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Improvement in Some Inequalities via Jensen–Mercer Inequality and Fractional Extended Riemann–Liouville Integrals

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Abstract: The primary intent of this study is to establish some important inequalities of the Hermite–Hadamard, trapezoid, and midpoint types under fractional extended Riemann–Liouville integrals (FERLIs). The proofs are constructed using the renowned Jensen–Mercer, power-mean, and Holder inequalities. Various equalities for the FERLIs and convex functions are construed to be the mainstay for finding new results. Some connections between our main findings and previous research on Riemann–Liouville fractional integrals and FERLIs are also discussed. Moreover, a number of examples are featured, with graphical representations to illustrate and validate the accuracy of the new findings.

Keywords: fractional integrals; fractional inequalities; Jensen–Merce inequality

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1. Introduction

In the context of nonconvex energy mappings, Panagiotopoulos proposed and expanded the theoretical framework of integral inequalities [1–3], in addition to their applicability in the fields of mechanical engineering and finance. We strongly suggest reading the article [4], which provides an analysis of this idea as well as further commentary. Following this helpful beginning of mathematical inequalities, the Bessel function and the improved Bessel function were linked to certain integral inequalities. They are also appropriate for use in different chemical engineering systems in which the Bessel function was first used. In regular-frequency radio waves, the zeros of the Bessel function have a great effect. Sine transporter waves and sine signal waves, both of which may be expressed mathematically using Bessel functions, are used in frequency-regulated transmission [5,6].

As an extension of traditional derivatives and integrals, the study of fractional calculus entails looking at derivatives and integrals with non-integer orders. It is as archaic as traditional calculus, but it has been garnering a lot of attention in the past twenty years thanks to its usefulness in numerous areas of science [7–10]. In recent years, researchers have been actively working to generalize previously discovered concepts by using modern ideas and innovative methods from the field of fractional calculus. Mathematical analysts are increasingly drawn to a technique known as "fractional operators" analysis. Integral inequalities with fractional integrals are very important because they may be used to check the solution advantages for many different types of integrodifferential fractional or fractal equations [11–13].

Fractional calculus was used by Sarikaya et al. [14] to prove many integral inequalities that are based on the inequality of Hermite–Hadamard. With this method, researchers now have a fresh avenue to explore this inequality. In addition, the authors in [15] covered important illustrations of a number of significant inequalities that were established by Set et al. [16]. Since that moment, a large number of researchers have put the concepts of fractional calculus to widespread use. As a result, they have obtained various novel and cutting-edge improvements of inequalities by using convexity and its extensions (see [17–23]).

Before getting into the main conclusions of the research, it may be helpful to look at a number of key terminology and obtained findings.

Definition 1 ([24])**.** *A function $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ that is considered to be convex is one that satisfies the following inequality.*

$$\mathcal{V}(\imath \mathbb{y}_1 + (1-\imath)\mathbb{y}_2) \leq \imath \mathcal{V}(\mathbb{y}_1) + (1-\imath)\mathcal{V}(\mathbb{y}_2), \ \ \imath \in [0,1], \text{ and } \mathbb{y}_1, \mathbb{y}_2 \in [\mathbb{p}, \mathbb{q}]. \tag{1}$$

If $(-\mathcal{V})$ is convex, then the function $\mathcal{V}$ is considered to have a concave form.

The following outcome offers an actual illustration of the convexity idea in a geometric manner.

Theorem 1 ([25])**.** *The subsequent inequality is found if we assume that $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ is a convex function.*

$$\mathcal{V}\left(\frac{\mathbb{p}+\mathbb{q}}{2}\right) \leq \frac{1}{\mathbb{q}-\mathbb{p}} \int_{\mathbb{p}}^{\mathbb{q}} \mathcal{V}(\imath) d\imath \leq \frac{\mathcal{V}(\mathbb{p}) + \mathcal{V}(\mathbb{q})}{2}. \tag{2}$$

The converse orientation of (2) *will also be valid if $\mathcal{V}$ is concave.*

In this study, we take into consideration the integral form inequality of the Hermite–Hadamard–Mercer kind, which is dependent on the inequalities of Hermite–Hadamard and Jensen–Mercer. Because of this, it is important to remember the Jensen–Mercer inequality.

Let $0 < \mathbb{x}_1 \leq \mathbb{x}_2 \leq \cdots \leq \mathbb{x}_n$ and $\mathbb{r} = (\mathbb{r}_1, \mathbb{r}_2, \ldots, \mathbb{r}_n)$ be weights such that $\mathbb{r}_i \geq 0$ and $\sum_{i=1}^{n} \mathbb{r}_i = 1$. For a convex mapp $\mathcal{V}$ on $[\mathbb{p}, \mathbb{q}]$, the inequality of Jensen is realized as follows [26,27]:

$$\mathcal{V}\left(\sum_{i=1}^{n} \mathbb{r}_i \mathbb{x}_i\right) \leq \sum_{i=1}^{n} \mathbb{c}_i \mathcal{V}(\mathbb{x}_i), \tag{3}$$

$\forall\, \mathbb{x}_i \in [\mathbb{p}, \mathbb{q}]$ and $\mathbb{r}_i \in [0,1]$, $i = 1, \ldots, n$.

In the literature, the following inequality is recognized as a Jensen–Mercer inequality.

Theorem 2 ([27,28])**.** *If the function $\mathcal{V}$ is convex over $[\mathbb{p}, \mathbb{q}]$, we obtain*

$$\mathcal{V}\left(\mathbb{p} + \mathbb{q} - \sum_{i=1}^{n} \mathbb{r}_i \mathbb{x}_i\right) \leq \mathcal{V}(\mathbb{p}) + \mathcal{V}(\mathbb{q}) - \sum_{i=1}^{n} \mathbb{c}_i \mathcal{V}(\mathbb{x}_i), \tag{4}$$

$\forall\, \mathbb{x}_i \in [\mathbb{p}, \mathbb{q}]$, $\mathbb{r}_i \in [0,1]$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \mathbb{r}_i = 1$.

For more findings regarding the Jensen–Mercer inequality, see [29–31].

Definition 2 ([32])**.** *For $0 < \mathbb{y}, \mathbb{z} < \infty$ and $\mathbb{y}, \mathbb{z} \in \mathbb{R}$, the gamma function, beta function, and incomplete beta function are described by*

$$\Gamma(\mathbb{y}) := \int_0^\infty \mathbb{u}^{\mathbb{y}-1} e^{-\mathbb{u}} d\mathbb{u},$$

$$\mathcal{B}(\mathbb{y},\mathbb{z}) := \int\limits_0^1 \mathbb{u}^{\mathbb{y}-1}(1-\mathbb{u})^{\mathbb{z}-1}d\mathbb{u},$$

and

$$\mathcal{B}(\mathbb{y},\mathbb{z},\mathbb{b}) := \int\limits_0^{\mathbb{b}} \mathbb{u}^{\mathbb{y}-1}(1-\mathbb{u})^{\mathbb{z}-1}d\mathbb{u},$$

respectively.

