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Exploring Explicit Definite Integral Formulae with Trigonometric and Hyperbolic Functions

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Abstract: Making use of integration by parts and variable replacement methods, we derive some interesting explicit definite integral formulae involving trigonometric or hyperbolic functions, whose results are expressed in terms of Catalan's constant, Dirichlet's beta function, and Riemann's zeta function, as well as π in the denominator.

Keywords: integral; Catalan's constant; Dirichlet's beta function; Riemann's zeta function

MSC: 33B10; 33E20

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1. Introduction and Motivation

For a complex number s, Dirichlet's beta function is defined by the alternating infinite series [1]

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

When setting $s = 2$, it becomes Catalan's constant G, expressed as

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \beta(2),$$

which was introduced by Eugène Charles Catalan in [2] (Equation (4)). There are many other famous representations of G (see [3–5]), for example,

$$G = \int_0^1 \frac{\arctan x}{x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx, \tag{1}$$

$$2\pi G = \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx + \frac{7}{2}\zeta(3). \tag{2}$$

Recently, Stewart [6] and Holland [7] proposed, respectively, the following problems requiring proof:

$$\int_0^{\frac{\pi}{2}} \frac{\sin(4x)}{\ln \tan x} dx = -14\frac{\zeta(3)}{\pi^2} \quad \text{and} \quad \int_0^{\infty} \frac{\tanh^2 x}{x^2} = 14\frac{\zeta(3)}{\pi^2},$$

where $\zeta(s)$ denotes the Riemann zeta function,

$$\zeta(s) = \begin{cases} \sum\limits_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum\limits_{n=1}^{\infty} \frac{1}{(2n-1)^s}, & (\Re(s) > 1); \\ \frac{1}{1-2^{1-s}} \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, & (\Re(s) > 0, s \neq 1). \end{cases}$$

Motivated by these two elegant formulae, we primarily investigate, in this paper, the following integrals involving trigonometric and hyperbolic functions with an integer parameter n:

$$\mathcal{H}_n := \int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^{2n+1}}{\ln \tan x} dx,\ n \in \mathbb{N}_0, \quad \mathcal{T}_n := \int_0^{\frac{\pi}{2}} \frac{(\sin 4x)^{2n+1}}{\ln \tan x} dx,\ n \in \mathbb{N}_0,$$
$$\mathcal{L}_n := \int_0^{\infty} \frac{(\tanh x)^{2n}}{x^2} dx,\ n \in \mathbb{N}, \qquad \mathcal{P}_n := \int_0^{\infty} \frac{(\tanh x)^{2n+1}}{x^3} dx,\ n \in \mathbb{N}.$$

Deriving explicit formulae involving trigonometric and hyperbolic functions is important for accurate computations in combinatorics, physics, computer science, and other fields. This study not only fills the gap in the literature for these types of integrals but also serves as a good inspiration for computing similar definite integrals.

To evaluate these integrals, we first give the following two lemmas (cf. [8] ([1.5.44 and 1.2.2])). And throughout the paper, $m \equiv_j n$ indicates that "m is congruent to n modulo j". $\chi(x)$ denotes the logical function defined by $\chi(true) = 1$ and $\chi(false) = 0$.

Lemma 1. *For each $n \in \mathbb{N}$, we have the representation*

$$\int_0^{\frac{\pi}{2}} \frac{x^n}{\sin x} dx = 2^{n+1} n \sum_{m=1}^{\infty} \frac{I(n,m)}{2m-1}, \tag{3}$$

where

$$I(n,m) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k+m+1} \frac{(n-1)!}{(n-1-2k)!} \cdot \frac{(\pi/4)^{n-1-2k}}{(4m-2)^{2k+1}}$$
$$- (-1)^{\frac{n-2}{2}} \frac{(n-1)!}{(4m-2)^n} \chi(n \equiv_2 0).$$

Proof. By means of the trigonometric function relations

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2 \tan \frac{x}{2}}{\sec^2 \frac{x}{2}},$$

we can rewrite the integral as

$$\int_0^{\frac{\pi}{2}} \frac{x^n}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{x^n}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx = \int_0^{\frac{\pi}{2}} \frac{x^n \sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} dx.$$

Setting $y = \tan \frac{x}{2}$, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{x^n \sec^2 \frac{x}{2}}{2 \tan \frac{x}{2}} dx &= 2^n \int_0^1 \frac{\arctan^n y}{y} dy \\
&= 2^n \ln y \arctan^n y \big|_0^1 - 2^n n \int_0^1 \frac{\ln y \cdot (\arctan y)^{n-1}}{1+y^2} dy \\
&= 0 - 2^n n \int_0^1 \frac{\ln y \cdot (\arctan y)^{n-1}}{1+y^2} dy.
\end{aligned}$$

Letting $y = \tan u$, we obtain

$$\int_0^1 \frac{\ln y \cdot (\arctan y)^{n-1}}{1+y^2} dy = \int_0^{\frac{\pi}{4}} u^{n-1} \ln \tan u du.$$

By means of the Fourier series [9] (Equation (1.7.1))

$$\ln\sin u = -\ln 2 - \sum_{m=1}^{\infty}\frac{\cos(2mu)}{m} \quad \text{and} \quad \ln\cos u = -\ln 2 - \sum_{m=1}^{\infty}(-1)^m\frac{\cos(2mu)}{m},$$

we obtain

$$\ln\tan u = \ln\sin u - \ln\cos u = -2\sum_{m=1}^{\infty}\frac{\cos(4m-2)u}{2m-1}.$$

Thus, we can evaluate the integral

$$\int_0^{\frac{\pi}{4}} u^{n-1}\ln\tan u du = -2\sum_{m=1}^{\infty}\frac{1}{2m-1}\int_0^{\frac{\pi}{4}} u^{n-1}\cos(4m-2)udu.$$

