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On the Generalized Hilfer Fractional Coupled Integro-Differential Systems with Multi-Point Ordinary and Fractional Integral Boundary Conditions

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Abstract: In this paper, we investigate a nonlinear coupled integro-differential system involving generalized Hilfer fractional derivative operators ((k,ψ)-Hilfer type) of different orders and equipped with non-local multi-point ordinary and fractional integral boundary conditions. The uniqueness results for the given problem are obtained by applying Banach's contraction mapping principle and the Boyd–Wong fixed point theorem for nonlinear contractions. Based on the Laray–Schauder alternative and the well-known fixed-point theorem due to Krasnosel'skiĭ, the existence of solutions for the problem at hand is established under different criteria. Illustrative examples for the main results are constructed.

Keywords: (k,ψ)-Hilfer fractional derivative; fractional differential system; existence; uniqueness; fixed point theorems

MSC: 26A33; 34A08; 34B10

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1. Introduction

The recent development of fractional calculus indicates that the fractional-order differential operators serve as a valuable tool in the modeling of many physical phenomena occurring in applied and social sciences. For theoretical and application details of fractional differential equations, we refer the reader to the books [1,2], while an up-to-date account of non-local and non-linear fractional boundary value problems can be found in the text [3]. Fractional derivative operators are usually defined in terms of fractional integral operators and there do exist a variety of such operators. Examples include Riemann–Liouville, Caputo, Hadamard, Erdélyi-Kober, Hilfer fractional derivatives, etc.—for details, see the text [1]. In [4], the concept of the Riemann–Liouville fractional integral operator was extended to the k–Riemann–Liouville fractional integral operator with the help of the generalized Euler's k-gamma function, which was used to define the k–Riemann–Liouville fractional derivative in [5]. For the explanation of the ψ–Riemann–Liouville fractional integral and derivative operators, see [1], while the details about ψ–Hilfer fractional derivative can be found in [6].

In a recent article [7], the authors discussed the concept of generalized and Caputo-type generalized fractional derivatives and integrals of a function with respect to another function. An initial value problem for generalized ψ-Hilfer type nonlinear implicit fractional differential equations was investigated in [8]. The authors in [9] discussed the attractivity of solutions for a problem involving the Hilfer fractional derivative operator.

Let us now dwell on the literature involving (k,ψ)–Hilfer fractional type initial and boundary value problems. The (k,ψ)–Riemann–Liouville fractional integral and derivative operators were respectively introduced in [5,10]. An initial value problem involving (k,ψ)–Hilfer fractional derivative operator was studied in [11]. In [12], the authors investigated the existence of solutions for the following (k,ψ)–Hilfer multi-point nonlocal fractional boundary value problem:

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}\pi(t) = \Pi_1(t,\pi(t)), \quad t \in (a_0,b_0], \\ \pi(a_0) = 0, \quad \pi(b_0) = \sum_{i=1}^{n} \mu_i \int_{a_0}^{\eta_i} \psi'(s)\pi(s)ds + \sum_{j=1}^{m} \zeta_j \, {}^{k}I^{\phi_j;\psi}\pi(z_j), \end{cases}$$

where $a_0, b_0 \in \mathbb{R}$, $a_0 < b_0$, ${}^{k,H}D^{\alpha,\beta;\psi}$ denotes the (k,ψ)–Hilfer type fractional derivative operator of order α, $1 < \alpha < 2$, $0 \le \beta \le 1$, $k > 0$, $\Pi_1 \in C([a_0,b_0] \times \mathbb{R}, \mathbb{R})$, ${}^{k}I^{\phi_j;\psi}$ is the (k,ψ)–Riemann–Liouville fractional integral operator of order $\phi_j > 0$, $\mu_i, \zeta_j \in \mathbb{R}$, and $a_0 < \eta_i, z_j < b_0$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

There has also been shown a significant interest in studying the fractional-order coupled systems, as such systems appear in the mathematical models associated with bio-engineering [13], financial economics [14], fractional dynamics [15], etc. In [16], the authors studied a coupled system of ψ–Hilfer fractional differential equations. In [17], nonlinear coupled hybrid systems involving generalized Hilfer fractional differential operators were studied. In [18], the authors derived existence and uniqueness results for Hilfer–Hadamard fractional differential equations supplemented with non-local coupled Hadamard fractional integral boundary conditions. In [19], the authors investigated a non-local coupled system of Hilfer-type generalized proportional fractional differential equations. A coupled system of fractional differential equations involving (k,φ)–Hilfer fractional derivative operators complemented with multi-point non-local boundary conditions was discussed in [20]. For some properties of (k,ψ)–Hilfer fractional differential operators, we refer the reader to the article [11]. However, it has been observed that the literature on systems of (k,ψ)–Hilfer fractional differential equations is scarce and needs to be developed further.

Motivated by the work presented in [20], in this paper, we study a (k,ψ)–Hilfer–type coupled system of nonlinear fractional differential equations equipped with nonlocal ordinary and fractional integral boundary conditions given by

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}\pi(t) = \Pi_1\left(t, \pi(t), \int_{a_0}^{t} \psi'(s)\varpi(s)ds\right), \quad t \in (a_0,b_0], \\ {}^{k,H}D^{p,q;\psi}\varpi(t) = \Pi_2\left(t, \varpi(t), \int_{a_0}^{t} \psi'(s)\pi(s)ds\right), \quad t \in (a_0,b_0], \\ \pi(a_0) = 0, \quad \pi(b_0) = \sum_{i=1}^{n} \mu_i \int_{a_0}^{\eta_i} \psi'(s)\varpi(s)ds + \sum_{j=1}^{m} \zeta_j \, {}^{k}I^{\phi_j;\psi}\varpi(z_j), \\ \varpi(a_0) = 0, \quad \varpi(b_0) = \sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l} \psi'(s)\pi(s)ds + \sum_{u=1}^{\lambda} \delta_u \, {}^{k}I^{\epsilon_u,\psi}\pi(\xi_u), \end{cases} \tag{1}$$

where ${}^{k,H}D^{\alpha,\beta;\psi}$, ${}^{k,H}D^{p,q;\psi}$ denote the (k,ψ)–Hilfer fractional derivative operator of orders α, p, $1 < \alpha, p < 2$ and parameters β, q, $0 \le \beta, q \le 1$, respectively, $\Pi_1, \Pi_2 : [a_0,b_0] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, ${}^{k}I^{\phi_j,\psi}$, ${}^{k}I^{\epsilon_u,\psi}$ are the (k,ψ)–Riemann–Liouville fractional integrals of order $\phi_j, \epsilon_u > 0$, respectively, $\mu_i, \zeta_j, r_l, \xi_u \in \mathbb{R}$, and $a_0 < \eta_i, z_j, \gamma_l, \xi_u < b_0$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, $l = 1,2,\ldots,\nu$, $u = 1,2,\ldots,\lambda$.

Here, we emphasize that the objective for considering the problem (1) is to enrich the literature on coupled systems of nonlinear (k,φ)–Hilfer type fractional differential equations. In [12], the authors discussed the existence of solutions for a (k,ψ)–Hilfer-type fractional differential equation supplemented with non-local integro-multi-point fractional boundary conditions. In the present work, we go a step further to address the investigation of a coupled system of (k, ψ)–Hilfer type nonlinear fractional differential equations complemented with coupled non-local ordinary and fractional integral boundary conditions. Moreover, the problem at hand is a generalized form of the fully coupled nonlinear nonlocal multi-point boundary value problem of (k, ψ)–Hilfer fractional differential equations discussed in [20]. Thus, the work established in this article fills the gap in the literature on (k, ψ)–Hilfer type non-linear fractional differential systems with coupled boundary conditions.

It is imperative to mention that the (k, ψ)–Hilfer fractional derivative is the most generalized version of the Hilfer derivative, and it specializes to (k, ψ)–Riemann–Liouville, (k, ψ)–Caputo, k–Hilfer–Katugampola, and k–Hilfer–Hadamard-type fractional derivative operators—for details, see [11,12]. An example of a physical system modeled by means of the Hilfer fractional derivative is described in [21], while the Hilfer fractional advection–diffusion equation with the power-law initial condition is studied in [22]. In [23,24], the Hilfer–Prabhakar and Hilfer fractional derivatives are used to model filtration processes. In a recent work [25], the authors discussed the attractivity for Hilfer fractional stochastic evolution equations. One can find the application of Hilfer fractional derivative operator in the cobweb economics model in [26]. The concept of the (k, ψ)–Hilfer fractional derivative operator is quite a recent one, and it is expected that the models based on the Hilfer fractional derivative operators will be considered with the (k, ψ)–Hilfer fractional derivatives to find more insight into these models. For the application of (k, ψ)–Hilfer fractional derivatives in variational problems, see [27].

We make use of Banach's contraction mapping principle and the Boyd–Wong fixed point theorem for nonlinear contractions to prove the uniqueness results for the problem (1) under different criteria, while the existence results for the given problem are established via alternative of Leray–Schauder and a fixed-point theorem due to Krasnosel'skiĭ. Examples have been constructed for illustrating the main results. It is worthwhile to mention that our results are novel in the given setting and produce several new results as special cases, for details, see the Section 5.

The remaining part of the paper is organized as follows. In Section 2, some known results are recalled. Also, an auxiliary result concerning the linear variant of the system (1) is proven, which facilitates transforming the nonlinear boundary value problem (1) into an equivalent fixed point problem. Section 3 contains the existence and uniqueness results for the problem (1), while Section 4 is devoted to the examples demonstrating the application of the results obtained in Section 3. In the last section, we discuss some interesting special cases of the problem under investigation.

2. Preliminaries

Let us begin this section with some definitions related to our study.

Definition 1 ([10])**.** *Assume that $\psi : [a_0, b_0] \to \mathbb{R}$ is a function with $\psi'(t) \neq 0$ for all $t \in [a_0, b_0]$ and increasing on $[a_0, b_0]$. Then, the (k, ψ)-fractional integral of Riemann–Liouville type of order $\alpha > 0$ ($\alpha \in \mathbb{R}$) of a function $w \in L^1([a_0, b_0], \mathbb{R})$ is defined by*

$$ {}^{k}I_{a+}^{\alpha;\psi} w(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t \psi'(u)(\psi(t) - \psi(u))^{\frac{\alpha}{k}-1} w(u)du,\ k > 0, $$

where $\Gamma_k(t) = \int_0^\infty s^{t-1} e^{-\frac{s^k}{k}} ds$.

Definition 2 ([11])**.** *Assume that $\psi \in C^n([a_0, b_0], \mathbb{R})$, $\psi'(t) \neq 0, t \in [a_0, b_0]$, $w \in C^n([a_0, b_0], \mathbb{R})$, and $\alpha, k \in \mathbb{R}^+ = (0, \infty)$, $\beta \in [0, 1]$. Then, the (k, ψ)-fractional derivative of the Hilfer type of the function w of order α and type β is given as*

$$ {}^{k,H}D^{\alpha,\beta;\psi} w(t) = {}^{k}I_{a+}^{\beta(nk-\alpha);\psi} \left(\frac{k}{\psi'(t)} \frac{d}{dt}\right)^n {}^{k}I_{a+}^{(1-\beta)(nk-\alpha);\psi} w(t),\ \ n = \left\lceil \frac{\alpha}{k} \right\rceil. $$

Now, an auxiliary result is proven for a linear variant of the (k,ψ)–Hilfer integro-multi-point nonlocal fractional system (1).

