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On Construction of Bounded Sets Not Admitting a General Type of Riesz Spectrum

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Abstract: We construct a bound set that does not admit a Riesz spectrum containing a nonempty periodic set for which the period is a rational multiple of a fixed constant. As a consequence, we obtain a bounded set V with an arbitrarily small Lebesgue measure such that for any positive integer N, the set of exponentials with frequencies in any union of cosets of $N\mathbb{Z}$ cannot be a frame for the space of square integrable functions over V. These results are based on the proof technique of Olevskii and Ulanovskii from 2008.

Keywords: complex exponentials; spectrum; exponential bases; Riesz bases; Riesz sequences; frames

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1. Introduction and Main Results

One of the fundamental research topics in Fourier analysis is the theory of exponential bases and frames. The elementary fact that $\{e^{2\pi i n\cdot x}\}_{n\in\mathbb{Z}^d}$ forms an orthogonal basis for $L^2[-\frac{1}{2},\frac{1}{2}]^d$ has far-reaching implications in many areas of mathematics and engineering. For instance, the celebrated Whittaker–Shannon–Kotel'nikov sampling theorem is an important consequence of this fact (see e.g., [1]).

As a natural generalization of the functions $\{e^{2\pi i n\cdot x}\}_{n\in\mathbb{Z}^d}$ in $L^2[-\frac{1}{2},\frac{1}{2}]^d$, one considers the set of exponentials $E(\Lambda) := \{e^{2\pi i\lambda\cdot x} : \lambda\in\Lambda\}$, where $\Lambda\subset\mathbb{R}^d$ is a discrete set consisting of the pure frequency components of exponentials (thus called the *frequency set* or *spectrum*), in the Hilbert space $L^2(S)$ for a finite positive measure set $S\subset\mathbb{R}^d$. That is, for each $\lambda\in\Lambda$, the map $x\mapsto e^{2\pi i\lambda\cdot x}$ restricted to the set S is considered as a function in $L^2(S)$. Characterizing the properties of $E(\Lambda)$ in the space $L^2(S)$, such as whether $E(\Lambda)$ forms an orthogonal/Riesz basis or a frame, has been an important problem in nonharmonic Fourier analysis. The problem has a close connection to the theory of entire functions of the exponential type in complex analysis through the celebrated work of Paley and Wiener [2]. For more details on this connection and for some historical background, we refer the reader to the excellent book by Young [3]. Below, we give a short overview of some known results on exponential bases and frames.

1.1. An Overview of Existing Work on Exponential Bases and Frames

Exponential orthogonal bases: For the case of orthogonal bases, Fuglede [4] posed a famous conjecture (also called the *spectral set conjecture*) that states that if $S\subset\mathbb{R}^d$ is a finite positive measure set, then there is an exponential orthogonal basis $E(\Lambda)$ (with $\Lambda\subset\mathbb{R}^d$) for $L^2(S)$ if and only if the set S tiles $\mathbb{R}^d$ by translations along a discrete set $\Gamma\subset\mathbb{R}^d$ in the sense that

$$\sum_{\gamma\in\Gamma}\chi_S(x+\gamma)=1 \quad \text{for a.e. } x\in\mathbb{R}^d, \tag{1}$$

where $\chi_S(x)=1$ for $x\in S$ and is 0 otherwise. The conjecture turned out to be false for $d\geq 3$ but is still open for $d=1,2$. Nevertheless, there are many special cases for which the conjecture is known to be true. For instance, the conjecture is true when Γ is a lattice of $\mathbb{R}^d$—in which case the set $\Lambda \subset \mathbb{R}^d$ can be chosen to be the dual lattice of Γ [4]—and also when $S \subset \mathbb{R}^d$ is a convex set of finite positive measure for all $d \in \mathbb{N}$ [5]. In particular, it was shown in [6] that there is no exponential orthogonal basis for $L^2(S)$ when S is the unit ball of $\mathbb{R}^d$ for $d \geq 2$, in contrast to the case $d = 1$, where the unit ball is simply $S = [-1, 1]$, and $E(\frac{1}{2}\mathbb{Z})$ is an orthogonal basis for $L^2[-1, 1]$. For more details on Fuglede's conjecture and its recent progress, we refer the reader to [5] and the references therein.

Exponential Riesz bases: The relaxed case of Riesz bases is yet more challenging. Certainly, relaxing the condition of orthogonal bases to Riesz bases allows for potentially much more feasible sets $S \subset \mathbb{R}^d$. However, there are only several classes of sets $S \subset \mathbb{R}^d$ that are known to *admit a Riesz spectrum*, meaning that there *exists an exponential Riesz basis for* $L^2(S)$. For instance, the class of convex symmetric polygons in $\mathbb{R}^2$ [7], the class of sets that are finite unions of intervals in $\mathbb{R}^d$ [8,9], and the class of certain symmetric convex polytopes in $\mathbb{R}^d$ for all $d \geq 1$ [10]. Moreover, the existence of exponential Riesz bases for disjoint intervals with hierarchical structure was proved in [11], and exponential Riesz bases with restricted supports were treated in [12]. Recently, Kozma, Nitzan, and Olevskii [13] constructed a bounded measurable set $S \subset \mathbb{R}$ such that no set of exponentials can be a Riesz basis for $L^2(S)$.

In search of an analogue to Fuglede's conjecture for Riesz bases, Grepstad and Lev [14] considered the sets $S \subset \mathbb{R}^d$ that satisfy for some discrete set $\Gamma \subset \mathbb{R}^d$ and some $k \in \mathbb{N}$

$$\sum_{\gamma \in \Gamma} \chi_S(x + \gamma) = k \quad \text{for a.e. } \ x \in \mathbb{R}^d.$$

Such a set $S \subset \mathbb{R}^d$ is called a *k-tile* with respect to Γ; in particular, the set S satisfying (1) is a 1-tile with respect to Γ. It was shown in [14] that if $S \subset \mathbb{R}^d$ is a bounded k-tile set with respect to a lattice $\Gamma \subset \mathbb{R}^d$ and has measure zero boundary, then the set S admits a Riesz spectrum $\Lambda \subset \mathbb{R}^d$, which is obtained using quasicrystals [15,16]. Later, Kolountzakis [17] removed the measure zero boundary condition of S and showed that Λ can be chosen to be a union of k translations of Γ^* (referred to as a (k, Γ^*)-*structured spectrum*), where $\Gamma^* := (A^{-1})^T \mathbb{Z}^d$ is the dual lattice of $\Gamma = A\,\mathbb{Z}^d$ with $A \in \mathrm{GL}(d, \mathbb{R})$. The converse of this statement was proved by Agora et al. [18], thus establishing the equivalence: given a lattice $\Gamma \subset \mathbb{R}^d$, a bounded set $S \subset \mathbb{R}^d$ is a k-tile with respect to Γ if and only if it admits a (k, Γ^*)-structured Riesz spectrum. They also showed that the *boundedness* of S is essential by constructing an unbounded 2-tile set $S \subset \mathbb{R}$ with respect to $\mathbb{Z}$ that does not admit a $(2, \mathbb{Z})$-structured Riesz spectrum. Nevertheless, for unbounded multi-tiles $S \subset \mathbb{R}^d$ with respect to a lattice Γ, Cabrelli and Carbajal [19] were able to provide a sufficient condition for S to admit a structured Riesz spectrum. Recently, Cabrelli et al. [20] found a necessary and sufficient condition for a multi-tile $S \subset \mathbb{R}^d$ of finite positive measure to admit a structured Riesz spectrum, which is given in terms of the Bohr compactification of the tiling lattice Γ.

Exponential frames: Since frames allow for redundancy, it is relatively easier to obtain exponential frames than exponential Riesz bases. For instance, the set of exponentials $\{e^{2\pi i n \cdot x}\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]^d$ and is thus a frame for $L^2(S)$ with frame bounds $A = B = 1$ whenever S is a measurable subset of $[-\frac{1}{2}, \frac{1}{2}]^d$.

Nitzan et al. [21] proved that if $S \subset \mathbb{R}^d$ is a finite positive measure set, then there exists an exponential frame $E(\Lambda)$ (with $\Lambda \subset \mathbb{R}^d$) for $L^2(S)$ with frame bounds $c\,|S|$ and $C\,|S|$, where $0 < c < C < \infty$ are absolute constants. The proof is based on a lemma from Marcus et al. [22] that resolved the famous Kadison–Singer problem in the affirmative.

Universality: In [23,24], Olevskii and Ulanovskii considered the interesting question of universality. They discovered some frequency sets $\Lambda \subset \mathbb{R}^d$ that have universal properties: namely, the so-called *universal* uniqueness/sampling/interpolation sets $\Lambda \subset \mathbb{R}^d$ for Paley–Wiener spaces $PW(S)$ with *all* sets $S \subset \mathbb{R}^d$ in a certain class. In our notation, this corresponds to the set of exponentials $E(\Lambda)$ being a complete sequence/frame/Riesz sequence in $L^2(S)$ for *all* sets $S \subset \mathbb{R}^d$ in a certain class. For the convenience of the readers, we include a short exposition on the relevant notions in Paley–Wiener spaces in Appendix A.

It has been shown that universal complete sets of exponentials exist: for instance, the system $E(\Lambda)$ with $\Lambda = \{\ldots, -6, -4, -2, 1, 3, 5, \ldots\}$ is complete in $L^2(S)$ for every measurable set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| \leq \frac{1}{2}$. Furthermore, any set $E(\Lambda)$ with $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ satisfying $0 < |\lambda_n - n| \leq 1/2^{|n|}$ for all $n \in \mathbb{Z}$ is complete in $L^2(S)$ whenever $S \subset \mathbb{R}$ is a bounded measurable set with $|S| < 1$.

On the other hand, the existence of universal exponential frames and universal exponential Riesz sequences depends on the topological properties of S. As a positive result, it has been shown that there is a perturbation Λ of $\mathbb{Z}$ such that $E(\Lambda)$ is a frame for $L^2(S)$ whenever $S \subset \mathbb{R}$ is a *compact* set with $|S| < 1$; a different construction of such a set $\Lambda \subset \mathbb{R}$ was given by Matei and Meyer [15,16] based on the theory of quasicrystals. Similarly, there is a perturbation Λ of $\mathbb{Z}$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(S)$ whenever $S \subset \mathbb{R}$ is an *open* set with $|S| > 1$. However, on the negative side, it has been shown that given any $0 < \epsilon < 2$ and a separated set $\Lambda \subset \mathbb{R}$ with $D^-(\Lambda) < 2$, there is a measurable set $S \subset [0, 2]$ with $|S| < \epsilon$ such that $E(\Lambda)$ is not a frame for $L^2(S)$, indicating that the *compactness* of S in the aforementioned result cannot be dropped. Similarly, it has been shown that given any $0 < \epsilon < 2$ and a separated set $\Lambda \subset \mathbb{R}$ with $D^+(\Lambda) > 0$, there is a measurable set $S \subset [0, 2]$ with $|S| > 2 - \epsilon$ such that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$, similarly indicating that the restriction to *open* sets cannot be dropped.