Keeping in mind the formula

$$\begin{aligned}\int u^n\cos(\lambda u)du = \sin(\lambda u)&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{n!}{(n-2k)!}\cdot\frac{x^{n-2k}}{\lambda^{2k+1}}\\ &+\cos(\lambda u)\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\frac{n!}{(n-2k-1)!}\cdot\frac{x^{n-2k-1}}{\lambda^{2k+2}}+C,\end{aligned}$$

we can evaluate the integral

$$\begin{aligned}\int_0^{\frac{\pi}{4}} u^{n-1}\cos(4m-2)udu = (-1)^{m+1}&\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\frac{(n-1)!}{(n-1-2k)!}\cdot\frac{(\pi/4)^{n-1-2k}}{(4m-2)^{2k+1}}\\ &-(-1)^{\frac{n-2}{2}}\frac{(n-1)!}{(4m-2)^n}\chi(n\equiv_2 0),\end{aligned}$$

which completes the proof. □

Letting $n = 3, 4, 5, 6$ in the above lemma, we can obtain the following corollary.

Corollary 1.

$$\int_0^{\frac{\pi}{2}}\frac{x^3}{\sin x}dx = \frac{3\pi^2}{2}G - 12\beta(4); \tag{4}$$

$$\int_0^{\frac{\pi}{2}}\frac{x^4}{\sin x}dx = \pi^3 G - 24\pi\beta(4) + \frac{93}{2}\zeta(5); \tag{5}$$

$$\int_0^{\frac{\pi}{2}}\frac{x^5}{\sin x}dx = \frac{5\pi^4}{8}G - 30\pi^2\beta(4) + 240\beta(6); \tag{6}$$

$$\int_0^{\frac{\pi}{2}}\frac{x^6}{\sin x}dx = \frac{3\pi^5}{8}G - 30\pi^3\beta(4) + 720\pi\beta(6) - \frac{5715}{4}\zeta(7). \tag{7}$$

The Beta function $B(p,q)$ is defined by the following integral [9] (Equation (1.1.12)):

$$B(p,q) := \int_0^1 z^{p-1}(1-z)^{q-1}dz, \quad (\Re(p) > 0; \Re(q) > 0). \tag{8}$$

Replacing z by $z/(1+z)$ in (8), we have an infinite integral representation of $B(p,q)$:

$$B(p,q) = \int_0^{\infty}\frac{z^{p-1}}{(1+z)^{p+q}}dz. \tag{9}$$

The Gamma function $\Gamma(s)$ is defined by [9] (Equation (1.1.18))

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} dt \ (\Re(s) > 0),$$

which is closely related to the Beta function. In fact, they satisfy the relation [9] (Equation (1.1.13))

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{with} \quad (\Re(p) > 0; \Re(q) > 0).$$

The following useful relationship between the Gamma and Circular functions [9] (Equation (1.2.1)) is well known as Euler's reflection formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{10}$$

Generally, for an integer $n \in \mathbb{N}_0$, by combining Equation (10) and $\Gamma(x+1) = x\Gamma(x)$, we can immediately obtain the following equations:

$$\Gamma(n+x)\Gamma(n-x) = \begin{cases} \frac{\pi x}{\sin(\pi x)}, & n = 1; \\ \frac{\pi x}{\sin \pi x}[(n-1)!]^2 \prod_{k=1}^{n-1}(1 - \frac{x^2}{k^2}), & 1 < n \in \mathbb{N}. \end{cases} \tag{11}$$

$$\Gamma(n + \frac{1}{2} + x)\Gamma(n + \frac{1}{2} - x) = \begin{cases} \frac{\pi}{\cos(\pi x)}, & n = 0; \\ \frac{\Gamma^2(n+\frac{1}{2})}{\cos(\pi x)} \prod_{k=1}^{n}[1 - \frac{4x^2}{(2k-1)^2}], & n \in \mathbb{N}. \end{cases} \tag{12}$$

Lemma 2. *For $\tau > 0$ and $v > 0$, the following identity holds:*

$$\int_0^{+\infty} \frac{z^{p-1}}{(1+\tau z^v)^{p+q}} = \frac{1}{v}(\frac{1}{\tau})^{\frac{p}{v}} B(\frac{p}{v}, p+q-\frac{p}{v}). \tag{13}$$

Proof. Setting $y = \tau z^v$, we can rewrite the integral as follows:

$$\int_0^{+\infty} \frac{z^{p-1}}{(1+\tau z^v)^{p+q}} = \frac{1}{v}(\frac{1}{\tau})^{\frac{p}{v}} \int_0^{\infty} \frac{y^{\frac{p}{v}} - 1}{(1+y)^{p+q}}.$$

Then, the proof follows by means of (9). □

The rest of this paper is organized as follows. In Section 2, we calculate $\mathcal{H}_n$ and $\mathcal{T}_n$. Then, in Section 3, we compute $\mathcal{L}_n$ and $\mathcal{P}_n$. Finally, in Section 4, the paper ends with two integral equations.

2. Integrations $\mathcal{H}_n$ and $\mathcal{T}_n$

In this section, we use the method of substitution and other techniques to evaluate the integrals $\mathcal{H}_n$ and $\mathcal{T}_n$.