Lemma 1. *Let* $a_0 < b_0$, $k > 0$, $1 < \alpha, p \leq 2$, $\beta, q \in [0,1]$, $t_k = \alpha + \beta(2k-\alpha)$, $w_k = p + q(2k-p)$ *and* $\pi, \varpi \in AC^2([a_0,b_0],\mathbb{R})$ *and* $h_1, h_2 \in AC([a_0,b_0],\mathbb{R})$. *If*

$$\mathbb{A} := A_1A_4 - A_2A_3 \neq 0, \tag{2}$$

then the unique solution of (k,ψ)*–Hilfer integro-multi-point nonlocal fractional system:*

$$\begin{cases} {}^{k,H}D^{\alpha,\beta;\psi}\pi(t) = h_1(t), \quad t \in (a_0,b_0], \\ {}^{k,H}D^{p,q;\psi}\varpi(t) = h_2(t), \quad t \in (a_0,b_0], \\ \pi(a_0) = 0, \quad \pi(b_0) = \sum_{i=1}^{n}\mu_i \int_{a_0}^{\eta_i}\psi'(s)\varpi(s)ds + \sum_{j=1}^{m}\zeta_j\, {}^kI^{\phi_j;\psi}\varpi(z_j), \\ \varpi(a_0) = 0, \quad \varpi(b_0) = \sum_{l=1}^{\nu} r_l \int_{a}^{\gamma_l}\psi'(s)\pi(s)ds + \sum_{u=1}^{\lambda}\delta_u\, {}^kI^{\epsilon_u,\psi}\pi(\xi_u), \end{cases} \tag{3}$$

is given by

$$\begin{aligned} \pi(t) &= {}^kI^{\alpha;\psi}h_1(t) + \frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{\mathbb{A}\Gamma_k(t_k)}\Bigg[A_4\Bigg(\sum_{i=1}^{n}\mu_i \int_{a_0}^{\eta_i}\psi'(s)^kI^{p;\psi}h_2(s)ds \\ &+ \sum_{j=1}^{m}\zeta_j\, {}^kI^{\phi_j+p;\psi}h_2(z_j) - {}^kI^{\alpha;\psi}h_1(b_0)\Bigg) + A_2\Bigg(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l}\psi'(s)^kI^{\alpha;\psi}h_1(s)ds \\ &+ \sum_{u=1}^{\lambda}\delta_u\, {}^kI^{\epsilon_u+\alpha;\psi}h_1(\xi_u) - {}^kI^{p;\psi}h_2(b_0)\Bigg)\Bigg], \end{aligned} \tag{4}$$

and

$$\begin{aligned} \varpi(t) &= {}^kI^{p;\psi}h_1(t) + \frac{(\psi(t)-\psi(a_0))^{\frac{w_k}{k}-1}}{\mathbb{A}\Gamma_k(w_k)}\Bigg[A_1\Bigg(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l}\psi'(s)^kI^{\alpha;\psi}h_1(s)ds \\ &+ \sum_{u=1}^{\lambda}\delta_u\, {}^kI^{\epsilon_u+\alpha;\psi}h_1(\xi_u) - {}^kI^{p;\psi}h_2(b_0)\Bigg) + A_3\Bigg(\sum_{i=1}^{n}\mu_i \int_{a_0}^{\eta_i}\psi'(s)^kI^{p;\psi}h_2(s)ds \\ &+ \sum_{j=1}^{m}\zeta_j\, {}^kI^{\phi_j+p;\psi}h_2(z_j) - {}^kI^{\alpha;\psi}h_1(b_0)\Bigg)\Bigg], \end{aligned} \tag{5}$$

where

$$\begin{aligned} A_1 &= \frac{(\psi(b_0)-\psi(a_0))^{\frac{t_k}{k}-1}}{\Gamma_k(t_k)}, \\ A_2 &= k\sum_{i=1}^{n}\mu_i \frac{(\psi(\eta_i)-\psi(a_0))^{\frac{w_k}{k}}}{\Gamma_k(w_k+k)} + \sum_{j=1}^{m}\zeta_j \frac{(\psi(z_j)-\psi(a_0))^{\frac{w_k+\phi_j}{k}-1}}{\Gamma(w_k+\phi_j)}, \\ A_3 &= k\sum_{l=1}^{\nu} r_l \frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{t_k}{k}}}{\Gamma_k(t_k+k)} + \sum_{u=1}^{\lambda}\delta_u \frac{(\psi(\xi_u)-\psi(a_0))^{\frac{t_k+\epsilon_u}{k}-1}}{\Gamma(t_k+\epsilon_u)}, \\ A_4 &= \frac{(\psi(b_0)-\psi(a_0))^{\frac{w_k}{k}-1}}{\Gamma_k(w_k)}. \end{aligned} \tag{6}$$

Proof. Let (π, ϖ) be a solution of the system (3). Applying fractional integral operators ${}^kI^{\alpha;\psi}$ and ${}^kI^{p;\psi}$, respectively, on both sides of the first and second equations in (3), we get

$$\pi(t) = {}^kI^{\alpha;\psi}h_1(t) + c_0\frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{\Gamma_k(t_k)} + c_1\frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-2}}{\Gamma_k(t_k-k)}, \tag{7}$$

and

$$\varpi(t) = {}^kI^{p;\psi}h_2(t) + d_0\frac{(\psi(t)-\psi(a_0))^{\frac{w_k}{k}-1}}{\Gamma_k(w_k)} + d_1\frac{(\psi(t)-\psi(a_0))^{\frac{w_k}{k}-2}}{\Gamma_k(w_k-k)}, \tag{8}$$

where c_0, c_1, d_0 and d_1 are unknown arbitrary constants.

By the conditions $\pi(a_0) = 0$ and $\varpi(a_0) = 0$ in (7) and (8), we get $c_1 = 0$ and $d_1 = 0$, since $\frac{t_k}{k} - 2 < 0$, $\frac{w_k}{k} - 2 < 0$. Inserting (7) and (8) with $c_1 = 0$ and $d_1 = 0$ in the boundary conditions:

$$\pi(b_0) = \sum_{i=1}^{n}\mu_i\int_{a_0}^{\eta_i}\psi'(s)\varpi(s)ds + \sum_{j=1}^{m}\zeta_j{}^kI^{\phi_j;\psi}\varpi(z_j),$$

and

$$\varpi(b_0) = \sum_{l=1}^{\nu}r_l\int_{a}^{\gamma_l}\psi'(s)\pi(s)ds + \sum_{u=1}^{\lambda}\delta_u\,{}^kI^{\epsilon_u,\psi}\pi(\xi_u),$$

together with the notation (6), we obtain

$$\begin{aligned} A_1c_0 - A_2d_0 &= \sum_{i=1}^{n}\mu_i\int_{a_0}^{\eta_i}\psi'(s){}^kI^{p;\psi}h_2(s)ds + \sum_{j=1}^{m}\zeta_j\,{}^kI^{\phi_j+p;\psi}h_2(z_j) - {}^kI^{\alpha;\psi}h_1(b_0), \\ -A_3c_0 + A_4d_0 &= \sum_{l=1}^{\nu}r_l\int_{a_0}^{\gamma_l}\psi'(s){}^kI^{\alpha;\psi}h_1(s)ds + \sum_{u=1}^{\lambda}\delta_u\,{}^kI^{\epsilon_u+\alpha;\psi}h_1(\xi_u) - {}^kI^{p;\psi}h_2(b_0). \end{aligned} \tag{9}$$

Solving system (9) for c_0 and d_0, we find that

$$\begin{aligned} c_0 &= \frac{1}{\mathbb{A}}\Bigg[A_4\Bigg(\sum_{i=1}^{n}\mu_i\int_{a_0}^{\eta_i}\psi'(s){}^kI^{p;\psi}h_2(s)ds + \sum_{j=1}^{m}\zeta_j\,{}^kI^{\phi_j+p;\psi}h_2(z_j) - {}^kI^{\alpha;\psi}h_1(b_0)\Bigg) \\ &\quad + A_2\Bigg(\sum_{l=1}^{\nu}r_l\int_{a_0}^{\gamma_l}\psi'(s){}^kI^{\alpha;\psi}h_1(s)ds + \sum_{u=1}^{\lambda}\delta_u\,{}^kI^{\epsilon_u+\alpha;\psi}h_1(\xi_u) - {}^kI^{p;\psi}h_2(b_0)\Bigg)\Bigg], \\ d_0 &= \frac{1}{\mathbb{A}}\Bigg[A_1\Bigg(\sum_{l=1}^{\nu}r_l\int_{a_0}^{\gamma_l}\psi'(s){}^kI^{\alpha;\psi}h_1(s)ds + \sum_{u=1}^{\lambda}\delta_u\,{}^kI^{\epsilon_u+\alpha;\psi}h_1(\xi_u) - {}^kI^{p;\psi}h_2(b_0)\Bigg) \\ &\quad + A_3\Bigg(\sum_{i=1}^{n}\mu_i\int_{a_0}^{\eta_i}\psi'(s){}^kI^{p;\psi}h_2(s)ds + \sum_{j=1}^{m}\zeta_j\,{}^kI^{\phi_j+p;\psi}h_2(z_j) - {}^kI^{\alpha;\psi}h_1(b_0)\Bigg)\Bigg]. \end{aligned}$$

Substituting the values of c_0, c_1 and d_0, d_1 in (7) and (8), respectively, we obtain the solution (4) and (5). By direct computation, we can easily show the converse. □

3. Existence and Uniqueness Results

Let the Banach space of all continuous functions π, ϖ from $[a_0, b_0]$ to $\mathbb{R}$ endowed with the norm $\|\pi\| = \max\{|\pi(t)|, t \in [a_0, b_0]\}$ and $\|\varpi\| = \max\{|\varpi(t)|, t \in [a_0, b_0]\}$ be denoted by $\mathbb{X} = C([a_0, b_0], \mathbb{R})$. Then, the product space $(\mathbb{X} \times \mathbb{X}, \|(\pi, \varpi)\|)$ is a Banach space with norm $\|(\pi, \varpi)\| = \|\pi\| + \|\varpi\|$.

In view of Lemma 1, we define an operator $\mathbb{W} : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \times \mathbb{X}$ associated with system (1) by

$$\mathbb{W}(\pi,\varpi)(t) = \begin{pmatrix} \mathbb{W}_1(\pi,\varpi)(t) \\ \mathbb{W}_2(\pi,\varpi)(t) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} \mathbb{W}_1(\pi,\varpi)(t) \quad = \quad & {}^kI^{\alpha;\psi}\Pi_1\left(t,\pi(t),\int_{a_0}^t \psi'(s)\varpi(s)ds\right) + \frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{\mathbb{A}\Gamma_k(t_k)} \\ & \times\Bigg[A_4\Bigg(\sum_{i=1}^n \mu_i \int_{a_0}^{\eta_i} \psi'(s)\, {}^kI^{p;\psi}\Pi_2\left(s,\varpi(s),\int_{a_0}^s \psi'(x)\pi(x)dx\right)ds \\ & + \sum_{j=1}^m \zeta_j\, {}^kI^{\phi_j+p;\psi}\Pi_2\left(z_j,\varpi(z_j),\int_{a_0}^{z_j} \psi'(s)\pi(s)ds\right) \\ & - {}^kI^{\alpha;\psi}\Pi_1\left(b_0,\pi(b_0),\int_{a_0}^{b_0} \psi'(s)\varpi(s)ds\right)\Bigg) \\ & + A_2\Bigg(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l} \psi'(s)\,{}^kI^{\alpha;\psi}\Pi_1\left(s,\pi(s),\int_{a_0}^s \psi'(x)\varpi(x)dx\right)ds \\ & + \sum_{u=1}^{\lambda} \delta_u\, {}^kI^{\epsilon_u+\alpha;\psi}\Pi_1\left(\xi_u,\pi(\xi_u),\int_{a_0}^{\xi_u} \psi'(s)\varpi(s)ds\right) \\ & - {}^kI^{p;\psi}\Pi_2\left(b_0,\varpi(b_0),\int_{a_0}^{b_0} \psi'(s)\pi(s)ds\right)\Bigg)\Bigg], \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathbb{W}_2(\pi,\varpi)(t) \quad = \quad & {}^kI^{p;\psi}\Pi_2\left(t,\varpi(t),\int_{a_0}^t \psi'(s)\pi(s)ds\right) + \frac{(\psi(t)-\psi(a_0))^{\frac{w_k}{k}-1}}{\mathbb{A}\Gamma_k(w_k)} \\ & \times\Bigg[A_1\Bigg(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l} \psi'(s)\, {}^kI^{\alpha;\psi}\Pi_1\left(s,\pi(s),\int_{a_0}^s \psi'(x)\varpi(x)dx\right)ds \\ & + \sum_{u=1}^{\lambda} \delta_u\, {}^kI^{\epsilon_u+\alpha;\psi}\Pi_1\left(\xi_u,\pi(\xi_u),\int_{a_0}^{\xi_u} \psi'(s)\varpi(s)ds\right) \\ & - {}^kI^{p;\psi}\Pi_2\left(b_0,\varpi(b_0),\int_{a_0}^{b_0} \psi'(s)\pi(s)ds\right)\Bigg) \\ & + A_3\Bigg(\sum_{i=1}^n \mu_i \int_{a_0}^{\eta_i} \psi'(s)\,{}^kI^{p;\psi}\Pi_2\left(s,\varpi(s),\int_{a_0}^s \psi'(x)\pi(x)dx\right)ds \\ & + \sum_{j=1}^m \zeta_j\, {}^kI^{\phi_j+p;\psi}\Pi_2\left(z_j,\varpi(z_j),\int_{a_0}^{z_j} \psi'(s)\pi(s)ds\right) \\ & - {}^kI^{\alpha;\psi}\Pi_1\left(b_0,\pi(b_0),\int_{a_0}^{b_0} \psi'(s)\varpi(s)ds\right)\Bigg)\Bigg]. \end{aligned} \tag{12}$$