For more details on the universality results, we refer the reader to Lectures 6 and 7 in the excellent lecture book by Olevskii and Ulanovskii [25].

1.2. Contribution of the Paper

This paper is motivated by the following problem.

Problem 1. *Is there a bounded/unbounded set $S \subset \mathbb{R}^d$ that does not admit a Riesz spectrum, meaning that for every $\Lambda \subset \mathbb{R}^d$, the set of exponentials $\{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is not a Riesz basis for $L^2(S)$?*

This problem was recently solved by Kozma, Nitzan, and Olevskii [13]. They constructed a bounded measurable set $S \subset \mathbb{R}$ such that no set of exponentials can be a Riesz basis for $L^2(S)$.

In this paper, we take a different approach to construct a bounded subset of $\mathbb{R}$ that does not admit a certain general type of Riesz spectrum. Through this, we offer diverse methods for constructing sets that do not admit Riesz spectra. In particular, our approach enables the design of specific spectra that we aim to exclude. To achieve this, we adapt the proof technique of Olevskii and Ulanovskii [24], which also works in higher dimensions (see Section 1 in [24]); thus, our results also extend to higher dimensions. However, for simplicity of presentation, we will only consider the dimension-one case ($d = 1$).

Before presenting our results, note that for any bounded set $S \subset \mathbb{R}$, there exist some parameters $\sigma > 0$ and $a \in \mathbb{R}$ such that $\frac{1}{\sigma}S + a \subset [-\frac{1}{2}, \frac{1}{2}]$. It is therefore enough to restrict our attention to sets $S \subset [-\frac{1}{2}, \frac{1}{2}]$ (see Lemma 1 below). Also, recall that a set $S \subset \mathbb{R}$ is said to *admit a Riesz spectrum* $\Lambda \subset \mathbb{R}$ if the system $E(\Lambda)$ is a Riesz basis for $L^2(S)$.

Our first main result is as follows.

Theorem 1. *Let $0 < \alpha \leq 1$ and $0 < \epsilon < 1$. There exists a measurable set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| > 1 - \epsilon$ satisfying the following property: if $\Lambda \subset \mathbb{R}$ contains arbitrarily long arithmetic progressions with a fixed common difference belonging in $\alpha\mathbb{N}$, then $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$. Moreover, such a set can be constructed explicitly as*

$$S = [-\tfrac{1}{2}, \tfrac{1}{2}] \backslash V \quad \textit{with} \quad V = [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \Big(\cup_{\ell=1}^{\infty} \cup_{m\in\mathbb{Z}} \tfrac{m}{\ell\alpha} + \big(-\tfrac{\epsilon}{\ell\cdot 2^{\ell+3}}, \tfrac{\epsilon}{\ell\cdot 2^{\ell+3}}\big) \Big). \qquad (2)$$

It should be noted that the set $V \subset [-\frac{1}{2}, \frac{1}{2}]$ is an open set containing $\frac{1}{\alpha}\mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$. This set has a small Lebesgue measure $|V| < \epsilon$ due to the exponentially decreasing length of the intervals. It is worth comparing the set V with a fat Cantor set that is a closed, nowhere dense subset of $[-\frac{1}{2}, \frac{1}{2}]$ with positive measure (see e.g., [26,27]), where a set is called *nowhere dense* if its closure has an empty interior. In contrast to the fat Cantor sets, the set V has a nonempty interior and is dense in $[-\frac{1}{2}, \frac{1}{2}]$ because it contains $\frac{1}{\alpha}\mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$.

To illustrate the *dense* set $V \subset [-\frac{1}{2}, \frac{1}{2}]$, we truncate the infinite union $\cup_{\ell=1}^{\infty}$ in its expression to the finite union over $\ell = 1, \ldots, 10$. The corresponding sets for $\alpha = 1$ and $\epsilon = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ are shown in Figure 1.

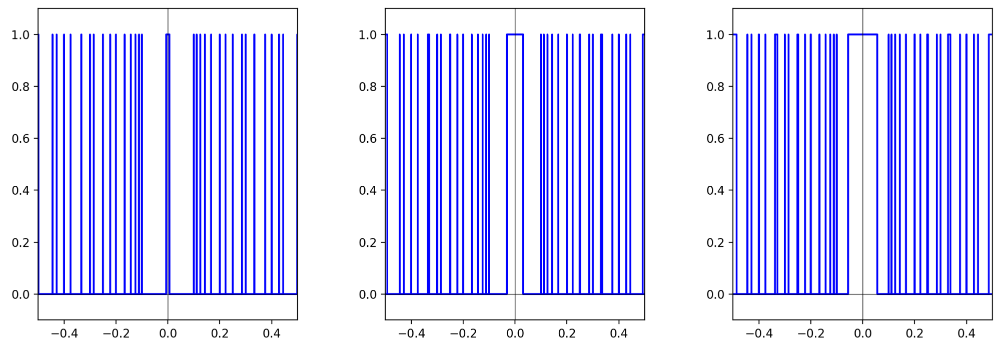

Figure 1. The characteristic function of the corresponding truncated set for $\alpha = 1$ and $\epsilon = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ (from left to right).

To help the understanding of the readers, we provide two sets $\Lambda \subset \mathbb{R}$: one which meets and the other which does not meet the condition stated in Theorem 1.

Example 1.

(a) *Let $M_1 < M_2 < \cdots$ be an increasing sequence in $\mathbb{N}$, and let $P \in \mathbb{N}$. Define the sequence $d_1 < d_2 < \cdots$ by $d_1 = 0$ and $d_k = 2\sum_{n=1}^{k-1} M_n P$ for $k \geq 2$. Clearly, we have $d_{k+1} - d_k = 2M_k P$ for all $k \in \mathbb{N}$. Consider the set*

$$\Lambda = \pm \bigcup_{k=1}^{\infty} \{d_k+P,\ d_k+2P,\ \ldots,\ d_k+M_kP\} \quad \subset \mathbb{Z}$$

where $\pm\Lambda_0 := \Lambda_0 \cup (-\Lambda_0)$ for any set $\Lambda_0 \subset \mathbb{R}$. This set contains arbitrarily long arithmetic progressions with common difference P and has lower and upper Beurling densities given by $D^-(\Lambda) = \frac{1}{2P}$ and $D^+(\Lambda) = \frac{1}{P}$, respectively (see Section 2.3 for the definition of the Beurling density).

(b) *Let $N \in \mathbb{N}$ and let $\{\sigma_k\}_{k=1}^{\infty} \subset (0,1)$ be a sequence of distinct irrational numbers between 0 and 1. Consider the set*

$$\Lambda = \pm \bigcup_{k=1}^{\infty} \Big(\sigma_k+Nk + \{0 \cdot 100^k,\ 1 \cdot 100^k,\ \ldots,\ (k-1) \cdot 100^k\}\Big) \quad \subset \mathbb{R}$$

that has a uniform Beurling density $D(\Lambda) = \frac{1}{N}$. For each $k \in \mathbb{N}$, the set Λ contains exactly one arithmetic progression with a common difference 100^k in the positive domain $(0, \infty)$: namely, the arithmetic progression $\sigma_k+Nk, \sigma_k+Nk+100^k, \ldots, \sigma_k+Nk+(k-1)\cdot 100^k$ of length k. Due to the $\pm$ mirror symmetry, the set Λ has another such arithmetic progression in the negative domain $(-\infty, 0)$. Note that all of these arithmetic progressions have integer-valued common differences and are distanced by some distinct irrational numbers, so none of them can be connected with another to form a longer arithmetic progression. Hence, there is no number $P \in \mathbb{N}$ for which the set Λ contains arbitrarily long arithmetic progressions with common difference P. Such a set $\Lambda \subset \mathbb{R}$ is not covered by the class of frequency sets considered in Theorem 1.

Our second main result is the following.

Theorem 2. *Let $0 < \epsilon < 1$, and let $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$ be a family of separated sets with $D^+(\Lambda_\ell) > 0$ for all $\ell \in \mathbb{N}$. One can construct a measurable set $S = S(\epsilon, \{\Lambda_\ell\}_{\ell=1}^\infty) \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| > 1 - \epsilon$ such that $E(\Lambda_\ell)$ is not a Riesz sequence in $L^2(S)$ for all $\ell \in \mathbb{N}$.*

Let us present some interesting implications of our main results.

By convention, a discrete set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ with $\lambda_n < \lambda_{n+1}$ is called *periodic with period* $t > 0$ (or *t-periodic*) if there is a number $N \in \mathbb{N}$ such that $\lambda_{n+N} - \lambda_n = t$ for all $n \in \mathbb{Z}$. Note that if $\Lambda \subset \mathbb{R}$ is a nonempty periodic set with period $\alpha \cdot \frac{P}{Q} \in \alpha\mathbb{Q}$, where $P, Q \in \mathbb{N}$ are coprime numbers, then it must contain a translated copy of $\alpha P\mathbb{Z}$: that is, $\alpha P\mathbb{Z}+d \subset \Lambda$ for some $d \in \mathbb{R}$. As a result, we have the following corollary of Theorem 1.

Corollary 1. *For any $0 < \alpha \leq 1$ and $0 < \epsilon < 1$, let $S \subset [-\frac{1}{2}, \frac{1}{2}]$ be the set given by (2). Then for any nonempty periodic set $\Lambda \subset \mathbb{R}$ with its period belonging in $\alpha\mathbb{Q}_+ = \alpha\mathbb{Q} \cap (0, \infty)$, the system $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$. Consequently, the set S does not admit a Riesz spectrum containing a nonempty periodic set with its period belonging in $\alpha\mathbb{Q}_+$.*

It is worth noting that the class of nonempty periodic sets with a rational period is *uncountable* because of the flexibility in the placement of elements in each period; hence, Corollary 1 cannot be deduced from Theorem 2.

As mentioned in Section 1.1, Agora et al. [18] constructed an unbounded 2-tile set $S \subset \mathbb{R}$ with respect to $\mathbb{Z}$ that does not admit a Riesz spectrum of the form $(\mathbb{Z}+\sigma_1) \cup (\mathbb{Z}+\sigma_2)$ with $\sigma_1, \sigma_2 \in \mathbb{R}$. By a dilation, one could easily generalize this example to an unbounded 2-tile set $W \subset \mathbb{R}$ with respect to $\frac{1}{\alpha}\mathbb{Z}$ for any fixed $\alpha > 0$ that does not admit a Riesz spectrum of the form $(\alpha\mathbb{Z}+\sigma_1) \cup (\alpha\mathbb{Z}+\sigma_2)$ with $\sigma_1, \sigma_2 \in \mathbb{R}$. Note that such a form of Riesz spectrum is α-periodic and thus not admitted by our set S given by (2) for any $0 < \epsilon < 1$. In fact, our set S has a much stronger property than W: namely, that S does not admit a periodic Riesz spectrum with its period belonging in $\alpha\mathbb{Q}_+$, and moreover, the set S is *bounded*.