As a warm-up, we first calculate two particular cases, $n = 0$ and $n = 1$:

$$\mathcal{H}_0 = \int_0^{\frac{\pi}{2}} \frac{\cos 2x}{\ln \tan x} dx \quad \text{and} \quad \mathcal{H}_1 = \int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^3}{\ln \tan x} dx.$$

Theorem 3.

$$\mathcal{H}_0 = \int_0^{\frac{\pi}{2}} \frac{\cos 2x}{\ln \tan x} dx = -\frac{4}{\pi} G. \tag{14}$$

Proof. Making the replacement $\ln\tan x \to y$, we have

$$\cos(2x) = -\tanh y \quad \text{and} \quad dx = \frac{1}{2\cosh y}dy,$$

which leads us to

$$\begin{aligned}\mathcal{H}_0 &= -\int_0^{\infty} \frac{\sinh y}{y\cosh^2 y}dy = -\int_0^{\infty} \frac{1}{\cosh^2 y}\Big(\int_0^1 \cosh(uy)du\Big)dy \\ &= -\int_0^1\int_0^{\infty} \frac{\cosh(uy)}{\cosh^2 y}dydu.\end{aligned}$$

By setting $z = e^t$ and using (13), the inner integral can be evaluated as

$$\int_0^{\infty} \frac{\cosh(uy)}{\cosh^2 y}dy = \int_0^{\infty} \frac{z^{u+1}+z^{-u+1}}{(1+z^2)^2}dz = \Gamma(1+\frac{u}{2})\Gamma(1-\frac{u}{2}). \tag{15}$$

Therefore, by making use of (1) and (11), we have

$$\mathcal{H}_0 = -\int_0^1 \Gamma(1+\frac{u}{2})\Gamma(1-\frac{u}{2})du = -\frac{2}{\pi}\int_0^1 \frac{u\pi/2}{\sin(u\pi/2)}d(u\pi/2) = -\frac{4}{\pi}G.$$

□

Theorem 4.

$$\mathcal{H}_1 = \int_0^{\frac{\pi}{2}} \frac{\cos^3 2x}{\ln\tan x}dx = \frac{16}{\pi^3}\beta(4) - \frac{10}{3\pi}G. \tag{16}$$

Proof. Setting $t = \ln\tan x$, we can manipulate the integral as follows:

$$\begin{aligned}\mathcal{H}_1 &= -\frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sinh^3 t}{t\cosh^4 t}dt = -\frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{\cosh^4 t}\Big(\int_0^1 3\cosh ut\sinh^2 utdu\Big)dt \\ &= -\frac{3}{2}\int_0^1\int_{-\infty}^{+\infty} \frac{\cosh ut\sinh^2 ut}{\cosh^4 t}dtdu.\end{aligned}$$

By replacing e^t by z and then using (11) and (13), we can compute the inner integral:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\cosh ut\sinh^2 ut}{\cosh^4 t}dt &= 2\int_0^{+\infty} \frac{z^{3u+3}+z^{-3u+3}-z^{u+3}-z^{-u+3}}{(1+z^2)^4}dz \\ &= \frac{u(1-9u^2/4)\pi}{2\sin(3u\pi/2)} - \frac{u(1-u^2/4)\pi}{6\sin(u\pi/2)},\end{aligned}$$

which leads us to

$$\mathcal{H}_1 = \frac{1}{2}\int_0^1 \frac{u(1-u^2/4)\pi}{2\sin(u\pi/2)}du - \frac{1}{2}\int_0^1 \frac{3u(1-9u^2/4)\pi}{2\sin(3u\pi/2)}du. \tag{17}$$

By means of Formulae (1) and (4), we can evaluate the first integral:

$$\int_0^1 \frac{u(1-u^2/4)\pi}{2\sin(u\pi/2)}du = \frac{G}{\pi} + \frac{24}{\pi^3}\beta(4). \tag{18}$$

For the second integral, by making the replacement $\frac{3}{2}u\pi \to t$, we obtain

$$\int_0^1 \frac{3u(1-9u^2/4)\pi}{2\sin(3u\pi/2)}du$$
$$= \frac{2}{3\pi}\Big(\int_0^{\frac{\pi}{2}} \frac{t-t^3/\pi^2}{\sin t}dt + \int_{\frac{\pi}{2}}^{\pi} \frac{t-t^3/\pi^2}{\sin t}dt + \int_{\pi}^{\frac{3\pi}{2}} \frac{t-t^3/\pi^2}{\sin t}dt\Big).$$

Then, replacing t by $-t+\pi$, the second integral in "( )" leads to

$$\int_{\frac{\pi}{2}}^{\pi} \frac{t-t^3/\pi^2}{\sin t}dt = \int_0^{\frac{\pi}{2}} \frac{\pi - t - (\pi-t)^3/\pi^2}{\sin t}dt.$$

And replacing t by $t+\pi$, the third integral in "( )" gives

$$\int_{\pi}^{\frac{3\pi}{2}} \frac{t-t^3/\pi^2}{\sin t}dt = -\int_0^{\frac{\pi}{2}} \frac{t+\pi-(t+\pi)^3/\pi^2}{\sin t}dt.$$

Introducing them into the above equation and making use of (1) and (4), we have

$$\int_0^1 \frac{3u(1-9u^2/4)\pi}{2\sin(3u\pi/2)}du = \frac{2}{3\pi}\int_0^{\frac{\pi}{2}} \frac{5t+t^3/\pi^2}{\sin t}dt = \frac{23}{3\pi}G - \frac{8}{\pi^3}\beta(4).$$

According to the above results, we can obtain the formula stated in the theorem. □

Generally, for an integer $n \in \mathbb{N}_0$, we have the following theorem.