Kilbas et al. [32] presented fractional integrals, also called Riemann–Liouville integrals, as follows:

Definition 3 ([32])**.** *Let $\mathcal{V} \in L_1[\mathbb{p},\mathbb{q}]$, $\mathbb{p},\mathbb{q} \in \mathbb{R}$ with $\mathbb{p} < \mathbb{q}$. The integrals of Riemann–Liouville $J^{\mathbb{d}}_{\mathbb{p}^+}\ \mathcal{V}$ and $J^{\mathbb{d}}_{\mathbb{q}^-}\ \mathcal{V}$ of order $\mathbb{d} > 0$ are presented as follows:*

$$J^{\mathbb{d}}_{\mathbb{p}^+}\mathcal{V}(\mathbb{y}) = \frac{1}{\Gamma(\mathbb{d})}\int_{\mathbb{p}}^{\mathbb{y}}(\mathbb{y}-\mathbb{u})^{\mathbb{d}-1}\mathcal{V}(\mathbb{u})d\mathbb{u}, \quad \mathbb{y} > \mathbb{p} \tag{5}$$

and

$$J^{\mathbb{d}}_{\mathbb{q}^-}\mathcal{V}(x) = \frac{1}{\Gamma(\mathbb{d})}\int_{\mathbb{y}}^{\mathbb{q}}(\mathbb{u}-\mathbb{y})^{\mathbb{d}-1}\mathcal{V}(\mathbb{u})d\mathbb{u}, \quad \mathbb{y} < \mathbb{q}, \tag{6}$$

respectively. Here, Γ denotes the Gamma integral form.

Jarad et al. [33] developed the fractional extended Riemann–Liouville integrals (FERLIs), gave their features, and compared them with many fractional tools studied before. Here is how to describe the FERLIs:

Definition 4 ([33])**.** *Let $\mathbb{d} > 0$ and $\mathbb{c} \in (0,1]$. For $\mathcal{V} \in L_1[\mathbb{p},\mathbb{q}]$, the FERLIs ${}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}$ and ${}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}$ are defined by*

$${}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{y}) = \frac{1}{\Gamma(\mathbb{d})}\int_{\mathbb{p}}^{\mathbb{y}}\left(\frac{(\mathbb{y}-\mathbb{p})^{\mathbb{c}}-(\mathbb{u}-\mathbb{p})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}-1}\frac{\mathcal{V}(\mathbb{u})}{(\mathbb{u}-\mathbb{p})^{1-\mathbb{c}}}d\mathbb{u}, \quad \mathbb{y} > \mathbb{p}, \tag{7}$$

and

$${}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{y}) = \frac{1}{\Gamma(\mathbb{d})}\int_{\mathbb{y}}^{\mathbb{q}}\left(\frac{(\mathbb{q}-\mathbb{y})^{\mathbb{c}}-(\mathbb{q}-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}-1}\frac{\mathcal{V}(\mathbb{u})}{(\mathbb{q}-\mathbb{u})^{1-\mathbb{c}}}d\mathbb{u}, \quad \mathbb{y} < \mathbb{q}, \tag{8}$$

respectively.

If we assign $\mathbb{c} = 1$, then the FERLIs in (7) and (8) equal the Riemann–Liouville fractional integrals in (5) and (6), respectively.

The following inequality of Hermite–Hadamard was proved for the FERLIs by Set et al. [34]

Theorem 3. *Assume that $\mathcal{V} : [\mathbb{p},\mathbb{q}] \to \mathbb{R}$ is convex and $\mathcal{V} \in L_1[\mathbb{p},\mathbb{q}]$. We obtain*

$$\mathcal{V}\left(\frac{\mathbb{p}+\mathbb{q}}{2}\right) \leq \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}_{+}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right] \leq \frac{\mathcal{V}(\mathbb{p})+\mathcal{V}(\mathbb{q})}{2} \tag{9}$$

for $\mathbb{d} > 0$ and $\mathbb{c} \in (0,1]$.

Set et al. [34] also proved the following Lemma:

Lemma 1. *Let the function $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ be a differentiable on $(\mathbb{p}, \mathbb{q})$ and $\mathcal{V}' \in L[\mathbb{p}, \mathbb{q}]$. Then, we have*

$$\begin{aligned} &\frac{\mathcal{V}(\mathbb{p}) + \mathcal{V}(\mathbb{q})}{2} - \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right] \\ = &\frac{(\mathbb{q}-\mathbb{p})\mathbb{c}^{\mathbb{d}}}{2}\int_0^1 \left[\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}} - \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\mathcal{V}'(\mathbb{u}\mathbb{p} + (1-\mathbb{u})\mathbb{q})d\mathbb{u} \end{aligned} \tag{10}$$

for $\mathbb{d} > 0$ and $\mathbb{c} \in (0, 1]$.

Hezenci and Budak proved the following result [35]:

Lemma 2. *If $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ is differentiable over $(\mathbb{p}, \mathbb{q})$ and $\mathcal{V}' \in L_1[\mathbb{p}, \mathbb{q}]$, the following equality is valid:*

$$\mathcal{V}\left(\frac{\mathbb{p}+\mathbb{q}}{2}\right) - \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q})\right] = \frac{(\mathbb{q}-\mathbb{p})\mathbb{c}^{\mathbb{d}}}{2}\{A_1 - A_2 - A_3 + A_4\}. \tag{11}$$

Here,

$$A_1 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}} \mathcal{V}'(\mathbb{u}\mathbb{q} + (1-\mathbb{u})\mathbb{p})d\mathbb{u},$$

$$A_2 = \int_0^{\frac{1}{2}} \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}} \mathcal{V}'(\mathbb{u}\mathbb{p} + (1-\mathbb{u})\mathbb{q})d\mathbb{u},$$

$$A_3 = \int_{\frac{1}{2}}^1 \left[\frac{1}{\mathbb{c}^{\mathbb{d}}} - \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right] \mathcal{V}'(\mathbb{u}\mathbb{q} + (1-\mathbb{u})\mathbb{p})d\mathbb{u},$$

$$A_4 = \int_{\frac{1}{2}}^1 \left[\frac{1}{\mathbb{c}^{\mathbb{d}}} - \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right] \mathcal{V}'(\mathbb{u}\mathbb{p} + (1-\mathbb{u})\mathbb{q})d\mathbb{u}.$$

Öğülmüs and Sarikaya [36] proved an inequality for Riemann–Liouville fractional integrals , which is known as the Hermite–Hadamard–Mercer inequality. Butt et al. proved the following Hermite–Hadamard–Mercer inequality for FERLIs [37]:

Theorem 4. *According to the convexity of $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$, the next inequalities are valid for the FERLIs in (7) and (8):*

$$\begin{aligned} \mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right) &\leq \mathcal{V}(\mathbb{p}) + \mathcal{V}(\mathbb{q}) - \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{y}+}\mathcal{V}(\mathbb{z}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{z}-}\mathcal{V}(\mathbb{y})\right] \\ &\leq \mathcal{V}(\mathbb{p}) + \mathcal{V}(\mathbb{q}) - \mathcal{V}\left(\frac{\mathbb{y}+\mathbb{z}}{2}\right), \end{aligned} \tag{12}$$

for all $\mathbb{y}, \mathbb{z} \in [\mathbb{p}, \mathbb{q}]$ with $\mathbb{y} < \mathbb{z}$.