Since the set S is contained in $[-\frac{1}{2}, \frac{1}{2}]$, it is particularly interesting to consider the frequency sets consisting of integers $\Omega \subset \mathbb{Z}$. Noting that a periodic subset of $\mathbb{Z}$ is necessarily N-periodic for some $N \in \mathbb{N}$, we immediately deduce the following result from Corollary 1.

Corollary 2. *Let $S \subset [-\frac{1}{2}, \frac{1}{2}]$ be the set given by (2) with $\alpha = 1$ and any $0 < \epsilon < 1$. Then for any nonempty periodic set $\Omega \subset \mathbb{Z}$, the system $E(\Omega)$ is not a Riesz sequence in $L^2(S)$.*

Alternatively, one could construct such a set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ from Theorem 2 by observing that the family of all nonempty periodic integer sets is *countable*; indeed, the one and only nonempty 1-periodic integer set is $\mathbb{Z}$, the nonempty 2-periodic integer sets are $2\mathbb{Z}$, $2\mathbb{Z}+1$, $\mathbb{Z}$, and so on.

Further, it is easy to deduce the following result from Corollary 2 and Proposition 2 below by setting $V := [-\frac{1}{2}, \frac{1}{2}] \backslash S$ and $\Omega' := \mathbb{Z} \backslash \Omega$.

Corollary 3. *Let $S \subset [-\frac{1}{2}, \frac{1}{2}]$ be the set given by (2) with $\alpha = 1$ and any $0 < \epsilon < 1$, and let $V := [-\frac{1}{2}, \frac{1}{2}] \backslash S$. Then for any proper periodic subset $\Omega' \subsetneq \mathbb{Z}$, the system $E(\Omega')$ is not a frame for $L^2(V)$.*

The significance of Corollary 3 is in the fact that for any $N \in \mathbb{N}$ and any proper subset $I \subsetneq \{0, \ldots, N-1\}$, the set of exponentials $E\big(\cup_{n\in I} (N\mathbb{Z}+n)\big)$ is not a frame for $L^2(V)$ even though the set V has a very small Lebesgue measure $|V| < \epsilon$. Note that $E(\mathbb{Z})$ is a frame for $L^2(V)$ with frame bounds $A = B = 1$ since it is an orthonormal basis for $L^2[0, 1]$.

2. Preliminaries

2.1. Sequences in Separable Hilbert Spaces

Definition 1. *A sequence $\{f_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called*

- *a Bessel sequence in $\mathcal{H}$ (with a Bessel bound B) if there is a constant $B > 0$ such that*
$$\sum_{n\in\mathbb{Z}} \left|\langle f, f_n\rangle\right|^2 \le B\,\|f\|^2 \quad \text{for all } f \in \mathcal{H};$$
- *a frame for $\mathcal{H}$ (with frame bounds A and B) if there are constants $0 < A \le B < \infty$ such that*
$$A\,\|f\|^2 \le \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \le B\,\|f\|^2 \quad \text{for all } f \in \mathcal{H};$$
- *a Riesz sequence in $\mathcal{H}$ (with Riesz bounds A and B) if there are constants $0 < A \le B < \infty$ such that*
$$A\,\|\mathbf{c}\|_{\ell_2}^2 \le \Big\|\sum_{n\in\mathbb{Z}} c_n f_n\Big\|^2 \le B\,\|\mathbf{c}\|_{\ell_2}^2 \quad \text{for all } \mathbf{c} = \{c_n\}_{n\in\mathbb{Z}} \in \ell_2(\mathbb{Z});$$
- *a Riesz basis for $\mathcal{H}$ (with Riesz bounds A and B) if it is a complete Riesz sequence in $\mathcal{H}$ (with Riesz bounds A and B);*
- *an orthogonal basis for $\mathcal{H}$ if it is a complete sequence of nonzero elements in $\mathcal{H}$ such that $\langle f_m, f_n\rangle = 0$ whenever $m \ne n$;*
- *an orthonormal basis for $\mathcal{H}$ if it is complete and $\langle f_m, f_n\rangle = \delta_{m,n}$ whenever $m \ne n$.*

The associated bounds A and B are said to be optimal if they are the tightest constants satisfying the respective inequality.

In general, an orthonormal basis is a Riesz basis with Riesz bounds $A = B = 1$, but an orthogonal basis is not necessarily norm-bounded below and thus is generally not a Riesz basis (for instance, consider the sequence $\{\frac{e_n}{n}\}_{n=1}^{\infty}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$). Nevertheless, exponential functions have a constant norm in $L^2(S)$ for any finite measure set $S \subset \mathbb{R}^d$: namely, $\|e^{2\pi i\lambda\cdot(\cdot)}\|_{L^2(S)} = |S|^{1/2}$ for all $\lambda \in \mathbb{R}^d$. Thus, an exponential orthogonal basis is simply an exponential orthonormal basis scaled by a constant.

Proposition 1. *Let $\mathcal{H}$ be a separable Hilbert space.*

(a) Corollary 3.7.2 in [28]: Every subfamily of a Riesz basis is a Riesz sequence with the same bounds (the optimal bounds may be tighter).

(b) Corollary 8.24 in [29]: If $\{f_n\}_{n\in\mathbb{Z}}$ is a Bessel sequence in $\mathcal{H}$ with Bessel bound B, then $\|f_i\|^2 \le B$ for all $i \in I$. If $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ with bounds $0 < A \le B < \infty$, then $A \le \|f_i\|^2 \le B$ for all $i \in I$.

(c) Lemma 3.6.9, Theorems 3.6.6, 5.4.1 and 7.1.1 in [28] (or see Theorems 7.13, 8.27 and 8.32 in [29]): Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$ and let $\{f_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$. The following are equivalent.
- *$\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$;*
- *$\{f_n\}_{n\in\mathbb{Z}}$ is an exact frame (i.e., a frame that ceases to be a frame whenever a single element is removed) for $\mathcal{H}$;*
- *$\{f_n\}_{n\in\mathbb{Z}}$ is an unconditional basis of $\mathcal{H}$ with $0 < \inf_{n\in\mathbb{Z}} \|f_n\| \le \sup_{n\in\mathbb{Z}} \|f_n\| < \infty$;*
- *There is a bijective bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that $Te_n = f_n$ for all $n \in \mathbb{Z}$.*

Moreover, in this case, the optimal frame bounds coincide with the optimal Riesz bounds.

Proposition 2 (Proposition 5.4 in [30]). *Let $\{e_n\}_{n\in I}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$, where I is a countable index set. Let $P : \mathcal{H} \to \mathcal{M}$ be the orthogonal projection from $\mathcal{H}$ onto a closed subspace $\mathcal{M}$. Let $J \subset I$ and $0 < \alpha \le 1$. The following are equivalent.*

(i) *$\{Pe_n\}_{n\in J} \subset \mathcal{M}$ is a frame for $\mathcal{M}$ with lower bound α;*
(ii) *$\{Pe_n\}_{n\in I\setminus J} \subset \mathcal{M}$ is a Bessel sequence with bound $1 - \alpha$;*
(iii) *$\{(\mathrm{Id} - P)e_n\}_{n\in I\setminus J} \subset \mathcal{M}^{\perp}$ is a Riesz sequence with lower bound α.*

2.2. Exponential Systems

As already introduced in Section 1, we define the exponential system $E(\Lambda) = \{e^{2\pi i\lambda\cdot(\cdot)} \colon \lambda \in \Lambda\}$ for a discrete set $\Lambda \subset \mathbb{R}^d$ (called a *frequency set* or a *spectrum*).

Lemma 1. *Assume that $E(\Lambda)$ is a Riesz basis for $L^2(S)$ with bounds $0 < A \leq B < \infty$, where $\Lambda \subset \mathbb{R}^d$ is a discrete set and $S \subset \mathbb{R}^d$ is a measurable set.*

(a) For any $a \in \mathbb{R}^d$, the system $E(\Lambda)$ is a Riesz basis for $L^2(S + a)$ with bounds A and B.

(b) For any $b \in \mathbb{R}^d$, the system $E(\Lambda + b)$ is a Riesz basis for $L^2(S)$ with bounds A and B.

(c) For any $\sigma > 0$, the system $\sqrt{\sigma}\, E(\sigma\Lambda)$ is a Riesz basis for $L^2(\frac{1}{\sigma}S)$ with bounds A and B; equivalently, $E(\sigma\Lambda)$ is a Riesz basis for $L^2(\frac{1}{\sigma}S)$ with bounds $\frac{A}{\sigma}$ and $\frac{B}{\sigma}$.

A proof of Lemma 1 is given in Appendix B.

Remark 1. *Lemma 1 remains valid if the term "Riesz basis" is replaced with one of the following: "Riesz sequence", "frame", or "frame sequence" (and also "Bessel sequence", in which case the lower bound is simply neglected).*

Theorem 3 (The Paley–Wiener stability theorem [2])**.** *Let $V \subset \mathbb{R}$ be a bounded set of positive measure and $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ be a sequence of real numbers such that $E(\Lambda)$ is a Riesz basis for $L^2(V)$ (respectively, a frame for $L^2(V)$, a Riesz sequence in $L^2(V)$). There exists a constant $\theta = \theta(\Lambda, V) > 0$ such that whenever $\Lambda' = \{\lambda'_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ satisfies*

$$|\lambda'_n - \lambda_n| \leq \theta, \quad n \in \mathbb{Z},$$

the set of exponentials $E(\Lambda')$ is a Riesz basis for $L^2(V)$ (respectively, a frame for $L^2(V)$, a Riesz sequence in $L^2(V)$).

For a proof of Theorem 3, we refer the reader to p. 160 in [3] for the case where V is a single interval and Section 2.3 in [8] for the general case. It is worth noting that the constant $\theta = \theta(\Lambda, V)$ depends on the Riesz bounds of the Riesz basis $E(\Lambda)$ for $L^2(V)$, which are determined once Λ and V are given. Also, it is pointed out in Section 2.3, Remark 2 in [8] that the theorem also holds for frames and Riesz sequences.

2.3. Density of Frequency Sets

The lower and upper (Beurling) densities of a discrete set $\Lambda \subset \mathbb{R}^d$ are defined respectively by (see e.g., [31])

$$D^-(\Lambda) = \liminf_{r\to\infty} \frac{\inf_{\mathbf{x}\in\mathbb{R}^d} |\Lambda \cap (\mathbf{x} + [0,r]^d)|}{r^d} \qquad \text{and}$$
$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\sup_{\mathbf{x}\in\mathbb{R}^d} |\Lambda \cap (\mathbf{x} + [0,r]^d)|}{r^d}.$$

If $D^-(\Lambda) = D^+(\Lambda)$, we say that Λ has a *uniform* (Beurling) density $D(\Lambda) := D^-(\Lambda) = D^+(\Lambda)$. A discrete set $\Lambda \subset \mathbb{R}^d$ is called *separated* (or *uniformly discrete*) if its separation constant $\Delta(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda\}$ is positive. For a separated set $\Lambda \subset \mathbb{R}$, we will always label its elements in increasing order: that is, $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ with $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{Z}$.

The following proposition is considered folklore. The corresponding statements for Gabor systems of $L^2(\mathbb{R}^d)$ are well-known (see Theorem 1.1 in [32] and also Lemma 2.2 in [33]), and the following proposition can be proved similarly.