Theorem 5. *For each $n \in \mathbb{N}_0$, we have the representation*

$$\mathcal{H}_n := \int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^{2n+1}}{\ln\tan x}dx = -\frac{(n!)^2}{2(2n)!}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^1 h(n,x)dx, \tag{19}$$

where

$$h(n,x) = \frac{\frac{2k-2n+1}{2}\pi x}{\sin(\frac{2k-2n+1}{2}\pi x)}\prod_{i=1}^{n}[1-(\frac{2k-2n+1}{2i}x)^2]$$
$$+ \frac{\frac{2k-2n-1}{2}\pi x}{\sin(\frac{2k-2n-1}{2}\pi x)}\prod_{i=1}^{n}[1-(\frac{2k-2n-1}{2i}x)^2].$$

Proof. Setting $t = \ln\tan x$, we can compute the integral

$$\int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^{2n+1}}{\ln\tan x}dx = -\frac{1}{2}\int_{-\infty}^{+\infty} \frac{(\sinh t)^{2n+1}}{t(\cosh t)^{2n+2}}dt$$
$$= -\frac{2n+1}{2}\int_{-\infty}^{+\infty} \frac{1}{(\cosh t)^{2n+2}}\Big(\int_0^1 (\sinh ut)^{2n}\cosh ut du\Big)dt$$
$$= -\frac{2n+1}{2}\int_0^1\int_{-\infty}^{+\infty} \frac{(\sinh ut)^{2n}\cosh ut}{(\cosh t)^{2n+2}}dtdu.$$

Letting $z = e^t$, the inner integral can be evaluated:

$$\int_{-\infty}^{+\infty} \frac{(\sinh ut)^{2n} \cosh ut}{(\cosh t)^{2n+2}} dt = 2 \int_0^{+\infty} \frac{z^{2n+1}(z^u - z^{-u})^{2n}(z^u + z^{-u})}{(1+z^2)^{2n+2}} dz$$
$$= 2 \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \int_0^{+\infty} \frac{z^{2n+1+(2k-2n+1)u} + z^{2n+1+(2k-2n-1)u}}{(1+z^2)^{2n+2}} dz$$
$$= \frac{(n!)^2}{(2n+1)!} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} h(n,u),$$

where

$$h(n,u) = \frac{\frac{2k-2n+1}{2}\pi u}{\sin(\frac{2k-2n+1}{2}\pi u)} \prod_{i=1}^{n} [1 - (\frac{2k-2n+1}{2i}u)^2]$$
$$+ \frac{\frac{2k-2n-1}{2}\pi u}{\sin(\frac{2k-2n-1}{2}\pi u)} \prod_{i=1}^{n} [1 - (\frac{2k-2n-1}{2i}u)^2],$$

which completes the proof. □

For a given parameter $n \in \mathbb{N}_0$, by making use of Lemma 1 and Theorem 5, we can evaluate the definite integral

$$\mathcal{H}_n := \int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^{2n+1}}{\ln \tan x} dx, \ n \in \mathbb{N}_0$$

whose results can be expressed in terms of π, Catalan's constant G, and Dirichlet's beta function $\beta(s)$. For instance, setting $n = 2$, we have the following formula:

Corollary 2.

$$\mathcal{H}_2 = \int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^5}{\ln \tan x} dx = \frac{24}{\pi^3}\beta(4) - \frac{64}{\pi^5}\beta(6) - \frac{89}{30\pi}G. \tag{20}$$

Proof. By setting $n = 2$ in Theorem 5, we have

$$\mathcal{H}_2 = -\frac{1}{12}(J_1 - J_2 + J_3),$$

where

$$J_1 = \int_0^1 \frac{5\pi x(1 - \frac{25}{4}x^2)(1 - \frac{25}{16}x^2)}{\sin(\frac{5}{2}\pi x)} dx,$$
$$J_2 = \int_0^1 \frac{9\pi x(1 - \frac{9}{4}x^2)(1 - \frac{9}{16}x^2)}{\sin(\frac{3}{2}\pi x)} dx,$$
$$J_3 = \int_0^1 \frac{2\pi x(1 - \frac{x^2}{4})(1 - \frac{x^2}{16})}{\sin(\frac{\pi}{2}x)} dx.$$

By means of variable replacement and (1), (4), and (6), we can ascertain that

$$J_1 = \frac{4}{5\pi} \int_0^{\frac{\pi}{2}} \frac{16x + \frac{55}{4\pi^2}x^3 + \frac{1}{4\pi^4}x^5}{\sin x} dx = \frac{1689}{40\pi}G - \frac{138}{\pi^3}\beta(4) + \frac{48}{\pi^5}\beta(6),$$
$$J_2 = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \frac{4x - \frac{15}{4\pi^2}x^3 - \frac{1}{4\pi^4}x^5}{\sin x} dx = \frac{71}{8\pi}G + \frac{210}{\pi^3}\beta(4) - \frac{240}{\pi^5}\beta(6),$$
$$J_3 = \frac{8}{\pi} \int_0^{\frac{\pi}{2}} \frac{x - \frac{5}{4\pi^2}x^3 + \frac{1}{4\pi^4}x^5}{\sin x} dx = \frac{9}{4\pi}G + \frac{60}{\pi^3}\beta(4) + \frac{480}{\pi^5}\beta(6),$$

which completes the proof. □

Similarly, we can establish the following formula for the integral $\mathcal{T}_n$.

Theorem 6. *For each $n \in \mathbb{N}_0$, we have the representation*

$$\begin{aligned}\mathcal{T}_n &:= \int_0^{\frac{\pi}{2}} \frac{(\sin 4x)^{2n+1}}{\ln \tan x} dx \\ &= -2^{4n+1}\Gamma^2(2n+\frac{3}{2})\frac{2n+1}{(4n+2)!}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^1 f(n,x)dx, \end{aligned} \tag{21}$$

where

$$f(n,x) = \frac{\prod_{i=1}^{2n+1}[1-(\frac{2k-2n+1}{2i-1}x)^2]}{\cos(\frac{2k-2n+1}{2}x\pi)} + \frac{\prod_{i=1}^{2n+1}[1-(\frac{2k-2n-1}{2i-1}x)^2]}{\cos(\frac{2k-2n-1}{2}x\pi)}.$$

Proof. Letting $y = \ln \tan x$, we have

$$dx = \frac{1}{2\cosh y}dy \quad \text{and} \quad \sin 4x = -2\frac{\tanh y}{\cosh y},$$

which yields

$$\begin{aligned}\mathcal{T}_n &:= \int_0^{\frac{\pi}{2}} \frac{(\sin 4x)^{2n+1}}{\ln \tan x} dx = -4^n \int_{-\infty}^{+\infty} \frac{(\sinh y)^{2n+1}}{y(\cosh y)^{4n+3}}dy \\ &= -4^n(2n+1)\int_0^1\int_{-\infty}^{+\infty}\frac{(\sinh uy)^{2n}\cosh uy}{(\cosh y)^{4n+3}}dydu.\end{aligned}$$