The essential target of this discussion is to offer Hermite–Hadamard-, trapezoid-, and midpoint-kind inequalities for FERLIs using the Jensen–Mercer inequality. In Section 2, an identity will be established by utilizing Lemma 1, as given by Set et al. in [34]. By employing this equality and the well-known Hölder and power-mean inequalities, various trapezoid-type inequalities for FERLIs will be derived using the Jensen–Mercer inequality. Additionally, Section 3 will introduce an equality based on Lemma 2, proven by Hezenci and Budak in [35]. Subsequently, utilizing this newly established identity and the Jensen–Mercer inequality, several midpoint-type inequalities will be obtained. In both

Sections 2 and 3, we will discuss the connections between our main findings and prior studies conducted on Riemann–Liouville fractional integrals and FERLIs. Section 4 will feature a number of examples aimed at illustrating the primary outcomes. Furthermore, graphical representations will be used to validate the accuracy of these findings. Section 5 provides an overview of the conclusions drawn from the study.

The following equalities will be used in the new sections:

Lemma 3. *We have the following equalities for* $\mathbb{d} > 0$ *and* $\mathbb{c} \in (0,1]$.

$$\Omega_1 = \int_0^{\frac{1}{2}} (1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 (1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \frac{1}{\mathbb{c}} \mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right),$$

$$\begin{aligned} \Omega_2 &= \int_0^{\frac{1}{2}} (1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 (1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} \\ &= \frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right], \end{aligned}$$

$$\Omega_3 = \int_0^{\frac{1}{2}} t(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 (1-\mathbb{u})(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \frac{1}{\mathbb{c}} \mathcal{B}\left(\frac{2}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right),$$

$$\begin{aligned} \Omega_4 &= \int_0^{\frac{1}{2}} (1-\mathbb{u})(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 \mathbb{u}(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} \\ &= \frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{2}{\mathbb{c}}, \mathbb{d}+1\right) - \mathcal{B}\left(\frac{2}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right], \end{aligned}$$

$$\Omega_5 = \int_0^{\frac{1}{2}} \mathbb{u}(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 (1-\mathbb{u})(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \Omega_2 - \Omega_4,$$

$$\Omega_6 = \int_0^{\frac{1}{2}} (1-\mathbb{u})(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \int_{\frac{1}{2}}^1 \mathbb{u}(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}} d\mathbb{u} = \Omega_1 - \Omega_3.$$

2. Trapezoid-Type Inequalities with Jensen–Mercer Inequality

In this section, we present some trapezoid-type inequalities proved using the Jensen–Mercer inequality.

Lemma 4. *Assume that* $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ *is differentiable on* $(\mathbb{p}, \mathbb{q})$ *and* $\mathcal{V}' \in L[\mathbb{p}, \mathbb{q}]$. *Then, we have*

$$\begin{aligned} & \frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y}) + \mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})}{2} \\ - & \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right] \\ = & \frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\int_0^1 \left[\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}} - \left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))d\mathbb{u} \end{aligned} \tag{13}$$

for $\mathbb{d} > 0$ *and* $\mathbb{c} \in (0,1]$.

Proof. If we write $\mathbb{p}+\mathbb{q}-\mathbb{z}$ and $\mathbb{p}+\mathbb{q}-\mathbb{y}$ instead of $\mathbb{p}$ and $\mathbb{q}$, respectively, in Lemma 1, then we obtain the desired result immediately. One can find a corresponding proof in [37]. □

Theorem 5. *Suppose that $\mathcal{V} : [\mathbb{p}, \mathbb{q}] \to \mathbb{R}$ is differentiable on $(\mathbb{p}, \mathbb{q})$ and $|\mathcal{V}'|$ is convex on $[\mathbb{p}, \mathbb{q}]$. For the FERLIs in (7) and (8), the following inequality is valid.*

$$\begin{aligned}
&\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
&\leq \frac{\mathbb{z}-\mathbb{y}}{\mathbb{c}}\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\frac{|\mathcal{V}'(\mathbb{z})|+|\mathcal{V}'(\mathbb{y})|}{2}\right] \\
&\times\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)\right]
\end{aligned} \tag{14}$$

for $\mathbb{y}, \mathbb{z} \in [\mathbb{p}, \mathbb{q}]$ with $\mathbb{y} < \mathbb{z}$ and for $\mathbb{d} > 0$, $\mathbb{c} \in (0, 1]$.

Proof. In view of Lemma 4, we have

$$\begin{aligned}
&\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
\leq\ &\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))\right|d\mathbb{u}.
\end{aligned}$$

Next, we use the Jensen–Mercer inequality to derive

$$\begin{aligned}
&\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
&\leq\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\mathbb{u}|\mathcal{V}'(\mathbb{z})|-(1-\mathbb{u})|\mathcal{V}'(\mathbb{y})|\right]d\mathbb{u} \\
&=\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\left[\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|\right]\Lambda_1-|\mathcal{V}'(\mathbb{z})|\Lambda_2-|\mathcal{V}'(\mathbb{y})|\Lambda_3\right].
\end{aligned} \tag{15}$$

Here, we have

$$\begin{aligned}
\Lambda_1 &= \int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|d\mathbb{u} \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}\left[\int_0^{\frac{1}{2}}\left[(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right. \\
&\quad\left.+\int_{\frac{1}{2}}^1\left[(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}-(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right] \\
&= \frac{2}{\mathbb{c}^{\mathbb{d}}}\int_0^{\frac{1}{2}}\left[(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u} \\
&= \frac{2}{\mathbb{c}^{\mathbb{d}}}[\Omega_1-\Omega_2] \\
&= \frac{2}{\mathbb{c}^{\mathbb{d}+1}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)\right].
\end{aligned} \tag{16}$$

$$\begin{aligned}
\Lambda_2 &= \int_0^1 \mathbb{u}\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|d\mathbb{u} \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}\left[\int_0^{\frac{1}{2}} \mathbb{u}\left[(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1 \mathbb{u}\left[(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}-(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right] \qquad (17) \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}[\Omega_3-\Omega_5+\Omega_6-\Omega_4] \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}[\Omega_1-\Omega_2] \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}+1}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_3 &= \int_0^1 (1-\mathbb{u})\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|d\mathbb{u} \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}\left[\int_0^{\frac{1}{2}} (1-\mathbb{u})\left[(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right. \\
&\quad \left.+\int_{\frac{1}{2}}^1 (1-\mathbb{u})\left[(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}-(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}\right]d\mathbb{u}\right] \qquad (18) \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}}[\Omega_6-\Omega_4+\Omega_3-\Omega_5] \\
&= \frac{1}{\mathbb{c}^{\mathbb{d}+1}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)\right].
\end{aligned}$$

By substituting the equalities (16)–(18) in (15), we obtain the desired result. □

Remark 1. *If we choose $\mathbb{y}=\mathbb{p}$ and $\mathbb{z}=\mathbb{q}$ in Theorem 5, then we obtain the inequality*

$$\begin{aligned}
&\left|\frac{\mathcal{V}(\mathbb{p})+\mathcal{V}(\mathbb{q})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right]\right| \\
&\leq \frac{(\mathbb{q}-\mathbb{p})}{2\mathbb{c}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)\right][|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|],
\end{aligned}$$

which was proved by Set et al. in [34].

