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A Self Adaptive Three-Step Numerical Scheme for Variational Inequalities

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Abstract: In this paper, we introduce a new three-step iterative scheme for finding the common solutions of the variational inequality using the technique of updating the solution. We suggest, iterative algorithms involving three-steps for the predictor-corrector method of variational inequality in real Hilbert spaces H. Our results include the Takahashi and Toyoda, extra gradient, Mann and Noor iterations as special cases. We also investigate the convergence criteria of the three-step iterative scheme. As special cases, the earlier findings are included in our results, which can be seen as an advancement and improvement over the previous investigation. This is a new refinement in our existing literature and previously known algorithms. A numerical example is given to illustrate the efficiency and performance of the proposed self-adaptive scheme.

Keywords: variational inequalities; iterative methods; Hilbert spaces; fixed-point problem; three-step iteration methods; convergence criteria; numerical results

MSC: 26A33; 26A51; 26D10

Citation: Sanaullah, K.; Ullah, S.; Aloraini, N.M. A Self Adaptive Three-Step Numerical Scheme for Variational Inequalities. *Axioms* **2024**, *13*, 57. https://doi.org/10.3390/axioms13010057

Academic Editor: Behzad Djafari-Rouhani

Received: 15 November 2023
Revised: 11 December 2023
Accepted: 20 December 2023
Published: 18 January 2024

1. Introduction

Making decisions is greatly impacted by uncertainty. It appears that as we get more connected, we uncover more sources of uncertainty. For instance, the price and cost functions for electricity are unpredictable, and transportation, communication, and financial systems have been caused by how these systems function and the way they interact with one another. This raises a number of intriguing issues related to modeling uncertainty and making decisions in the face of ambiguity. Variational principles have practical importance in a variety of industries and commercial situations due to their natural modeling capability for numerous real-world applications. Variational principles provide a framework for optimization, where the target is to get the optimum solution or approximation for a set of possibilities. For more than two centuries, this has been the main area of mathematics, particularly in engineering sciences. Newton, Fermat, Leibniz, Bernoulli, and Lagrange are credited with developing the variational principles, see [1].

Variational inequalities theory is an important and well-established field, representing an enrichment of variational principles. The Variational Inequality Problem (VIP) encompasses several well-known mathematical issues, such as optimization problems, complementarity problems, and fixed-point problems. As a result, the VIP offers a methodological framework for studying various equilibrium issues in engineering and economics. Consequently, it is possible to consider the theory of VIPs as a methodology that facilitates the modeling, analysis, and computations for a variety of real-world applications.

Hartman and Stampacchia first developed variational inequalities in 1966 [2] as a tool for studying partial differential equations, particularly in mechanics, and defined within infinite-dimensional fields.

The finite-dimensional Variational Inequality Problem (VIP) was subsequently generalized from the nonlinear complementarity problem (NCP), which was initially identified by Cottle in his research thesis in 1964 [3]. Since then, this field has evolved into a flourishing branch of mathematical programming, offering a wealth of theory, efficient algorithms for problem solving, connections to a wide range of fields, and numerous significant applications in engineering and economics. VIPs find applications in representing frictional contact problems, traffic equilibrium challenges, engineering scenarios, and economic contexts. For instance, VIPs are employed to gain insights into strategic resources, including wireless and wireline systems within the realm of communication technology and networks [4–6].

We applied it to interactions in cognitive radio systems and networks [7]. Furthermore, we tackled contact problems within the engineering field and traffic network equilibria [8,9]. By employing the VIP, we establish the existence of a traffic equilibrium pattern, develop an algorithm for pattern creation, and estimate its convergence rate [10,11].

Notably, these concepts are interconnected with variational inequalities, the static traffic equilibrium model, the market equilibrium problem in dynamics, and extend to continuum cases. Furthermore, a computational method is outlined for identifying solutions to the equilibrium problem and formulating the VI, particularly tailored to the traffic equilibrium system [12]. The principles of variational inequalities are leveraged to construct dynamic and other control-theoretical systems, establishing models for urban network flows that anticipate time-varying conditions [13,14]. Employing the auxiliary principal technique, inequalities derived from variations propose and explore iterative solutions for trifunctional equilibrium challenges [15]. Many of these applications often involve a certain degree of uncertainty, possibly stemming from incomplete data or inherent unpredictability within the problem. Consequently, the investigation of the variational inequality problem (VIP) has captured the attention of mathematicians and, more broadly, the operations research community in recent years.

Later on, the concept of VIP was extended to vector variational inequality problems [16]. Due to the diverse nature of problems across various fields of mathematics, particularly in engineering sciences, VIs have been expanded in numerous directions. GVI, introduced by Noor [17] in 1988, extends the theory of VI and finds numerous applications across various fields. Another class of VI, known as mixed variational inequalities (MVI), incorporates four operators and was introduced by Noor in 2011, as seen in [16–19]. This class is also referred to as the extended general mixed variational inequality. Cottle, Pang, and Stone conducted a comprehensive analysis of the Linear Complementarity Problem (LCP) see [3]. In the realm of numerical methods development for variational inequalities, Scarf pioneered the first constructive iterative method to approximate a fixed point of continuous mapping and the computation of economic equilibria [20].

