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Oettli-Théra Theorem and Ekeland Variational Principle in Fuzzy *b*-Metric Spaces

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Abstract

The purpose of this paper is to establish the Oettli–Théra theorem in the setting of KM-type fuzzy *b*-metric spaces. To achieve this, we first prove a lemma that removes the constraints on the space coefficients, which significantly simplifies the proof process. Based on the Oettli–Théra theorem, we further demonstrate the equivalence of Ekeland variational principle, Caristi's fixed point theorem, and Takahashi's nonconvex minimization theorem in fuzzy *b*-metric spaces. Notably, the results obtained in this paper are consistent with the conditions of the corresponding theorems in classical fuzzy metric spaces, thereby extending the existing theories to the broader framework of fuzzy *b*-metric spaces.

Keywords: fuzzy *b*-metric space; Oettli-Théra theorem; Ekeland variational principle; Caristi's fixed point theorem; Takahashi's nonconvex minimization theorem

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1. Introduction

In 1993, Oettli and Théra [1] established a new theorem that links equilibrium problems with variational principles, which is now widely recognized as the Oettli–Théra theorem. This result not only demonstrated equivalence to the famous Ekeland variational principle (EVP) [2] but also generalized a new nonconvex minimization principle proposed by Takahashi (see [3]). This theorem is a pivotal result in nonlinear analysis, particularly in the study of vector equilibrium problems and their generalizations. It provides conditions for the existence of solutions to systems of inequalities and has profound connections to variational principles, optimization, and fixed point theory. Many scholars have studied this theorem and its equivalences, as shown in references [4–8]. Currently, research on the Oettli–Théra theorem and EVP primarily focuses on functional forms and domain spaces. The explored functional forms include vector-valued, set-valued, and interval-valued functions. Additionally, the considered spaces encompass uniform spaces, *b*-metric spaces, fuzzy metric spaces and others (see [5–19]).

On the other hand, Kramosil and Michalek [19] first proposed the concept of fuzzy metric space (FMS) in 1975, which later became known as KM-type FMS. In KM-type FMS, a fuzzy set (typically represented by a value between 0 and 1 indicating the degree of membership) is used to replace the classical distance function. This membership value can be interpreted as "the degree to which two points exhibit a certain distance (or similarity)". Subsequently, fuzzy metric spaces have been extensively studied in the literature [16,17,20,21] and its references. The significance of KM-type fuzzy metric spaces lies in their role as one of the important cornerstones of fuzzy analysis. They successfully generalize the classical concept of metric into fuzzy set theory, providing a powerful mathematical tool and analytical framework for dealing with uncertainty and fuzziness in the real world. In 1994, George and Veeramani [22] improved the concept of KM-type FMS and defined the Hausdorff topology in this FMS, later known as GV-FMS. Afterwards, many classic results were discussed in such spaces, such as [16,23]. It is worth noting that in 2016, Abbasi et al. [16] studied the Caristi's fixed point theorem (CFPT) in complete GV space and provided the corresponding variational principle. After that, Wu et al. [17] also discussed CFPT in fuzzy quasi-metric spaces. Additionally, Zhu et al. [24], Xiao [25] and Qiu [15] extended fixed point theorems and EVP to fuzzy metric spaces. However, the fuzzy metric spaces in these papers are fundamentally different from the KM-type. Furthermore, Czerwik [26] introduced the concept of b-metric spaces in 1993, which provides a broader type of spatial framework than metric spaces. Many scholars have conducted extensive research in this space and provided various types of fixed point theorems and numerous examples; see [18,26,27].

Based on the concepts proposed in references [19,22,26], Sedghi and Shobe [28] combined the b-metric space with the KM-type FMS, introducing a new type of space called the KM-type fuzzy b-metric space (in short, Fb-MS) in 2012. The KM-type Fb-MS, building upon fuzzy metrics, further relaxes the fuzzy form of the triangle inequality by introducing a constant s, thereby effectively combining fuzzy sets with b-metrics. This provides a more flexible and broader mathematical framework for describing and analyzing real-world systems that are both fuzzy and do not fully satisfy strict metric axioms, while also expanding the application boundaries of fuzzy mathematical theory. Many scholars discussed the properties of such spaces and established some fixed point theorems in them, as shown in [29–32].

To the best of our knowledge, the Oettli–Théra theorem and its related topics have not yet been thoroughly investigated within the framework of KM-type fuzzy b-metric spaces [28,30]. The aim of this paper is then to present the versions of the Oettli–Théra theorem, EVP, Caristi-Kirk's fixed point theorem (CKFPT), and Takahashi's nonconvex minimization theorem (in short, TMT) in fuzzy b-metric spaces, as well as the equivalence chain of these principles. Before presenting these theorems, we establish a key lemma that removes the influence of the coefficients in the triangle inequality of fuzzy b-metric spaces. This influence is eliminated for the first time by revealing the essential local properties of fuzzy function. Furthermore, we provide specific examples to illustrate the feasibility and effectiveness of the Oettli–Théra theorem. It is worth noting that our results generalize classical theorems from fuzzy metric spaces to a broader range of fuzzy b-metric spaces, while maintaining consistency in their conditions.