Setting $z = e^y$ and using (12), the inner integral can be rewritten as

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{(\sinh uy)^{2n}\cosh uy}{(\cosh y)^{4n+3}}dy \\ &= 4^{n+1}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\int_0^{+\infty}\frac{z^{4n+2+(2k-2n+1)u}+z^{4n+2+(2k-2n-1)u}}{(1+z^2)^{4n+3}}dz \\ &= 4^{n+1}\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}f(n,u),\end{aligned}$$

where

$$f(n,u) = \frac{\prod_{i=1}^{2n+1}[1-(\frac{2k-2n+1}{2i-1}u)^2]}{\cos(\frac{2k-2n+1}{2}u\pi)} + \frac{\prod_{i=1}^{2n+1}[1-(\frac{2k-2n-1}{2i-1}u)^2]}{\cos(\frac{2k-2n-1}{2}u\pi)},$$

which concludes the proof. □

Corollary 3.

$$\mathcal{T}_0 = \int_0^{\frac{\pi}{2}} \frac{\sin(4x)}{\ln \tan x}dx = -\frac{14}{\pi^2}\zeta(3). \tag{22}$$

Proof. Letting $n = 0$ in Theorem 6, we have

$$\mathcal{T}_0 = \int_0^{\frac{\pi}{2}} \frac{\sin 4x}{\ln \tan x}dx = -\frac{\pi}{2}\int_0^1 \frac{1-u^2}{\cos(\frac{u}{2}\pi)}du. \tag{23}$$

Letting $u = 1 - \frac{2}{\pi}v$ and making use of (1) and (2), we obtain

$$\mathcal{T}_0 = \frac{4}{\pi^2}\int_0^{\frac{\pi}{2}} \frac{v^2}{\sin v}dv - \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{v}{\sin v}dv = -\frac{14}{\pi^2}\zeta(3).$$

□

The above formula resolves the problem proposed by Stewart [6].

Corollary 4.

$$\mathcal{T}_1 = \int_0^{\frac{\pi}{2}} \frac{(\sin 4x)^3}{\ln\tan x}dx = \frac{1016}{\pi^6}\zeta(7) - \frac{248}{3\pi^4}\zeta(5) - \frac{392}{45\pi^2}\zeta(3). \tag{24}$$

Proof. Letting $n = 1$ in Theorem 6, we have

$$\mathcal{T}_1 = -\frac{15}{16}\pi\int_0^1 \left(\frac{(1-x^2)(1-9x^2)(1-\frac{9}{25}x^2)}{\cos(\frac{3}{2}\pi x)} - \frac{(1-x^2)(1-\frac{x^2}{9})(1-\frac{x^2}{25})}{\cos(\frac{\pi}{2}x)}\right)dx.$$

By means of variable replacement, we can ascertain that

$$\begin{aligned}\mathcal{T}_1 = &-\frac{32}{15\pi}\int_0^{\frac{\pi}{2}} \frac{z}{\sin z}dz + \frac{112}{45\pi^2}\int_0^{\frac{\pi}{2}} \frac{z^2}{\sin z}dz - \frac{16}{9\pi^4}\int_0^{\frac{\pi}{2}} \frac{z^4}{\sin z}dz \\ &+ \frac{32}{15\pi^5}\int_0^{\frac{\pi}{2}} \frac{z^5}{\sin z}dz - \frac{32}{45\pi^6}\int_0^{\frac{\pi}{2}} \frac{z^6}{\sin z}dz.\end{aligned}$$

Then, the proof follows by making use of (1), (2), (5), (6), and (7). □

3. Integrations $\mathcal{L}_n$ and $\mathcal{P}_n$

In this section, by making use of the method of substitution and integrating by parts, as well as other techniques, we will compute the integrals $\mathcal{L}_n$ and $\mathcal{P}_n$.

First of all, we compute two simple cases, $n = 1$ and $n = 2$:

$$\mathcal{L}_1 = \int_0^{+\infty} \frac{\tanh^2 x}{x^2}dx \quad \text{and} \quad \mathcal{L}_2 = \int_0^{+\infty} \frac{\tanh^4 x}{x^2}dx.$$

Theorem 7.

$$\mathcal{L}_1 = \int_0^{+\infty} \frac{\tanh^2 x}{x^2}dx = \frac{14}{\pi^2}\zeta(3). \tag{25}$$

Proof. For the second integral, integrating by parts, we can rewrite it as

$$\begin{aligned}\int_0^{+\infty} \frac{\tanh^2 x}{x^2}dx &= -\int_0^{+\infty} \tanh^2 x d\frac{1}{x} \\ &= -\frac{1}{x}\cdot\tanh^2 x\Big|_0^{+\infty} + 2\int_0^{+\infty} \frac{\tanh x\cdot\operatorname{sech}^2 x}{x}dx.\end{aligned}$$

Obviously, it is not difficult to obtain

$$\frac{1}{x}\cdot\tanh^2 x\Big|_0^{+\infty} = \lim_{x\to+\infty}\frac{\tanh^2 x}{x} - \lim_{x\to 0}\frac{\tanh^2 x}{x} = 0 - 0 = 0,$$

where the second limit can be evaluated by means of L'Hospital's rule. Then, by making the replacement $x \to \ln\tan y$, we have

$$\tanh x\cdot\operatorname{sech}^2 x = -\cos 2y\cdot\sin^2 2y \quad \text{and} \quad dx = d\ln\tan y = \frac{2}{\sin 2y}dy.$$

Therefore, we can rewrite the integral as

$$\int_0^{+\infty} \frac{\tanh^2 x}{x^2} dx = -2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 4y}{\ln \tan y} dy.$$

In addition, replacing y by $\frac{\pi}{2} - y$ yields

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 4y}{\ln \tan y} dy = \int_0^{\frac{\pi}{4}} \frac{\sin 4y}{\ln \tan y} dy,$$

which leads us to

$$\int_0^{+\infty} \frac{\tanh^2 x}{x^2} dx = -\int_0^{\frac{\pi}{2}} \frac{\sin 4x}{\ln \tan x} dx.$$

Then, the proof follows by using Corollary 3 and resolves the problem proposed recently by Holland [7]. □

Theorem 8.