Proposition 3. *Let $\Lambda \subset \mathbb{R}^d$ be a discrete set, and let $S \subset \mathbb{R}^d$ be a finite positive measure set that is not necessarily bounded.*

(i) If $E(\Lambda)$ is a Bessel sequence in $L^2(S)$, then $D^+(\Lambda) < \infty$.

(ii) *If $E(\Lambda)$ is a Riesz sequence in $L^2(S)$, then Λ is separated, i.e., $\Delta(\Lambda) > 0$.*

A proof of Proposition 3 is given in Appendix B.

Theorem 4 ([34,35])**.** *Let $\Lambda \subset \mathbb{R}^d$ be a discrete set, and let $S \subset \mathbb{R}^d$ be a finite positive measure set.*

(i) *If $E(\Lambda)$ is a frame for $L^2(S)$, then $|S| \le D^-(\Lambda) \le D^+(\Lambda) < \infty$.*
(ii) *If $E(\Lambda)$ is a Riesz sequence in $L^2(S)$, then Λ is separated and $D^+(\Lambda) \le |S|$.*

Corollary 4. *Let $\Lambda \subset \mathbb{R}^d$ be a discrete set, and let $S \subset \mathbb{R}^d$ be a finite positive measure set. If $E(\Lambda)$ is a Riesz basis for $L^2(S)$, then Λ is separated and has a uniform Beurling density $D(\Lambda) = |S|$.*

3. A Result of Olevskii and Ulanovskii

As our main results (Theorems 1 and 2) hinge on the proof technique of Olevskii and Ulanovskii [24], we will briefly review the relevant result from [24].

Theorem 5 (Theorem 4 in [24])**.** *Let $0 < \epsilon < 1$, and let $\Lambda \subset \mathbb{R}$ be a separated set with $D^+(\Lambda) > 0$. One can construct a measurable set $S = S(\epsilon, \Lambda) \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| > 1 - \epsilon$ such that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$.*

The proof of Theorem 5 relies on a technical lemma (Lemma 2 below) that is based on the celebrated Szemerédi's theorem [36] asserting that any integer set $\Omega \subset \mathbb{Z}$ with a positive upper Beurling density $D^+(\Omega) > 0$ contains at least one arithmetic progression of length M for all $M \in \mathbb{N}$. Here, an arithmetic progression of length M means a sequence of the form

$$d,\ d{+}P,\ d{+}2P,\ \dots,\ d{+}(M{-}1)P \qquad \text{with } d \in \mathbb{Z} \text{ and } P \in \mathbb{N}.$$

As a side remark, we mention that the common difference $P \in \mathbb{N}$ of the arithmetic progression resulting from Szemerédi's theorem can be restricted to a fairly sparse subset of positive integers $\mathcal{C} \subset \mathbb{N}$. For instance, one can ensure that P is a multiple of any prescribed number $L \in \mathbb{N}$ by passing to a subset of Ω that is contained in $L\mathbb{Z}{+}u$ for some $u \in \{0, 1, \dots, L-1\}$ and has a positive upper Beurling density. This allows us to take $\mathcal{C} = L\mathbb{N}$, which clearly satisfies $D^+(\mathcal{C}) = 1/L$. Further, one can even choose $\mathcal{C} = \{1^q, 2^q, 3^q, \dots\}$ for any $q \in \mathbb{N}$, which satisfies

$$D^+(\mathcal{C}) = \begin{cases} 1 & \text{if } q = 1, \\ 0 & \text{if } q > 1. \end{cases}$$

More generally, one may choose $\mathcal{C} = \{p(n) : n \in \mathbb{N}\}$ for any polynomial p with rational coefficients such that $p(0) = 0$ and $p(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}\backslash\{0\}$ (see [37] p. 733). On the other hand, it was shown by (Theorem 7 in [38]) that $\mathcal{C} \subset \mathbb{N}$ cannot be a lacunary sequence, i.e., a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying $\liminf_{n\to\infty} a_{n+1}/a_n > 1$ (for instance, $\{2^n : n = 0, 1, 2, \dots\}$). Note that the aforementioned set $\mathcal{C} = \{p(n) : n \in \mathbb{N}\}$ can be sparse but not lacunary since $\lim_{n\to\infty} p(n+1)/p(n) = 1$ for any polynomial p. We refer to Section 2 in [39] for a short review of the possible choice of (deterministic) sets $\mathcal{C} \subset \mathbb{N}$ and also for the situation for which $\mathcal{C}$ is chosen randomly.

Lemma 2 (Lemma 5.1 in [24])**.** *Let $\Lambda \subset \mathbb{R}$ be a separated set with $D^+(\Lambda) > 0$. For any $M \in \mathbb{N}$ and $\delta > 0$, there exist constants $c = c(M, \delta, \Lambda) \in \mathbb{N}$, $d = d(M, \delta, \Lambda) \in \mathbb{R}$ and an increasing sequence $s(-M) < s(-M{+}1) < \ldots < s(M)$ in Λ such that*

$$\big|s(j) - cj - d\big| \le \delta \quad \text{for } j = -M, \dots, M. \tag{3}$$

Moreover, the constant $c = c(M, \delta, \Lambda) \in \mathbb{N}$ can be chosen to be a multiple of any prescribed number $L \in \mathbb{N}$.

As Lemma 2 will be used in the proof of Theorem 2, we include a short proof of Lemma 2 in Appendix B for self-contained nature of the paper.

4. Proof of Theorem 1

Before proving Theorem 1, we note that Theorem 2 is an extension of Theorem 5 from a single set $\Lambda \subset \mathbb{R}$ to a countable family of sets $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$. We first consider a particular choice of sets $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = 2\alpha\mathbb{Z}$, $\Lambda_3 = 3\alpha\mathbb{Z}, \cdots$ for any fixed $0 < \alpha \leq 1$, from which a desired set for Theorem 1 will be acquired.

Proposition 4. *Let $0 < \alpha \leq 1$ and $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = 2\alpha\mathbb{Z}$, $\Lambda_3 = 3\alpha\mathbb{Z}, \cdots$; that is, $\Lambda_\ell = \ell\alpha\mathbb{Z}$ for $\ell \in \mathbb{N}$. Given any $0 < \epsilon < 1$, one can construct a measurable set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| > 1 - \epsilon$ such that $E(\Lambda_\ell)$ is not a Riesz sequence in $L^2(S)$ for all $\ell \in \mathbb{N}$.*

Proof. Fix any $0 < \epsilon < 1$ and choose an integer $R > \frac{1}{1-\epsilon}$ so that $0 < \epsilon < \frac{R-1}{R}$. We claim that for each $0 < \eta < \frac{R-1}{R}$ there exists a set $V_\eta \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|V_\eta| < \eta$ satisfying the following property: for each $\ell \in \mathbb{N}$, there is a finitely supported sequence $b^{(\eta,\ell)} = \{b_j^{(\eta,\ell)}\}_{j\in\mathbb{Z}}$ satisfying

$$\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \Big| \sum_{j\in\mathbb{Z}} b_j^{(\eta,\ell)}\, e^{2\pi i \ell\alpha j x} \Big|^2 dx \;\leq\; R\, \frac{\eta}{2^\ell} \sum_{j\in\mathbb{Z}} |b_j^{(\eta,\ell)}|^2. \tag{4}$$

If this claim is proved, one could take $V := \cup_{k=1}^{\infty} V_{\epsilon/2^k}$ and $S := [-\frac{1}{2}, \frac{1}{2}]\setminus V$. Indeed, we have $|V| \leq \sum_{k=1}^{\infty} |V_{\epsilon/2^k}| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$ so that $|S| > 1 - \epsilon$. Also, it holds for any $k, \ell \in \mathbb{N}$ that

$$\begin{aligned}\int_S \Big| \sum_{j\in\mathbb{Z}} b_j^{(\epsilon/2^k,\ell)}\, e^{2\pi i \ell\alpha j x} \Big|^2 dx \;&\leq\; \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_{\epsilon/2^k}} \Big| \sum_{j\in\mathbb{Z}} b_j^{(\epsilon/2^k,\ell)}\, e^{2\pi i \ell\alpha j x} \Big|^2 dx \\ &\overset{(4)}{\leq}\; R\, \frac{\epsilon}{2^{k+\ell}} \sum_{j\in\mathbb{Z}} |b_j^{(\epsilon/2^k,\ell)}|^2.\end{aligned}$$

By fixing any $\ell \in \mathbb{N}$ and letting $k \to \infty$, we conclude that $E(\ell\alpha\mathbb{Z})$ is not a Riesz sequence in $L^2(S)$.

To prove claim (4), fix any $0 < \eta < \frac{R-1}{R}$. For each $\ell \in \mathbb{N}$, let $\widetilde{a}^{(\eta\alpha/2^\ell)} = \{\widetilde{a}_j^{(\eta\alpha/2^\ell)}\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ be the sequence given by

$$\widetilde{a}_j^{(\eta\alpha/2^\ell)} := \begin{cases} \sqrt{\frac{\eta\alpha}{2^{\ell+1}}} & \text{if } j = 0, \\ \sqrt{\frac{2^{\ell+1}}{\eta\alpha}}\, \frac{1}{\pi j} \sin\left(\frac{\pi j \eta\alpha}{2^{\ell+1}}\right) & \text{if } j \neq 0, \end{cases} \tag{5}$$

which is the Fourier coefficient of the 1-periodic function

$$\widetilde{p}_{\eta\alpha/2^\ell}(x) := \begin{cases} \sqrt{\frac{2^{\ell+1}}{\eta\alpha}} & \text{for } x \in [-\frac{\eta\alpha}{4\cdot 2^\ell}, \frac{\eta\alpha}{4\cdot 2^\ell}], \\ 0 & \text{for } x \in [-\frac{1}{2}, \frac{1}{2})\setminus[-\frac{\eta\alpha}{4\cdot 2^\ell}, \frac{\eta\alpha}{4\cdot 2^\ell}]; \end{cases} \tag{6}$$

that is, $\widetilde{p}_{\eta\alpha/2^\ell}(x) = \sum_{j\in\mathbb{Z}} \widetilde{a}_j^{(\eta\alpha/2^\ell)}\, e^{2\pi i j x}$ for almost every $x \in [-\frac{1}{2}, \frac{1}{2}]$. Note that $\|\widetilde{a}^{(\eta\alpha/2^\ell)}\|_{\ell_2} = \|\widetilde{p}_{\eta\alpha/2^\ell}(x)\|_{L^2[-\frac{1}{2},\frac{1}{2}]} = 1$. Choose a number $\widetilde{M} = \widetilde{M}(\eta\alpha/2^\ell) \in \mathbb{N}$ satisfying

$$\sum_{|j|>\widetilde{M}} |\widetilde{a}_j^{(\eta\alpha/2^\ell)}|^2 \;<\; \frac{1}{\alpha}\cdot\left(\frac{\eta\alpha}{2^\ell}\right) \;=\; \frac{\eta}{2^\ell}$$

so that

$$\sum_{j=-\widetilde{M}}^{\widetilde{M}} |\widetilde{a}_j^{(\eta\alpha/2^\ell)}|^2 \;>\; 1 - \frac{\eta}{2^\ell} \;\geq\; 1 - \eta \;>\; \frac{1}{R}. \tag{7}$$