Remark 2. *If we choose $\mathbb{c}=1$ in Theorem 5, then we obtain the inequality*

$$\begin{aligned}
&\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{d}}}\left[J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
&\leq \frac{\mathbb{z}-\mathbb{y}}{\mathbb{d}+1}\left(1-\frac{1}{2^{\mathbb{d}}}\right)\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\frac{|\mathcal{V}'(\mathbb{z})|-|\mathcal{V}'(\mathbb{y})|}{2}\right],
\end{aligned}$$

which was proved by Öğülmüş and Sarikaya in [36].

Theorem 6. *Assume that $\mathcal{V}:[\mathbb{p},\mathbb{q}]\to\mathbb{R}$ is differentiable on $(\mathbb{p},\mathbb{q})$ and $|\mathcal{V}'|^{\mu}$ is convex over $[\mathbb{p},\mathbb{q}]$ for $\mu>1$. Then, for the FERLIs in (7) and (8), the following inequality is valid.*

$$\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq\frac{\mathbb{z}-\mathbb{y}}{2}\left(\int_0^1\left|(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}$$
$$\times\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{2}\right)^{\frac{1}{\mu}}$$

where $\nu^{-1}+\mu^{-1}=1$.

Proof. According to Lemma 4 and Hölder's inequality, we conclude

$$\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\left(\int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}$$
$$\times\left(\int_0^1|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))|^{\mu}d\mathbb{u}\right)$$

Utilizing the inequality of Jensen–Mercer and convexity of $|\mathcal{V}'\mathbb{u}|^{\mu}$, we obtain

$$\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\left(\int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}$$
$$\times\left(\int_0^1\left[|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\left(\mathbb{u}|\mathcal{V}'(\mathbb{z})|^{\mu}+(1-\mathbb{u})|\mathcal{V}'(\mathbb{y})|^{\mu}\right)\right]d\mathbb{u}\right)^{\frac{1}{\mu}}$$
$$=\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\left(\int_0^1\left|\left(\frac{1-\mathbb{u}^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}$$
$$\times\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{2}\right)^{\frac{1}{\mu}}$$

which completes the proof of Theorem 6. □

Remark 3. *If we choose* $\mathbb{y}=\mathbb{p}$ *and* $\mathbb{z}=\mathbb{q}$ *in Theorem 6, then we obtain the inequality*

$$\left|\frac{\mathcal{V}(\mathbb{p})+\mathcal{V}(\mathbb{q})}{2}-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right]\right|$$
$$\leq\frac{(\mathbb{q}-\mathbb{p})}{2}\left(\int_0^1\left|(1-\mathbb{u}^{\mathbb{c}})^{\mathbb{d}}-(1-(1-\mathbb{u})^{\mathbb{c}})^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{2}\right)^{\frac{1}{\mu}}.$$

Remark 4. *If we choose* $\mathbb{c}=1$ *in Theorem 6, then we obtain the inequality*

$$\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})+\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2}-\frac{\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{d}}}\left[J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq\frac{\mathbb{z}-\mathbb{y}}{2}\left(\int_0^1\left|(1-\mathbb{u})^{\mathbb{d}}-\mathbb{u}^{\mathbb{d}}\right|^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}$$
$$\times\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{2}\right)^{\frac{1}{\mu}}.$$

3. Midpoint-Type Inequalities with Jensen–Mercer Inequality

Here, we use the Jensen–Mercer inequality to set up certain inequalities of the midpoint kind.

Lemma 5. *Note that $\mathcal{V}:[\mathbb{p},\mathbb{q}]\to\mathbb{R}$ is differentiable on $(\mathbb{p},\mathbb{q})$ and $\mathcal{V}'\in L_1[\mathbb{p},\mathbb{q}]$. For the FERLIs in (7) and (8), the following equality is valid.*

$$\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]$$
$$=\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\{B_1-B_2-B_3+B_4\}.$$

Here,

$$\begin{cases}B_1=\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))d\mathbb{u},\\ B_2=\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))d\mathbb{u},\\ B_3=\int_{\frac{1}{2}}^1\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))d\mathbb{u},\\ B_4=\int_{\frac{1}{2}}^1\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))d\mathbb{u}.\end{cases}$$

Proof. If we write $\mathbb{p}+\mathbb{q}-\mathbb{z}$ and $\mathbb{p}+\mathbb{q}-\mathbb{y}$ instead of $\mathbb{p}$ and $\mathbb{q}$, respectively, in Lemma 2, then we obtain the intended outcome right away. □

Theorem 7. *Assume that $\mathcal{V}:[\mathbb{p},\mathbb{q}]\to\mathbb{R}$ is differentiable on $(\mathbb{p},\mathbb{q})$ and $|\mathcal{V}'|$ is convex over $[\mathbb{p},\mathbb{q}]$. Consequently, the FERLIs satisfy the following inequality:*

$$\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq(\mathbb{z}-\mathbb{y})\left[\frac{1}{2}+\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)-2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right] \tag{19}$$
$$\times\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\frac{|\mathcal{V}'(\mathbb{y})|+|\mathcal{V}'(\mathbb{z})|}{2}\right].$$

Proof. By Lemma 5, we have

$$\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
\leq\ &\frac{(\mathbb{z}-\mathbb{y})\mathbb{c}^{\mathbb{d}}}{2}\left\{\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))\right|d\mathbb{u}\right. \\
&+\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))\right|d\mathbb{u} \\
&+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))\right|d\mathbb{u} \\
&\left.+\int_{\frac{1}{2}}^{1}\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))\right|d\mathbb{u}\right\}.
\end{aligned} \tag{20}$$

Then, using the Jensen–Mercer inequality, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\left|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))\right|d\mathbb{u} \\
\leq\ &\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\left[\left|\mathcal{V}'(\mathbb{p})\right|+\left|\mathcal{V}'(\mathbb{q})\right|-t\left|\mathcal{V}'(\mathbb{y})\right|-(1-\mathbb{u})\left|\mathcal{V}'(\mathbb{z})\right|\right]d\mathbb{u} \\
=\ &\frac{1}{\mathbb{c}^{\mathbb{d}}}\left[\left(\left|\mathcal{V}'(\mathbb{p})\right|+\left|\mathcal{V}'(\mathbb{q})\right|\right)\Omega_2-\left|\mathcal{V}'(\mathbb{y})\right|\Omega_5-\left|\mathcal{V}'(\mathbb{z})\right|\Omega_4\right].
\end{aligned} \tag{21}$$