Numerous studies have explored numerical approaches for VIs, with many of the pre-2003 findings available in the books of Facchinei and Pang see [21]. Since the publication of Facchinei and Pang's works, the field has witnessed numerous methodological innovations. There are a large number of publications on self-adaptive methods applied to solving variational inequalities by other authors. For a comprehensive review of these works, please refer to the references therein [22–33]. Inertial type self-adaptive iterative algorithms for pseudomonotone equilibrium problems and fixed point problems are mentioned in [25], and convergence analysis of an inertial Tseng's extragradient algorithm [34] for solving pseudomonotone variational inequalities proved in [24]. In this study, we employ a three-step iterative scheme to propose a modified self-adaptive algorithm for solving VIs. This novel technique builds upon previously established algorithms. Convergence analysis is a crucial step to validate the authenticity of these new algorithms, and we have undertaken this analysis, providing proof under appropriate conditions.

Motivating Examples and Background

Examples of what motivates us to continue our studies include:

(i) In Nash games, participants engage in noncooperative competition, and the Nash equilibrium is a stable point that denotes a set of strategies where unilateral deviation is undesirable.
(ii) The analysis of supplies, demand, and prices of commodities in a network of physically distinct marketplaces is a component of problems involving spatial price equilibrium.
(iii) In addition to being employed in traffic planning and toll collection policy decisions, traffic equilibrium problems seek to anticipate steady-state traffic flows in a crowded network.
(iv) Market arrangements with a few firms are captured by oligopolistic market equilibrium issues, which also allow for strategic interactions between the enterprises. Financial markets, electrical markets, department stores, computer companies, and the automotive, chemical, and mineral extraction industries are a few examples.

In order to tackle these difficult problems we need a standard approach for obtaining approximate solutions to VIs. In the next section, we shall discuss source problems and some basic results of variational inequality theory.

2. Preliminary Results

Systems of equations: Observe that a VI(R^n, F) can be used to model the issue of solving a system of nonlinear equations with the solution $F(q) = 0$. It is clear that $F's$ zeros perfectly satisfy the variational inequality problem.
Problems related to Optimization: An optimization problem is identified by a set of constraints as well as its objective function, which must either be maximized (profit) or minimized (loss), depending on the task. A problem containing optimization involving objective function f and constraint in a set K is denoted by minimize $f(q)$ subject to the constraint $q \in K$.
Complementarity problems: Consider the complementarity condition $q.q^* = 0$, which indicates that if we take q as a positive, it is understood that q^* must be 0 and vice versa. The sets present in the decision variables that represent the equilibrium of supply and demand in economic systems usually interact in complementary ways. The complementarity problem (CP) is also included in the VIP as a special instance; if the VIP's underlying set K is defined as a cone, then the VIP can be equivalently represented as a Complementarity Problem (CP).
Problems related to fixed points: A fixed point of a function is a point which the function maps to itself. These functions are closely related to the VIP solutions based on projection mapping. Specifically, all VIP solutions can be represented as fixed points on a designed projection map.
Variational inequality and its generalization: Let's examine a real Hilbert space H, where the inner product of two vectors is represented as $\langle\, .\, ,\, .\, \rangle$, and the norm is denoted by $\|\, .\, \|$. Given a set, typically denoted as K, and a mapping, denoted as $T, g : H \to H$, the goal is to find $q \in K$ such that:

$$\langle \rho T(q),\ q^* - q\rangle \geq 0, \quad \forall\, q^* \in K,\ \rho > 0. \tag{1}$$

Here, we assume that the map T is continuous and set K is closed and convex. The given inequality is denoted by VI$(K,\ T)$, and called the classical variational inequality problem. A general-variational inequality, or GVI$(K,\ g)$, is the generalization of VI$(K,\ T)$, and it appears when two operators T and g are used in VI$(K,\ T)$, such that:

$$\langle \rho T(q),\ g(q^*) - g(q)\rangle \geq 0, \quad \forall\, q^* \in H,\ g(q), g(q^*) \in K,\ \rho > 0.$$

Lemma 1 ([4])**.** *The inequality is satisfied by $q \in K$ for $z \in H$, such that;*

$$\langle q - z,\ q^* - q\rangle \geq 0, \quad \forall\, q^* \in K, \tag{2}$$

iff

$$q = P_K z, \tag{3}$$

here P_K is in closed convex set K and nonexpansive projection of H, that is

$$\|P_K(q) - P_K(q^*)\| \leq \|q - q^*\|, \forall q\,,\ q^* \in H.$$

As a result of Lemma (1), we have

$$\langle z - P_K(z),\ P_K(z) - q^* \rangle \geq 0, \forall\ q^* \in K \tag{4}$$

and

$$\|P_K(z) - q^*\| \leq \|z - q^*\|, \forall z \in R^n,\ q^* \in K.$$

Lemma 1 makes it simple to demonstrate that the Problem 1 is equal to the fixed-point problem.

Definition 1 ([35]). *P_K is called the metric projection of H onto K, if for every point $q \in K$, there exists a unique nearest point in K, such that*

$$P_K(z) = argmin\{q - y, y \in K\}$$

where the metric projection is denoted by $P_K(z)$.