Now, the set Λ_ℓ comes into play. We write $\Lambda_\ell = \ell\alpha\mathbb{Z} = \{s_\ell(j) : j \in \mathbb{Z}\}$ with

$$s_\ell(j) := \ell\alpha j \quad \text{for all} \;\; j \in \mathbb{Z}.$$

For $x \in [-\frac{1}{2}, \frac{1}{2}]$, we define

$$\widetilde{f}_{\eta\alpha/2^\ell,\Lambda_\ell}(x) := \sum_{j=-\widetilde{M}}^{\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^\ell)} \, e^{2\pi i s_\ell(j) x} \tag{8}$$

and observe that

$$\widetilde{f}_{\eta\alpha/2^\ell,\Lambda_\ell}(x) - \widetilde{p}_{\eta\alpha/2^\ell}(\ell\alpha x) = -\sum_{|j|>\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^\ell)} \, e^{2\pi i \ell\alpha j x}. \tag{9}$$

Setting $V_\eta^{(\ell)} := [-\frac{1}{2}, \frac{1}{2}] \cap \operatorname{supp} \widetilde{p}_{\eta\alpha/2^\ell}(\ell\alpha x)$ for $\ell \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(\ell)}} \big|\widetilde{f}_{\eta\alpha/2^\ell,\Lambda_\ell}(x)\big|^2 \, dx \\
&\leq \int_{-1/2}^{1/2} \Big| \sum_{|j|>\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^\ell)} \, e^{2\pi i \ell\alpha j x} \Big|^2 \, dx = \sum_{|j|>\widetilde{M}} \big|\widetilde{a}_j^{(\eta\alpha/2^\ell)}\big|^2 \\
&< \frac{\eta}{2^\ell} \overset{(7)}{<} R\,\frac{\eta}{2^\ell} \sum_{j=-\widetilde{M}}^{\widetilde{M}} \big|\widetilde{a}_j^{(\eta\alpha/2^\ell)}\big|^2.
\end{aligned}$$

Note from (6) that $\operatorname{supp} \widetilde{p}_{\eta\alpha/2^\ell}(\ell\alpha x) = \frac{1}{\ell\alpha} \cup_{m\in\mathbb{Z}} \big(m + [-\frac{\eta\alpha}{4\cdot 2^\ell}, \frac{\eta\alpha}{4\cdot 2^\ell}]\big) = \cup_{m\in\mathbb{Z}} \big(\frac{m}{\ell\alpha} + [-\frac{\eta}{4\ell\cdot 2^\ell}, \frac{\eta}{4\ell\cdot 2^\ell}]\big)$, which implies $|V_\eta^{(\ell)}| < \frac{\eta}{2^\ell}$. Indeed, the set $[-\frac{1}{2}, \frac{1}{2}] \cap \cup_{m\in\mathbb{Z}} \big(m + [-\frac{\eta}{4\cdot 2^\ell}, \frac{\eta}{4\cdot 2^\ell}]\big) = [-\frac{\eta}{4\cdot 2^\ell}, \frac{\eta}{4\cdot 2^\ell}]$ is of length $\frac{\eta}{2^{\ell+1}}$, and the dilated set $\frac{1}{\ell} \cup_{m\in\mathbb{Z}} \big(m + [-\frac{\eta}{4\cdot 2^\ell}, \frac{\eta}{4\cdot 2^\ell}]\big) = \cup_{m\in\mathbb{Z}} \big(\frac{m}{\ell} + [-\frac{\eta}{4\ell\cdot 2^\ell}, \frac{\eta}{4\ell\cdot 2^\ell}]\big)$ restricted to $[-\frac{1}{2}, \frac{1}{2}]$ has a Lebesgue measure $\frac{\eta}{2^{\ell+1}}$ as well, so the set $V_\eta^{(\ell)} = [-\frac{1}{2}, \frac{1}{2}] \cap \cup_{m\in\mathbb{Z}} \big(\frac{m}{\ell\alpha} + [-\frac{\eta}{4\ell\cdot 2^\ell}, \frac{\eta}{4\ell\cdot 2^\ell}]\big)$ with $0 < \alpha \leq 1$ has a Lebesgue measure of at most $\frac{\eta}{2^{\ell+1}}$, which is strictly less than $\frac{\eta}{2^\ell}$. Finally, define $V_\eta := \cup_{\ell=1}^\infty V_\eta^{(\ell)}$, which clearly satisfies $|V_\eta| \leq \sum_{\ell=1}^\infty |V_\eta^{(\ell)}| < \sum_{\ell=1}^\infty \frac{\eta}{2^\ell} = \eta$. Then for each $\ell \in \mathbb{N}$,

$$\begin{aligned}
&\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \Big| \sum_{j=-\widetilde{M}}^{\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^\ell)} \, e^{2\pi i s_\ell(j) x} \Big|^2 \, dx = \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \big|\widetilde{f}_{\eta\alpha/2^\ell,\Lambda_\ell}(x)\big|^2 \, dx \\
&\leq \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(\ell)}} \big|\widetilde{f}_{\eta\alpha/2^\ell,\Lambda_\ell}(x)\big|^2 \, dx \; < \; R\,\frac{\eta}{2^\ell} \sum_{j=-\widetilde{M}}^{\widetilde{M}} \big|\widetilde{a}_j^{(\eta\alpha/2^\ell)}\big|^2
\end{aligned}$$

which establishes claim (4). This completes the proof. □

Remark 2 (The construction of S for $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = 2\alpha\mathbb{Z}$, $\Lambda_3 = 3\alpha\mathbb{Z}, \cdots$)**.** *In the proof above, the set S is constructed as follows. Given any $0 < \epsilon < 1$, choose an integer $R > \frac{1}{1-\epsilon}$ so that $0 < \epsilon < \frac{R-1}{R}$. The set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ is then given by $S := [-\frac{1}{2}, \frac{1}{2}] \setminus V$ with $V := \cup_{k=1}^\infty V_{\epsilon/2^k}$, where*

$$
\begin{aligned}
V_\eta &:= \cup_{\ell=1}^{\infty} V_\eta^{(\ell)} \qquad \textit{and} \\
V_\eta^{(\ell)} &:= [-\tfrac{1}{2},\tfrac{1}{2}] \cap \operatorname{supp} \widetilde{p}_{\eta\alpha/2^\ell}(\ell\alpha x) \\
&= [-\tfrac{1}{2},\tfrac{1}{2}] \cap \tfrac{1}{\ell\alpha}\Big(\cup_{m\in\mathbb{Z}} \big(m + [-\tfrac{\eta\alpha}{4\cdot 2^\ell}, \tfrac{\eta\alpha}{4\cdot 2^\ell}]\big)\Big) \\
&= [-\tfrac{1}{2},\tfrac{1}{2}] \cap \Big(\cup_{m\in\mathbb{Z}} \big(\tfrac{m}{\ell\alpha} + [-\tfrac{\eta}{4\ell\cdot 2^\ell}, \tfrac{\eta}{4\ell\cdot 2^\ell}]\big)\Big) \\
&\textit{for any } \ 0<\eta<\tfrac{R-1}{R} \ \textit{ and } \ \ell\in\mathbb{N}.
\end{aligned}
\tag{10}
$$

In short,

$$
\begin{aligned}
S &:= [-\tfrac{1}{2},\tfrac{1}{2}] \backslash V \quad \textit{with} \\
V &:= \cup_{k=1}^{\infty} V_{\epsilon/2^k} = \cup_{k=1}^{\infty}\cup_{\ell=1}^{\infty} V_{\epsilon/2^k}^{(\ell)} \\
&= [-\tfrac{1}{2},\tfrac{1}{2}] \cap \Big(\cup_{k=1}^{\infty}\cup_{\ell=1}^{\infty}\cup_{m\in\mathbb{Z}} \big(\tfrac{m}{\ell\alpha} + [-\tfrac{\epsilon}{4\ell\cdot 2^{k+\ell}}, \tfrac{\epsilon}{4\ell\cdot 2^{k+\ell}}]\big)\Big) \\
&= [-\tfrac{1}{2},\tfrac{1}{2}] \cap \Big(\cup_{\ell=1}^{\infty}\cup_{m\in\mathbb{Z}} \big(\tfrac{m}{\ell\alpha} + [-\tfrac{\epsilon}{4\ell\cdot 2^{\ell+1}}, \tfrac{\epsilon}{4\ell\cdot 2^{\ell+1}}]\big)\Big),
\end{aligned}
\tag{11}
$$

where the set V satisfies $|V| < \epsilon$, and thus $|S| > 1-\epsilon$. Note that the two sets S given in (2) and (11) are identical up to a countable set. Since a measure zero set is negligible in integration, these sets can be used interchangeably.

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let $\Lambda \subset \mathbb{R}$ be a set containing arbitrarily long arithmetic progressions with a fixed common difference $P\alpha$ for some $P \in \mathbb{N}$. To prove that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$, it suffices to show that the set $V_\eta \subset [-\frac{1}{2},\frac{1}{2}]$ given by (10) for $0<\eta<\frac{R-1}{R}$ and $\ell\in\mathbb{N}$ (with a fixed integer $R > \frac{1}{1-\epsilon}$) satisfies the following property: there is a finitely supported sequence $b^{(\eta,\Lambda)} = \{b_\lambda^{(\eta,\Lambda)}\}_{\lambda\in\Lambda}$ with

$$
\int_{[-\frac{1}{2},\frac{1}{2}]\backslash V_\eta} \Big| \sum_{\lambda\in\Lambda} b_\lambda^{(\eta,\Lambda)}\, e^{2\pi i\lambda x}\Big|^2 dx \ \le\ R\,\tfrac{\eta}{2^P} \sum_{\lambda\in\Lambda} \big|b_\lambda^{(\eta,\Lambda)}\big|^2. \tag{12}
$$

Indeed, since $S := [-\frac{1}{2},\frac{1}{2}] \backslash \cup_{k=1}^{\infty} V_{\epsilon/2^k}$ (see Remark 2), it then holds for any $k\in\mathbb{N}$ that

$$
\begin{aligned}
\int_S \Big| \sum_{\lambda\in\Lambda} b_\lambda^{(\epsilon/2^k,\Lambda)}\, e^{2\pi i\lambda x}\Big|^2 dx &\le \int_{[-\frac{1}{2},\frac{1}{2}]\backslash V_{\epsilon/2^k}} \Big| \sum_{\lambda\in\Lambda} b_\lambda^{(\epsilon/2^k,\Lambda)}\, e^{2\pi i\lambda x}\Big|^2 dx \\
&\overset{(12)}{\le} R\,\tfrac{\epsilon}{2^{k+P}} \sum_{\lambda\in\Lambda} \big|b_\lambda^{(\epsilon/2^k,\Lambda)}\big|^2
\end{aligned}
$$

which implies that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$.