Similarly, by the Jensen–Mercer inequality, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))|d\mathbb{u} \\
\leq\ &\frac{1}{\mathbb{c}^{\mathbb{d}}}[(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\Omega_2-|\mathcal{V}'(\mathbb{z})|\Omega_5-|\mathcal{V}'(\mathbb{y})|\Omega_4],
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{y}+(1-\mathbb{u})\mathbb{z}))|d\mathbb{u} \\
\leq\ &\frac{1}{\mathbb{c}^{\mathbb{d}}}\left[(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\left(\frac{1}{2}-\Omega_1\right)-|\mathcal{V}'(\mathbb{y})|\left(\frac{3}{8}-\Omega_6\right)-|\mathcal{V}'(\mathbb{z})|\left(\frac{1}{8}-\Omega_3\right)\right],
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left[\frac{1}{\mathbb{c}^{\mathbb{d}}}-\left(\frac{1-(1-\mathbb{u})^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}}\right]|\mathcal{V}'(\mathbb{p}+\mathbb{q}-(\mathbb{u}\mathbb{z}+(1-\mathbb{u})\mathbb{y}))|d\mathbb{u} \\
\leq\ &\frac{1}{\mathbb{c}^{\mathbb{d}}}\left[(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\left(\frac{1}{2}-\Omega_1\right)-|\mathcal{V}'(\mathbb{z})|\left(\frac{3}{8}-\Omega_6\right)-|\mathcal{V}'(\mathbb{y})|\left(\frac{1}{8}-\Omega_3\right)\right].
\end{aligned} \tag{24}$$

By substituting the inequalities (21)–(24) in (20), we obtain

$$
\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
\leq\ & \frac{(\mathbb{z}-\mathbb{y})}{2}\{(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\Omega_2-|\mathcal{V}'(\mathbb{y})|\Omega_5-|\mathcal{V}'(\mathbb{z})|\Omega_4 \\
&+(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\Omega_2-|\mathcal{V}'(\mathbb{z})|\Omega_5-|\mathcal{V}'(\mathbb{y})|\Omega_4 \\
&+(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\left(\frac{1}{2}-\Omega_1\right)-|\mathcal{V}'(\mathbb{y})|\left(\frac{3}{8}-\Omega_6\right)-|\mathcal{V}'(\mathbb{z})|\left(\frac{1}{8}-\Omega_3\right) \\
&+(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)\left(\frac{1}{2}-\Omega_1\right)-|\mathcal{V}'(\mathbb{z})|\left(\frac{3}{8}-\Omega_6\right)-|\mathcal{V}'(\mathbb{y})|\left(\frac{1}{8}-\Omega_3\right)\Bigg\} \\
=\ & \frac{(\mathbb{z}-\mathbb{y})}{2}\left[(|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|)(1-2\Omega_1+2\Omega_2)-(|\mathcal{V}'(\mathbb{y})|+|\mathcal{V}'(\mathbb{z})|)\left(\frac{1}{2}-\Omega_1+\Omega_2\right)\right] \\
=\ & (\mathbb{z}-\mathbb{y})\left(\frac{1}{2}-\Omega_1+\Omega_2\right)\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\frac{|\mathcal{V}'(\mathbb{y})|+|\mathcal{V}'(\mathbb{z})|}{2}\right].
\end{aligned}
$$

This wraps up the proof. □

Remark 5. *If we choose $\mathbb{y}=\mathbb{p}$ and $\mathbb{z}=\mathbb{q}$ in Theorem 7, then we obtain the inequality*

$$
\begin{aligned}
&\left|\mathcal{V}\left(\frac{\mathbb{p}+\mathbb{q}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right]\right| \\
&\leq\frac{(\mathbb{q}-\mathbb{p})}{2}\left[\frac{1}{2}+\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1\right)-2\mathcal{B}\left(\frac{1}{\mathbb{c}},\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right][|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|],
\end{aligned}
$$

which was proved by Hezenci and Budak in [35].

Remark 6. *If we choose $\mathbb{c}=1$ in Theorem 7, then we obtain the inequality*

$$
\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{d}}}\left[J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
&\leq(\mathbb{z}-\mathbb{y})\left[\frac{1}{2}-\frac{1}{\mathbb{d}+1}\left(1-\frac{1}{2^{\mathbb{d}}}\right)\right]\left[|\mathcal{V}'(\mathbb{p})|+|\mathcal{V}'(\mathbb{q})|-\frac{|\mathcal{V}'(\mathbb{y})|+|\mathcal{V}'(\mathbb{z})|}{2}\right]
\end{aligned}
$$

Theorem 8. *Note that $\mathcal{V}:[\mathbb{p},\mathbb{q}]\to\mathbb{R}$ is differentiable on $(\mathbb{p},\mathbb{q})$ and $|\mathcal{V}'|^{\mu}$ is convex over $[\mathbb{p},\mathbb{q}]$ for some $\mu>1$. Then, the FERLIs fulfill the inequality below:*

$$
\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
\leq\ & \frac{(\mathbb{z}-\mathbb{y})}{2^{1+\frac{1}{\mu}}}\left[\left(\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right)^{\frac{1}{\nu}}+\Omega_7\right] \\
&\times\left[\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+3|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right. \\
&\left.+\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{3|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right],
\end{aligned}
$$

where $\nu^{-1} + \mu^{-1} = 1$ *and*

$$\Omega_7 = \left(\int_{\frac{1}{2}}^{1} \left[1 - (1 - (1 - \mathbb{u})^{\mathbb{c}})^{\mathbb{d}} \right]^{\nu} d\mathbb{u} \right)^{\frac{1}{\nu}}$$

Proof. By the well-known Hölder inequality and Jensen–Mercer inequality, we have

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left(\frac{1 - (1 - \mathbb{u})^{\mathbb{c}}}{\mathbb{c}} \right)^{\mathbb{d}} \left| \mathcal{V}'(\mathbb{p} + \mathbb{q} - (\mathbb{u}\mathbb{y} + (1 - \mathbb{u})\mathbb{z})) \right| d\mathbb{u} \\
\leq & \left(\int_0^{\frac{1}{2}} \left(\frac{1 - (1 - \mathbb{u})^{\mathbb{c}}}{\mathbb{c}} \right)^{\nu\mathbb{d}} d\mathbb{u} \right)^{\frac{1}{\nu}} \left(\int_0^{\frac{1}{2}} \left| \mathcal{V}'(\mathbb{p} + \mathbb{q} - (\mathbb{u}\mathbb{y} + (1 - \mathbb{u})\mathbb{z})) \right|^{\mu} d\mathbb{u} \right)^{\frac{1}{\mu}} \\
\leq & \frac{1}{\mathbb{c}^{\mathbb{d}}} \left(\frac{1}{\mathbb{c}} \left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1 \right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1, \left(\frac{1}{2} \right)^{\mathbb{c}} \right) \right] \right)^{\frac{1}{\nu}} \\
& \times \left(\int_0^{\frac{1}{2}} \left[\left| \mathcal{V}'(\mathbb{p}) \right|^{\mu} + \left| \mathcal{V}'(\mathbb{q}) \right|^{\mu} - \left(\mathbb{u} \left| \mathcal{V}'(\mathbb{y}) \right|^{\mu} + (1 - \mathbb{u}) \left| \mathcal{V}'(\mathbb{z}) \right|^{\mu} \right) \right] d\mathbb{u} \right)^{\frac{1}{\mu}} \\
= & \frac{1}{\mathbb{c}^{\mathbb{d}}} \left(\frac{1}{\mathbb{c}} \left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1 \right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1, \left(\frac{1}{2} \right)^{\mathbb{c}} \right) \right] \right)^{\frac{1}{\nu}} \\
& \times \left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu} + |\mathcal{V}'(\mathbb{q})|^{\mu}}{2} - \frac{|\mathcal{V}'(\mathbb{y})|^{\mu} + 3|\mathcal{V}'(\mathbb{z})|^{\mu}}{8} \right)^{\frac{1}{\mu}}.
\end{aligned} \tag{25}$$