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2(2x)\sin(4x)}{\ln \tan x} dx = \frac{62}{\pi^4}\zeta(5) - \frac{28}{3\pi^2}\zeta(3). \tag{26}$$

Proof. Replacing $\ln \tan x$ by t, we have

$$dy = \frac{1}{2\cosh t} dt, \quad \cos 2y = -\tanh t \quad \text{and} \quad \sin 4y = -2\frac{\tanh t}{\cosh t},$$

by means of which we obtain

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos^2(2x)\sin(4x)}{\ln \tan x} dx &= -\int_{-\infty}^{+\infty} \frac{\sinh^3 t}{t\cosh^5 t} dt \\ &= -\int_{-\infty}^{+\infty} \frac{1}{\cosh^5 t} \int_0^1 3\sin^2(ut)\cos(ut) du dt \\ &= -3\int_0^1 \int_{-\infty}^{+\infty} \frac{\sin^2(ut)\cos(ut)}{\cosh^5 t} dt du. \end{aligned}$$

By letting $z = e^t$ and using (13), the inner integral can be rewritten as

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{\sin^2(ut)\cos(ut)}{\cosh^5 t} dt &= 4\int_0^{+\infty} \frac{z^{3u+4} + z^{-3u+4} - z^{u+4} - z^{-u+4}}{(1+z^2)^5} dz \\ &= \frac{(9/4 - 9u^2/4)(1/4 - 9u^2/4)\pi}{6\cos(3u\pi/2)} \\ &\quad - \frac{(9/4 - u^2/4)(1/4 - u^2/4)\pi}{6\cos(u\pi/2)}. \end{aligned}$$

Therefore, the integral can be rewritten as

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\cos^2(2x)\sin(4x)}{\ln \tan x} dx = &\frac{1}{2}\int_0^1 \frac{(\frac{9}{4} - \frac{u^2}{4})(\frac{1}{4} - \frac{u^2}{4})\pi}{\cos(u\pi/2)} du \\ &- \frac{1}{2}\int_0^1 \frac{(\frac{9}{4} - \frac{9u^2}{4})(\frac{1}{4} - \frac{9u^2}{4})\pi}{\cos(3u\pi/2)} du \end{aligned}$$

Letting $u = 1 - \frac{2t}{\pi}$ and then using (1), (2), (4), and (5), the first integral on the right-hand side can be rewritten as

$$\begin{aligned}\frac{1}{2}\int_0^1 \frac{(\frac{9}{4}-\frac{u^2}{4})(\frac{1}{4}-\frac{u^2}{4})\pi}{\cos(u\pi/2)}du &= \frac{1}{\pi^4}\int_0^{\frac{\pi}{2}} \frac{2\pi^3 t - \pi^2 t^2 - 2\pi t^3 + t^4}{\sin t}dt \\ &= \frac{7}{2\pi^2}\zeta(3) + \frac{93}{2\pi^4}\zeta(5).\end{aligned}$$

Setting $v = \frac{3\pi}{2}u$, the second integral on the right-hand side can be rewritten as

$$\begin{aligned}&\frac{1}{2}\int_0^1 \frac{(\frac{9}{4}-\frac{9u^2}{4})(\frac{1}{4}-\frac{9u^2}{4})\pi}{\cos(3u\pi/2)}du \\ =&\frac{1}{3}\int_0^{\frac{\pi}{2}} f(v)dv + \frac{1}{3}\int_{\frac{\pi}{2}}^{\pi} f(v)dv + \frac{1}{3}\int_{\pi}^{\frac{3\pi}{2}} f(v)dv,\end{aligned}$$

where

$$f(v) = \frac{\frac{9}{16} - \frac{5}{2\pi^2}v^2 + \frac{1}{\pi^4}v^4}{\cos v}.$$

Letting $v = \frac{\pi}{2} - z$ and then using (1), (2), (4), and (5), we have

$$\begin{aligned}\frac{1}{3}\int_0^{\frac{\pi}{2}} f(v)dv &= \frac{1}{3\pi^4}\int_0^{\frac{\pi}{2}} \frac{z^4 - 2\pi z^3 - \pi^2 z^2 + 2\pi^3 z}{\sin z}dz \\ &= \frac{31}{2\pi^4}\zeta(5) + \frac{7}{6\pi^2}\zeta(3).\end{aligned}$$

Similarly, by setting $v = \frac{\pi}{2} + z$ and $v = \frac{3\pi}{2} - z$, we can obtain

$$\begin{aligned}\frac{1}{3}\int_{\frac{\pi}{2}}^{\pi} f(v)dv &= -\frac{1}{3\pi^4}\int_0^{\frac{\pi}{2}} \frac{z^4 + 2\pi z^3 - \pi^2 z^2 - 2\pi^3 z}{\sin z}dz \\ &= \frac{2}{3\pi}G + \frac{16}{\pi^3}\beta(4) - \frac{31}{2\pi^4}\zeta(5) - \frac{7}{6\pi^2}\zeta(3)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{3}\int_{\pi}^{\frac{3\pi}{2}} f(v)dv &= -\frac{1}{3\pi^4}\int_0^{\frac{\pi}{2}} \frac{z^4 - 6\pi z^3 + 11\pi^2 z^2 - 11\pi^3 z}{\sin z}dz \\ &= \frac{77}{6\pi^2}\zeta(3) - \frac{31}{2\pi^4}\zeta(5) - \frac{16}{\pi^3}\beta(4) - \frac{2}{3\pi}G\end{aligned}$$

respectively. Based on the above results, we can confirm the first formula. □

Theorem 9.