The remaining paper is organized as follows: In Section 2, we introduce some basic definitions and properties of Fb-MS. Besides, we provide some specific examples to illustrate these properties. In Section 3, we present the Oettli–Théra theorem using the newly established lemmas. Moreover, we provide an example to demonstrate the feasibility of the theorem. In Section 4, as applications of the Oettli–Théra theorem, we establish EVP, CKFPT, and TMT in fuzzy b-metric spaces. Furthermore, we demonstrate the equivalence between these theorems and the Oettli–Théra theorem. Notably, the above results we obtained are improvements to the relevant conclusions in [1–3,8,16].

2. Preliminaries

This section first reviews the fundamental concepts of t-norm and KM-type Fb-MS and then discusses their basic topological properties, including convergence, completeness, and continuity. Throughout the paper, we denote the set of all non-negative integers by $\mathbb{N}$ and the set of all positive integers by $\mathbb{N}^+$.

Definition 1 ([16,19,21]). *A binary operation* $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(1) $\diamond$ *is associative and commutative,*
(2) $\diamond$ *is continuous,*
(3) $u \diamond 1 = u$ *for all* $u \in [0,1]$,
(4) $u \diamond v \leq p \diamond q$ *whenever* $u \leq p$ *and* $v \leq q$, *for each* $u, v, p, q \in [0,1]$.

From [16,17,19], we can see some paradigmatic examples of continuous t-norms as follows:

(1) $u \wedge v = \min\{u, v\}$ (minimum t-norm);
(2) $u \cdot v = uv$ (product t-norm);
(3) $u \diamond_L v = \max\{u + v - 1, 0\}$ (the Lukasiewicz t-norm);
(4) $u \diamond_{SW(\lambda)} v = \max\left\{\frac{u+v-1+\lambda uv}{1+\lambda}, 0\right\}$ ($\lambda \in (-1, +\infty)$, the Sugeno-Weber t-norm).

Archimedean condition: The t-norm $\diamond$ is called Archimedean if for any pair $u, v \in (0,1)$, there is $n \in \mathbb{N}$ such that

$$u^n = \underbrace{u \diamond u \diamond \cdots \diamond u}_{n} < v.$$

This condition can be simplified to: if $u, v \in (0,1]$ and $u \diamond v \geq u$, then $v = 1$. It is easy to see that $\cdot$, $\diamond_L$ and $\diamond_{SW(\lambda)}$ are Archimedean; however, $\wedge$ is not Archimedean.

Definition 2 ([28,30,31]). *Let* $\mathcal{F}$ *be a nonempty set and* $s \geq 1$ *be a real number. A fuzzy set* $\mathcal{L}$ *in* $\mathcal{F} \times \mathcal{F} \times (0, +\infty) \to (0,1]$ *is a fuzzy b-metric on* $\mathcal{F}$ *if for all* $u, v, w \in \mathcal{F}$, *and* $\alpha, \beta > 0$ *the following conditions hold:*

(1) $\mathcal{L}(u, v, \alpha) > 0$,
(2) $\mathcal{L}(u, v, \alpha) = 1$ *if and only if* $u = v$,
(3) $\mathcal{L}(u, v, \alpha) = \mathcal{L}(v, u, \alpha)$,
(4) $\mathcal{L}(u, v, \cdot) : (0, \infty) \to [0,1]$ *is continuous,*
(5) $\mathcal{L}(u, v, \alpha + \beta) \geq \mathcal{L}(u, w, \frac{\alpha}{s}) \diamond \mathcal{L}\left(w, v, \frac{\beta}{s}\right)$.

Then $(\mathcal{F}, \mathcal{L}, \diamond)$ *is called to be a fuzzy b-metric space.*

Remark 1. *If* $\mathcal{L}(u, v, \alpha + \beta) \geq \mathcal{L}(u, w, \alpha) \diamond \mathcal{L}(w, v, \beta)$ *is used instead of (5), then* $(\mathcal{F}, \mathcal{L}, \diamond)$ *is a FMS. It is important to note that the class of fuzzy b-metric spaces is indeed broader than that of fuzzy metric spaces, as a fuzzy b-metric reduces to a fuzzy metric when the parameter* $s = 1$.

Specifically, we construct examples that show a fuzzy b-metric on $\mathcal{F}$ (with $s \geq 1$) does not necessarily reduce to a standard fuzzy metric on $\mathcal{F}$, as shown below.

Example 1 ([28]). *Let* $\mathcal{L}(u, v, \alpha) = e^{\frac{-|u-v|^p}{\alpha}}$ *and* $u \diamond v = uv$, *where* $p \geq 1$ *is a real number. Then* $\mathcal{L}$ *is a fuzzy b-metric with* $s = 2^{p-1}$. *But for any* $p > 1$, *it is easy to see that* $\mathcal{L}$ *is not a fuzzy metric.*

Example 2. *Let* $\mathcal{F} = [0,2]$ *and*

$$\mathcal{L}(u, v, \alpha) = e^{-|u-v| e^{-\frac{\alpha}{|u-v|^2+1}}},$$

for all $u, v \in \mathcal{F}$ *and* $\alpha > 0$. *We show that* $\mathcal{L}$ *is a fuzzy b-metric with* $s = 5$. *However,* $\mathcal{L}(u, v, \alpha)$ *is not a fuzzy metric.*