Assumption 1. *H is a finite dimension space.*

Assumption 2. *A mapping $T, g : H \to H$ is said to be strongly monotone that is,*

$$\langle Tq - Tq^*,\ q - q^* \rangle \geq \delta \|q - q^*\|^2, \quad \forall\ q,\ q^* \in H.$$

Assumption 3. *Lipschitz continuity for operator T is defined as:*

$$\|Tq - Tq^*\| \leq \sigma \|q - q^*\|,\ \ \forall\ q,\ q^* \in H.$$

where $\sigma > 0$ is a constant. In particular, from the definitions we have, $\delta \leq \sigma$.

Assumption 4. *If $\sigma = 1$, then T is non-expensive operator such that:*

$$\|Tq - Tq^*\| \leq \|q - q^*\|,\ \ \forall\ q,\ q^* \in H.$$

Lemma 2 ([22]). *Let us consider $\{q_n\}$ is sequence of nonnegative real numbers such that:*

$$q_{n+1} \leq (1 - \gamma_n) q_n + \varsigma_n$$

where $\{\gamma_n\} \in (0,1)$ and $\{\varsigma_n\}$ is a sequence such that; if $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $limsup_{n\to\infty} \frac{\varsigma_n}{\gamma_n}$ ≤ 0, or $\sum_{n=1}^{\infty} |\varsigma_n| < \infty$, then $lim_{n\to\infty} q_n = 0$.

Lemma 3 ([17]). *If $q \in K$ satisfies the relation*

$$q = P_k[q - \rho T q],$$

then the function $q \in K$ is a solution of the $VI(K,\ g)$,(1), where $\rho > 0$ is a constant. From Lemma 3 we can see that, Variational inequalities and fixed point problems are related concepts in mathematical analysis and optimization theory. They both deal with finding a solution to a given problem. Now we state and prove our main results in the next section.

3. Main Results

It is evident from Lemma 3 that the fixed-point problems and VI are comparable. In the VI studies, this alternative equivalence has played a significant role to suggest the following three-step predictor-corrector iterative technique for finding a common solution of the variational inequalities. To be precise, Noor [32,33] has suggested the three-step iterative scheme for variational inequalities. Let us consider

$$y = P_K[q - \rho Tq]. \tag{5}$$
$$w = (1-\beta_n)y + \beta_n P_K[y - \rho Ty]. \tag{6}$$
$$q = (1-\gamma_n)w + \gamma_n P_K[w - \rho Tw] \tag{7}$$

where $0 \leq \beta_n,\ \gamma_n \leq 1$, for all $n \geq 0$.

And we define the residue vector, $R(q)$ as

$$\begin{aligned} R(q) &= q - w, \\ &= q - (1-\beta_n)y - \beta_n P_K[y - \rho Ty], \\ &= q - (1-\beta_n)P_K[q-\rho Tq] - \beta_n P_K[P_K[q-\rho Tq] - \rho T P_K[q - \rho Tq]]. \end{aligned}$$

It is clear $q \in H$ is solution of (VI) iff $q \in H$ is zero of the equation.

$$R(q) = 0.$$

As we know that K is a convex set, for all $\eta \in [0,1]$ then $q, w \in K$, we have

$$\begin{aligned} x &= (1-\eta)q + \eta w\ , \\ &= q - \eta(q - w), \\ &= q - \eta R(q). \end{aligned}$$

We can rewrite for a positive step, α,

$$\begin{aligned} q &= (1-\gamma_n)q + \gamma_n P_K[q - \rho Tq], \\ &= (1-\gamma_n)q + \gamma_n P_K[q - \alpha(\eta R(q) + \rho Tx)], \\ &= (1-\gamma_n)q + \gamma_n P_K[q - \alpha d(q)], \end{aligned}$$

where $d(q)$ is extragradient such that;

$$\begin{aligned} d(q) &= \eta R(q) + \rho Tx, \\ &= \eta R(q) + \rho T(q - \eta R(q)). \end{aligned}$$

In this paper, motivated by the iterative schemes (5), (6) and (7), we introduced an iterative process for updating the solution to suggested two-step and three-step projection iterative schemes for solving variational inequalities and related optimization problems. Three-step iterative schemes also known as Noor iterations Algorithm 1. Clearly Noor iterations include Ishikawa (two-step) Algorithm 2 and Mann (one-step) Algorithm 3 iterative methods as special cases. We also use the technique of updating the solution to suggest novel Self-adaptive Iterative Scheme Algorithm 4 for solving the variational inequalities (1).

Algorithm 1 Three-step predictor–corrector method

For a given $q_0 \in H$, compute the approximate solution q_n by the iterative scheme.

$$\begin{aligned} y_n &= P_K[q_n - \rho T q_n]. \\ w_n &= (1-\beta_n) y_n + \beta_n P_K[y_n - \rho T y_n]. \\ q_{n+1} &= (1-\gamma_n) w_n + \gamma_n P_K[w_n - \rho T w_n]. \end{aligned}$$

where $\beta_n, \gamma_n \in [0,1]$ for all $n \geq 0$. This Algorithm is known as a three-step predictor-corrector method. For $\beta_n = 0$, Algorithm 1 reduces to Algorithm 2 [32].