$$\mathcal{L}_2 = \int_0^{+\infty} \frac{\tanh^4 x}{x^2}dx = \frac{56}{3\pi^2}\zeta(3) - \frac{124}{\pi^4}\zeta(5). \tag{27}$$

Proof. Integrating by parts, we can rewrite it as

$$\begin{aligned}\int_0^{+\infty} \frac{\tanh^4 x}{x^2}dx &= -\frac{\tanh^4 x}{x}\Big|_0^{+\infty} + 4\int_0^{+\infty} \frac{\tanh^3 x\operatorname{sech}^2 x}{x}dx \\ &= 4\int_0^{+\infty} \frac{\tanh^3 x\operatorname{sech}^2 x}{x}dx.\end{aligned}$$

Replacing x by $\ln\tan y$, we have

$$\int_0^{+\infty} \frac{\tanh^3 x\operatorname{sech}^2 x}{x}dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2(2y)\sin(4y)}{\ln\tan y}dy$$
$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2(2y)\sin(4y)}{\ln\tan y}dy,$$

which completes the proof by using Theorem 8. □

Generally, for an integer $n \in \mathbb{N}$, we have the following formula.

Theorem 10. *For each $n \in \mathbb{N}$, we have the representation*

$$\int_0^{+\infty} \frac{(\tanh x)^{2n}}{x^2}dx = -n\int_0^{\frac{\pi}{2}} \frac{\cos(2x)^{2n-2}\sin(4x)}{\ln\tan x}dx$$
$$= \frac{\Gamma^2(n+\frac{1}{2})}{(2n-2)!}\sum_{k=0}^{2n-2}(-1)^k\binom{2n-2}{k}\int_0^1 g(n,x)dx, \tag{28}$$

where

$$g(n,x) = \frac{\prod_{i=1}^n[1-(\frac{2k-2n+3}{2i-1}x)^2]}{\cos(\frac{2k-2n+3}{2}\pi x)} + \frac{\prod_{i=1}^n[1-(\frac{2k-2n+1}{2i-1}x)^2]}{\cos(\frac{2k-2n+1}{2}\pi x)}.$$

Proof. Integrating by parts, we can rewrite the integral as

$$\int_0^{+\infty} \frac{(\tanh x)^{2n}}{x^2}dx = -\int_0^{+\infty}(\tanh x)^{2n}d\frac{1}{x}$$
$$= -\frac{1}{x}\cdot(\tanh x)^{2n}\Big|_0^{+\infty} + 2n\int_0^{+\infty}\frac{(\tanh x)^{2n-1}\operatorname{sech}^2 x}{x}dx.$$

Obviously, it is not difficult to obtain

$$\frac{1}{x}\cdot\tanh^2 x\Big|_0^{+\infty} = \lim_{x\to+\infty}\frac{(\tanh x)^{2n}}{x} - \lim_{x\to 0}\frac{(\tanh x)^{2n}}{x} = 0-0=0,$$

where the second limit can be evaluated by means of L'Hospital's rule.

Letting $x = \ln\tan y$, we can evaluate

$$\tanh x = -\cos(2y), \quad \operatorname{sech} x = \sin(2y) \quad \text{and} \quad dx = \frac{2}{\sin(2y)}dy,$$

which yields

$$\int_0^{+\infty}\frac{(\tanh x)^{2n-1}\operatorname{sech}^2 x}{x}dx = -\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos(2y)^{2n-2}\sin(4y)}{\ln\tan y}dy.$$

By setting $t = \ln\tan y$, the above integral becomes

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos(2y)^{2n-2}\sin(4y)}{\ln\tan y}dy = -\int_0^{+\infty}\frac{(\sinh t)^{2n-1}}{t(\cosh t)^{2n+1}}dt$$
$$= -\int_0^{+\infty}\frac{1}{(\cosh t)^{2n+1}}\Big(\int_0^1(2n-1)(\sinh ut)^{2n-2}\cosh ut\,du\Big)dt$$
$$= -(2n-1)\int_0^1\int_0^{+\infty}\frac{(\sinh ut)^{2n-2}\cosh ut}{(\cosh t)^{2n+1}}dtdu.$$

Then, letting $z = e^t$, the inner integral can be evaluated as

$$\int_0^{+\infty} \frac{(\sinh ut)^{2n-2}\cosh ut}{(\cosh t)^{2n+1}} dt = 2\int_0^{+\infty} \frac{z^{2n}(z^u - z^{-u})^{2n-2}(z^u + z^{-u})}{(1+z^2)^{2n+1}} dz$$
$$= 2\sum_{k=0}^{2n-2} (-1)^k \binom{2n-2}{k} \int_0^{+\infty} \frac{z^{2n+(2k-2n+3)u} + z^{2n+(2k-2n+1)u}}{(1+z^2)^{2n+1}} dz$$
$$= \frac{\Gamma^2(n+\frac{1}{2})}{(2n)!} \sum_{k=0}^{2n-2} (-1)^k \binom{2n-2}{k} \left\{ \frac{\prod_{i=1}^{n}[1-(\frac{2k-2n+3}{2i-1}u)^2]}{\cos(\frac{2k-2n+3}{2}\pi u)} + \frac{\prod_{i=1}^{n}[1-(\frac{2k-2n+1}{2i-1}u)^2]}{\cos(\frac{2k-2n+1}{2}\pi u)} \right\}.$$

Therefore, we obtain

$$\int_0^{+\infty} \frac{(\tanh x)^{2n}}{x^2} dx = \frac{\Gamma^2(n+\frac{1}{2})}{(2n-2)!} \sum_{k=0}^{2n-2} (-1)^k \binom{2n-2}{k} \int_0^1 g(n,u) du,$$

where

$$g(n,u) = \frac{\prod_{i=1}^{n}[1-(\frac{2k-2n+3}{2i-1}u)^2]}{\cos(\frac{2k-2n+3}{2}\pi u)} + \frac{\prod_{i=1}^{n}[1-(\frac{2k-2n+1}{2i-1}u)^2]}{\cos(\frac{2k-2n+1}{2}\pi u)},$$

which completes the proof. □

For a given integer $n \in \mathbb{N}$, we can evaluate $\mathcal{L}_n$ based on the above Theorem 10 and Lemma 1.