Obviously, conditions (1)–(4) in Definition 2 are satisfied. Next, we prove the triangle inequality in fuzzy b-metric spaces. For all $u, v, w \in \mathcal{F}$, we will show that

$$e^{-|u-v|e^{-\frac{\alpha+\beta}{|u-v|^2+1}}} \geq e^{-|u-w|e^{-\frac{\alpha}{5[|u-w|^2+1]}}} \diamond e^{-|w-v|e^{-\frac{\beta}{5[|w-v|^2+1]}}}.$$

Here, we take $u \diamond v = uv$. Then, we only need to prove

$$|u-v|e^{-\frac{\alpha+\beta}{|u-v|^2+1}} \leq |u-w|e^{-\frac{\alpha}{5[|u-w|^2+1]}} + |w-v|e^{-\frac{\beta}{5[|w-v|^2+1]}}.$$

In fact, for all $u, v, w \in \mathcal{F}$, we have $|u-v|^2+1 \leq 5\big[|u-w|^2+1\big]$. Thus, we can infer that $\frac{\alpha+\beta}{|u-v|^2+1} \geq \frac{\alpha}{5[|u-w|^2+1]}$. Similarly, we can infer that

$$\begin{aligned} |u-v|e^{-\frac{\alpha+\beta}{|u-v|^2+1}} &\leq |u-w|e^{-\frac{\alpha+\beta}{|u-v|^2+1}} + |w-v|e^{-\frac{\alpha+\beta}{|u-v|^2+1}} \\ &\leq |u-w|e^{-\frac{\alpha}{5[|u-w|^2+1]}} + |w-v|e^{-\frac{\beta}{5[|w-v|^2+1]}}. \end{aligned}$$

Hence, for each $u, v, w \in \mathcal{F}$ and $\alpha, \beta > 0$, we obtain that

$$\begin{aligned} \mathcal{L}(u, v, \alpha+\beta) &= e^{-|u-v|e^{-\frac{\alpha+\beta}{|u-v|^2+1}}} \\ &\geq e^{-|u-w|e^{-\frac{\alpha}{5[|u-w|^2+1]}}} \cdot e^{-|w-v|e^{-\frac{\beta}{5[|w-v|^2+1]}}} \\ &= \mathcal{L}\left(u, w, \frac{\alpha}{5}\right) \diamond \mathcal{L}\left(w, v, \frac{\beta}{5}\right). \end{aligned}$$

Thus condition (5) of Definition 2 holds, and $\mathcal{L}(u, v, \alpha)$ is a fuzzy b-metric.

Especially, let $u = 0, v = 2, w = 1$ and $\alpha = \beta = 1$. Then, we know that $\mathcal{L}(u, v, \alpha+\beta) = e^{-2e^{-\frac{\alpha+\beta}{5}}} = e^{-2e^{-\frac{2}{5}}}$, $\mathcal{L}(u, w, \alpha) = e^{-e^{-\frac{1}{2}}}$ and $\mathcal{L}(w, v, \beta) = e^{-e^{-\frac{1}{2}}}$. Since $2e^{-\frac{2}{5}} > 2e^{-\frac{1}{2}}$, we have

$$\mathcal{L}(u, v, \alpha+\beta) = e^{-2e^{-\frac{2}{5}}} < e^{-2e^{-\frac{1}{2}}} = e^{-(e^{-\frac{1}{2}}+e^{-\frac{1}{2}})} = \mathcal{L}(u, w, \alpha) \cdot \mathcal{L}(w, v, \beta).$$

Therefore, $\mathcal{L}(u, v, \alpha)$ does not satisfy the triangle inequality of fuzzy metric spaces.

Theorem 1 (Refer to [28,30,31])**.** *Suppose that $(\mathcal{F}, \mathcal{L}, \diamond)$ is a Fb-MS. A sequence $\{u_n\}$ in $\mathcal{F}$ converges to u if and only if $\lim_{n\to\infty} \mathcal{L}(u_n, u, \alpha) = 1$, for all $\alpha > 0$. And u is unique.*

Definition 3 (Refer to [28,30])**.**

(1) A sequence $\{u_n\}$ in a Fb-MS $(\mathcal{F}, \mathcal{L}, \diamond)$ is a Cauchy sequence if for each $\varepsilon \in (0,1)$ and each $\alpha > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{L}(u_m, u_n, \alpha) > 1 - \varepsilon$ for all $m, n > n_0$.

(2) We say that a Fb-MS $(\mathcal{F}, \mathcal{L}, \diamond)$ is complete if any Cauchy sequence in $\mathcal{F}$ is convergent.

It worth noting that if $(\mathcal{F}, \mathcal{L}, \diamond)$ is a FMS, then $\mathcal{L}$ is a continuous function on $\mathcal{F} \times \mathcal{F} \times (0, +\infty)$ (See [28]). On the other hand, a fuzzy b-metric can be discontinuous, as shown in the following example.