Algorithm 2 Two-step iterative method

For an arbitrarily chosen $q_0 \in H$, compute the sequence $\{q_n\}$ by the iterative scheme:

$$y_n = P_K[q_n - \rho T q_n].$$

$$q_{n+1} = (1-\gamma_n) y_n + \gamma_n P_K[y_n - \rho T y_n].$$

In the context of this work, Algorithm 2 is referred to as the two-step iterative method. When the parameter $\gamma_n = 1$ then, Algorithm 2 simplifies to become Algorithm 3, demonstrating a specific case within the broader algorithmic framework such that: [4]

Algorithm 3 One-step iteration method

Compute the sequence $\{q_n\}$, for an arbitrarily chosen initial point $q_0 \in H$, by the iterative scheme;

$$y_n = P_K[q_n - \rho T q_n].$$

$$q_{n+1} = P_K[y_n - \rho T y_n].$$

or,

$$q_{n+1} = P_K[P_K(q_n - \rho T q_n) - \rho T[P_K(q_n - \rho T q_n)]].$$

Which is called extragradient Algorithm.

Algorithm 4 Self-adaptive Iterative Scheme

Now compute the approximate solution q_{n+1}, for a given $q_0 \in H$ by the iterative schemes. We use the technique of updating the solution to suggest predictor corrector techniques for solving the variational inequalities (1).

$$\begin{aligned} y_n &= P_K[q_n - \rho T q_n]. \\ w_n &= (1-\beta_n) y_n + \beta_n P_K[y_n - \rho T y_n]. \\ q_n &= (1-\gamma_n) w_n + \gamma_n P_K[w_n - \rho T w_n]. \end{aligned}$$

Step 0. Given $\epsilon > 0$, $\rho > 0$, $\gamma \in [1,2)$, $\mu \in [0,1)$ and $q_0 \in H$, set $n = 0$.

Step 1. **Stopping criteria:** Let ρ_n be defined as ρ. To proceed, check if the norm of $R(q_n)$ is less than ϵ.If this condition holds true, terminate the process. Otherwise, seek the smallest non-negative integer m_n, such that, $\rho_n = \rho\mu^{m_n}$, that satisfies,

$$\rho_n \eta_n \langle T q_n - T(q_n - \eta_n R(q_n)),\ R(q_n) \rangle \leq \sigma \|R(q_n)\|^2,\ \sigma \in [0,1].$$

Step 2. Compute

$$d(q_n) = \eta R(q_n) + \rho_n T(q_n - \eta_n R(q_n)),$$

$$\alpha_n = \frac{(\eta_n - \sigma)}{\gamma_n} \frac{\| R(q_n) \|^2}{\| d(q_n) \|^2},$$

Step 3. Get next iteration

$$q_{n+1} = (1 - \gamma_n) q_n + \gamma_n P_K[q_n - \alpha_n d(q_n)], n = 0, 1, 2, 3.......$$

Otherwise, go to the step 1.

In the upcoming section, we delve into the convergence criteria of the above algorithm, which forms the primary driving force behind our findings and outcomes. The analysis of convergence holds significant importance in establishing the existence of a solution under specific conditions. This theorem encapsulates the convergent aspect of the newly established outcomes.

Theorem 1. *If* $q^\alpha \in H$ *is a solution of variational inequality and operator* $T : H \to H$ *is pseudomonotone then*

$$\langle q - q^\alpha,\ d(q) \rangle \geq (\eta - \sigma) \| R(q) \|^2 .$$

Proof of Theorem 1. let $q^\alpha \in H$ is a solution of variational inequality and operator $T : H \to H$ is pseudomonotone operator, then

$$\langle Tq^\alpha,\ q^* - q^\alpha \rangle \geq 0 \quad \forall q^* \in K,$$

this implies that

$$\langle Tq^*,\ q^* - q^\alpha \rangle \geq 0 \quad \forall q^* \in K,$$

since T is pseudomonotone.
Taking $q^* = q - \eta R(q) = x$ in above equation.

$$\langle Tq^*,\ q - \eta R(q) - q^\alpha \rangle \geq 0, \tag{8}$$

$$\langle \rho Tx,\ q - q^\alpha \rangle \geq \eta \langle \rho Tx,\ R(q) \rangle. \tag{9}$$

From Lemma 1 we have

$$\langle q - z, q^* - q \rangle \geq 0. \tag{10}$$

Taking

$$\begin{aligned} z &= q - \rho Tq, \\ q &= (1 - \beta_n) y + \beta_n P_K[y - \rho Ty] = w, \\ q^* &= q^\alpha \end{aligned}$$

in Equation (10) we have,

$$\langle (1 - \beta_n) y + \beta_n P_K[y - \rho Ty] - (q - \rho Tq),\ q^\alpha - (1 - \beta_n) y - \beta_n P_K[y - \rho Ty] \rangle \geq\ 0.$$
$$\langle w - q + \rho Tq,\ q^\alpha - w \rangle \geq\ 0.$$
$$\langle \rho Tq - (q - w),\ q^\alpha - q + (q - w) \rangle \geq 0.$$

Substituting $R(q) = q - w$.

$$\begin{aligned} &\langle \rho Tq - R(q),\ q^\alpha - q + R(q) \rangle \geq 0, \\ &\langle R(q) - \rho Tq,\ q - q^\alpha - R(q) \rangle \geq 0, \\ &\langle R(q) - \rho Tq,\ q - q^\alpha \rangle \geq \langle R(q) - \rho Tq,\ R(q) \rangle, \\ &\langle R(q),\ q - q^\alpha \rangle \geq \langle \rho Tq,\ q - q^\alpha \rangle + \langle R(q) - \rho Tq,\ R(q) \rangle. \end{aligned}$$