For the sake of computational convenience, we use the notation:

$$\begin{aligned} Q_0 \quad &= \quad \psi(b_0)-\psi(a_0), \\ Q_1 \quad &= \quad \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \end{aligned}$$

$$
\begin{aligned}
& +|A_2|\left(k\sum_{l=1}^{\nu}|\gamma_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)}\right)\Bigg],\\
Q_2 &= \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_2|\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}\\
& +|A_4|\left(k\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)}+\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)}\right)\Bigg],\\
Q_3 &= \frac{Q_0^{\frac{w_k}{k}-1}}{|\mathbb{A}|\Gamma_k(w_k)}\Bigg[|A_3|\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\\
& +|A_1|\left(k\sum_{l=1}^{\nu}|r_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)}\right)\Bigg],\\
Q_4 &= \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}+\frac{Q_0^{\frac{w_k}{k}-1}}{|\mathbb{A}|\Gamma_k(w_k)}\Bigg[|A_1|\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}\\
& +|A_3|\left(k\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)}+\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)}\right)\Bigg],\\
Q_1^* &= Q_1-\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)},\ Q_4^*=Q_4-\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}. \qquad (13)
\end{aligned}
$$

3.1. Uniqueness Results

Firstly, we prove the uniqueness of solutions for nonlocal (k,ψ)–Hilfer fractional system (1) by applying Banach's contraction mapping principle [28].

Theorem 1. *Let $\Pi_1,\Pi_2:[a_0,b_0]\times\mathbb{R}^2\to\mathbb{R}$ satisfy the following condition:*
(H_1) there exist constants $m_i,n_i,i=1,2$, such that, for all $t\in[a_0,b_0]$ and $x_i,y_i\in\mathbb{R},i=1,2$,

$$|\Pi_1(t,x_1,x_2)-\Pi_1(t,y_1,y_2)|\le m_1|x_1-y_1|+m_2|x_2-y_2|,$$

and

$$|\Pi_2(t,x_1,x_2)-\Pi_2(t,y_1,y_2)|\le n_1|x_1-y_1|+n_2|x_2-y_2|.$$

Then, the nonlocal (k,ψ)–Hilfer fractional system (1) has a unique solution on $[a_0,b_0]$, provided that

$$Q_0[(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)]<1, \qquad (14)$$

where $Q_i,i=0,1,2,3,4$, are given in (13).

Proof. Let us first show that $\mathbb{W}B_r\subset B_r$, where the operator $\mathbb{W}$ is defined in (10) and $B_r=\{(\pi,\varpi)\in\mathbb{X}\times\mathbb{X}:\|(\pi,\varpi)\|\le r\}$, with

$$r\ge\frac{(Q_1+Q_3)N+(Q_2+Q_4)N_1}{1-[(Q_1+Q_3)(m_1+Q_0m_2)+(Q_2+Q_4)(n_1+Q_0n_2)]},$$

$\sup_{t\in[a_0,b_0]}\Pi_1(t,0,0)=N<\infty$, $\sup_{t\in[a_0,b_0]}\Pi_2(t,0,0)=N_1<\infty$. Then, we have

$$
\begin{aligned}
\left|\Pi_1\left(t,\pi(t),\int_{a_0}^{t}\psi'(s)\varpi(s)ds\right)\right| &\le \left|\Pi_1\left(t,\pi(t),\int_{a_0}^{t}\psi'(s)\varpi(s)ds\right)-\Pi_1(t,0,0)\right|+|\Pi_1(t,0,0)|\\
&\le m_1\|\pi\|+m_2\left|\int_{a_0}^{t}\psi'(s)\varpi(s)ds\right|+N
\end{aligned}
$$

$$\leq \quad m_1\|\pi\| + m_2 Q_0\|\varpi\| + N,$$

and similarly

$$\left|\Pi_2\left(t, \varpi(t), \int_{a_0}^{t} \psi'(s)\pi(s)ds\right)\right| \leq n_1\|\varpi\| + n_2 Q_0\|\pi\| + N_1.$$

Next, for $(\pi, \varpi) \in B_r$, we obtain

$$\begin{aligned}
& |\mathbb{W}_1(\pi,\varpi)(t)| \\
\leq \quad & {}^{k}I^{\alpha;\psi}\left|\Pi_1\left(t, \pi(t), \int_{a_0}^{t}\psi'(s)\varpi(s)ds\right)\right| + \frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)} \\
& \times\Bigg[|A_4|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\, {}^{k}I^{p;\psi}\left|\Pi_2\left(s, \varpi(s), \int_{a_0}^{s}\psi'(x)\pi(x)dx\right)\right|ds \\
& + \sum_{j=1}^{m}|\zeta_j|\, {}^{k}I^{\phi_j+p;\psi}\left|\Pi_2\left(z_j, \varpi(z_j), \int_{a_0}^{z_j}\psi'(s)\pi(s)ds\right)\right| \\
& + {}^{k}I^{\alpha;\psi}\left|\Pi_1\left(b_0, \pi(b_0), \int_{a_0}^{b_0}\psi'(s)\varpi(s)ds\right)\right|\Bigg) \\
& + |A_2|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s)^{k}I^{\alpha;\psi}\left|\Pi_1\left(s, \pi(s), \int_{a_0}^{s}\psi'(x)\varpi(x)dx\right)\right|ds \\
& + \sum_{u=1}^{\lambda}|\delta_u|\, {}^{k}I^{\epsilon_u+\alpha;\psi}\left|\Pi_1\left(\xi_u, \pi(\xi_u), \int_{a_0}^{\xi_u}\psi'(s)\varpi(s)ds\right)\right| \\
& + {}^{k}I^{p;\psi}\left|\Pi_2\left(b_0, \varpi(b_0), \int_{a_0}^{b_0}\psi'(s)\pi(s)ds\right)\right|\Bigg)\Bigg] \\
\leq \quad & \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}[m_1\|\pi\| + m_2 Q_0\|\varpi\| + N] \\
& + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\Bigg(k[n_1\|\varpi\| + n_2 Q_0\|\pi\| + N_1]\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)} \\
& + [n_1\|\varpi\| + n_2 Q_0\|\pi\| + N_1]\sum_{j=1}^{m}|\zeta_j|\,\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)} \\
& + \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}[m_1\|\pi\| + m_2 Q_0\|\varpi\| + N]\Bigg) \\
& + |A_2|\Bigg(k[m_1\|\pi\| + m_2 Q_0\|\varpi\| + N]\sum_{l=1}^{\nu}|r_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)} \\
& + [m_1\|\pi\| + m_2 Q_0\|\varpi\| + N]\sum_{u=1}^{\lambda}|\delta_u|\,\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)} \\
& + [n_1\|\varpi\| + n_2 Q_0\|\pi\| + N_1]\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}\Bigg)\Bigg] \\
\leq \quad & \Bigg\{\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
& + |A_2|\Bigg(k\sum_{l=1}^{\nu}|\gamma_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)} + \sum_{u=1}^{\lambda}|\delta_u|\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)}\Bigg)\Bigg]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
& \times[m_1\|\pi\|+m_2Q_0\|\varpi\|+N] \\
& +\left\{\frac{Q_0^{\frac{t_k}{k}-1}}{|A|\Gamma_k(t_k)}\left[|A_2|\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}+|A_4|\left(k\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)}\right.\right.\right. \\
& \left.\left.\left.+\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)}\right)\right]\right\}[n_1\|\varpi\|+n_2Q_0\|\pi\|+N_1] \\
= \ & Q_1[m_1\|\pi\|+Q_0m_2\|\varpi\|+N]+Q_2[n_1\|\varpi\|+Q_0n_2\|\pi\|+N_1] \\
= \ & (Q_1m_1+Q_0Q_2n_2)\|\pi\|+(Q_2n_1+Q_0Q_1m_2)\|\varpi\|+Q_1N+Q_2N_1 \\
\leq \ & (Q_1m_1+Q_0Q_2n_2+Q_2n_1+Q_0Q_1m_2)r+Q_1N+Q_2N_1.
\end{aligned}
$$

Similarly, one can find that

$$|\mathbb{W}_2(\pi,\varpi)(t)| \leq (Q_3m_1+Q_0Q_4n_2+Q_4n_1+Q_0Q_3m_2)r+Q_3N+Q_4N_1.$$

In view of the foregoing inequalities, we obtain

$$
\begin{aligned}
\|\mathbb{W}(\pi,\varpi)\| &= \|\mathbb{W}_1(\pi,\varpi)\|+\|\mathbb{W}_2(\pi,\varpi)\| \\
&\leq [(Q_1+Q_3)(m_1+Q_0m_2)+(Q_2+Q_4)(n_1+Q_0n_2)]r \\
&\quad +(Q_1+Q_3)N+(Q_2+Q_4)N_1 \leq r,
\end{aligned}
$$

which implies that $\mathbb{W}B_r \subset B_r$.