Example 3. *For all $u, v \in [0,2]$ and $\alpha > 0$, define*

$$\mathcal{L}(u, v, \alpha) = \begin{cases} \frac{\alpha}{\alpha+|u-v|}, & uv \neq 0, \\ \frac{\alpha}{\alpha+5|u-v|}, & uv = 0. \end{cases}$$

Then, from [30], we can easily know that $\mathcal{L}(u,v,\alpha)$ is a fuzzy b-metric. Next, we will show that the fuzzy b-metric is not continuous about u and v. Let $u_n = \frac{1}{n}(n \in \mathbb{N}^+)$; thus, for any $n \in \mathbb{N}^+$, we have $u_n \neq 0$. Since for all $\alpha > 0$,

$$\lim_{n\to\infty} \mathcal{L}(u_n, 0, \alpha) = \lim_{n\to\infty} \mathcal{L}\left(\frac{1}{n}, 0, \alpha\right) = \lim_{n\to\infty} \frac{\alpha}{\alpha + |\frac{1}{n} - 0|} = 1,$$

we obtain that u_n converges to $u = 0$. Let $v = 1$, for any $\alpha > 0$, we have

$$\lim_{n\to\infty} \mathcal{L}(u_n, 1, \alpha) = \lim_{n\to\infty} \frac{\alpha}{\alpha + |\frac{1}{n} - 1|} = \frac{\alpha}{\alpha+1} > \frac{\alpha}{\alpha+5} = \lim_{n\to\infty} \mathcal{L}(0, 1, \alpha).$$

Therefore, $\mathcal{L}(u,v,\alpha)$ is not continuous.

Definition 4 (Refer to [28,30])**.** *A function $l : \mathbb{R} \to \mathbb{R}$ is called to be s-nondecreasing if $l(\beta) \geq l(\alpha)$ for all $\beta > s\alpha$.*

Lemma 1 (Refer to[28])**.** *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a Fb-MS. Then $\mathcal{L}$ is s-nondecreasing with respect to α, for all $u, v \in \mathcal{F}$. Hence*

$$\mathcal{L}(u, v, s^n\alpha) \geq \mathcal{L}(u, v, \alpha), \ \forall n \in \mathbb{N}.$$

Definition 5. *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a Fb-MS. A mapping ϑ is called upper semicontinuous (lower semicontinuous) if and only if $\limsup_{n\to\infty} \vartheta(u_n) \leq \vartheta(u)$ $(\liminf_{n\to\infty} \vartheta(u_n) \geq \vartheta(u))$ for any sequence $\{u_n\}$ which converges to $u \in \mathcal{F}$.*

3. Oettli–Théra Theorem in KM-Type Fuzzy *b*-Metric Spaces

In this section, we establish the Oettli–Théra theorem within the framework of complete KM-type fuzzy *b*-metric spaces. Before that, we present several key lemmas used to prove our theorems as below.

Lemma 2. *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a Fb-MS with $s \geq 1$, and $u, v \in \mathcal{F}$ are given. Then we have $\lim_{\alpha\to 0^+} \mathcal{L}(u, v, \alpha) = \inf_{\alpha>0} \mathcal{L}(u, v, \alpha)$.*

Proof. First, we put $\alpha_n = (\frac{1}{s})^n = \frac{1}{s}, \frac{1}{s^2}, \cdots (n = 1, 2, \cdots)$. From Lemma 1 we can conclude that

$$\mathcal{L}(u, v, \alpha_{n+1}) \leq \mathcal{L}(u, v, s \cdot \alpha_{n+1}) = \mathcal{L}(u, v, \alpha_n).$$

Obviously, $\{\mathcal{L}(u, v, \alpha_n)\}_{n\in\mathbb{N}^+}$ is a decreasing sequence. Note that $\mathcal{L}(u, v, \alpha)$ has a lower bound, hence, $\lim_{n\to\infty} \mathcal{L}(u, v, \alpha_n)$ exists. For brevity, we denote $\lim_{n\to\infty} \mathcal{L}(u, v, \alpha_n)$ by $\mathcal{L}_{inf}(u, v)$.

Next, we will prove $\mathcal{L}_{inf}(u, v) := \lim_{n\to\infty} \mathcal{L}(u, v, \alpha_n) = \inf_{\alpha>0} \mathcal{L}(u, v, \alpha)$. If $\alpha \geq 1$, obviously, we have $\mathcal{L}(u, v, \alpha) \geq \mathcal{L}_{inf}(u, v)$. If $0 < \alpha < 1$, then there exists $n \in \mathbb{N}$, such that $\alpha \in \left(\frac{1}{s^n}, \frac{1}{s^{n-1}}\right]$. Since $\alpha > \frac{1}{s^n} = s \cdot \frac{1}{s^{n+1}} = s \cdot \alpha_{n+1}$, we get

$$\mathcal{L}(u, v, \alpha) \geq \mathcal{L}\left(u, v, \frac{1}{s^{n+1}}\right) = \mathcal{L}(u, v, \alpha_{n+1}).$$

Hence, for all $\alpha > 0$, $\mathcal{L}(u, v, \alpha) \geq \mathcal{L}_{inf}(u, v)$. By the definition of $\mathcal{L}_{inf}(u, v)$, it can be inferred that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$, $|\mathcal{L}(u, v, \alpha_n) - \mathcal{L}_{inf}(u, v)| < \varepsilon$, and so, $\mathcal{L}(u, v, \alpha_n) < \mathcal{L}_{inf}(u, v) + \varepsilon$. Then from the definition of infimum, we can obtain that $\mathcal{L}_{inf}(u, v) = \inf_{\alpha>0} \mathcal{L}(u, v, \alpha)$.