Taking
$$\langle \rho Tq,\ q - q^{\alpha} \rangle \geq 0.$$
We have
$$\langle R(q),\ q - q^{\alpha} \rangle \geq \langle R(q) - \rho Tq,\ R(q) \rangle.$$
For $\eta \geq 0$, we have
$$\eta \langle R(q),\ q - q^{\alpha} \rangle \geq \eta \langle R(q) - \rho Tq,\ R(q) \rangle. \tag{11}$$
By adding (9) and (11), we get;
$$\langle \eta R(q),\ q - q^{\alpha} \rangle + \langle \rho Tx,\ q - q^{\alpha} \rangle \geq \eta \langle R(q) - \rho Tq,\ R(q) \rangle + \eta \langle \rho Tx,\ R(q) \rangle,$$
$$\langle \eta R(q) + \rho Tx,\ q - q^{\alpha} \rangle \geq \eta \langle R(q) - \rho Tq + \rho Tx,\ R(q) \rangle,$$
then we have
$$\begin{aligned} \langle q - q^{\alpha},\ d(q) \rangle &\geq \eta \langle R(q),\ R(q) \rangle - \eta\rho \langle R(q),\ Tq - Tx \rangle, \\ &\geq \eta \|R(q)\|^2 - \eta\rho \langle R(q),\ Tq - Tx \rangle. \end{aligned}$$
We know that $x = q - \eta R(q)$, so
$$\rho\eta \langle Tq - T(q - \eta R(q)),\ R(q) \rangle \leq \sigma \parallel R(q) \parallel^2, \quad \sigma \in [0,\ 1]$$
$$\begin{aligned} \langle q - q^{\alpha},\ d(q) \rangle &\geq \eta \|R(q)\|^2 - \sigma \parallel R(q) \parallel^2, \\ &\geq (\eta - \sigma) \|R(q)\|^2. \end{aligned}$$
Which is the required result. □

Theorem 2. *If $q^{\alpha} \in H$ is a solution of variational inequality (1) and q_{n+1} be the approximate solution obtained from Algorithm 4 then*
$$\|q_{n+1} - q^{\alpha}\|^2 \leq \|q_n - q^{\alpha}\|^2 - (\eta_n - \sigma)^2 \frac{\|R(q_n)\|^4}{\|d(q_n)\|^2}.$$
where η_n is the smallest nonnegative integer that satisfied the given inequality in such a way:
$$\rho_n \eta_n \langle Tq_n - T(q_n - \eta_n R(q_n)),\ R(q_n) \rangle \leq \sigma \parallel R(q_n) \parallel^2, \quad \sigma \in [0,\ 1].$$

Proof of Theorem 2. From Algorithm 4, we have
$$q_{n+1} = (1 - \gamma_n) q_n + \gamma_n P_K[q_n - \alpha_n d(q_n)], \tag{12}$$
so,
$$\|q_{n+1} - q^{\alpha}\|^2 = \|(1 - \gamma_n) q_n + \gamma_n P_K[q_n - \alpha_n d(q_n)] - q^{\alpha}\|^2,$$
as P_K is non-expansive, then we have
$$\begin{aligned} \|q_{n+1} - q^{\alpha}\|^2 &\leq \|(1 - \gamma_n) q_n + \gamma_n (q_n - \alpha_n d(q_n)) - q^{\alpha}\|^2, \\ &\leq \|q_n - \gamma_n q_n + \gamma_n q_n - \gamma_n \alpha_n d(q_n) - q^{\alpha}\|^2, \\ &\leq \|q_n - \gamma_n \alpha_n d(q_n) - q^{\alpha}\|^2, \\ &\leq \|q_n - q^{\alpha}\|^2 - 2\gamma_n \alpha_n \langle q_n - q^{\alpha},\ d(q_n) \rangle + \gamma_n^2 \alpha_n^2 \|d(q_n)\|^2. \end{aligned}$$
As we know that from (Theorem 1),
$$\langle q - q^{\alpha},\ d(q) \rangle \geq (\eta - \sigma) \|R(q)\|^2, \quad \forall\, q^{\alpha} \in H.$$

so,

$$\|q_{n+1} - q^{\alpha}\|^2 \leq \|q_n - q^{\alpha}\|^2 - 2\gamma_n\alpha_n(\eta - \sigma)\|R(q_n)\|^2 + \gamma_n^2\alpha^2\|d(q_n)\|^2.$$

Since,

$$\alpha_n = \frac{(\eta_n - \sigma)}{\gamma_n}\frac{\| R(q_n) \|^2}{\| d(q_n) \|^2},$$

$$\begin{aligned}\|q_{n+1} - q^{\alpha}\|^2 &\leq \|q_n - q^{\alpha}\|^2 - 2(\eta - \sigma)^2\frac{\| R(q_n) \|^4}{\| d(q_n) \|^2} + (\eta - \sigma)^2\frac{\| R(q_n) \|^4}{\| d(q_n) \|^2}, \\ &\leq \|q_n - q^{\alpha}\|^2 - (\eta - \sigma)^2\frac{\| R(q_n) \|^4}{\| d(q_n) \|^2}.\end{aligned}$$