Now, we show that the operator $\mathbb{W}$ given in (10) is a contraction. For (π_2,ϖ_2), $(\pi_1,\varpi_1) \in \mathbb{X}\times\mathbb{X}$, and $t\in[a_0,b_0]$, we get

$$
\begin{aligned}
& |\mathbb{W}_1(\pi_2,\varpi_2)(t)-\mathbb{W}_1(\pi_1,\varpi_1)(t)| \\
\leq \ & {}^kI^{\alpha;\psi}\left|\Pi_1\left(t,\pi_2(t),\int_{a_0}^{t}\psi'(s)\varpi_2(s)ds\right)-\Pi_1\left(t,\pi_1(t),\int_{a_0}^{t}\psi'(s)\varpi_1(s)ds\right)\right| \\
& +\frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{|A|\Gamma_k(t_k)}\left[|A_4|\left(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\,{}^kI^{p;\psi}\left|\Pi_2\left(s,\varpi_2(s),\int_{a_0}^{s}\psi'(x)\pi_2(x)dx\right)\right.\right.\right. \\
& \left.-\Pi_2\left(s,\varpi_1(s),\int_{a_0}^{s}\psi'(x)\pi_1(x)dx\right)\right|ds \\
& +\sum_{j=1}^{m}|\zeta_j|\,{}^kI^{\phi_j+p;\psi}\left|\Pi_2\left(z_j,\varpi_2(z_j),\int_{a_0}^{z_j}\psi'(s)\pi_2(s)ds\right)\right. \\
& \left.-\Pi_2\left(z_j,\varpi_1(z_j),\int_{a_0}^{z_j}\psi'(s)\pi_1(s)ds\right)\right| \\
& \left.+{}^kI^{\alpha;\psi}\left|\Pi_1\left(b_0,\pi_2(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi_2(s)ds\right)-\Pi_1\left(b_0,\pi_1(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi_1(s)ds\right)\right|\right) \\
& +|A_2|\left(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s){}^kI^{\alpha;\psi}\left|\Pi_1\left(s,\pi_2(s),\int_{a_0}^{s}\psi'(x)\varpi_2(x)dx\right)\right.\right. \\
& \left.-\Pi_1\left(s,\pi_1(s),\int_{a_0}^{s}\psi'(x)\varpi_1(x)dx\right)\right|ds \\
& +\sum_{u=1}^{\lambda}|\delta_u|\,{}^kI^{\epsilon_u+\alpha;\psi}\left|\Pi_1\left(\xi_u,\pi_2(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi_2(s)ds\right)\right. \\
& \left.-\Pi_1\left(\xi_u,\pi_1(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi_1(s)ds\right)\right| \\
& \left.\left.+{}^kI^{p;\psi}\left|\Pi_2\left(b_0,\varpi_2(b_0),\int_{a_0}^{b_0}\psi'(s)\pi_2(s)ds\right)-\Pi_2\left(b_0,\varpi_1(b_0),\int_{a_0}^{b_0}\psi'(s)\pi_1(s)ds\right)\right|\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left\{\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}+\frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\left[|A_4|\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\right.\right.\\
&\quad\left.\left.+|A_2|\left(k\sum_{l=1}^{\nu}|\gamma_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)}\right)\right]\right\}\\
&\quad\times Q_0(m_1\|\pi_2-\pi_1\|+m_2\|\varpi_2-\varpi_1\|)\\
&\quad+\left\{\frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\left[|A_2|\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}+|A_4|\left(k\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)}\right.\right.\right.\\
&\quad\left.\left.\left.+\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)}\right)\right]\right\}Q_0(n_1\|\varpi_2-\varpi_1\|+n_2\|\pi_2-\pi_1\|)\\
&= Q_1(m_1\|\pi_2-\pi_1\|+Q_0m_2\|\varpi_2-\varpi_1\|)+Q_2(n_1\|\varpi_2-\varpi_1\|+Q_0n_2\|\pi_2-\pi_1\|)\\
&= (Q_1m_1+Q_0Q_2n_2)\|\pi_2-\pi_1\|+(Q_2n_1+Q_0Q_1m_1)\|\varpi_2-\varpi_1\|,
\end{aligned}
$$

which yields

$$
\|\mathbb{W}_1(\pi_2,\varpi_2)-\mathbb{W}_1(\pi_1,\varpi_1)\|\leq (Q_1m_1+Q_0Q_2n_2+Q_2n_1+Q_0Q_1m_2)[\|\pi_2-\pi_1\|+\|\varpi_2-\varpi_1\|]. \quad (15)
$$

Similarly, one can find that

$$
\|\mathbb{W}_2(\pi_2,\varpi_2)-\mathbb{W}_2(\pi_1,\varpi_1)\|\leq (Q_3m_1+Q_0Q_4n_2+Q_4n_1+Q_0Q_3m_2)[\|\pi_2-\pi_1\|+\|\varpi_2-\varpi_1\|]. \quad (16)
$$

From (15) and (16), we have

$$
\begin{aligned}
&\|\mathbb{W}(\pi_2,\varpi_2)-\mathbb{W}(\pi_1,\varpi_1)\|\\
\leq\ & Q_0[(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)](\|\pi_2-\pi_1\|+\|\varpi_2-\varpi_1\|),
\end{aligned}
$$

which, in view of the condition (14), shows that the operator $\mathbb{W}$ is a contraction. Thus, by Banach's contraction mapping principle, the operator $\mathbb{W}$ has a unique fixed point. In consequence, there exists a unique solution to the (k,ψ)–Hilfer fractional system (1). □

Next, we give our second uniqueness result for (k,ψ)–Hilfer fractional system (1), which is based on a fixed point theorem for nonlinear contractions due to Boyd and Wong [29].

Definition 3. *Let E be a Banach space. Then, the mapping $B : E \to E$ is called a nonlinear contraction if there exists a continuous and non-decreasing function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Omega(0) = 0$ and $\Omega(t) < t$ for all $t > 0$ with the property:*

$$
\|Bx-By\|\leq \Omega(\|x-y\|), \qquad \forall x,y\in E.
$$

Lemma 2 (Boyd and Wong [29])**.** *Assume that $A : E \to E$ is a nonlinear contraction in the Banach space E. Then, there exists a unique fixed point of A in E.*

Theorem 2. *Suppose that $\Pi_1, \Pi_2 : [a_0, b_0] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions such that*

$$
|\Pi_1(t,x_1,x_2)-\Pi_1(t,y_1,y_2)|\leq F(t)\frac{|x_1-y_1|+|x_2-y_2|}{F^*+|x_1-y_1|+|x_2-y_2|},
$$

and

$$
|\Pi_2(t,x_1,x_2)-\Pi_2(t,y_1,y_2)|\leq G(t)\frac{|x_1-y_1|+|x_2-y_2|}{G^*+|x_1-y_1|+|x_2-y_2|},
$$

for $t \in [a_0, b_0]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, where $F, G : [a_0, b_0] \to \mathbb{R}^+$ are continuous functions, and the positive constants F^* and G^* are given by

$$\begin{aligned} F^* &= {}^kI^{\alpha;\psi}\Pi_1(b_0) + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\,{}^kI^{p;\psi}\Pi_2(s)ds \\ &+\sum_{j=1}^{m}|\zeta_j|\,{}^kI^{\phi_j+p;\psi}\Pi_2(z_j) + {}^kI^{\alpha;\psi}\Pi_1(b_0)\Bigg) + |A_2|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s){}^kI^{\alpha;\psi}\Pi_1(s)ds \\ &+\sum_{u=1}^{\lambda}|\delta_u|\,{}^kI^{\epsilon_u+\alpha;\psi}\Pi_1(\xi_u) + {}^kI^{p;\psi}\Pi_2(b_0)\Bigg)\Bigg], \end{aligned}$$

$$\begin{aligned} G^* &= {}^kI^{p;\psi}\Pi_2(b_0) + \frac{Q_0^{\frac{w_k}{k}-1}}{|\mathbb{A}|\Gamma_k(w_k)}\Bigg[|A_1|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s)\,{}^kI^{\alpha;\psi}\Pi_1(s)ds \\ &+\sum_{u=1}^{\lambda}|\delta_u|\,{}^kI^{\epsilon_u+\alpha;\psi}\Pi_1(\xi_u) + {}^kI^{p;\psi}\Pi_2(b_0)\Bigg) + |A_3|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s){}^kI^{p;\psi}\Pi_2(s)ds \\ &+\sum_{j=1}^{m}|\zeta_j|\,{}^kI^{\phi_j+p;\psi}\Pi_2(z_j) + {}^kI^{\alpha;\psi}\Pi_1(b_0)\Bigg)\Bigg], \end{aligned}$$

and

$$\begin{aligned} {}^kI^{x,\psi}\Pi_1(y) &= {}^kI^{x,\psi}\Pi_1\left(y, \pi(y), \int_{a_0}^{y}\psi'(s)\varpi(s)ds\right), \\ {}^kI^{x,\psi}\Pi_2(y) &= {}^kI^{x,\psi}\Pi_2\left(y, \varpi(y), \int_{a_0}^{y}\psi'(s)\pi(s)ds\right), \end{aligned}$$

${}^kI^{x,\psi}$ denotes the Riemann–Liouville (k,ψ)-fractional integral operator of order x.

Then, the (k,ψ)–Hilfer fractional system (1) has a unique solution on $[a_0, b_0]$.

Proof. Let us introduce the continuous and non-decreasing functions $\Omega_1, \Omega_2 : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Omega_1(t) = \frac{F^*t}{F^*+t}, \qquad \Omega_2(t) = \frac{G^*t}{G^*+t}, \qquad \Omega = \Omega_1 + \Omega_2, \qquad \forall t \geq 0.$$

Observe that $\Omega_i(0) = 0$ and $\Omega_i(t) < t$ $(i = 1,2)$ for all $t > 0$. For any $(\pi_2, \varpi_2), (\pi_1, \varpi_1) \in \mathbb{X} \times \mathbb{X}$ and for each $t \in [a_0, b_0]$, we have

$$\begin{aligned} &|\mathbb{W}_1(\pi_2,\varpi_2)(t) - \mathbb{W}_1(\pi_1,\varpi_1)(t)| \\ \leq\ & {}^kI^{\alpha;\psi}\left|\Pi_1\left(t, \pi_2(t), \int_{a_0}^{t}\psi'(s)\varpi_2(s)ds\right) - \Pi_1\left(t, \pi_1(t), \int_{a_0}^{t}\psi'(s)\varpi_1(s)ds\right)\right| \\ &+\frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{|A|\Gamma_k(t_k)}\Bigg[|A_4|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\,{}^kI^{p;\psi}\Bigg|\Pi_2\left(s, \varpi_2(s), \int_{a_0}^{s}\psi'(x)\pi_2(x)dx\right) \\ &-\Pi_2\left(s, \varpi_1(s), \int_{a_0}^{s}\psi'(x)\pi_1(x)dx\right)\Bigg|ds \\ &+\sum_{j=1}^{m}|\zeta_j|\,{}^kI^{\phi_j+p;\psi}\Bigg|\Pi_2\left(z_j, \varpi_2(z_j), \int_{a_0}^{z_j}\psi'(s)\pi_2(s)ds\right) \\ &-\Pi_2\left(z_j, \varpi_1(z_j), \int_{a_0}^{z_j}\psi'(s)\pi_1(s)ds\right)\Bigg| \end{aligned}$$