To prove claim (12), consider the sequence $\widetilde{a}^{(\eta\alpha/2^P)} = \{\widetilde{a}_j^{(\eta\alpha/2^P)}\}_{j\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, the function $\widetilde{p}_{\eta\alpha/2^P}$, and the number $\widetilde{M} = \widetilde{M}(\eta\alpha/2^P) \in \mathbb{N}$ taken, respectively, from (5)–(7) with $\ell = P$. By the assumption, the set $\Lambda\subset\mathbb{R}$ contains an arithmetic progression of length $2\widetilde{M}+1$ with common difference $P\alpha$, which can be expressed as

$$
s_\Lambda(j) := P\alpha j + d, \qquad j = -\widetilde{M},\dots,\widetilde{M}
$$

for some $d\in\mathbb{Z}$. Similarly to (8) and (9), we define

$$
\widetilde{f}_\Lambda(x) := \sum_{j=-\widetilde{M}}^{\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^P)}\, e^{2\pi i s_\Lambda(j) x} \quad \text{for } \ x\in\mathbb{R}
$$

and observe that

$$\widetilde{f}_\Lambda(x) - \widetilde{p}_{\eta\alpha/2^P}(P\alpha x)\, e^{2\pi i d x} = -\sum_{|j|>\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^P)}\, e^{2\pi i (P\alpha j + d)x} \quad \text{for all} \quad x \in \mathbb{R}.$$

Recalling that $V_\eta^{(P)} := [-\frac{1}{2}, \frac{1}{2}] \cap \operatorname{supp} \widetilde{p}_{\eta\alpha/2^P}(P\alpha x)$ (see Remark 2), we have

$$\begin{aligned} &\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(P)}} \big|\widetilde{f}_\Lambda(x)\big|^2\, dx \\ &\le \int_{-1/2}^{1/2} \Big| \sum_{|j|>\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^P)}\, e^{2\pi i (P\alpha j + d)x} \Big|^2 dx = \sum_{|j|>\widetilde{M}} \big|\widetilde{a}_j^{(\eta\alpha/2^P)}\big|^2 \\ &< \tfrac{\eta}{2^P} \;<\; R\, \tfrac{\eta}{2^P} \end{aligned}$$

where the inequality (7) for $\ell = P$ is used in the last step. Since $V_\eta := \cup_{\ell=1}^{\infty} V_\eta^{(\ell)}$, we have

$$\begin{aligned} \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \Big| \sum_{j=-\widetilde{M}}^{\widetilde{M}} \widetilde{a}_j^{(\eta\alpha/2^P)}\, e^{2\pi i s_\Lambda(j) x} \Big|^2 dx &= \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \big|\widetilde{f}_\Lambda(x)\big|^2\, dx \\ &\le \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(P)}} \big|\widetilde{f}_\Lambda(x)\big|^2\, dx \;<\; R\, \tfrac{\eta}{2^P} \end{aligned}$$

which establishes claim (12). □

5. Proof of Theorem 2

We will now prove Theorem 2, which generalizes Proposition 4 from $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = 2\alpha\mathbb{Z}$, $\Lambda_3 = 3\alpha\mathbb{Z}, \cdots$ to arbitrary separated sets $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$ with positive upper Beurling densities. The proof is similar to the proof of Proposition 4, but since an arbitrary separated set is in general non-periodic, we need the additional step of extracting an *approximate* arithmetic progression from each set Λ_ℓ with the help of Lemma 2.

Proof of Theorem 2. Fix any $0 < \epsilon < 1$ and choose an integer $R > \frac{1}{1-\epsilon}$ so that $0 < \epsilon < \frac{R-1}{R}$. We claim that for each $0 < \eta < \frac{R-1}{R}$, there exists a set $V_\eta \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|V_\eta| < \eta$ satisfying the following property: for each $\ell \in \mathbb{N}$, there is a finitely supported sequence $b^{(\eta,\Lambda_\ell)} = \{b_\lambda^{(\eta,\Lambda_\ell)}\}_{\lambda\in\Lambda_\ell}$ with

$$\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \Big| \sum_{\lambda\in\Lambda_\ell} b_\lambda^{(\eta,\Lambda_\ell)}\, e^{2\pi i \lambda x} \Big|^2 dx \;\le\; R\,\eta^2 \sum_{\lambda\in\Lambda_\ell} \big|b_\lambda^{(\eta,\Lambda_\ell)}\big|^2. \tag{13}$$

To prove claim (13), fix any $0 < \eta < \frac{R-1}{R}$. For each $\ell \in \mathbb{N}$, let $a^{(\eta/2^\ell)} = \{a_j^{(\eta/2^\ell)}\}_{j\in\mathbb{Z}}$ be an ℓ_1-sequence with unit ℓ_2-norm $\|a^{(\eta/2^\ell)}\|_{\ell_2} = 1$ such that

$$p_{\eta/2^\ell}(x) := \sum_{j\in\mathbb{Z}} a_j^{(\eta/2^\ell)}\, e^{2\pi i j x} \quad \text{satisfies} \quad p_{\eta/2^\ell}(x) = 0 \quad \text{for} \quad \tfrac{\eta}{4\cdot 2^\ell} \le |x| \le \tfrac{1}{2}. \tag{14}$$

Since the sequence $a^{(\eta/2^\ell)} \in \ell_1(\mathbb{Z})$ is not finitely supported, there is a number $M = M(\eta/2^\ell) \in \mathbb{N}$ with $0 < \sum_{|j|>M} |a_j^{(\eta/2^\ell)}| < \frac{\eta}{2^\ell}$. Note that since $|a_j^{(\eta/2^\ell)}| \le \|a^{(\eta/2^\ell)}\|_{\ell_2} = 1$ for all $j \in \mathbb{Z}$, we have

$$\sum_{|j|>M} \big|a_j^{(\eta/2^\ell)}\big|^2 = \sum_{|j|>M} \big|a_j^{(\eta/2^\ell)}\big| < \tfrac{\eta}{2^\ell}$$

so that

$$\sum_{j=-M}^{M} \big|a_j^{(\eta/2^\ell)}\big|^2 \;>\; 1 - \tfrac{\eta}{2^\ell} \;\ge\; 1 - \eta \;>\; \tfrac{1}{R}. \tag{15}$$

We then choose a small parameter $0 < \delta = \delta(\eta/2^\ell) < 1$ satisfying

$$\sin(\pi\delta/2) < \frac{\eta/2^\ell}{2\sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}|}$$

so that $\sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}| \cdot |e^{i\pi\delta} - 1| = \sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}| \cdot 2\sin(\pi\delta/2) < \frac{\eta}{2^\ell}$. Note that all the terms up to this point depend only on the parameters η and ℓ: in fact, only on the value $\eta/2^\ell$.

Now, the set Λ_ℓ comes into play. Applying Lemma 2 to the set Λ_ℓ with the parameters M and δ chosen above, we deduce that there exist constants $c = c(\eta/2^\ell, \Lambda_\ell) \in \mathbb{N}$ and $d = d(\eta/2^\ell, \Lambda_\ell) \in \mathbb{R}$ and an increasing sequence $s_{\eta/2^\ell,\Lambda_\ell}(-M) < s_{\eta/2^\ell,\Lambda_\ell}(-M+1) < \ldots < s_{\eta/2^\ell,\Lambda_\ell}(M)$ in Λ_ℓ satisfying

$$\left|s_{\eta/2^\ell,\Lambda_\ell}(j) - cj - d\right| \le \delta \quad \text{for } j = -M, \ldots, M.$$

For $x \in [-\frac{1}{2}, \frac{1}{2}]$, we define

$$f_{\eta/2^\ell,\Lambda_\ell}(x) := \sum_{j=-M}^{M} a_j^{(\eta/2^\ell)} \exp\left(2\pi i s_{\eta/2^\ell,\Lambda_\ell}(j) x\right)$$

and observe that

$$\begin{aligned}
&\left|f_{\eta/2^\ell,\Lambda_\ell}(x) - p_{\eta/2^\ell}(cx)\, e^{2\pi i d x}\right| \\
&\le \left|\sum_{j=-M}^{M} a_j^{(\eta/2^\ell)} \Big(\exp\left(2\pi i s_{\eta/2^\ell,\Lambda_\ell}(j)x\right) - e^{2\pi i (cj+d)x}\Big)\right| + \left|\sum_{|j|>M} a_j^{(\eta/2^\ell)}\, e^{2\pi i (cj+d)x}\right| \\
&\le \sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}| \cdot \left|\exp\Big(2\pi i (s_{\eta/2^\ell,\Lambda_\ell}(j) - cj - d)x\Big) - 1\right| + \sum_{|j|>M} |a_j^{(\eta/2^\ell)}| \\
&< \tfrac{\eta}{2^\ell} + \tfrac{\eta}{2^\ell} = \tfrac{\eta}{2^{\ell-1}} \;\le\; \eta.
\end{aligned}$$

Setting $V_\eta^{(\ell)} := [-\frac{1}{2}, \frac{1}{2}] \cap \operatorname{supp} p_{\eta/2^\ell}(c(\eta/2^\ell, \Lambda_\ell)x)$, we have

$$\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(\ell)}} |f_{\eta/2^\ell,\Lambda_\ell}(x)|^2\, dx \;\le\; \eta^2 \;\overset{(15)}{<}\; R\,\eta^2 \sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}|^2.$$

Similarly to the proof of Proposition 4, we have $|V_\eta^{(\ell)}| < \frac{\eta}{2^\ell}$, and therefore, the set $V_\eta := \cup_{\ell=1}^{\infty} V_\eta^{(\ell)}$ satisfies $|V_\eta| < \eta$. It then holds for each $\ell \in \mathbb{N}$ that

$$\begin{aligned}
&\int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} \left|\sum_{j=-M}^{M} a_j^{(\eta/2^\ell)} \exp\left(2\pi i s_{\eta/2^\ell,\Lambda_\ell}(j)x\right)\right|^2 dx \;=\; \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta} |f_{\eta/2^\ell,\Lambda_\ell}(x)|^2\, dx \\
&\le \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_\eta^{(\ell)}} |f_{\eta/2^\ell,\Lambda_\ell}(x)|^2\, dx \;<\; R\,\eta^2 \sum_{j=-M}^{M} |a_j^{(\eta/2^\ell)}|^2
\end{aligned}$$

which proves claim (13).