Finally, we demonstrate $\lim_{\alpha\to 0^+}\mathcal{L}(u,v,\alpha)=\mathcal{L}_{inf}(u,v)$. For any $\varepsilon>0$, as

$$\lim_{n\to\infty}\mathcal{L}(u,v,\alpha_n)=\mathcal{L}_{inf}(u,v),$$

there exists $N'\in\mathbb{N}$, such that $\mathcal{L}(u,v,\alpha_{N'})<\mathcal{L}_{inf}(u,v)+\varepsilon$. Take $\delta=\frac{\alpha_{N'}}{s}$, then for any $\alpha\in(0,\delta)$, we have $\alpha_{N'}>s\alpha$, and

$$\mathcal{L}_{inf}(u,v)\leq\mathcal{L}(u,v,\alpha)\leq\mathcal{L}(u,v,\alpha_{N'})<\mathcal{L}_{inf}(u,v)+\varepsilon.$$

It implies that $\left|\mathcal{L}(u,v,\alpha)-\mathcal{L}_{inf}(u,v)\right|<\varepsilon$. Hence,

$$\lim_{\alpha\to 0^+}\mathcal{L}(u,v,\alpha)=\mathcal{L}_{inf}(u,v)=\lim_{n\to\infty}\mathcal{L}(u,v,\alpha_n)=\inf_{\alpha>0}\mathcal{L}(u,v,\alpha).$$

□

In the sequel, we use $\mathcal{L}_{inf}(u,v)=\lim_{\alpha\to 0^+}\mathcal{L}(u,v,\alpha)=\inf_{\alpha>0}\mathcal{L}(u,v,\alpha)$ to prove that $\mathcal{L}_{inf}(u,v)$ satisfies the triangle inequality in a Fb-MS.

Lemma 3. *Let $(\mathcal{F},\mathcal{L},\diamond)$ be a Fb-MS. For every $u,v,w\in\mathcal{F}$,*

$$\mathcal{L}_{inf}(u,v)\geq\mathcal{L}_{inf}(u,w)\diamond\mathcal{L}_{inf}(w,v).$$

Proof. For all $u,v,w\in\mathcal{F}$ and $\alpha>0$,

$$\mathcal{L}(u,v,2\alpha)\geq\mathcal{L}\Big(u,w,\frac{\alpha}{s}\Big)\diamond\mathcal{L}\Big(w,v,\frac{\alpha}{s}\Big).$$

Letting $\alpha\to 0^+$, we have

$$\begin{aligned}\mathcal{L}_{inf}(u,v)=\lim_{\alpha\to 0^+}\mathcal{L}(u,v,2\alpha)&\geq\lim_{\alpha\to 0^+}\Big[\mathcal{L}\Big(u,w,\frac{\alpha}{s}\Big)\diamond\mathcal{L}\Big(w,v,\frac{\alpha}{s}\Big)\Big]\\&=\lim_{\alpha\to 0^+}\mathcal{L}\Big(u,w,\frac{\alpha}{s}\Big)\diamond\lim_{\alpha\to 0^+}\mathcal{L}\Big(w,v,\frac{\alpha}{s}\Big)\\&=\mathcal{L}_{inf}(u,w)\diamond\mathcal{L}_{inf}(w,v).\end{aligned}$$

□

Lemma 4. *Suppose that $(\mathcal{F},\mathcal{L},\diamond)$ is a Fb-MS and $\vartheta:\mathcal{F}\to[0,1]$ is a mapping. For any $u\in\mathcal{F}$, define the set*

$$S(u):=\{v\in\mathcal{F}|\vartheta(v)\diamond\mathcal{L}(u,v,\alpha)\geq\vartheta(u),\ \forall\alpha>0\}.$$

Then for every $v\in S(u)$, $S(v)\subset S(u)$.

Proof. In fact, by the definition of $\mathcal{L}_{inf}$, we notice that

$$S(u):=\{v\in\mathcal{F}|\vartheta(v)\diamond\mathcal{L}(u,v,\alpha)\geq\vartheta(u),\ \forall\alpha>0\}$$

$$\Leftrightarrow S'(u):=\{v\in\mathcal{F}|\vartheta(v)\diamond\mathcal{L}_{inf}(u,v)\geq\vartheta(u)\}.$$

Since $v\in S(u)$, we have

$$\vartheta(v)\diamond\mathcal{L}_{inf}(u,v)\geq\vartheta(u).$$

For any $w\in S(v)$,

$$\vartheta(w)\diamond\mathcal{L}_{inf}(v,w)\geq\vartheta(v).$$

Hence, for every $\alpha > 0$, we can deduce that

$$\begin{aligned}\vartheta(w) \diamond \mathcal{L}_{inf}(u,w) &\geq \vartheta(w) \diamond \mathcal{L}_{inf}(u,v) \diamond \mathcal{L}_{inf}(v,w) \\ &\geq \vartheta(v) \diamond \mathcal{L}_{inf}(u,v) \\ &\geq \vartheta(u).\end{aligned}$$

Therefore, we can conclude that $w \in S(u)$, that is, $S(v) \subset S(u)$. □

In the following text, for all $u \in \mathcal{F}$ and $\alpha > 0$, we define the set

$$S(u) := \{v \in \mathcal{F} | \vartheta(v) \diamond \mathcal{L}(u,v,\alpha) \geq \vartheta(u), \forall \alpha > 0\}.$$

We also have $S(u) = \{v \in \mathcal{F} | \vartheta(v) \diamond \mathcal{L}_{inf}(u,v) \geq \vartheta(u)\}$. Next, using the above lemmas, we present the Oettli–Théra theorem in the setting of KM-type fuzzy b-metric spaces.