$$
\begin{aligned}
&+{}^{k}I^{\alpha;\psi}\left|\Pi_1\left(b_0,\pi_2(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi_2(s)ds\right)-\Pi_1\left(b_0,\pi_1(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi_1(s)ds\right)\right|\Bigg)\\
&+|A_2|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s){}^{k}I^{\alpha;\psi}\bigg|\Pi_1\left(s,\pi_2(s),\int_{a_0}^{s}\psi'(x)\varpi_2(x)dx\right)\\
&-\Pi_1\left(s,\pi_1(s),\int_{a_0}^{s}\psi'(x)\varpi_1(x)dx\right)\bigg|ds\\
&+\sum_{u=1}^{\lambda}|\delta_u|\,{}^{k}I^{\epsilon_u+\alpha;\psi}\bigg|\Pi_1\left(\xi_u,\pi_2(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi_2(s)ds\right)\\
&-\Pi_1\left(\xi_u,\pi_1(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi_1(s)ds\right)\bigg|\\
&+{}^{k}I^{p;\psi}\left|\Pi_2\left(b_0,\varpi_2(b_0),\int_{a_0}^{b_0}\psi'(s)\pi_2(s)ds\right)-\Pi_2\left(b_0,\varpi_1(b_0),\int_{a_0}^{b_0}\psi'(s)\pi_1(s)ds\right)\right|\Bigg)\Bigg]\\
\leq\;& {}^{k}I^{\alpha;\psi}\left[F(t)\frac{|\pi_2(t)-\pi_1(t)|+Q_0|\varpi_2(t)-\varpi_1(t)|}{F^*+|\pi_2(t)-\pi_1(t)|+Q_0|\varpi_2(t)-\varpi_1(t)|}\right]\\
&+\frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\,{}^{k}I^{p;\psi}\\
&\times\left[G(s)\frac{|\varpi_2(s)-\varpi_1(s)|+Q_0|\pi_2(s)-\pi_1(s)|}{G^*+|\varpi_2(s)-\varpi_1(s)|+Q_0|\pi_2(s)-\pi_1(s)|}\right]ds\\
&+\sum_{j=1}^{m}|\zeta_j|\,{}^{k}I^{\phi_j+p;\psi}\left[G(z_j)\frac{|\varpi_2(z_j)-\varpi_1(z_j)|+Q_0|\pi_2(z_j)-\pi_1(z_j)|}{G^*+|\varpi_2(z_j)-\varpi_1(z_j)|+Q_0|\pi_2(z_j)-\pi_1(z_j)|}\right]\\
&+{}^{k}I^{\alpha;\psi}\left[F(b_0)\frac{|\pi_2(b_0)-\pi_1(b_0)|+Q_0|\varpi_2(b_0)-\varpi_1(b_0)|}{F^*+|\pi_2(b_0)-\pi_1(b_0)|+Q_0|\varpi_2(b_0)-\varpi_1(b_0)|}\right]\Bigg)\\
&+|A_2|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s){}^{k}I^{\alpha;\psi}\left[F(s)\frac{|\pi_2(s)-\pi_1(s)|+Q_0|\varpi_2(s)-\varpi_1(s)|}{F^*+|\pi_2(s)-\pi_1(s)|+Q_0|\varpi_2(s)-\varpi_1(s)|}\right]ds\\
&+\sum_{u=1}^{\lambda}|\delta_u|\,{}^{k}I^{\epsilon_u+\alpha;\psi}\left[F(\xi_u)\frac{|\pi_2(\xi_u)-\pi_1(\xi_u)|+Q_0|\varpi_2(\xi_u)-\varpi_1(\xi_u)|}{F^*+|\pi_2(\xi_u)-\pi_1(\xi_u)|+Q_0|\varpi_2(\xi_u)-\varpi_1(\xi_u)|}\right]\\
&+{}^{k}I^{p;\psi}\left[G(b_0)\frac{|\varpi_2(b_0)-\varpi_1(b_0)|+|\pi_2(b_0)-\pi_1(b_0)|}{G^*+|\varpi_2(b_0)-\varpi_1(b_0)|+Q_0|\pi_2(b_0)-\pi_1(b_0)|}\right]\Bigg)\Bigg]\\
\leq\;& Q_0{}^{k}I^{\alpha;\psi}\left[F(t)\frac{|\pi_2(t)-\pi_1(t)|+Q_0|\varpi_2(t)-\varpi_1(t)|}{F^*+|\pi_2(t)-\pi_1(t)|+Q_0|\varpi_2(t)-\varpi_1(t)|}\right]\\
&+\frac{Q_0^{\frac{t_k}{k}}}{|\mathbb{A}|\Gamma_k(t_k)}\Bigg[|A_4|\Bigg(\sum_{i=1}^{n}|\mu_i|\int_{a_0}^{\eta_i}\psi'(s)\,{}^{k}I^{p;\psi}\\
&\times\left[G(s)\frac{|\varpi_2(s)-\varpi_1(s)|+Q_0|\pi_2(s)-\pi_1(s)|}{G^*+|\varpi_2(s)-\varpi_1(s)|+Q_0|\pi_2(s)-\pi_1(s)|}\right]ds\\
&+\sum_{j=1}^{m}|\zeta_j|\,{}^{k}I^{\phi_j+p;\psi}\left[G(z_j)\frac{|\varpi_2(z_j)-\varpi_1(z_j)|+Q_0|\pi_2(z_j)-\pi_1(z_j)|}{G^*+|\varpi_2(z_j)-\varpi_1(z_j)|+Q_0|\pi_2(z_j)-\pi_1(z_j)|}\right]\\
&+{}^{k}I^{\alpha;\psi}\left[F(b_0)\frac{|\pi_2(b_0)-\pi_1(b_0)|+Q_0|\varpi_2(b_0)-\varpi_1(b_0)|}{F^*+|\pi_2(b_0)-\pi_1(b_0)|+Q_0|\varpi_2(b_0)-\varpi_1(b_0)|}\right]\Bigg)\\
&+|A_2|\Bigg(\sum_{l=1}^{\nu}|r_l|\int_{a_0}^{\gamma_l}\psi'(s){}^{k}I^{\alpha;\psi}\left[F(s)\frac{|\pi_2(s)-\pi_1(s)|+Q_0|\varpi_2(s)-\varpi_1(s)|}{F^*+|\pi_2(s)-\pi_1(s)|+Q_0|\varpi_2(s)-\varpi_1(s)|}\right]ds
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{u=1}^{\lambda} |\delta_u| \, {}^k I^{\epsilon_u+\alpha;\psi} \left[F(\xi_u) \frac{|\pi_2(\xi_u) - \pi_1(\xi_u)| + Q_0|\varpi_2(\xi_u) - \varpi_1(\xi_u)|}{F^* + |\pi_2(\xi_u) - \pi_1(\xi_u)| + Q_0|\varpi_2(\xi_u) - \varpi_1(\xi_u)|} \right] \\
& + {}^k I^{p;\psi} \left[G(b_0) \frac{|\varpi_2(b_0) - \varpi_1(b_0)| + |\pi_2(b_0) - \pi_1(b_0)|}{G^* + |\varpi_2(b_0) - \varpi_1(b_0)| + Q_0|\pi_2(b_0) - \pi_1(b_0)|} \right] \Bigg) \Bigg] \\
\leq \; & \frac{\Omega_1(\|\pi_2 - \pi_1\| + \|\varpi_2 - \varpi_1\|)}{F^*} \Bigg\{ {}^k I^{\alpha;\psi}[F(b_0)] \\
& + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_4| \Bigg(\sum_{i=1}^{n} |\mu_i| \int_{a_0}^{\eta_i} \psi'(s) \, {}^k I^{p;\psi}[G(s)] ds \\
& + \sum_{j=1}^{m} |\zeta_j| \, {}^k I^{\phi_j+p;\psi}[G(z_j)] + {}^k I^{\alpha;\psi}[F(b_0)] \Bigg) \\
& + |A_2| \Bigg(\sum_{l=1}^{\nu} |r_l| \int_{a_0}^{\gamma_l} \psi'(s) {}^k I^{\alpha;\psi}[F(s)] ds + \sum_{u=1}^{\lambda} |\delta_u| \, {}^k I^{\epsilon_u+\alpha;\psi}[F(\xi_u)] \\
& + {}^k I^{p;\psi}[G(b_0)] \Bigg) \Bigg] \Bigg\} \\
= \; & \Omega_1(\|\pi_2 - \pi_1\| + \|\varpi_2 - \varpi_1\|).
\end{aligned}
$$

Hence, we get

$$\|\mathbb{W}_1(\pi_2, \varpi_2) - \mathbb{W}_1(\pi_1, \varpi_1)\| \leq \Omega_1(\|\pi_2 - \pi_1\| + \|\varpi_2 - \varpi_1\|).$$

In a similar manner, one can find that

$$\|\mathbb{W}_2(\pi_2, \varpi_2) - \mathbb{W}_2(\pi_1, \varpi_1)\| \leq \Omega_2(\|\pi_2 - \pi_1\| + \|\varpi_2 - \varpi_1\|).$$

Then, it follows from the foregoing inequalities that

$$\|\mathbb{W}(\pi_2, \varpi_2) - \mathbb{W}(\pi_1, \varpi_1)\| \leq \Omega(\|\pi_2 - \pi_1\| + \|\varpi_2 - \varpi_1\|),$$

which shows that $\mathbb{W}$ is a nonlinear contraction. Hence, by Lemma 2, the operator $\mathbb{W}$ has a unique fixed point, which is a unique solution to the (k, ψ)–Hilfer fractional system (1). □

3.2. Existence Results

In this subsection, we provide the criteria ensuring the existence of solutions for the (k, ψ)–Hilfer fractional system (1). Our first result relies on the Leray–Schauder alternative [30].

Theorem 3. *Suppose that:*

(H_2) $\Pi_1, \Pi_2 : [a_0, b_0] \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous functions and there exist real constants* $k_i, \nu_i \geq 0$, $i = 1, 2$, *and* $k_0, \nu_0 > 0$, *such that,* $\forall w_i \in \mathbb{R}$, $(i = 1, 2)$,

$$|\Pi_1(t, w_1, w_2)| \leq k_0 + k_1|w_1| + k_2|w_2|,$$

$$|\Pi_2(t, w_2, w_1)| \leq \nu_0 + \nu_1|w_2| + \nu_2|w_1|;$$

(H_3) $(Q_1 + Q_3)k_1 + Q_0(Q_2 + Q_4)\nu_2 < 1$ *and* $Q_0(Q_1 + Q_3)k_2 + (Q_2 + Q_4)\nu_1 < 1$, *where* $Q_i, i = 0, 1, 2, 3, 4$, *are given in (13).*

Then, the (k, ψ)*–Hilfer fractional system (1) has at least one solution on* $[a_0, b_0]$.

Proof. Evidently, $\mathbb{W}$ is continuous by the continuity of Π_1 and Π_2. Nex, we show that the operator $\mathbb{W}$ is completely continuous. For that, let $\mathbb{O}_r = \{(\pi, \varpi) \in \mathbb{X} \times \mathbb{X} : \|(\pi, \varpi)\| \leq r\} \subset \mathbb{X} \times \mathbb{X}$ be a bounded set. Then, by (H_2), we have

$$\begin{aligned}
|\Pi_1(t, w_1, w_2)| &\leq k_0 + k_1|w_1| + k_2|w_2| \\
&\leq k_0 + k_1(\|\pi\| + \|\varpi\|) + k_2 Q_0(\|\pi\| + \|\varpi\|) \\
&\leq k_0 + (k_1 + k_2 Q_0)r = L_1.
\end{aligned}$$

Likewise, we have that $|\Pi_2(t, w_2, w_1)| \leq \nu_0 + (\nu_1 + \nu_2 Q_0)r = L_2$.
Thus, for any $(\pi, \varpi) \in \mathbb{O}$, we have

$$\begin{aligned}
& |\mathbb{W}_1(\pi, \varpi)(t)| \\
\leq\ & \Bigg\{ \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} + \frac{Q_0^{\frac{t_k}{k} - 1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_4| \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha + k)} \\
& + |A_2| \left(k \sum_{l=1}^{\nu} |\gamma_l| \frac{(\psi(\gamma_l) - \psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha + 2k)} + \sum_{u=1}^{\lambda} |\delta_u| \frac{(\psi(\xi_u) - \psi(a_0))^{\frac{\epsilon_u + \alpha}{k}}}{\Gamma_k(\epsilon_u + \alpha + k)} \right) \Bigg] \Bigg\} \mathbb{L}_1 \\
& + \Bigg\{ \frac{Q_0^{\frac{t_k}{k} - 1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_2| \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p + k)} \\
& + |A_4| \left(k \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i) - \psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p + 2k)} + \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(z_j) - \psi(a_0))^{\frac{\phi_j + p}{k}}}{\Gamma_k(\phi_j + p + k)} \right) \Bigg] \Bigg\} \mathbb{L}_2 \\
=\ & Q_1 \mathbb{L}_1 + Q_2 \mathbb{L}_2,
\end{aligned}$$

and hence

$$\|\mathbb{W}_1(\pi, \varpi)\| \quad \leq \quad Q_1 \mathbb{L}_1 + Q_2 \mathbb{L}_2.$$