Similarly, we obtain

$$\begin{aligned}
& \int_0^{\frac{1}{2}} \left(\frac{1 - (1 - \mathbb{u})^{\mathbb{c}}}{\mathbb{c}} \right)^{\mathbb{d}} \left| \mathcal{V}'(\mathbb{p} + \mathbb{q} - (\mathbb{u}\mathbb{z} + (1 - \mathbb{u})\mathbb{y})) \right| d\mathbb{u} \\
\leq & \frac{1}{\mathbb{c}^{\mathbb{d}}} \left(\frac{1}{\mathbb{c}} \left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1 \right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \nu\mathbb{d} + 1, \left(\frac{1}{2} \right)^{\mathbb{c}} \right) \right] \right)^{\frac{1}{\nu}} \\
& \times \left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu} + |\mathcal{V}'(\mathbb{q})|^{\mu}}{2} - \frac{|\mathcal{V}'(\mathbb{z})|^{\mu} + 3|\mathcal{V}'(\mathbb{y})|^{\mu}}{8} \right)^{\frac{1}{\mu}},
\end{aligned} \tag{26}$$

$$\begin{aligned}
& \int_{\frac{1}{2}}^{1} \left[\frac{1}{\mathbb{c}^{\mathbb{d}}} - \left(\frac{1 - (1 - \mathbb{u})^{\mathbb{c}}}{\mathbb{c}} \right)^{\mathbb{d}} \right] |\mathcal{V}'(\mathbb{p} + \mathbb{q} - (\mathbb{u}\mathbb{y} + (1 - \mathbb{u})\mathbb{z}))| d\mathbb{u} \\
\leq & \frac{\Omega_7}{\mathbb{c}^{\mathbb{d}}} \left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu} + |\mathcal{V}'(\mathbb{q})|^{\mu}}{2} - \frac{3|\mathcal{V}'(\mathbb{y})|^{\mu} + |\mathcal{V}'(\mathbb{z})|^{\mu}}{8} \right)^{\frac{1}{\mu}}
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
& \int_{\frac{1}{2}}^{1} \left[\frac{1}{\mathbb{c}^{\mathbb{d}}} - \left(\frac{1 - (1 - \mathbb{u})^{\mathbb{c}}}{\mathbb{c}} \right)^{\mathbb{d}} \right] |\mathcal{V}'(\mathbb{p} + \mathbb{q} - (\mathbb{u}\mathbb{z} + (1 - \mathbb{u})\mathbb{y}))| d\mathbb{u} \\
\leq & \frac{\Omega_7}{\mathbb{c}^{\mathbb{d}}} \left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu} + |\mathcal{V}'(\mathbb{q})|^{\mu}}{2} - \frac{3|\mathcal{V}'(\mathbb{z})|^{\mu} + |\mathcal{V}'(\mathbb{y})|^{\mu}}{8} \right)^{\frac{1}{\mu}}.
\end{aligned} \tag{28}$$

If we substitute the inequalities (25)–(28) in (20), we obtain

$$
\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[\,{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+\,{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
\leq\;& \frac{(\mathbb{z}-\mathbb{y})}{2}\Bigg\{\left(\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right)^{\frac{1}{\nu}} \\
&\times\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{2}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+3|\mathcal{V}'(\mathbb{z})|^{\mu}}{8}\right)^{\frac{1}{\mu}} \\
&+\left(\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right)^{\frac{1}{\nu}} \\
&\times\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{2}-\frac{|\mathcal{V}'(\mathbb{z})|^{\mu}+3|\mathcal{V}'(\mathbb{y})|^{\mu}}{8}\right)^{\frac{1}{\mu}} \\
&+\Omega_7\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{2}-\frac{3|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{8}\right)^{\frac{1}{\mu}} \\
&+\Omega_7\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{2}-\frac{3|\mathcal{V}'(\mathbb{z})|^{\mu}+|\mathcal{V}'(\mathbb{y})|^{\mu}}{8}\right)^{\frac{1}{\mu}}\Bigg\} \\
=\;& \frac{(\mathbb{z}-\mathbb{y})}{2^{1+\frac{1}{\mu}}}\left(\left(\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right)^{\frac{1}{\nu}}+\Omega_7\right) \\
&\times\Bigg[\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+3|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}} \\
&+\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{3|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\Bigg].
\end{aligned}
$$

This concludes the proof. □

Remark 7. *If we choose $\mathbb{y}=\mathbb{p}$ and $\mathbb{z}=\mathbb{q}$ in Theorem 8, the following inequality is valid:*

$$
\begin{aligned}
&\left|\mathcal{V}\left(\frac{\mathbb{p}+\mathbb{q}}{2}\right)-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{q}-\mathbb{p})^{\mathbb{c}\mathbb{d}}}\left[\,{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{p}+}\mathcal{V}(\mathbb{q})+\,{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{q}-}\mathcal{V}(\mathbb{p})\right]\right| \\
&\leq\frac{(\mathbb{q}-\mathbb{p})}{2^{1+\frac{1}{\mu}}}\left[\left(\frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1\right)-\mathcal{B}\left(\frac{1}{\mathbb{c}},\nu\mathbb{d}+1,\left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right)^{\frac{1}{\nu}}+\Omega_7\right] \\
&\times\left[\left(\frac{3|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}}{4}\right)^{\frac{1}{\mu}}+\left(\frac{|\mathcal{V}'(\mathbb{p})|^{\mu}+3|\mathcal{V}'(\mathbb{q})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right]
\end{aligned}
$$

which was proved by Hezenci and Budak in [35].