This is the required result. □

Theorem 3. *Let T be a nonlinear operator having Lipschitz continuous property with constant $\sigma > 0$, and strongly monotonic property with constant $\delta > 0$. If $0 \leq \gamma_n,\ \beta_n \leq 1$, for all $n \geq 0$ and $\Sigma_{n=0}^{\infty}\gamma_n = 0$ then there exist a constant $\rho > 0$ such that $0 < \rho < \frac{2\delta}{\sigma^2}$. Afterward, the iterative scheme yields an approximate solution q_{n+1}, which ultimately converges to the exact solution q of the variational inequality.*

Proof of Theorem 3. Given that T is strongly monotonic with a constant $\delta > 0$, and Lipschitz continuous with a constant $\sigma > 0$. Now from Algorithm 4 (Updated self-adaptive Iterative algorithm) we can ascertain the existence of a fixed point $q \in H$, satisfying the following conditions:

$$q = (1 - \gamma_n)w + \gamma_n P_K[w - \rho Tw]. \tag{13}$$

$$w = (1 - \beta_n)y + \beta_n P_K[y - \rho Ty]. \tag{14}$$

$$y = P_K[q - \rho Tq]. \tag{15}$$

Since q_{n+1} and q are the solutions of VI (1), it follows from Algorithm 4 that

$$\begin{aligned}\|q_{n+1} - q\| &= \|(1 - \gamma_n)w_n + \gamma_n P_K[w_n - \rho Tw_n] - (1 - \gamma_n)w - \gamma_n P_K[w - \rho Tw]\| \\ &= \|(1 - \gamma_n)(w_n - w) + \gamma_n(P_K[w_n - \rho Tw_n] - P_K[w - \rho Tw])\|.\end{aligned}$$

Using the property of non-expansive of P_K, we obtain the following result:

$$\|q_{n+1} - q\| \leq (1 - \gamma_n)\|w_n - w\| + \gamma_n\|w_n - w - \rho(Tw_n - Tw)\|. \tag{16}$$

Consider

$$\|w_n - w - \rho(Tw_n - Tw)\|^2 = \|(w_n - w\|^2 - 2\rho\langle w_n - w,\ Tw_n - Tw\rangle + \rho^2\|Tw_n - Tw\|^2$$

As T is strongly monotonic with constant $\delta > 0$ and Lipschitz continuous with constant $\sigma > 0$, we have,

$$\|w_n - w - \rho(Tw_n - Tw)\|^2 \leq (1 - 2\rho\delta + \rho^2\sigma^2)\|w_n - w\|^2$$

After back subsitution into Equation (16), we obtained the following result

$$\|q_{n+1} - q\| \leq (1 - \gamma_n)\|w_n - w\| + \gamma_n\theta\|w_n - w\|, \tag{17}$$

$$\leq (1 - \gamma_n + \gamma_n\theta)\|w_n - w\|, \tag{18}$$

where $\theta = \sqrt{1 - 2\rho\delta + \rho^2\sigma^2}$. Now consider

$$\begin{aligned} \|w_n - w\| &= \|(1-\beta_n)y_n + \beta_n P_K[y_n - \rho T y_n] - (1-\beta_n)y - \beta_n P_K[y - \rho T y]\|, \\ &= \|(1-\beta_n)(y_n - y) + \beta_n(P_K[y_n - \rho T y_n] - P_K[y - \rho T y])\|. \end{aligned}$$

As P_K is non-expensive, then we have

$$\|w_n - w\| \leq (1-\beta_n)\|y_n - y\| + \beta_n\|y_n - y - \rho(Ty_n - Ty)\|. \tag{19}$$

Similarly consider

$$\|y_n - y - \rho(Ty_n - Ty)\|^2 = \|(y_n - w\|^2 - 2\rho\langle y_n - y,\ Ty_n - Ty\rangle + \rho^2\|Ty_n - Ty\|^2$$

As T is strongly monotonic and Lipschitz continuous, so we have.

$$\|y_n - w - \rho(Ty_n - Ty)\|^2 \leq (1 - 2\rho\delta + \rho^2\sigma^2)\|y_n - y\|^2$$

After back subsitution into Equation (19), we obtained the following result

$$\begin{aligned} \|w_n - w\| &\leq (1-\beta_n)\|y_n - y\| + \beta_n\theta\|y_n - y\|, \\ &\leq (1 - \beta_n + \beta_n\theta)\|y_n - y\|, \end{aligned} \tag{20}$$

for $0 < \rho < \dfrac{2\delta}{\sigma^2}$, we have $\theta < 1$ which implies:

$$(1 - \beta_n + \beta_n\theta) < 1.$$

Thus,

$$\|w_n - w\| \leq \|y_n - y\|. \tag{21}$$

Similarly from (15) and Algorithm 1, we have

$$\begin{aligned} \|y_n - y\| &= \|P_K[q_n - \rho T q_n] - P_K[q - \rho T q]\|, \\ &\leq \|q_n - q - \rho(Tq_n - Tq)\|, \end{aligned}$$

as P_K is non-expensive and T is strongly monotonic and Lipschitz continuous, so we have,

$$\|y_n - y\| \leq \theta\|q_n - q\|$$

For convergence criteria $0 < \rho < \dfrac{2\delta}{\sigma^2}$, so we have $\theta < 1$ using this fact, then we have

$$\|y_n - y\| \leq \|q_n - q\|. \tag{22}$$

By subsituting Equations (20) and (22) into Equation (18), we get

$$\begin{aligned} \|q_{n+1} - q\| &\leq (1 - \gamma_n + \gamma_n\theta)(1 - \beta_n + \beta_n\theta)\|q_n - q\| \\ \|q_{n+1} - q\| &\leq \Pi_{i,j=0}^{\infty}(1 - \gamma_i(1-\theta))(1 - \beta_j(1-\theta)\|q_n - q\| \end{aligned}$$