In a similar manner, we can find that

$$\|\mathbb{W}_2(\pi, \varpi)\| \leq Q_3 \mathbb{L}_1 + Q_4 \mathbb{L}_2.$$

Therefore, we get

$$\|\mathbb{W}(\pi, \varpi)\| = \|\mathbb{W}_1(\pi, \varpi)\| + \|\mathbb{W}_2(\pi, \varpi)\| \leq (Q_1 + Q_3)\mathbb{L}_1 + (Q_2 + Q_4)\mathbb{L}_2,$$

which means that the operator $\mathbb{W}$ is uniformly bounded. Next, let $t_1, t_2 \in [a_0, b_0]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
& |\mathbb{W}_1(\pi(t_2), \varpi(t_2)) - \mathbb{W}_1(\pi(t_1), \varpi(t_1))| \\
\leq\ & \frac{1}{\Gamma_k(\alpha)} \Bigg| \int_a^{t_1} \psi'(s)[(\psi(t_2) - \psi(s))^{\frac{\alpha}{k} - 1} - (\psi(t_1) - \psi(s))^{\frac{\alpha}{k} - 1}] \\
& \times \Pi_1\left(s, \pi(s), \int_{a_0}^{s} \psi'(r)\varpi(r)dr\right) ds \\
& + \int_{t_1}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\frac{\alpha}{k} - 1} \Pi_1\left(s, \pi(s), \int_{a_0}^{s} \psi'(r)\varpi(r)dr\right) ds \Bigg| \\
& + \frac{(\psi(t_2) - \psi(a_0))^{\frac{t_k}{k} - 1} - (\psi(t_1) - \psi(a_0))^{\frac{t_k}{k} - 1}}{|\mathbb{A}|\Gamma_k(t_k)} \\
& \times \Bigg[|A_4| \Bigg(\sum_{i=1}^{n} |\mu_i| \int_{a_0}^{\eta_i} \psi'(s)\, {}^{k}I^{p;\psi} \Big| \Pi_2\left(s, \varpi(s), \int_{a_0}^{s} \psi'(r)\pi(r)dr\right) \Big| ds
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{j=1}^{m} |\zeta_j| \, {}^{k}I^{\phi_j+p;\psi} \left| \Pi_2\left(z_j, \varpi(z_j), \int_{a_0}^{z_j} \psi'(s)\pi(s)ds \right) \right| \\
& + {}^{k}I^{\alpha;\psi} \left| \Pi_1\left(b_0, \pi(b_0), \int_{a_0}^{b_0} \psi'(s)\varpi(s)ds \right) \right| \Bigg) \\
& + |A_2| \left(\sum_{l=1}^{\nu} |r_l| \int_{a_0}^{\gamma_l} \psi'(s) {}^{k}I^{\alpha;\psi} \left| \Pi_1\left(s, \pi(s), \int_{a_0}^{s} \psi'(r)\varpi(r)dr \right) \right| ds \right. \\
& + \sum_{u=1}^{\lambda} |\delta_u| \, {}^{k}I^{\epsilon_u+\alpha;\psi} \left| \Pi_1\left(\xi_u, \pi(\xi_u), \int_{a_0}^{\xi_u} \psi'(s)\varpi(s)ds \right) \right| \\
& \left. + {}^{k}I^{p;\psi} \left| \Pi_2\left(b_0, \varpi(b_0), \int_{a_0}^{b_0} \psi'(s)\pi(s)ds \right) \right| \right) \Bigg] \\
\leq \; & \frac{\mathbb{L}_1}{\Gamma_k(\alpha+k)} [2(\psi(t_2)-\psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2)-\psi(a_0))^{\frac{\alpha}{k}} - (\psi(t_1)-\psi(a_0))^{\frac{\alpha}{k}}|] \\
& + \frac{(\psi(t_2)-\psi(a_0))^{\frac{t_k}{k}-1} - (\psi(t_1)-\psi(a_0))^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_4| \Bigg(k\mathbb{L}_2 \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)} \\
& + \mathbb{L}_2 \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)} + \mathbb{L}_1 \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \Bigg) \\
& + |A_2| \Bigg(k\mathbb{L}_1 \sum_{l=1}^{\nu} |r_l| \frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)} + \mathbb{L}_1 \sum_{u=1}^{\lambda} |\delta_u| \frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)} \\
& + \mathbb{L}_2 \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)} \Bigg) \Bigg],
\end{aligned}
$$

which tends to zero as $t_2 - t_1 \to 0$, independent of $(\pi, \varpi) \in \mathbb{O}_r$. Hence, $\mathbb{W}_1(\pi, \varpi)$ is equicontinuous. The equicontinuity of the operator $\mathbb{W}_2(\pi, \varpi)$ can be established in an analogous manner. In consequence, we deduce that the operator $\mathbb{W}(\pi, \varpi)$ is completely continuous.

To apply the conclusion of Laray–Schauder alternative, we need to show that the set $\mathbb{E} = \{(\pi, \varpi) \in \mathbb{X} \times \mathbb{X} : (\pi, \varpi) = \lambda \mathbb{W}(\pi, \varpi), 0 \leq \lambda \leq 1\}$ is bounded. Let $(\pi, \varpi) \in \mathbb{E}$, then $(\pi, \varpi) = \lambda \mathbb{W}(\pi, \varpi)$. For any $t \in [a_0, b_0]$, we have

$$
\pi(t) = \lambda \mathbb{W}_1(\pi, \varpi)(t), \quad \varpi(t) = \lambda \mathbb{W}_2(\pi, \varpi)(t).
$$

Then, we obtain

$$
\begin{aligned}
|\pi(t)| \;\leq\; & \Bigg\{ \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} + \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_4| \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
& + |A_2| \Bigg(k \sum_{l=1}^{\nu} |\gamma_l| \frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)} + \sum_{u=1}^{\lambda} |\delta_u| \frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)} \Bigg) \Bigg] \Bigg\} \\
& \times (k_0 + k_1\|\pi\| + Q_0 k_2 \|\varpi\|) \\
& + \Bigg\{ \frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)} \Bigg[|A_2| \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)} + |A_4| \Bigg(k \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)} \\
& + \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)} \Bigg) \Bigg] \Bigg\} (\nu_0 + \nu_1\|\varpi\| + Q_0 \nu_2 \|\pi\|),
\end{aligned}
$$

and

$$
\begin{aligned}
|\varpi(t)| \quad \leq \quad & \Bigg\{ \frac{Q_0^{\frac{w_k}{k}-1}}{|\mathbb{A}|\Gamma_k(w_k)} \Bigg[A_3 \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
& + |A_1| \Bigg(k \sum_{l=1}^{\nu} |r_l| \frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)} + \sum_{u=1}^{\lambda} |\delta_u| \frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)} \Bigg) \Bigg] \Bigg\} \\
& \times (k_0 + k_1\|\pi\| + Q_0 k_2 \|\varpi\|) \\
& + \Bigg\{ \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)} + \frac{Q_0^{\frac{w_k}{k}-1}}{|\mathbb{A}|\Gamma_k(w_k)} \Bigg[|A_1| \frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)} \\
& + |A_3| \Bigg(k \sum_{i=1}^{n} |\mu_i| \frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)} + \sum_{j=1}^{m} |\zeta_j| \frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)} \Bigg) \Bigg] \Bigg\} \\
& \times (\nu_0 + \nu_1 \|\varpi\| + Q_0 \nu_2 \|\pi\|).
\end{aligned}
$$

In consequence, we get

$$\|\pi\| \leq Q_1(k_0 + k_1\|\pi\| + Q_0 k_2\|\varpi\|) + Q_2(\nu_0 + \nu_1\|\varpi\| + Q_0\nu_2\|\pi\|),$$

and

$$\|\varpi\| \leq Q_3(k_0 + k_1\|\pi\| + Q_0 k_2\|\varpi\|) + Q_4(\nu_0 + \nu_1\|\varpi\| + Q_0\nu_2\|\pi\|),$$

which imply that

$$
\begin{aligned}
\|\pi\| + \|\varpi\| \quad \leq \quad & (Q_1+Q_3)k_0 + (Q_2+Q_4)\nu_0 + [(Q_1+Q_3)k_1 + Q_0(Q_2+Q_4)\nu_2]\|\pi\| \\
& + [Q_0(Q_1+Q_3)k_2 + (Q_2+Q_4)\nu_1]\|\varpi\|.
\end{aligned}
$$

Thus, for any $t \in [a_0, b_0]$, we have

$$\|(\pi, \varpi)\| \leq \frac{(Q_1+Q_3)k_0 + (Q_2+Q_4)\nu_0}{M_0},$$

where M_0 is defined by

$$M_0 = \min\{1 - [(Q_1+Q_3)k_1 + Q_0(Q_2+Q_4)\nu_2], 1 - [Q_0(Q_1+Q_3)k_2 + (Q_2+Q_4)\nu_1]\}.$$

Therefore, the set $\mathbb{E}$ is bounded. Hence, the (k, ψ)–Hilfer fractional system (1) has at least one solution on $[a_0, b_0]$. □

Finally, we present our second existence result for (k, ψ)–Hilfer fractional system (1), which is based on Krasnosel'skiĭ's fixed point theorem [31].

Theorem 4. *Suppose that (H_1) and the following condition hold:*
(H_4) There exist continuous functions $\mathbb{P}_1, \mathbb{P}_2 \in C([a_0, b_0], \mathbb{R}^+)$ such that

$$|\Pi_1(t,x,y)| \leq \mathbb{P}_1(t), \ |\Pi_2(t,y,x)| \leq \mathbb{P}_2(t), \ \textit{for each} \ (t,x,y) \in [a_0,b_0] \times \mathbb{R} \times \mathbb{R}.$$

Then, the (k, ψ)–Hilfer fractional system (1) has at least one solution on $[a_0, b_0]$, provided that

$$Q_0[(Q_1^* + Q_3)(m_1+m_2) + (Q_2 + Q_4^*)(n_1+n_2)] < 1, \tag{17}$$

where $Q_0, Q_2, Q_3, Q_1^*, Q_4^*$ are given in (13).

Proof. We decompose the operator $\mathbb{W}$ defined by (10) as $\mathbb{W} = \mathbb{W}_{1,1} + \mathbb{W}_{1,2} + \mathbb{W}_{2,1} + \mathbb{W}_{2,2}$, where

$$
\begin{aligned}
\mathbb{W}_{1,1}(\pi,\varpi)(t) &= {}^{k}I^{\alpha;\psi}\Pi_1(t,\pi(t),\varpi(t)), \quad t \in [a_0,b_0], \\
\mathbb{W}_{1,2}(\pi,\varpi)(t) &= \frac{(\psi(t)-\psi(a_0))^{\frac{t_k}{k}-1}}{\mathbb{A}\Gamma_k(t_k)} \\
&\quad \times \Bigg[A_4 \Big(\sum_{i=1}^{n} \mu_i \int_{a_0}^{\eta_i} \psi'(s)\, {}^{k}I^{p;\psi}\Pi_2\Big(s,\varpi(s),\int_{a_0}^{s}\psi'(r)\pi(r)dr\Big)ds \\
&\quad + \sum_{j=1}^{m} \zeta_j\, {}^{k}I^{\phi_j+p;\psi}\Pi_2\Big(z_j,\varpi(z_j),\int_{a_0}^{z_j}\psi'(s)\pi(s)ds\Big) \\
&\quad - {}^{k}I^{\alpha;\psi}\Pi_1\Big(b_0,\pi(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi(s)ds\Big)\Big) \\
&\quad + A_2 \Big(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l} \psi'(s)\, {}^{k}I^{\alpha;\psi}\Pi_1\Big(s,\pi(s),\int_{a_0}^{s}\psi'(r)\varpi(r)dr\Big)ds \\
&\quad + \sum_{u=1}^{\lambda} \delta_u\, {}^{k}I^{\epsilon_u+\alpha;\psi}\Pi_1\Big(\xi_u,\pi(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi(s)ds\Big) \\
&\quad - {}^{k}I^{p;\psi}\Pi_2\Big(b_0,\varpi(b_0),\int_{a_0}^{b_0}\psi'(s)\pi(s)ds\Big)\Big)\Bigg], \ t \in [a_0,b_0], \\
\mathbb{W}_{2,1}(\pi,\varpi)(t) &= {}^{k}I^{p;\psi}\Pi_2\Big(t,\varpi(t),\int_{a_0}^{t}\psi'(s)\pi(s)ds\Big), \quad t \in [a_0,b_0], \\
\mathbb{W}_{2,2}(\pi,\varpi)(t) &= \frac{(\psi(t)-\psi(a_0))^{\frac{w_k}{k}-1}}{\mathbb{A}\Gamma_k(w_k)} \\
&\quad \times \Bigg[A_1 \Big(\sum_{l=1}^{\nu} r_l \int_{a_0}^{\gamma_l} \psi'(s)\, {}^{k}I^{\alpha;\psi}\Pi_1\Big(s,\pi(s),\int_{a_0}^{s}\psi'(r)\varpi(r)dr\Big)ds \\
&\quad + \sum_{u=1}^{\lambda} \delta_u\, {}^{k}I^{\epsilon_u+\alpha;\psi}\Pi_1\Big(\xi_u,\pi(\xi_u),\int_{a_0}^{\xi_u}\psi'(s)\varpi(s)ds\Big) \\
&\quad - {}^{k}I^{p;\psi}\Pi_2\Big(b_0,\varpi(b_0),\int_{a_0}^{b_0}\psi'(s)\pi(s)ds\Big)\Big) \\
&\quad + A_3 \Big(\sum_{i=1}^{n} \mu_i \int_{a_0}^{\eta_i} \psi'(s)\, {}^{k}I^{p;\psi}\Pi_2\Big(s,\varpi(s),\int_{a_0}^{s}\psi'(r)\pi(r)dr\Big)ds \\
&\quad + \sum_{j=1}^{m} \zeta_j\, {}^{k}I^{\phi_j+p;\psi}\Pi_2\Big(z_j,\varpi(z_j),\int_{a_0}^{z_j}\psi'(s)\pi(s)ds\Big) \\
&\quad - {}^{k}I^{\alpha;\psi}\Pi_1\Big(b_0,\pi(b_0),\int_{a_0}^{b_0}\psi'(s)\varpi(s)ds\Big)\Big)\Bigg], \ t \in [a_0,b_0].
\end{aligned}
$$