Remark 8. *If we choose $\mathbb{c}=1$ in Theorem 7, then we obtain the inequality*

$$\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)-\frac{\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{d}}}\left[J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})+J^{\mathbb{d}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right|$$
$$\leq \frac{(\mathbb{z}-\mathbb{y})}{2^{1+\frac{1}{\mu}}}\left[\left(\frac{1}{(\nu\mathbb{d}+1)2^{\nu\mathbb{d}+1}}\right)^{\frac{1}{\nu}}+\left(\int_{\frac{1}{2}}^{1}\left[1-\mathbb{u}^{\mathbb{d}}\right]^{\nu}d\mathbb{u}\right)^{\frac{1}{\nu}}\right]$$
$$\times\left[\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{|\mathcal{V}'(\mathbb{y})|^{\mu}+3|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right.$$
$$\left.+\left(|\mathcal{V}'(\mathbb{p})|^{\mu}+|\mathcal{V}'(\mathbb{q})|^{\mu}-\frac{3|\mathcal{V}'(\mathbb{y})|^{\mu}+|\mathcal{V}'(\mathbb{z})|^{\mu}}{4}\right)^{\frac{1}{\mu}}\right].$$

4. Some Examples

Example 1. *Consider the convex function $\mathcal{V}:[0,5]\to\mathbb{R}$, $\mathcal{V}(\mathbb{u})=\mathbb{u}^3$. For $\mathbb{y}=1$ and $\mathbb{z}=3$, the left- and right-hand sides of (12) can be computed as*

$$\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right)=\mathcal{V}(3)=27$$

and

$$\mathcal{V}(\mathbb{p})+\mathcal{V}(\mathbb{q})-\mathcal{V}\left(\frac{\mathbb{y}+\mathbb{z}}{2}\right)=\mathcal{V}(0)+\mathcal{V}(5)-\mathcal{V}(2)=117$$

respectively. By (7), we have

$$\begin{aligned}
{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{y}+}\mathcal{V}(\mathbb{z}) &= {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{1+}\mathcal{V}(3)=\frac{1}{\Gamma(\mathbb{d})}\int_{1}^{3}\left(\frac{2^{\mathbb{c}}-(\mathbb{u}-1)^{\mathbb{c}}}{\mathbb{c}}\right)^{\mathbb{d}-1}(\mathbb{u}-1)^{\mathbb{c}-1}\mathbb{u}^3d\mathbb{u}\\
&= \frac{1}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\int_{1}^{3}(2^{\mathbb{c}}-(\mathbb{u}-1)^{\mathbb{c}})^{\mathbb{d}-1}\left[(\mathbb{u}-1)^{\mathbb{c}+2}+3(\mathbb{u}-1)^{\mathbb{c}+1}+3(\mathbb{u}-1)^{\mathbb{c}}+(\mathbb{u}-1)^{\mathbb{c}-1}\right]d\mathbb{u}\\
&= \frac{2^{\mathbb{c}\mathbb{d}}}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\int_{0}^{1}\mathbb{u}^{\mathbb{d}-1}\left[8(1-\mathbb{u})^{\frac{3}{\mathbb{c}}}+12(1-\mathbb{u})^{\frac{2}{\mathbb{c}}}+6(1-\mathbb{u})^{\frac{1}{\mathbb{c}}}+1\right]d\mathbb{u}\\
&= \frac{2^{\mathbb{c}\mathbb{d}}}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\left[8\mathcal{B}\left(\mathbb{d},\frac{3}{\mathbb{c}}+1\right)+12\mathcal{B}\left(\mathbb{d},\frac{2}{\mathbb{c}}+1\right)+6\mathcal{B}\left(\mathbb{d},\frac{1}{\mathbb{c}}+1\right)+\frac{1}{\mathbb{d}}\right]
\end{aligned}$$

and similarly, by (8), we have

$${}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{z}-}\mathcal{V}(\mathbb{y})={}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{3-}\mathcal{V}(1)=\frac{2^{\mathbb{c}\mathbb{d}}}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\left[\frac{27}{\mathbb{d}}-54\mathcal{B}\left(\mathbb{d},\frac{1}{\mathbb{c}}+1\right)+36\mathcal{B}\left(\mathbb{d},\frac{2}{\mathbb{c}}+1\right)-8\mathcal{B}\left(\mathbb{d},\frac{3}{\mathbb{c}}+1\right)\right].$$

Thus, the mid-term of (12) can be calculated as

$$\begin{aligned}
&\mathcal{V}(\mathbb{p})+\mathcal{V}(\mathbb{q})-\frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{y}+}\mathcal{V}(\mathbb{z})+{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{\mathbb{z}-}\mathcal{V}(\mathbb{y})\right]\\
&= 125-2\mathbb{d}\left[12\mathcal{B}\left(\mathbb{d},\frac{2}{\mathbb{c}}+1\right)-12\mathcal{B}\left(\mathbb{d},\frac{1}{\mathbb{c}}+1\right)+\frac{7}{\mathbb{d}}\right].
\end{aligned}$$

Consequently, we have the following inequality from inequality (12):

$$27 \leq 125 - 2\mathbb{d}\left[12\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}} + 1\right) - 12\mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}} + 1\right) + \frac{7}{\mathbb{d}}\right] \leq 117. \tag{29}$$

One can see the validity of the inequality (29) in Figure 1.

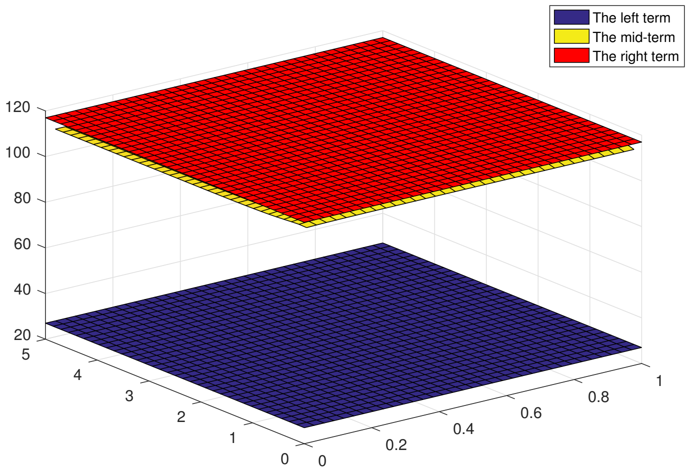

Figure 1. An example of Theorem 4, depending on $\mathbb{c} \in (0,1]$ and $\mathbb{d} \in (0,5]$, analysed and visualized with MATLAB.

Example 2. *Let us consider the same function in Example 1. It is clear that the function $|\mathcal{V}'|$ is convex. By (7) and (8), we have*

$$\begin{aligned} {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y}) &= {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{2+}\mathcal{V}(4) \\ &= \frac{2^{\mathbb{c}\mathbb{d}}}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\left[8\mathcal{B}\left(\mathbb{d}, \frac{3}{\mathbb{c}} + 1\right) + 24\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}} + 1\right) + 24\mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}} + 1\right) + \frac{1}{\mathbb{d}}\right] \end{aligned}$$

and

$$\begin{aligned} {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z}) &= {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{4-}\mathcal{V}(2) \\ &= \frac{2^{\mathbb{c}\mathbb{d}}}{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d})}\left[64/\mathbb{d} - 96\mathcal{B}(\mathbb{d}, 1/\mathbb{c}+1) + 48\mathcal{B}(\mathbb{d}, 2/\mathbb{c}+1) - 8\mathcal{B}\left(\mathbb{d}, \frac{3}{\mathbb{c}} + 1\right)\right]. \end{aligned}$$

Thus, the left-hand side of the inequality (14) in Theorem 5 can be written as

$$\begin{aligned} &\left|\frac{\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z}) + \mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y})}{2} - \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\ = &\left|\frac{8+64}{2} - \frac{\mathbb{d}}{2}\left[72\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}} + 1\right) - 72\mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}} + 1\right) + \frac{72}{\mathbb{d}}\right]\right| \\ = &\left|36 - 36\mathbb{d}\left[\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}} + 1\right) - \mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}} + 1\right) + \frac{1}{\mathbb{d}}\right]\right|. \end{aligned}$$