Finally, based on the established claim (13), we define $V := \cup_{k=1}^{\infty} V_{\epsilon/2^k}$ and $S := [-\frac{1}{2}, \frac{1}{2}]\setminus V$. Clearly, we have $|V| \le \sum_{k=1}^{\infty} |V_{\epsilon/2^k}| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$ so that $|S| > 1 - \epsilon$. Also, it holds for any $k, \ell \in \mathbb{N}$ that

$$\begin{aligned}
\int_S \left|\sum_{\lambda\in\Lambda_\ell} b_\lambda^{(\epsilon/2^k,\Lambda_\ell)}\, e^{2\pi i \lambda x}\right|^2 dx \;&\le\; \int_{[-\frac{1}{2},\frac{1}{2}]\setminus V_{\epsilon/2^k}} \left|\sum_{\lambda\in\Lambda_\ell} b_\lambda^{(\epsilon/2^k,\Lambda_\ell)}\, e^{2\pi i \lambda x}\right|^2 dx \\
&\overset{(13)}{\le}\; R\left(\tfrac{\epsilon}{2^k}\right)^2 \sum_{\lambda\in\Lambda_\ell} |b_\lambda^{(\epsilon/2^k,\Lambda_\ell)}|^2.
\end{aligned}$$

By fixing any $\ell \in \mathbb{N}$ and letting $k \to \infty$, we conclude that $E(\Lambda_\ell)$ is not a Riesz sequence in $L^2(S)$. $\square$

Remark 3 (The construction of S for arbitrary separated sets $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$)**.** *In the proof above, the set S for arbitrary separated sets $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$ is constructed as follows. Given any $0 < \epsilon < 1$, choose an integer $R > \frac{1}{1-\epsilon}$ so that $0 < \epsilon < \frac{R-1}{R}$. The set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ is then given by $S := [-\frac{1}{2}, \frac{1}{2}] \backslash V$ with $V := \cup_{k=1}^{\infty} V_{\epsilon/2^k}$, where*

$$\begin{aligned}
V_\eta \ &:= \ \cup_{\ell=1}^{\infty} V_\eta^{(\ell)} \qquad \textit{and} \\
V_\eta^{(\ell)} \ &:= \ [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \operatorname{supp} p_{\eta/2^\ell}\big(c(\eta/2^\ell, \Lambda_\ell) x\big) \ = \ [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \Big(\tfrac{1}{c(\eta/2^\ell, \Lambda_\ell)} \operatorname{supp} p_{\eta/2^\ell} \Big) \\
&\subset \ [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \Big(\cup_{m \in \mathbb{Z}} \big(\tfrac{m}{c(\eta/2^\ell, \Lambda_\ell)} + \big[-\tfrac{\eta}{4 \cdot c(\eta/2^\ell, \Lambda_\ell) \cdot 2^\ell}, \tfrac{\eta}{4 \cdot c(\eta/2^\ell, \Lambda_\ell) \cdot 2^\ell} \big] \big) \Big) \\
&\textit{for any} \ \ 0 < \eta < \tfrac{R-1}{R} \ \ \textit{and} \ \ \ell \in \mathbb{N}.
\end{aligned}$$

Here, $c(\eta/2^\ell, \Lambda_\ell)$ is a positive integer that depends on the value $\eta/2^\ell$ and the set Λ_ℓ. In short,

$$\begin{aligned}
S &:= [-\tfrac{1}{2}, \tfrac{1}{2}] \backslash V \quad \textit{with} \\
V &:= \cup_{k=1}^{\infty} V_{\epsilon/2^k} \ = \ \cup_{k=1}^{\infty} \cup_{\ell=1}^{\infty} V_{\epsilon/2^k}^{(\ell)} \\
&= [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \Big(\cup_{k=1}^{\infty} \cup_{\ell=1}^{\infty} \tfrac{1}{c(\epsilon/2^{k+\ell}, \Lambda_\ell)} \operatorname{supp} p_{\epsilon/2^{k+\ell}} \Big) \\
&\subset [-\tfrac{1}{2}, \tfrac{1}{2}] \cap \\
&\quad \Big(\cup_{k=1}^{\infty} \cup_{\ell=1}^{\infty} \cup_{m \in \mathbb{Z}} \big(\tfrac{m}{c(\epsilon/2^{k+\ell}, \Lambda_\ell)} + \big[-\tfrac{\epsilon}{4 \cdot c(\epsilon/2^{k+\ell}, \Lambda_\ell) \cdot 2^{k+\ell}}, \tfrac{\epsilon}{4 \cdot c(\epsilon/2^{k+\ell}, \Lambda_\ell) \cdot 2^{k+\ell}} \big] \big) \Big).
\end{aligned}$$

6. Conclusions

In this paper, we constructed a bounded subset of $\mathbb{R}$ that does not admit a certain general type of Riesz spectrum. Specifically, we constructed a set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ that does not admit a Riesz spectrum containing a nonempty periodic set with its period belonging in $\alpha\mathbb{Q}_+$ for any fixed constant $\alpha > 0$, where $\mathbb{Q}_+$ denotes the set of all positive rational numbers. In particular, this led to a set $V \subset [-\frac{1}{2}, \frac{1}{2}]$ with an arbitrarily small Lebesgue measure such that for any $N \in \mathbb{N}$ and any proper subset I of $\{0, \ldots, N-1\}$, the set of exponentials $e^{2\pi i k x}$ with $k \in \cup_{n \in I}(N\mathbb{Z}+n)$ is not a frame for $L^2(V)$. The obtained results have immediate consequences in sampling theory and frame theory and have potential applicability in practical problems such as OFDM (orthogonal frequency division multiplexing) based communications that involve the design of exponential bases.

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Appendix A. Related Notions in Paley–Wiener Spaces

The Fourier transform is defined densely on $L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f) := \widehat{f}(\omega) = \int f(x)\, e^{2\pi i x \cdot \omega}\, dx \quad \text{for} \ \ f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

This is a nonstandard but equivalent definition of the Fourier transform that has no negative sign in the exponent; this definition is employed only to justify relation (A1). Alternatively, as in [23,24] one could use the standard definition of the Fourier transform, which has negative sign in the exponent, and define the Paley–Wiener space $PW(S)$ to be the image of $L^2(S)$ under the Fourier transform. It is easily seen that $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a unitary operator satisfying $\mathcal{F}^2 = \mathcal{I}$, where $\mathcal{I} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the reflection operator defined by $\mathcal{I}f(x) = f(-x)$, and thus, $\mathcal{F}^4 = \mathrm{Id}_{L^2(\mathbb{R}^d)}$. The *Paley–Wiener space* over a measurable set $S \subset \mathbb{R}^d$ is defined by

$$PW(S) := \{f \in L^2(\mathbb{R}^d) : \operatorname{supp} \widehat{f} \subset S\} \; = \; \mathcal{F}^{-1}\big[L^2(S)\big]$$

equipped with the norm $\|f\|_{PW(S)} := \|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(S)}$, where $L^2(S)$ is embedded into $L^2(\mathbb{R}^d)$ by the trivial extension. Denoting the Fourier transform of $f \in PW(S)$ by $F \in L^2(S)$, we see that for almost all $x \in \mathbb{R}^d$,

$$f(x) = \big(\mathcal{F}^{-1}F\big)(x) = \int_S F(\omega)\, e^{-2\pi i x\cdot\omega}\, d\omega = \big\langle F, e^{2\pi i x\cdot(\cdot)}\big\rangle_{L^2(S)}. \tag{A1}$$

Moreover, if the set $S \subset \mathbb{R}^d$ has a finite measure, then f is continuous, and thus, (A1) holds for all $x \in \mathbb{R}^d$.

Definition A1. *Let $S \subset \mathbb{R}^d$ be a measurable set. A discrete set $\Lambda \subset \mathbb{R}^d$ is called*

- a uniqueness set (a set of uniqueness) *for $PW(S)$ if the only function $f \in PW(S)$ satisfying $f(\lambda) = 0$ for all $\lambda \in \Lambda$ is the trivial function $f = 0$;*
- a sampling set (a set of sampling) *for $PW(S)$ if there are constants $0 < A \le B < \infty$ such that*

$$A\,\|f\|^2_{PW(S)} \;\le\; \sum_{\lambda\in\Lambda} \big|f(\lambda)\big|^2 \;\le\; B\,\|f\|^2_{PW(S)} \quad \textit{for all} \;\; f \in PW(S);$$

- an interpolating set (a set of interpolation) *for $PW(S)$ if for each $\{c_\lambda\}_{\lambda\in\Lambda} \in \ell_2(\Lambda)$ there exists a function $f \in PW(S)$ satisfying $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.*

It follows immediately from (A1) that

- Λ is a uniqueness set for $PW(S)$ if and only if $E(\Lambda)$ is complete in $L^2(S)$;
- Λ is a sampling set for $PW(S)$ if and only if $E(\Lambda)$ is a frame for $L^2(S)$.

Also, we have the following characterization of interpolation sets for $PW(S)$ (see p. 129, Theorem 3 in [3]):

- Λ is an interpolating set for $PW(S)$ if and only if there is a constant $A > 0$ such that

$$A\,\|c\|^2_{\ell_2} \;\le\; \Big\| \sum_{n\in\mathbb{Z}} c_\lambda\, e^{2\pi i\lambda\cdot(\cdot)} \Big\|^2_{L^2(S)} \quad \text{for all} \;\; \{c_\lambda\}_{\lambda\in\Lambda} \in \ell_2(\Lambda),$$

meaning that the lower Riesz inequality of $E(\Lambda)$ for $L^2(S)$ holds.

Combining with the Bessel inequality (which corresponds to the upper Riesz inequality), we obtain a more convenient statement:

- If $E(\Lambda)$ is a Bessel sequence in $L^2(S)$, then Λ is an interpolating set for $PW(S)$ if and only if $E(\Lambda)$ is a Riesz sequence in $L^2(S)$.

In fact, this statement can be proved by elementary functional analytic arguments. Indeed, if $E(\Lambda)$ is Bessel, i.e., if the synthesis operator $T : \ell_2(\Lambda) \to L^2(S)$ defined by $T(\{c_\lambda\}_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} c_\lambda\, e^{2\pi i\lambda\cdot(\cdot)}$ is a bounded linear operator (equivalently, the analysis operator $T^* : L^2(S) \to \ell_2(\Lambda)$ defined by $T^*F = \{\langle F, e^{2\pi i\lambda\cdot(\cdot)}\rangle_{L^2(S)}\}_{\lambda\in\Lambda}$ is a bounded linear operator), then T is bounded below (that is, the lower Riesz inequality holds) if and only if T is injective and has closed range, if and only if T^* has dense and closed range, i.e., T^* is surjective, which means that $E(\Lambda)$ is an interpolating set for $PW(S)$ by (A1).

The statement above is often useful because $E(\Lambda)$ is necessarily a Bessel sequence in $L^2(S)$ whenever $\Lambda \subset \mathbb{R}^d$ is separated and $S \subset \mathbb{R}^d$ is bounded (p. 135, Theorem 4 in [3]). Note that $\Lambda \subset \mathbb{R}^d$ is necessarily separated if $E(\Lambda)$ is a Riesz sequence in $L^2(S)$ (see Proposition 3).

Appendix B. Proof of Some Auxiliary Results

Proof of Lemma 1. To prove (a), note that for any $a \in \mathbb{R}^d$,

$$T_{-a}[E(\Lambda)] = \{e^{2\pi i\lambda\cdot(x+a)} : \lambda \in \Lambda\} = \{e^{2\pi i\lambda\cdot a}\, e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}.$$

Since the phase factor $e^{2\pi i\lambda\cdot a} \in \mathbb{C}$ for $\lambda \in \Lambda$ does not affect the Riesz basis property and the Riesz bounds, it follows that $T_{-a}[E(\Lambda)]$ is a Riesz basis for $L^2(S)$ with bounds A and B. Consequently, $E(\Lambda)$ is a Riesz basis for $L^2(S+a)$ with bounds A and B.