Since $\gamma, \beta \in [0,\ 1]$ and $\Sigma_{i=1}^{\infty}\gamma_i, \beta_j < \infty$, so

$$\underset{n\to\infty}{Lim}\ \Pi_{i,j=0}^{\infty}(1 - \gamma_i(1-\theta))(1 - \beta_j(1-\theta) = 0, \tag{23}$$

$$\underset{n\to\infty}{Lim}\ \|q_n - q\| \to 0 \tag{24}$$

$$\underset{n\to\infty}{Lim}\ q_{n+1} = q. \tag{25}$$

The above result shows that the general solution q_{n+1} of Algorithm 4 converges to approximate solution q . Since $\theta < 1$ and from Equation (23), the Problem (1) has a unique solution consequently q_{n+1}, which is the required result. It also follows from (21) and (22) that the sequences w_n and y_n strongly converge to q in H. These results show that under certain conditions, the solution exists and it is unique. This was the main motivation of the results. In the next section, we provide the numerical example for the solution of the problem. This is the implementation of the defined results. □

4. Numerical Example

In this section, numerical results are presented for the proposed three-step predictor-corrector method. We consider the inequality complimentary problem of finding, $q \in K$ such that:

$$\langle\, T(q),\, q^* - q\rangle \geq 0, \quad \forall\, q^* \in K. \tag{26}$$

where, $T(q) = D(q) + Mq + t$, the nonlinear part $D(q) = d^* arctan(q)$ and linear part $M = A^T A + B$ of $T(q)$ respectively. The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 500)$(easy problems) and $(-500, 0)$(hard problems), respectively. In $D(q)$, the nonlinear part of $T(q)$, the components are $D_j(q) = d_j * \arctan(q_j\,)$, here d_j is a random variable in $(0, 1)$. With the starting point $u_0 = (0,\ 0,\ 0, \ldots, 0)^T$, we take, $\mu = \frac{2}{3}$, $\delta = 0.95$, and $\gamma = 1.95$ for the given test. Computation of codes written in matlab begins with $\rho_0 = 0$ and stop as soon as $\|D(q^n)\| \leq 10^{-7}$. The test results for hard problems $(q \in (-500,\ 0\,))$ are reported in Table 1.

Table 1. Numerical Results for $q \in (-500, 0)$.

Size of Matrix	[19]	Self-Adaptive Iterative Scheme
n	**No. Iterations**	**No. Iterations**
ine 100	54	45
150	77	67
200	34	29
300	43	32
500	96	85

We have compared our numerical results with published work by Noor. et al. (2017) [19]. It has been observed that the number of iterations is less in the three-step predictor-corrector self-adaptive method. So, the new Self-adaptive Iterative Scheme is a better and improved version of previous published work.

5. Conclusions

The three-step method, in particular, is quite generic and includes several innovative and well-known approaches for resolving variational inequalities. This work aims to establish a theoretical foundation for these iterations. Through our analysis, we observed that three-step iterations are considerably more practical and manageable for calculations compared to two- and one-step iterations when dealing with nonlinear problems arising in elasticity and mechanics. Furthermore, we have provided proof for the convergence analysis, which constitutes the primary motivation behind this paper. An example is given to illustrate the efficiency of the proposed method. Comparison with other methods shows that the three-step predictor-corrector self-adaptive method perform better.

Author Contributions: Conceptualization, methodology, and analysis, K.S.; funding acquisition, N.M.A.; investigation, S.U.; supervision, S.U.; visualization, N.M.A.; writing—review and editing, K.S.; proofreading and editing, S.U. and K.S; resources, N.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to acknowledge the support of Qassim University for providing the Article Processing Charges (APC) of this publication.

Data Availability Statement: The manuscript included all required data and implementing information.