Theorem 2 (Oettli–Théra Theorem)**.** *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and let $\vartheta : \mathcal{F} \to [0,1]$ be a non-trivial and upper semicontinuous mapping. Suppose that $\diamond$ is a continuous and Archimedean t-norm. Consider $u_0 \in \mathcal{F}$ such that $\vartheta(u_0) \neq 0$. Assuming that $D \subset \mathcal{F}$ satisfies the following property:*

for every $u \in S(u_0) \setminus D$, there exists $u' \neq u$ such that $\vartheta(u') \diamond \mathcal{L}(u,u',\alpha) \geq \vartheta(u)$ for all $\alpha > 0$.

Then $S(u_0) \cap D \neq \emptyset$.

Proof. First, by $u_0 \in S(u_0)$, we have $S(u_0) \neq \emptyset$. Suppose that $S(u_0) \cap D = \emptyset$, then for every $u \in S(u_0)$, there exists $u' \neq u$ such that $\vartheta(u') \diamond \mathcal{L}(u,u',\alpha) \geq \vartheta(u)$ for all $\alpha > 0$. Now, choose $u_1 \in S(u_0)$ such that

$$\vartheta(u_1) \geq \sup_{u \in S(u_0)} \vartheta(u) - \frac{1}{2}.$$

As $u_1 \in S(u_0)$, by Lemma 4 we have $S(u_1) \subset S(u_0)$. Similarly, we take $u_{n+1} \in S(u_n)$ such that

$$\vartheta(u_{n+1}) \geq \sup_{u \in S(u_n)} \vartheta(u) - \frac{1}{2^n}.$$

By the property of $\diamond$ and $u_{n+1} \in S(u_n)$, we have

$$\vartheta(u_{n+1}) \geq \vartheta(u_{n+1}) \diamond \mathcal{L}_{inf}(u_{n+1},u_n) \geq \vartheta(u_n), \ \forall n \in \mathbb{N}.$$

Hence, we know that $\{\vartheta(u_n)\}_{n \in \mathbb{N}}$ is a increasing sequence. As

$$\sup_{u \in S(u_n)} \vartheta(u) - \frac{1}{2^n} \leq \vartheta(u_{n+1}) \leq \sup_{u \in S(u_n)} \vartheta(u),$$

letting $n \to \infty$, we can deduce that

$$\lim_{n \to \infty} \vartheta(u_n) = p = \lim_{n \to \infty} \sup_{u \in S(u_n)} \vartheta(u). \tag{1}$$

Next, we will show that $\{u_n\}$ is a Cauchy sequence in $\mathcal{F}$. Before that, we first prove the following statement holds true:

$$\forall\, \varepsilon \in (0,1), \exists\, N \in \mathbb{N} \text{ such that } \mathcal{L}_{inf}(u_m,u_n) > 1 - \varepsilon, \ \forall m,n > N. \tag{2}$$

If the statement is not valid, then there exists $\varepsilon_0 \in (0,1)$ such that for every $N \in \mathbb{N}$, there exists $m_k > n_k > N$, such that

$$\mathcal{L}_{inf}(u_{m_k}, u_{n_k}) \leq 1 - \varepsilon_0.$$

Note that $\{\vartheta(u_n)\}$ is a increasing sequence, hence, for all $n \in \mathbb{N}$, we have $\vartheta(u_n) \leq \lim_{m\to\infty} \vartheta(u_m) = p$. Using the definition of limit and Equation (1), we know that for every $\varepsilon' \in (0,1)$, there exists $N' \in \mathbb{N}$ such that for any $n > N'$, $p(1-\varepsilon') \leq \vartheta(u_n) \leq p$. Hence, let the above $N = N'$. Since $u_{m_k} \in S(u_{n_k})$, we have

$$p \diamond (1-\varepsilon_0) \geq \vartheta(u_{m_k}) \diamond \mathcal{L}_{inf}(u_{m_k}, u_{n_k}) \geq \vartheta(u_{n_k}) \geq p(1-\varepsilon').$$

As ε' is arbitrary, we get $p \diamond (1-\varepsilon_0) \geq p$. This is contradicted by Archimedean condition. Hence, the statement (2) holds true.

From Lemma 2, we have $\mathcal{L}_{inf}(u_m, u_n) = \inf_{\alpha>0} \mathcal{L}(u_m, u_n, \alpha)$. This implies that for each $\varepsilon \in (0,1)$ and $\alpha > 0$, there exists $N \in \mathbb{N}$ such that $\mathcal{L}(u_m, u_n, \alpha) > 1 - \varepsilon$ for each $m, n > N$. Therefore, $\{u_n\}$ is a Cauchy sequence. By the completeness of $(\mathcal{F}, \mathcal{L}, \diamond)$, there exists $\bar{u} \in \mathcal{F}$ such that $u_n \to \bar{u}(n \to \infty)$, i.e., $\lim_{n\to\infty} \mathcal{L}(u_n, \bar{u}, \alpha) = 1$, for each $\alpha > 0$.