On the other hand, since $\mathcal{V}'(\mathbb{u}) = 3\mathbb{u}^2$, we can calculate the right-hand side of (14) as

$$\begin{aligned}
&\frac{\mathbb{z}-\mathbb{y}}{\mathbb{c}}\left[|\mathcal{V}'(\mathbb{p})| + |\mathcal{V}'(\mathbb{q})| - \frac{|\mathcal{V}'(\mathbb{z})| + |\mathcal{V}'(\mathbb{y})|}{2}\right] \\
&\times \left[2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right)\right] \\
&= \frac{120}{\mathbb{c}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right)\right].
\end{aligned}$$

Therefore, we obtain the inequality

$$\begin{aligned}
&\left|36 - 36\mathbb{d}\left[\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}}+1\right) - \mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}}+1\right) + \frac{1}{\mathbb{d}}\right]\right| \\
\leq\ & \frac{120}{\mathbb{c}}\left[2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right) - \mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right)\right].
\end{aligned} \tag{30}$$

One can see the validity of the inequality (30) in Figure 2.

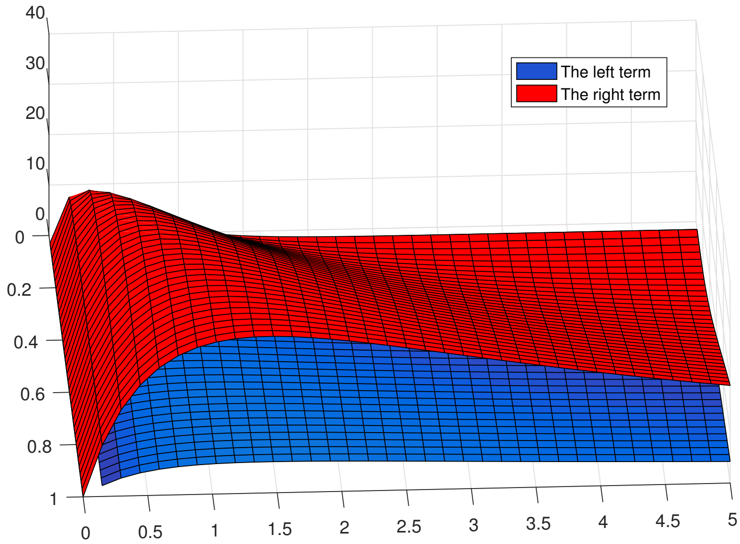

Figure 2. An example of Theorem 5, depending on $\mathbb{c} \in (0,1]$ and $\mathbb{d} \in (0,5]$, analysed and visualized with MATLAB.

Similarly, we can calculate the left and right sides of the inequality (19) in Theorem 7 as follows:

$$\begin{aligned}
&\left|\mathcal{V}\left(\mathbb{p}+\mathbb{q}-\frac{\mathbb{y}+\mathbb{z}}{2}\right) - \frac{\mathbb{c}^{\mathbb{d}}\Gamma(\mathbb{d}+1)}{2(\mathbb{z}-\mathbb{y})^{\mathbb{c}\mathbb{d}}}\left[{}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{z})+}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{y}) + {}^{\mathbb{d}}\mathcal{Y}^{\mathbb{c}}_{(\mathbb{p}+\mathbb{q}-\mathbb{y})-}\mathcal{V}(\mathbb{p}+\mathbb{q}-\mathbb{z})\right]\right| \\
=\ &\left|27 - 36\mathbb{d}\left[\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}}+1\right) - \mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}}+1\right) + \frac{1}{\mathbb{d}}\right]\right|
\end{aligned}$$

and

$$\begin{aligned}
&(\mathbb{z}-\mathbb{y})\left[\frac{1}{2} + \frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right) - 2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right] \\
&\times \left[|\mathcal{V}'(\mathbb{p})| + |\mathcal{V}'(\mathbb{q})| - \frac{|\mathcal{V}'(\mathbb{y})| + |\mathcal{V}'(\mathbb{z})|}{2}\right] \\
&= 120\left[\frac{1}{2} + \frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right) - 2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right].
\end{aligned}$$

Thus, we have the inequality

$$\left|27 - 36\mathbb{d}\left[\mathcal{B}\left(\mathbb{d}, \frac{2}{\mathbb{c}} + 1\right) - \mathcal{B}\left(\mathbb{d}, \frac{1}{\mathbb{c}} + 1\right) + \frac{1}{\mathbb{d}}\right]\right| \le 120\left[\frac{1}{2} + \frac{1}{\mathbb{c}}\left[\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1\right) - 2\mathcal{B}\left(\frac{1}{\mathbb{c}}, \mathbb{d}+1, \left(\frac{1}{2}\right)^{\mathbb{c}}\right)\right]\right]. \tag{31}$$

One can see the validity of the inequality (31) in Figure 3.

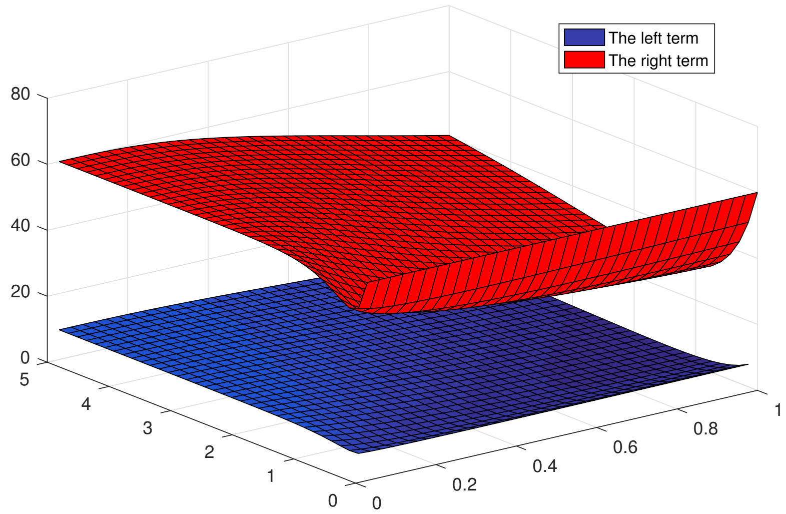

Figure 3. An example of Theorem 7, depending on $\mathbb{c} \in (0, 1]$ and $\mathbb{d} \in (0, 5]$, analysed and visualized with MATLAB.

5. Conclusions

This study aimed to investigate some important inequalities via the FERLIs and Jensen–Mercer inequality. The proven inequalities included the Hermite–Hadamard, trapezoid, and midpoint types. In order to obtain these new results, some equalities have been established with the FERLIs and the features of the convex functions. Also, in an easy way, one can see the connections between our main findings and prior studies conducted on Riemann–Liouville fractional integrals and FERLIs. To validate the accuracy of the new findings, a number of examples have been illustrated with graphical representations.

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