Let $B_\rho = \{(\pi,\varpi) \in \mathbb{X}\times\mathbb{X} : \|(\pi,\varpi)\| \leq \rho\}$ be a closed and bounded ball with $\rho \geq (Q_1+Q_3)\|\mathbb{P}_1\| + (Q_2+Q_4)\|\mathbb{P}_2\|$. For any $(\pi_2,\varpi_2),(\pi_1,\varpi_1) \in B_\rho$, as in the proof of Theorem 3, we have

$$|\mathbb{W}_{1,1}(\pi_1,\pi_2)(t) + \mathbb{W}_{1,2}(\varpi_1,\varpi_2)(t)| \leq Q_1\|\mathbb{P}_1\| + Q_2\|\mathbb{P}_2\|,$$

$$|\mathbb{W}_{2,1}(\pi_1,\pi_2)(t) + \mathbb{W}_{2,2}(\varpi_1,\varpi_2)(t)| \leq Q_3\|\mathbb{P}_1\| + Q_4\|\mathbb{P}_2\|.$$

Therefore, we get

$$\|(\mathbb{W}_{1,1}+\mathbb{W}_{2,1})(\pi_1,\pi_2)+(\mathbb{W}_{1,2}+\mathbb{W}_{2,2})(\varpi_1,\varpi_2)\| \le (Q_1+Q_3)\|\mathbb{P}_1\|+(Q_2+Q_4)\|\|\mathbb{P}_2\| < \rho,$$

which means that $(\mathbb{W}_{1,1}+\mathbb{W}_{2,1})(\pi_1,\pi_2)+(\mathbb{W}_{1,2}+\mathbb{W}_{2,2})(\varpi_1,\varpi_2) \in B_\rho$.

Next, we show that $(\mathbb{W}_{1,2},\mathbb{W}_{2,2})$ is a contraction mapping. Let $(\pi_1,\pi_2),(\varpi_1,\varpi_2) \in B_\rho$. Then, we have

$$\begin{aligned}
&|\mathbb{W}_{1,2}(\pi_2,\varpi_2)(t)-\mathbb{W}_{1,2}(\pi_1,\varpi_1)(t)| \\
\le\;& \Bigg\{\frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Big[|A_4|\frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \\
&+|A_2|\Big(k\sum_{l=1}^{\nu}|\gamma_l|\frac{(\psi(\gamma_l)-\psi(a_0))^{\frac{\alpha}{k}+1}}{\Gamma_k(\alpha+2k)}+\sum_{u=1}^{\lambda}|\delta_u|\frac{(\psi(\xi_u)-\psi(a_0))^{\frac{\epsilon_u+\alpha}{k}}}{\Gamma_k(\epsilon_u+\alpha+k)}\Big)\Big]\Bigg\} \\
&\times(m_1\|\pi_2-\pi_1\|+Q_0m_2\|\varpi_2-\varpi_1\|) \\
&+\Bigg\{\frac{Q_0^{\frac{t_k}{k}-1}}{|\mathbb{A}|\Gamma_k(t_k)}\Big[|A_2|\frac{Q_0^{\frac{p}{k}}}{\Gamma_k(p+k)}+|A_4|\Big(k\sum_{i=1}^{n}|\mu_i|\frac{(\psi(\eta_i)-\psi(a_0))^{\frac{p}{k}+1}}{\Gamma_k(p+2k)} \\
&+\sum_{j=1}^{m}|\zeta_j|\frac{(\psi(z_j)-\psi(a_0))^{\frac{\phi_j+p}{k}}}{\Gamma_k(\phi_j+p+k)}\Big)\Big]\Bigg\}(n_1\|\varpi_2-\varpi_1\|+Q_0n_2\|\pi_2-\pi_1\|) \\
=\;& Q_1^*(m_1\|\pi_2-\pi_1\|+Q_0m_2\|\varpi_2-\varpi_1\|)+Q_2(n_1\|\varpi_2-\varpi_1\|+Q_0n_2\|\pi_2-\pi_1\|) \\
=\;& (Q_1^*m_1+Q_0Q_2n_2)\|\pi_2-\pi_1\|+(Q_0Q_1^*m_2+Q_2n_1)\|\varpi_2-\varpi_1\|.
\end{aligned} \tag{18}$$

Likewise, one can obtain that

$$\begin{aligned}
&|\mathbb{W}_{2,2}(\pi_2,\varpi_2)(t)-\mathbb{W}_{2,2}(\pi_1,\varpi_1)(t)| \\
\le\;& (Q_3m_1+Q_0Q_4^*n_2)\|\pi_2-\pi_1\|+(Q_0Q_3m_2+Q_4^*n_1)\|\varpi_2-\varpi_1\|].
\end{aligned} \tag{19}$$

It follows from (18) and (19) that

$$\begin{aligned}
&\|(\mathbb{W}_{1,2},\mathbb{W}_{2,2})(\pi_2,\varpi_2)-(\mathbb{W}_{1,2},\mathbb{W}_{2,2})(\pi_1,\varpi_1)\| \\
\le\;& Q_0\Big\{(Q_1^*+Q_3)(m_1+m_2)+(Q_2+Q_4^*)(n_1+n_2)\Big\}(\|\pi_1-\pi_2\|+\|\varpi_1-\varpi_2\|),
\end{aligned}$$

which means that $(\mathbb{W}_{1,2},\mathbb{W}_{2,2})$ is a contraction in view of the constraint (17).

Note that the operator $(\mathbb{W}_{1,1},\mathbb{W}_{2,1})$ is continuous, since Π_1 and Π_2 are continuous, and uniformly bounded on B_ρ as

$$\|(\mathbb{W}_{1,1},\mathbb{W}_{2,1})(\pi,\varpi)\| \le \frac{Q_0^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\|\mathbb{P}_1\|+\frac{Q_0^{\frac{\alpha_1}{k}}}{\Gamma_k(\alpha_1+k)}\|\mathbb{P}_2\|.$$

In the next step, we will show that the set $(\mathbb{W}_{1,1},\mathbb{W}_{2,1})B_\rho$ is equicontinuous. For $t_1,t_2 \in [a_0,b_0], t_1 < t_2$, and for any $(\pi,\varpi) \in B_\rho$, we have

$$\begin{aligned}
&|\mathbb{W}_{1,1}(\pi,\varpi)(t_2)-\mathbb{W}_{1,1}(\pi,\varpi)(t_1)| \\
\le\;& \frac{1}{\Gamma_k(\alpha)}\Bigg|\int_c^{t_1}\psi'(s)[(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1}-(\psi(t_1)-\psi(s))^{\frac{\alpha}{k}-1}] \\
&\times\Pi_1\Big(s,\pi(s),\int_{a_0}^{s}\psi'(r)\varpi(r)dr\Big)ds \\
&+\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\frac{\alpha}{k}-1}\Pi_1\Big(s,\pi(s),\int_{a_0}^{s}\psi'(r)\varpi(r)dr\Big)ds\Bigg|
\end{aligned}$$

$$\leq \quad \frac{\|\mathbb{P}_1\|}{\Gamma_k(\alpha+k)}[2(\psi(t_2)-\psi(t_1))^{\frac{\alpha}{k}} + |(\psi(t_2)-\psi(a_0))^{\frac{\alpha}{k}} - (\psi(t_1)-\psi(a_0))^{\frac{\alpha}{k}}|],$$

which tends to zero as $t_1 \to t_2$ independently of $(\pi, \varpi) \in B_\rho$. Similarly, we can show that $|\mathbb{W}_{2,1}(\pi,\varpi)(t_2) - \mathbb{W}_{2,1}(\pi,\varpi)(t_1)| \to 0$ as $t_1 \to t_2$ independently of $(\pi,\varpi) \in B_\rho$. Thus, $|(\mathbb{W}_{1,1}, \mathbb{W}_{2,1})(\pi,\varpi)(t_2) - (\mathbb{W}_{1,1}, \mathbb{W}_{2,1})(\pi,\varpi)(t_1)|$ tends to zero, as $t_1 \to t_2$. Therefore, $(\mathbb{W}_{1,1}, \mathbb{W}_{2,1})$ is equicontinuous, and hence $(\mathbb{W}_{1,1}, \mathbb{W}_{2,1})$ is compact on B_ρ by the Arzelá–Ascoli theorem.

From the preceding steps, we deduce that all the assumptions of Krasnosel'skiǐ's fixed point theorem are satisfied and hence, it follows by its conclusion that the (k,ψ)–Hilfer fractional system (1) has at least one solution on $[a_0, b_0]$. □

4. Illustrative Examples

Consider the following non-local coupled system for (k,ψ)–Hilfer fractional differential equations with nonlocal multipoint integral boundary conditions:

$$\begin{cases} {}^{\frac{5}{4},H}D^{\frac{4}{3},\frac{2}{3};t^2e^{-t/2}}\pi(t) = \Pi_1\Big(t, \pi(t), \int_{2/9}^{t} \psi'(s)\varpi(s)ds\Big), \qquad t \in \left(\frac{2}{9}, \frac{14}{9}\right], \\ {}^{\frac{5}{4},H}D^{\frac{5}{3},\frac{1}{3};t^2e^{-t/2}}\varpi(t) = \Pi_2\Big(t, \varpi(t), \int_{2/9}^{t} \psi'(s)\pi(s)ds\Big), \qquad t \in \left(\frac{2}{9}, \frac{14}{9}\right], \\ x\left(\frac{2}{9}\right) = 0, \quad x\left(\frac{14}{9}\right) = \frac{1}{25}\int_{2/9}^{5/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\varpi(s)ds \\ \qquad + \frac{2}{35}\int_{2/9}^{8/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\varpi(s)ds \\ \qquad + \frac{3}{45}\int_{2/9}^{11/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\varpi(s)ds \\ \qquad + \frac{2}{54}\,{}^{k}I^{3/4;t^2e^{-t/2}}\varpi(1/3) + \frac{3}{65}\,{}^{k}I^{4/5;t^2e^{-t/2}}\varpi(13/9), \\ y\left(\frac{2}{9}\right) = 0, \quad y\left(\frac{14}{7}\right) = \frac{3}{37}\int_{2/9}^{4/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\pi(s)ds \\ \qquad + \frac{4}{49}\int_{2/9}^{7/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\varpi(s)ds \\ \qquad + \frac{5}{61}\int_{2/9}^{13/9}\left[-\frac{(s^2e^{-s/2})}{2} + 2se^{-s/2}\right]\varpi(s)ds \\ \qquad + \frac{4}{51}\,{}^{k}I^{5/6;t^2e^{-t/2}}\varpi(2/3) + \frac{5}{62}\,{}^{k}I^{6/7;t^2e^{-t/2}}\varpi(1). \end{cases} \tag{20}$$