Corollary 5.

$$\int_0^{\frac{\pi}{2}} \frac{\sin(8x)}{\ln \tan x} dx = \frac{248}{\pi^4}\zeta(5) - \frac{28}{3\pi^2}\zeta(3). \tag{29}$$

Proof. By means of the relation

$$\sin(8x) = 2\sin(4x)\cos(4x) = 4(\cos 2x)^2 \sin(4x) - 2\sin(4x)$$

and Corollary 3 and the formula of Theorem 8, we can obtain the desired formula. □

Theorem 11.

$$\mathcal{P}_1 = \int_0^{\infty} \frac{\tanh^3 x}{x^3} dx = \frac{186}{\pi^4}\zeta(5) - \frac{7}{\pi^2}\zeta(3). \tag{30}$$

Proof. Integrating by parts two times, we can manipulate the integral:

$$\mathcal{P}_1 = -\frac{1}{2}\int_0^{\infty} \tanh^3 x d\frac{1}{x^2} = \frac{3}{2}\int_0^{\infty} \frac{\tanh^2 x \cdot \mathrm{sech}^2 x}{x^2} dx$$
$$= -\frac{3}{2}\int_0^{\infty} \tanh^2 x \cdot \mathrm{sech}^2 x d\frac{1}{x}$$
$$= \frac{3}{2}\int_{-\infty}^{+\infty} \frac{\tanh x \cdot \mathrm{sech}^4 x - \tanh^3 x \cdot \mathrm{sech}^2 x}{x} dx.$$

Letting $x = \ln \tan y$, we have

$$\tanh x = -\cos(2y), \quad \mathrm{sech} x = \sin(2y) \quad \text{and} \quad dx = \frac{2}{\sin(2y)} dy,$$

which leads us to

$$\begin{aligned}\mathcal{P}_1 &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2(2y)\sin(4y) - \sin^2(2y)\sin(4y)}{\ln\tan y}dy \\ &= \frac{3}{2}\int_0^{\frac{\pi}{2}} \frac{2\cos^2(2y)\sin(4y) - \sin(4y)}{\ln\tan y}dy.\end{aligned}$$

Then, the proof follows by means of the first identities of Corollary 3 and Theorem 8. □

Generally, for any integer $n \in \mathbb{N}$, we have the following theorem.

Theorem 12 ($n \in \mathbb{N}$)**.**

$$\mathcal{P}_n = \int_0^{\infty} \frac{\tanh^{2n+1} x}{x^3}dx = \frac{2n+1}{2}(\mathcal{L}_n - \mathcal{L}_{n+1}). \tag{31}$$

Proof. Similar to the proof of Theorem 11, integrating by parts two times, we obtain

$$\mathcal{P}_n = \frac{2n+1}{2}\int_{-\infty}^{+\infty} \frac{n\tanh^{2n-1} x \cdot \operatorname{sech}^4 x - \tanh^{2n+1} x \cdot \operatorname{sech}^2 x}{x}dx.$$

By making the replacement $x \to \ln\tan y$, we have

$$\mathcal{P}_n = \frac{2n+1}{2}\int_0^{\frac{\pi}{2}} \frac{(n+1)\cos^{2n}(2y)\sin(4y) - n\cos^{2n-2}(2y)\sin(4y)}{\ln\tan y}dy.$$

By making use of Theorem 10, we can complete the proof. □

4. Conclusions

In this paper, by means of the method of substitution and integration by parts, integrals containing trigonometric and hyperbolic functions are transformed into integrals of rational fractions or into known integrals, and by employing Beta and Gamma functions, we then derive their explicit results. The methods have been shown to be efficient in dealing with the types of definite integrals mentioned in previous sections, and it is possible to derive more similar formulae. For instance, we can obtain the following identity:

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2x)(\sin 4x)^2}{\ln\tan x}dx = \frac{256}{\pi^5}\beta(6) - \frac{32}{\pi^3}\beta(4) - \frac{22}{15\pi}G.$$

Finally, we point out that, for two integers $m, n \in \mathbb{N}_0$ and $m \equiv_2 n$, we have

$$\int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^m(\sin 4x)^n}{\ln\tan x}dx = 0.$$

In fact, setting $t = \ln\tan x$, we have

$$\int_0^{\frac{\pi}{2}} \frac{(\cos 2x)^m(\sin 4x)^n}{\ln\tan x}dx = 2^{n-1}(-1)^{m+n}\int_{-\infty}^{+\infty} \frac{(\tanh t)^{m+n}}{t(\cosh t)^{m+1}}dt.$$

The integrand $\frac{(\tanh t)^{m+n}}{t(\cosh t)^{m+1}}$ is an odd function, so the value of the integral is 0.

In this paper, we focus on the computation of four types of definite integrals whose results can be expressed in terms of special functions, such as Dirichlet's beta function, Riemann's zeta function, and Catalan's constant. Our results not only provide methods for the computation of similar definite integrals but also are important for the study of Dirichlet's beta function, Riemann's zeta function, and Catalan's constant.

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