_k]\right) &= \bigwedge_{n\in N}\left[\overline{apr}_{P^{+}}\left(\sum_k a_k\lambda_k n\right)\right]\\ &= \bigwedge_{n\in N}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_{P}\left(\sum_K a_k\lambda_k n,\ \vartheta\right)\right]\right]\\ &= \bigwedge_{n\in N}\left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_{P}\left(\sum_K[a_k,\lambda_k]\right)[n,\vartheta]\right]\right]\\ &\geq \overline{apr}_{P}\left(\sum_k\ [a_k,\lambda_k]\right)\\ \left(\underline{apr}_{P^{+}}\right)^{+\prime}\left(\sum_k[\lambda_k,\ a_k]\right) &= \bigvee_{n\in N}\left[\underline{apr}_{P^{+}}\left(\sum_k a_k\lambda_k n\right)\right]\\ &= \bigvee_{n\in N}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P}\left(\sum_k a_k\lambda_k n,\ \vartheta\right)\right]\right]\\ &= \bigvee_{n\in N}\left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_{P}\left(\sum_K[a_k,\lambda_k]\right)[n,\vartheta]\right]\right]\\ &\leq \underline{apr}_{P}\left(\sum_k\ [a_k,\lambda_k]\right).\end{aligned}$$

So, $\mathrm{P} \subseteq \left(\mathrm{P}^{+}\right)^{+'}$. let $\sum\limits_{\mathrm{j}}[\delta_{\mathrm{j}}', \mathrm{e}_{\mathrm{j}}']$ be the element of N with right unity. Then

$$\begin{aligned}
&\overline{apr}_P\left(\sum_k [a_k, \lambda_k]\right) = \overline{apr}_P\left(\sum_j [\delta_j', \ e_j']\sum_k [a_k, \ \lambda_k \ ]\right) \\
&\geq \bigwedge_j \left[\overline{apr}_P \ \left([\delta_1{}', e_1']\sum_k [a_k, \ \lambda_k \ ] \ \right), \overline{apr}_P \ \left([\delta_2{}', e_2']\sum_k [a_k, \ \lambda_k \ ] \ \right) \ldots\right] \\
&\geq \bigwedge_{n\in N} \left[\bigwedge_{\vartheta\in\Gamma}\left[\overline{apr}_P\left(\sum_k [a_k, \ \lambda_k \ ][n, \vartheta]\right)\right]\right] = (\overline{apr}_{P^+})^{+'}\left(\sum_k \ [a_k, \ \lambda_k \ ]\right) \\
&\underline{apr}_P\left(\sum_k \ [a_k, \ \lambda_k \ ]\right) = \underline{apr}_P\left(\sum_j [\delta_j', \ e_j']\sum_k \ [a_k, \ \lambda_k \ ]\right) \\
&\leq \bigvee_j \left[\underline{apr}_P\left([\delta_1{}', e_1']\sum_k [a_k, \ \lambda_k \ ]\right), \ \underline{apr}_P\left([\delta_2{}', e_2']\sum_k [a_k, \ \lambda_k \ ]\right) \ldots\right] \\
&\leq \bigvee_{n\in N} \left[\bigvee_{\vartheta\in\Gamma}\left[\underline{apr}_P\left(\sum_k \ [a_k, \lambda_k]\right)[n, \vartheta]\right]\right] = \left(\underline{apr}_{P^+}\right)^{+'}\left(\sum_k \ [a_k, \lambda_k]\right).
\end{aligned}$$

So, $\mathrm{P} \supseteq \left(P^{+}\right)^{+'}$. Thus $\mathrm{P} = \left(P^{+}\right)^{+'}$. Thus the correspondence $\mathrm{P} \rightarrow P^{+'}$ is bijection. Now, $\mathrm{P}_1, \mathrm{P}_2 \in \mathrm{RFI(N)}$ such that $\mathrm{P}_1 \subseteq \mathrm{P}_2$

$$\begin{aligned}
\overline{apr}_{P_1{}^{+'}}\left(\sum_i [a_i, \lambda_i]\right) &= \bigwedge_{n\in N} \overline{apr}_{P_1}\left(\sum_i a_i\lambda_i n\right) \\
&\leq \bigwedge_{n\in N} \overline{apr}_{P_2}\left(\sum_i a_i\lambda_i n\right) \\
&= \overline{apr}_{P_2{}^{+'}}\left(\sum_i [a_i, \lambda_i]\right) \ for \ all \ \sum_i [a_i, \lambda_i] \in L.
\end{aligned}$$

$$\begin{aligned}
\underline{apr}_{P_1{}^{+'}}\left(\sum_i [a_i, \lambda_i]\right) &= \bigvee_{n\in N} \underline{apr}_{P_1}\left(\sum_i a_i\lambda_i n\right) \\
&\geq \bigvee_{n\in N} \underline{apr}_{P_2}\left(\sum_i a_i\lambda_i n\right) \\
&= \underline{apr}_{P_2{}^{+'}}\left(\sum_i [a_i, \lambda_i]\right), \ for \ all \ \sum_i [a_i, \lambda_i] \in L.
\end{aligned}$$

So, $P_1{}^{+'} \subseteq P_2{}^{+'}$. Similarly, we can show that if $P_1 \subseteq P_2$, where $P_1, P_2 \in RFI(L)$, then $P_1{}^{+'} \subseteq P_2{}^{+'}$. So, the mapping $\mathrm{P} \rightarrow P^{+'}$ is a lattice isomorphism. □

5. Conclusions

The contribution of RS theory is more to pure and applied mathematics. Combining RS theory with algebraic structures produces interesting results and research topics. Researchers are interested in the study of RS in algebraic structures such as groups, rings, and fields. An interesting field of research in RS theory is the study of RS on rings. In this paper, we have discussed several interesting theorems in rough fuzzy environment. The limitation of RS is its dependence on equivalence relations to partition the universe of objects in information systems. The extension of this proposed work will try to implement a cryptography protocol using left and right operator rings. In our future studies, we will concentrate on rough fuzzy approximations on various algebraic structures, such as gamma near ring, gamma field and gamma near field.

Author Contributions: Conceptualization, validation, investigation: E.D.; Methodology: D.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the editor and reviewers for the valuable comments, suggestions and ideas.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

N	Gamma Ring
RS	Rough Set
$\overline{\text{apr}}$	Upper approximation
$\underline{\text{apr}}$	Lower approximation
IFS	Intuitionistic Fuzzy Set
RFS	Rough Fuzzy Set
RFI	Rough Fuzzy Ideal
RFLI	Rough Fuzzy Left Ideal
RFRI	Rough Fuzzy Right Ideal
RFLI (N)	Set of all Rough Fuzzy Left Ideals of N
RFRI (N)	Set of all Rough Fuzzy Right Ideals of N
LOR	Left Operator Ring
ROR	Right Operator Ring.

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