Acknowledgments: Researchers would like to thank the Deanship of scientific Research, Qassim University for funding publication of this project.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jacques-Louis, L.; Stampacchia, G. Variational inequalities. *Commun. Pure Appl. Math.* **1967**, *20*, 493–519.
2. Hartman, P.; Stampacchia, G. *On Some Non-Linear Elliptic Differential-Functional Equations*; Project Euclid: Durham, NC, USA, 1966; pp. 271–310.
3. Cottle, R.W.; Pang, J.S.; Stone, R.E. *The Linear Complementarity Problems*; Academic Press: New York, NY, USA, 1992.
4. Noor, M.A. Some developments in general variational inequalities. *Appl. Math. Comput.* **2004**, *152*, 199–277.
5. Alpcan, T.; Başar, T. Distributed algorithms for Nash equilibria of flow control games. In *Advances in Dynamic Games: Applications to Economics, Finance, Optimization, and Stochastic Control*; Birkhäuser: Boston, MA, USA, 2005; pp. 473–498.
6. Yin, H.; Shanbhag, U.V.; Mehta, P.G. Nash equilibrium problems with scaled congestion costs and shared constraints. *IEEE Trans. Autom. Control.* **2011**, *56*, 1702–1708.
7. Scutari, G.; Palomar, D.P.; Facchinei, F.; Pang, J.S. Monotone games for cognitive radio systems. In *Distributed Decision Making and Control*; Springer: London, UK, 2011; pp. 83–112.
8. Noor, M.A. New approximation schemes for general variational inequalities. *J. Math. Anal. Appl.* **2000**, *251*, 217–229. [CrossRef]
9. Yao, Y.; Shahzad, N.; Yao, J.C. Convergence of Tseng-type self-adaptive algorithms for variational inequalities and fixed point problems. *Carpathian J. Math.* **2021**, *37*, 541–550. [CrossRef]
10. Daniele, P.; Giannessi, F.; Maugeri, A. *Equilibrium Problems and Variational Models*; Kluwer Academic Publishers: London, UK, 2003.
11. Kinderlehrer, D.; Stampacchia, G. *An Introduction to Variational Inequalities and Their Applications*; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 2000.
12. Somyot, P.; Punpaeng, R. A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.* **2007**, *336*, 455–469.
13. Colao, V.; López, G.; Marino, G.; Martín-Márquez, V. Equilibrium problems in Hadamard manifolds. *J. Math. Anal. Appl.* **2012**, *388*, 61–77. [CrossRef]
14. Antipin, A.S.; Jacimovic, V.; Jacimovic, M. Dynamics and variational inequalities. *Comput. Math. Math. Phy.* **2017**, *57*, 784–801. [CrossRef]
15. Noor, M.A.; Noor, K.I. Equilibrium problems and variational inequalities. *Mathematica* **2005**, *47*, 89–100.
16. Noor, M.A.; Ullah, S.; Noor, K.I.; Al-Said, E. Iterative methods for solving extended general mixed variational inequalities. *Comput. Math. Appl.* **2011**, *62*, 804–813. [CrossRef]
17. Noor, M.A. General variational inequalities. *Appl. Math. Lett.* **1988**, *1*, 119–121. [CrossRef]
18. Baiocchi, C.; Capelo, A. *Variational and Quasi Variationally Inequalities*; J. Wiley and Sons: New York, NY, USA, 1984.
19. Noor, M.A.; Ullah, S. Predictor-corrector self-adaptive methods for variational inequalities. *Transylv. Rev.* **2017**, *25*, 4147–4152.
20. Scarf, H.E.; Hansen, T.B. *The Computation of Economic Equilibria*; Yale University Press: New Haven, CT, USA , 1973.
21. Francisco, F.; Pang, J. (Eds.) *Finite-Dimensional Variational Inequalities and Complementarity Problems*; Springer: New York, NY, USA, 2003.
22. Xu, H.K. Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.* **2004**, *298*, 279–291. [CrossRef]
23. Noor, M.A. Generalized Wiener-Hopf Equations and Onlinear Quasi Variational Inequalities. *Korean J. Comput. Appl. Math.* **1998**, *5*, 73–89. [CrossRef]
24. Liya, L.; Cho, S.Y.; Yao, J. Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications. *J. Nonlinear Var. Anal.* **2021**, *5*, 627–644.
25. Ogbuisi, F.U.; Iyiola, O.S.; Ngnotchouye, J.M.T.; Shumba, T.M.M. On inertial type self-adaptive iterative algorithms for solving pseudomonotone equilibrium problems and fixed point problems. *J. Nonlinear Funct. Anal.* **2021**, *4*, 1–18.
26. Bux, M.; Ullah, S.; Arif, M.S.; Abodayeh, K. A self-Adaptive Technique for Solving Variational Inequalities: A New Approach to the Problem. *J. Funct. Spaces* **2022**, *2022*, 7078707. [CrossRef]
27. Noor, M.A.; Bnouhachem, A.; Ullah, S. Self-adaptive methods for general variational inequalities. *Nonlinear Anal. Theory Methods Appl.* **2009**, *71*, 3728–3738. [CrossRef]

28. Bnouhachem, A.; Noor, M.A.; Al-Shemas, E.H. On self-adaptive method for general mixed variational inequalities. *Math. Probl. Eng.* **2008**, *2008*, 280956. [CrossRef]
29. Ge, Z.; Han, D. Self-adaptive implicit methods for monotone variant variational inequalities. *J. Inequalities Appl.* **2009**, *2009*, 458134. [CrossRef]
30. Noor, M.A.; Noor, K.I.; Treanta, S.; Nonlaopon, K. On three-step iterative schemes associated with general quasi-variational inclusions. *Alex. Eng. J.* **2022**, *61*, 12051–12059. [CrossRef]
31. Mukheimer, A.; Ullah, S.; Bux, M.; Arif, M.S.; Abodayeh, K. New Iterative Scheme Involving Self-Adaptive Method for Solving Mixed Variational Inequalities. *Axioms* **2023**, *12*, 310. [CrossRef]
32. Huang, Z.; Noor, M.A. Strong convergence criteria for nonexpansive and related (r,c)-cocoercive mappings in Hilbert spaces. *Preprint* **2006**.
33. Noor, M.A. General variational inequalities and nonexpansive mappings. *J. Math. Anal. Appl.* **2007**, *331*, 810–822. [CrossRef]
34. Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **2011**, *148*, 318–335. [CrossRef]
35. Goebel, K.; Reich, S. *Uniform Convexity, Hyperbolic Geometry, and Non-Expansive Mappings*; Wiley: Hoboken, NJ, USA, **1984**.