Then, we will show that $\lim_{n\to\infty} \mathcal{L}_{inf}(u_n, \bar{u}) = 1$. Let $\varepsilon > 0$. By (2), there exists $N \in \mathbb{N}$ such that

$$\mathcal{L}_{inf}(u_p, u_q) > 1 - \varepsilon, \ \forall p, q > N.$$

For every $m, n > N$ and $\alpha > 0$, by combining Lemma 2 and (2), we can conclude that

$$\begin{aligned}
\mathcal{L}(u_n, \bar{u}, \alpha) &\geq \mathcal{L}\Big(u_n, u_m, \frac{\alpha}{2s}\Big) \diamond \mathcal{L}\Big(u_m, \bar{u}, \frac{\alpha}{2s}\Big) \\
&\geq \mathcal{L}_{inf}(u_n, u_m) \diamond \mathcal{L}\Big(u_m, \bar{u}, \frac{\alpha}{2s}\Big) \\
&\geq (1-\varepsilon) \diamond \mathcal{L}\Big(u_m, \bar{u}, \frac{\alpha}{2s}\Big).
\end{aligned} \tag{3}$$

Since $\{u_n\}$ converges to $\bar{u}$, we have $\lim_{m\to\infty} \mathcal{L}(u_m, \bar{u}, \alpha) = 1$ for all $\alpha > 0$. Hence, by letting $m \to \infty$ in (3), we get

$$\mathcal{L}(u_n, \bar{u}, \alpha) \geq (1-\varepsilon) \diamond 1 \geq 1 - \varepsilon, \ \forall \alpha > 0.$$

Therefore, we conclude that $\lim_{n\to\infty} \mathcal{L}_{inf}(u_n, \bar{u}) = 1$.

Finally, we prove $\bar{u} \in S(u_n)$, and the conclusion holds true. From Lemma 3, we have $\mathcal{L}_{inf}(u_m, u_n) \geq \mathcal{L}_{inf}(u_m, \bar{u}) \diamond \mathcal{L}_{inf}(\bar{u}, u_n)$. Letting $m \to \infty$,

$$\lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n) \geq \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, \bar{u}) \diamond \mathcal{L}_{inf}(\bar{u}, u_n) = \mathcal{L}_{inf}(\bar{u}, u_n).$$

Similarly, we have $\mathcal{L}_{inf}(\bar{u}, u_n) \geq \mathcal{L}_{inf}(u_m, \bar{u}) \diamond \mathcal{L}_{inf}(u_m, u_n)$ and

$$\mathcal{L}_{inf}(\bar{u}, u_n) \geq \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, \bar{u}) \diamond \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n) = \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n).$$

Thus,

$$\lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n) = \mathcal{L}_{inf}(\bar{u}, u_n). \tag{4}$$

Moreover, for any $m > n$, we have $\vartheta(u_m) \diamond \mathcal{L}_{inf}(u_m, u_n) \geq \vartheta(u_n)$. Letting $m \to \infty$, we get $\lim_{m\to\infty} \vartheta(u_m) \diamond \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n) \geq \vartheta(u_n)$. From ϑ is upper semicontinuous and Equation (4), we have

$$\vartheta(\bar{u}) \diamond \mathcal{L}_{inf}(\bar{u}, u_n) \geq \limsup_{m\to\infty} \vartheta(u_m) \diamond \lim_{m\to\infty} \mathcal{L}_{inf}(u_m, u_n) \geq \vartheta(u_n).$$

Hence, $\bar{u} \in S(u_n)$ for all $n \in \mathbb{N}$. On the other hand, from the property of D, there exists $u' \neq \bar{u}$ such that $\vartheta(u') \diamond \mathcal{L}(u', \bar{u}, \alpha) \geq \vartheta(\bar{u})$ for all $\alpha > 0$, that is, $u' \in S(\bar{u})$. Then, $u' \in S(u_n)$ for all $n \in \mathbb{N}$. From (1) we have $\vartheta(u') \leq \lim_{n\to\infty} \sup_{u \in S(u_n)} \vartheta(u) = p$ and $\vartheta(u') \diamond \mathcal{L}(u', \bar{u}, \alpha) \geq \vartheta(\bar{u}) = p \geq \vartheta(u')$. Then from the Archimedean condition, we have $\mathcal{L}(u', \bar{u}, \alpha) = 1$, which contradicts $u' \neq \bar{u}$. Therefore, we conclude that $S(u_0) \cap D \neq \emptyset$. □

Example 4. *Let $(\mathcal{F}, \mathcal{L}, \cdot)$ be the same space in Example 2. Obviously, $(\mathcal{F}, \mathcal{L}, \cdot)$ is complete. Suppose that $\vartheta : \mathcal{F} \to [0,1]$ defined by $\vartheta(u) = e^{-u}$, for all $u \in \mathcal{F}$. In fact, from Theorem 2, we know that for each $u_0 \in \mathcal{F}$,*

$$S(u_0) = \left\{ u \in \mathcal{F} | e^{-u} \cdot e^{-|u_0 - u|} \geq e^{-u_0} \right\} = \{ u \in \mathcal{F} | |u_0 - u| \leq u_0 - u \},$$

that is, $S(u_0) = [0, u_0]$. Let $D \subset \mathcal{F}$ be a subset satisfying the condition that for every $u \in S(u_0) \backslash D$, there exists $u' \neq u$ such that $\vartheta(u') \diamond \mathcal{L}(u, u', \alpha) \geq \vartheta(u)$, for all $\alpha > 0$. Assume that $\hat{D} = \{u \in \mathcal{F} | S(u) = \{u\}\}$. It is obviously that $\hat{D} \subset D$. Since $S(u_0) \cap \hat{D} = \{0\} \neq \emptyset$, thus $S(u_0) \cap D \neq \emptyset$.