For (b) and (c), note that the modulation $F(x) \mapsto e^{2\pi ib\cdot x}F(x)$ is a unitary operator on $L^2(S)$ and that the dilation $F(x) \mapsto \sqrt{\sigma}F(\sigma x)$ is also a unitary operator from $L^2(S)$ onto $L^2(\frac{1}{\sigma}S)$. It is easily seen from Proposition 1(c) that if $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and if $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}_1$, then $\{Uf_n\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}_2$. Parts (b) and (c) follow immediately from this statement. □

Proof of Proposition 3. For simplicity, we will only consider the case $d = 1$.

(i) Assume that $D^+(\Lambda) = \infty$. This means that there is a real-valued sequence $1 \le r_1 < r_2 < \cdots \to \infty$ such that

$$\frac{\sup_{x\in\mathbb{R}} |\Lambda \cap [x, x+r_n]|}{r_n} > n \quad \text{for all} \;\; n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, there exists some $x_n \in \mathbb{R}$ satisfying

$$\frac{|\Lambda \cap [x_n, x_n+r_n]|}{r_n} \ge n.$$

For each $k \in \mathbb{N}$, we partition the interval $[x_n, x_n+r_n]$ into k subintervals of equal length $\frac{r_n}{k}$: namely, the intervals $\big[x_n, x_n+\frac{r_n}{k}\big], \ldots, \big[x_n+\frac{(k-1)r_n}{k}, x_n+r_n\big]$. Then at least one of the subintervals, which we denote by $I_{n,k}$, must satisfy

$$\frac{|\Lambda \cap I_{n,k}|}{|I_{n,k}|} \ge n, \tag{A2}$$

where $|I_{n,k}| = \frac{r_n}{k}$. Letting $k \to \infty$, we see that

$$\limsup_{r\to 0} \frac{\sup_{x\in\mathbb{R}} |\Lambda \cap [x, x+r]|}{r} = \infty.$$

Define the function $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = |S|^{-1/2}\chi_S(x)$ for $x \in \mathbb{R}$. Then $\|g\|_{L^2(\mathbb{R})} = \|g\|_{L^2(S)} = 1$ and $\widehat{g}(0) = \int_S g(x)\,dx = |S|^{1/2}$. Since $g \in L^1(\mathbb{R})$, its Fourier transform $\widehat{g}$ is continuous on $\mathbb{R}$ and therefore there exists $0 < \delta < \frac{1}{2}$ such that $|\widehat{g}(\omega)| \ge \frac{1}{2}|S|^{1/2}$ for all $\omega \in [-\frac{\delta}{2}, \frac{\delta}{2}]$. For each $n \in \mathbb{N}$, we set $k_n := \lceil \frac{r_n}{\delta} \rceil \ge 2$ so that $k_n - 1 < \frac{r_n}{\delta} \le k_n$ and thus $\frac{\delta}{2} < \frac{r_n}{2(k_n-1)} \le \frac{r_n}{k_n} \le \delta$. It then follows from (A2) that

$$|\Lambda \cap I_{n,k_n}| \;\ge\; n \cdot |I_{n,k_n}| \;\ge\; n \cdot \tfrac{r_n}{k_n} \;>\; n \cdot \tfrac{\delta}{2}\,.$$

For each $n \in \mathbb{N}$, we denote the center of the interval I_{n,k_n} by $c_n \in \mathbb{R}$ and let $f_n \in L^2(S)$ be defined by $f_n(x) := e^{2\pi i c_n x}\, g(x)$ for $x \in S$. Then

$$\begin{aligned} \sum_{\lambda\in\Lambda} \big|\langle f_n, e^{2\pi i\lambda(\cdot)}\rangle_{L^2(S)}\big|^2 \;&\ge\; \sum_{\lambda\in\Lambda\cap I_{n,k_n}} \big|\langle g, e^{2\pi i(\lambda-c_n)(\cdot)}\rangle_{L^2(S)}\big|^2 \\ &=\; \sum_{\lambda\in\Lambda\cap I_{n,k_n}} \big|\widehat{g}(c_n-\lambda)\big|^2 \;>\; \big(n\cdot\tfrac{\delta}{2}\big)\cdot\tfrac{|S|}{4}, \end{aligned} \tag{A3}$$

where we used the fact that $c_n - \lambda \in [-\frac{\delta}{2}, \frac{\delta}{2}]$ for all $\lambda \in \Lambda \cap I_{n,k_n}$ since I_{n,k_n} is an interval of length $\frac{r_n}{k_n} \le \delta$. While $\|f_n\|_{L^2(S)} = \|g\|_{L^2(S)} = 1$ for all $n \in \mathbb{N}$, the right-hand side of (A3) tends to infinity as $n \to \infty$. Hence, we conclude that $E(\Lambda)$ is not a Bessel sequence in $L^2(S)$ if $D^+(\Lambda) = \infty$.

(ii) Suppose to the contrary that $E(\Lambda)$ is a Riesz sequence in $L^2(S)$ with Riesz bounds A and B, but the set $\Lambda \subset \mathbb{R}$ is not separated. Then there are two sequences $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ in Λ such that $|\lambda_n - \lambda'_n| \to 0$ as $n \to \infty$. Note that $S \subset \mathbb{R}$ is a finite measure set, and for each $x \in S$, we have $|e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}| \le 2$ and $e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x} \to 0$ as $n \to \infty$. Thus, we have $\lim_{n\to\infty} \int_S \left|e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}\right|^2 dx = 0$ by the dominated convergence theorem. For $\lambda \in \Lambda$, let $\delta_\lambda \in \ell_2(\Lambda)$ be the Kronecker delta sequence supported at λ; that is, $\delta_\lambda(\lambda') = 1$ if $\lambda' = \lambda$ and is 0 otherwise. Then, since $E(\Lambda)$ is a Riesz sequence in $L^2(S)$, we have

$$\begin{aligned} 2 &= \|\delta_{\lambda_n} - \delta_{\lambda'_n}\|^2_{\ell_2(\Lambda)} \\ &\le \frac{1}{A} \left\|e^{2\pi i \lambda_n (\cdot)} - e^{2\pi i \lambda'_n (\cdot)}\right\|^2_{L^2(S)} = \frac{1}{A} \int_S \left|e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}\right|^2 dx \;\to\; 0, \end{aligned}$$

yielding a contradiction.

□

Proof of Lemma 2. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ with $\lambda_n < \lambda_{n+1}$ for all n, and fix any $\delta > 0$. Choose a sufficiently large number $N \in \mathbb{N}$ so that $\frac{1}{N} < \tau := \min\{\Delta(\Lambda), 2\delta\}$, where $\Delta(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \ne \lambda' \in \Lambda\}$ is the separation constant of Λ (see Section 2.3). Consider the perturbation $\widetilde{\Lambda} \subset \frac{1}{N}\mathbb{Z}$ of Λ, obtained by rounding each element of Λ to the nearest point in $\frac{1}{N}\mathbb{Z}$ (if $\lambda \in \Lambda$ is exactly the midpoint of $\frac{k}{N}$ and $\frac{k+1}{N}$, then we choose $\frac{k}{N}$). Since $\Delta(\Lambda) > \frac{1}{N}$, all elements in Λ are rounded to distinct points in $\frac{1}{N}\mathbb{Z}$, i.e., the set $\widetilde{\Lambda} = \{\widetilde{\lambda}_n\}_{n\in\mathbb{Z}} \subset \frac{1}{N}\mathbb{Z}$ has no repeated elements. Clearly, there is a 1:1 correspondence between λ_n and $\widetilde{\lambda}_n$, and we have $|\lambda_n - \widetilde{\lambda}_n| \le \frac{1}{2N} < \frac{\tau}{2} \le \delta$ for all $n \in \mathbb{Z}$.

We claim that for any $M \in \mathbb{N}$, there exist constants $c \in \mathbb{N}$, $d \in \frac{1}{N}\mathbb{Z}$ and an increasing sequence $\widetilde{s}(-M) < \widetilde{s}(-M+1) < \ldots < \widetilde{s}(M)$ in $\widetilde{\Lambda} \subset \frac{1}{N}\mathbb{Z}$ satisfying

$$\widetilde{s}(j) = cj + d \quad \text{for} \quad j = -M, \ldots, M.$$

Once this claim is proved, it follows that the sequence $\{s(j)\}_{j=-M}^M \subset \Lambda$ corresponding to $\{\widetilde{s}(j)\}_{j=-M}^M \subset \widetilde{\Lambda}$, satisfies the condition (3) as desired.

To prove the claim, consider the partition of $N\widetilde{\Lambda}$ $(\subset \mathbb{Z})$ based on residue modulo N: that is, consider the sets $N\widetilde{\Lambda} \cap N\mathbb{Z}, N\widetilde{\Lambda} \cap (N\mathbb{Z}+1), \ldots, N\widetilde{\Lambda} \cap (N\mathbb{Z}+N-1)$. Since $D^+(\widetilde{\Lambda}) = D^+(\Lambda) > 0$, at least one of these N sets must have a positive upper density, i.e., $D^+(N\widetilde{\Lambda} \cap (N\mathbb{Z}+u)) > 0$ for some $u \in \{0, \ldots, N-1\}$. Then, Szemerédi's theorem implies that for any $M \in \mathbb{N}$, the set $N\widetilde{\Lambda} \cap (N\mathbb{Z}+u)$ contains an arithmetic progression of length $2M+1$: that is, $\{c_0 j + d_0 : j = -M, \ldots, M\} \subset N\widetilde{\Lambda} \cap (N\mathbb{Z}+u)$ for some $c_0 \in \mathbb{N}$ and $d_0 \in \mathbb{Z}$. This means that there is an increasing sequence $\widetilde{s}(-M) < \widetilde{s}(-M+1) < \ldots < \widetilde{s}(M)$ in $\widetilde{\Lambda}$ satisfying

$$N\,\widetilde{s}(j) = c_0 j + d_0 \quad \text{for} \quad j = -M, \ldots, M.$$

Since the numbers $c_0 j + d_0$, $j = -M, \ldots, M$ are in $N\mathbb{Z}+u$, it is clear that $c_0 \in N\mathbb{N}$ and $d_0 \in N\mathbb{Z}+u$. Thus, setting $c := \frac{1}{N}c_0 \in \mathbb{N}$ and $d := \frac{1}{N}d_0 \in \mathbb{Z}+\frac{u}{N} \subset \frac{1}{N}\mathbb{Z}$, we have $\widetilde{s}(j) = cj + d$ for $j = -M, \ldots, M$, as claimed.

Finally, one can easily force the constant $c \in \mathbb{N}$ to be a multiple of any prescribed number $L \in \mathbb{N}$. This is achieved by considering the partition of $N\widetilde{\Lambda}$ $(\subset \mathbb{Z})$ based on residue modulo LN instead of modulo N. □

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