Here, $k = 5/4$, $\alpha = 4/3$, $\beta = 2/3$, $p = 5/3$, $q = 1/3$, $\psi(t) = t^2e^{-t/2}$, $a = 2/9$, $b = 14/9$, $n = 3$, $m = 2$, $\nu = 3$, $\lambda = 2$, $\mu_1 = 1/25$, $\mu_2 = 2/35$, $\mu_3 = 3/45$, $\eta_1 = 5/9$, $\eta_2 = 8/9$, $\eta_3 = 11/9$, $\zeta_1 = 2/54$, $\zeta_2 = 3/65$, $z_1 = 1/3$, $z_2 = 13/9$, $\phi_1 = 3/4$, $\phi_2 = 4/5$, $r_1 = 3/37$, $r_2 = 4/49$, $r_3 = 5/61$, $\gamma_1 = 4/9$, $\gamma_2 = 7/9$, $\gamma_3 = 13/9$, $\delta_1 = 4/51$, $\delta_2 = 5/62$, $\xi_1 = 2/3$, $\xi_2 = 1$, $\epsilon_1 = 5/6$, $\epsilon_2 = 6/7$. Using the given values, it is found that $t_{\frac{5}{4}} = 19/9$, $w_{\frac{5}{4}} = 29/18$, $A_1 \approx 0.9894137797$, $A_2 \approx 0.0737375203$, $A_3 \approx 0.0833890039$, $A_4 \approx 1.062508916$, $\mathbb{A} \approx 1.045112065$, $Q_1 \approx 1.067507459$, $Q_1 \approx 1.649053094$, $Q_2 \approx 0.0681355524$, $Q_3 \approx 0.1370135866$, $Q_4 \approx 1.425499425$, $Q_1^* \approx 0.8286319543$, $Q_4^* \approx 0.7156209862$.

(a) For illustrating Theorem 1, we take $\Pi_1, \Pi_2 : [(2/9),(14/9)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$\begin{aligned} \Pi_1\Big(t, \pi(t), \int_{2/9}^{t}\psi'(s)\varpi(s)ds\Big) &= \frac{e^{-(9t-2)^3}}{2(9t+1)^3}\left(\frac{8\pi^2 + 9|\pi|}{1+|\pi|}\right) + \frac{1}{3}t + \frac{3}{5} \\ &\quad + \frac{1}{7}\sin^2\left|\int_{2/9}^{t}\left(\frac{-s^2e^{-s/2}}{2} + 2se^{-s/2}\right)\varpi(s)ds\right|, \end{aligned} \tag{21}$$

$$\Pi_2\left(t,\varpi(t),\int_{2/9}^t \psi'(s)\pi(s)ds\right) = \frac{\sin\left|\int_{2/9}^t\left(\dfrac{-s^2e^{-s/2}}{2}+2se^{-s/2}\right)\pi(s)ds\right|}{9t+7} + \frac{\cos^2(t+\pi/2)}{8}\left(\frac{\varpi^2+|\varpi|}{1+|\varpi|}\right)+\frac{1}{4}t+\frac{1}{3}. \tag{22}$$

Notice that $m_1 = 1/6$, $m_2 = 1/7$, $n_1 = 1/9$ and $n_2 = 1/8$ as

$$|\Pi_1(t,\pi_1,\varpi_1) - \Pi_1(t,\pi_2,\varpi_2)| \le \frac{1}{6}|\pi_1-\pi_2| + \frac{1}{7}|\varpi_1-\varpi_2|,$$

and

$$|\Pi_2(t,\varpi_1,\pi_1) - \Pi_2(t,\varpi_2,\pi_2)| \le \frac{1}{9}|\pi_1-\pi_2| + \frac{1}{8}|\varpi_1-\varpi_2|,$$

for all $\pi_1,\pi_2,\varpi_1,\varpi_2 \in \mathbb{R}$. Moreover, we find that

$$Q_0[(Q_1+Q_3)(m_1+m_2)+(Q_2+Q_4)(n_1+n_2)] \approx 0.966621574 < 1.$$

Thus, by Theorem 1, the (k,ψ)–Hilfer fractional differential (20) with Π_1 and Π_2 defined by (21) and (22) respectively, has a unique solution on $[2/9, 14/9]$.

(b) Let the nonlinear bounded functions $\Pi_1, \Pi_2 : [(2/9),(14/9)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by

$$\Pi_1\left(t,\pi(t),\int_{2/9}^t \psi'(s)\varpi(s)ds\right) = \frac{e^{-(9t-2)}}{15t}\tan^{-1}|\pi| + \frac{4}{5} + \frac{1}{81t^2} \times\left(\frac{\left|\int_{2/9}^t\left(\dfrac{-s^2e^{-s/2}}{2}+2se^{-s/2}\right)\varpi(s)ds\right|}{1+\left|\int_{2/9}^t\left(\dfrac{-s^2e^{-s/2}}{2}+2se^{-s/2}\right)\varpi(s)ds\right|}\right), \tag{23}$$

$$\Pi_2\left(t,\varpi(t),\int_{2/9}^t \psi'(s)\pi(s)ds\right) = \frac{3}{20}\cos^2|\varpi| + \frac{2}{3} + \frac{1}{5}\sin(t+\pi/2) \times\left(\frac{\left|\int_{2/9}^t\left(\dfrac{-s^2e^{-s/2}}{2}+2se^{-s/2}\right)\pi(s)ds\right|}{1+\left|\int_{2/9}^t\left(\dfrac{-s^2e^{-s/2}}{2}+2se^{-s/2}\right)\pi(s)ds\right|}\right). \tag{24}$$

Note that Π_1 and Π_2 are bounded as

$$|\Pi_1(t,\pi,\varpi)| \le \frac{e^{-(9t-2)}}{15t} + \frac{1}{81t^2} + \frac{4}{5}, \quad |\Pi_2(t,\varpi,\pi)| \le \frac{1}{5}\sin(t+\pi/2) + \frac{3}{20} + \frac{2}{3},$$

and

$$|\Pi_1(t,\pi_1,\varpi_1) - \Pi_1(t,\pi_2,\varpi_2)| \le \frac{3}{10}|\pi_1-\pi_2| + \frac{1}{4}|\varpi_1-\varpi_2|,$$

$$|\Pi_2(t,\varpi_1,\pi_1) - \Pi_2(t,\varpi_2,\pi_2)| \le \frac{1}{5}|\pi_1-\pi_2| + \frac{3}{20}|\varpi_1-\varpi_2|.$$

Setting $m_1 = 3/10$, $m_2 = 1/4$, $n_1 = 1/5$ and $n_2 = 3/20$, we find that $Q_0[(Q_1 + Q_3)((3/10)+(1/4)) + (Q_2+Q_4)((1/5)+(3/20))] \approx 1.606714996 > 1$, which means that Theorem 1 does not apply to the problem at hand. On the other hand,

$$Q_0[(Q_1^*+Q_3)(m_1+m_2)+(Q_2+Q_4^*)(n_1+n_2)] \approx 0.8597916826 < 1.$$

Therefore, it follows by the conclusion of Theorem 4 that there exists at least one solution for the (k,ψ)–Hilfer fractional differential system (20) with Π_1 and Π_2 given by (23) and (24) respectively, on $[2/9, 14/9]$.

(c) Let the nonlinear functions $\Pi_1, \Pi_2 : [(2/9), (14/9)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be expressed by

$$\begin{aligned} \Pi_1\left(t, \pi(t), \int_{2/9}^{t} \psi'(s)\varpi(s)ds\right) &= \frac{1}{9t+1} + \frac{1}{3}|\pi| + \frac{2}{5}\cos^2\pi \\ &\times \frac{\left|\int_{2/9}^{t}\left(\frac{-s^2e^{-s/2}}{2} + 2se^{-s/2}\right)\varpi(s)ds\right|^{150}}{1 + \left|\int_{2/9}^{t}\left(\frac{-s^2e^{-s/2}}{2} + 2se^{-s/2}\right)\varpi(s)ds\right|^{149}}, \qquad (25) \\ \Pi_2\left(t, \varpi(t), \int_{2/9}^{t} \psi'(s)\pi(s)ds\right) &= \frac{1}{18t+1} + \frac{1}{7}\varpi + \frac{1}{4}\tan^{-1}|\varpi| \\ &\times \left(\frac{\left|\int_{2/9}^{t}\left(\frac{-s^2e^{-s/2}}{2} + 2se^{-s/2}\right)\pi(s)ds\right|^{17}}{1 + \left|\int_{2/9}^{t}\left(\frac{-s^2e^{-s/2}}{2} + 2se^{-s/2}\right)\pi(s)ds\right|^{16}}\right). \qquad (26) \end{aligned}$$

Observe that

$$|\Pi_1(t, \pi, \varpi)| \leq \frac{1}{3} + \frac{1}{3}|\pi| + \frac{2}{5}|\varpi| \quad \text{and} \quad |\Pi_2(t, \varpi, \pi)| \leq \frac{1}{5} + \frac{1}{4}|\pi| + \frac{1}{7}|\varpi|.$$

Fixing $k_0 = 1/3$, $k_1 = 1/3$, $k_2 = 2/5$, $\nu_0 = 1/5$, $\nu_1 = 1/4$, $\nu_2 = 1/7$, we have $(Q_1 + Q_3)k_1 + Q_0(Q_2 + Q_4)\nu_1 \approx 0.9939721801 < 1$ and $Q_0(Q_1 + Q_3)k_2 + (Q_2 + Q_4)\nu_2 \approx 0.9760322269 < 1$. Therefore, by Theorem 3, the nonlocal coupled system for the (k,ψ)–Hilfer fractional differential system (20) with Π_1 and Π_2 defined in (25) and (26) respectively, has at least one solution on $[2/9, 14/9]$.

5. Conclusions

In this paper, we have presented the existence and uniqueness criteria for solutions of a system of (k,ψ)–Hilfer fractional differential equations complemented with non-local multi-point integral boundary conditions. We first converted the given nonlinear problem into a fixed-point problem and then applied the tools of the fixed point theory to prove the existence and uniqueness results for it. Our results are not only new in the given configuration, but also specialize to several new results for the given (k,ψ)–Hilfer fractional differential system by setting different combinations of the terms in the given nonlocal multipoint integral boundary conditions. Here are two examples for the specialized non-local integral boundary conditions.

(i) Our results correspond to the boundary conditions: $\pi(a_0) = 0, \varpi(a_0) = 0$, $\pi(b_0) = \sum_{i=1}^{n} \mu_i \int_{a_0}^{\eta_i} \psi'(s)\varpi(s)ds, \varpi(b_0) = \sum_{l=1}^{\nu} r_l \int_{a}^{\gamma_l} \psi'(s)\pi(s)ds$ if we take $\zeta_j = 0, \forall j = 1, \ldots, m$ and $\delta_u = 0, \forall u = 1, \ldots, \lambda$, in the results of this paper.

(ii) By taking $\mu_i = 0, \forall i = 1, \ldots, n$ and $r_l = 0, \forall l = 1, \ldots, \nu$ in the obtained results, we get the ones associated with the boundary conditions of the form:

$$\pi(a_0) = 0, \varpi(a_0) = 0, \pi(b_0) = \sum_{j=1}^{m} \zeta_j \, {}^{k}I^{\phi_j;\psi}\varpi(z_j), \varpi(b_0) = \sum_{u=1}^{\lambda} \delta_u \, {}^{k}I^{\epsilon_u,\psi}\pi(\xi_u).$$

Thus, our work is significant in the sense that it not only enriches the literature on nonlocal boundary value problems of (k,ψ)–Hilfer fractional differential systems equipped with non-local coupled multi-point integral boundary conditions, but also it covers a variety of special cases. In future, we plan to investigate the existence of solutions for a system of (k, ψ)–Hilfer fractional differential inclusions subject to non-local multi-point integral boundary conditions and impulsive variant of the problem (1).

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