Lemma 5. *Consider a KM-type Fb-MS endowed with Sugeno–Weber t-norm $\diamond_{SW(\lambda)}$, $\lambda \in (-1, +\infty)$. Then the triangle inequality holds if and only if*

$$(1+\lambda)\mathcal{L}(u, v, \alpha + \beta) + 1 \geq \mathcal{L}\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}\left(w, v, \frac{\beta}{s}\right) + \lambda \mathcal{L}\left(u, w, \frac{\alpha}{s}\right) \cdot \mathcal{L}\left(w, v, \frac{\beta}{s}\right)$$

for all $u, v, w \in \mathcal{F}$ and $\alpha, \beta > 0$.

Proof. The desired result follows directly from the definition of Sugen–Weber t-norms. □

The following two corollaries are different from Theorem 2, as the t-norm no longer requires the Archimedean condition.

Corollary 1. *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and let $\vartheta : \mathcal{F} \to \mathbb{R}^*$ be an u.s.c. and upper bounded function. Also assume that $\diamond$ is a continuous t-norm such that $u \diamond v \geq u \diamond_L v$ for all $u, v \in [0,1]$. In addition, consider $u_0 \in \mathcal{F}$ such that $\vartheta(u_0) \neq 0$. Suppose that $D \subset \mathcal{F}$ such that for every $u \in S(u_0) \backslash D$, there exists $u' \neq u$ such that $\mathcal{L}(u, u', \alpha) \geq 1 + \vartheta(u) - \vartheta(u')$ for all $\alpha > 0$. Then $S(u_0) \cap D \neq \emptyset$.*

Proof. Consider a F*b*-MS with t-norm $\diamond_L$. Define $\mathcal{L}'(u, v, \alpha) = e^{\mathcal{L}(u,v,\alpha)-1}$ on $\mathcal{F} \times \mathcal{F} \times (0, +\infty)$. Next, we claim that $(\mathcal{F}, \mathcal{L}', \cdot)$ is a complete F*b*-MS. Then we only need to show that the triangle inequality holds. Since $(\mathcal{F}, \mathcal{L}, \diamond_L)$ is a complete F*b*-MS, from Lemma 5, for all $u, v, w \in \mathcal{F}$ and $\alpha, \beta > 0$, we can obtain that

$$\begin{aligned}
\mathcal{L}(u, v, \alpha + \beta) &\geq \mathcal{L}\left(u, w, \frac{\alpha}{s}\right) \diamond \mathcal{L}\left(w, v, \frac{\beta}{s}\right) \\
&\geq \mathcal{L}\left(u, w, \frac{\alpha}{s}\right) \diamond_L \mathcal{L}\left(w, v, \frac{\beta}{s}\right) \\
&\geq \mathcal{L}\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}\left(w, v, \frac{\beta}{s}\right) - 1,
\end{aligned}$$

which implies that

$$\mathcal{L}'(u, v, \alpha + \beta) = e^{\mathcal{L}(u,v,\alpha+\beta)-1} \geq e^{\mathcal{L}\left(u,w,\frac{\alpha}{s}\right)+\mathcal{L}\left(w,v,\frac{\beta}{s}\right)-2} = \mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) \cdot \mathcal{L}'\left(w, v, \frac{\beta}{s}\right).$$

On the other hand, by the definition of $\mathcal{L}'$, it is easy to see that if $\{u_n\}$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{L}, \diamond_L)$ then it is a Cauchy in $(\mathcal{F}, \mathcal{L}', \cdot)$, so $(\mathcal{F}, \mathcal{L}', \cdot)$ is a complete Fb-MS. Next, define $\vartheta'(u) = e^{\vartheta(u)}$, for all $u \in \mathcal{F}$. One can deduce that ϑ' is a nontrivial u.s.c. and upper bounded function from $\mathcal{F}$ into $[0, +\infty)$. Hence,

$$S'(u_0) = \{u \in \mathcal{F} | \vartheta'(u) \cdot \mathcal{L}'(u, u_0, \alpha) \geq \vartheta'(u_0),\ \forall \alpha > 0\} \subset S(u_0).$$

From the assumption of Corollary 1, we know that for every $u \in S(u_0) \backslash D$, there exists $u' \neq u$ such that $\mathcal{L}(u, u', \alpha) \geq 1 + \vartheta(u) - \vartheta(u')$, $\forall \alpha > 0$. Thus, for every $u \in S'(u_0) \backslash D$ we have $u' \neq u$ and $\vartheta'(u') \cdot \mathcal{L}(u', u, \alpha) \geq \vartheta'(u)$ for all $\alpha > 0$. Without loss of generality we can assume that $p \leq 1$. Since otherwise we can consider ϑ'/p, where $p = \sup\{\vartheta'(u) : u \in \mathcal{F}\}$. Therefore, Theorem 2 concludes that $S'(u_0) \cap D \neq \emptyset$, and so, $S(u_0) \cap D \neq \emptyset$. □

Corollary 2. *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and let $\vartheta : \mathcal{F} \to \mathbb{R}^*$ be an u.s.c. and upper bounded function. Also assume that $\diamond$ is a continuous t-norm such that $u \diamond v \geq u \diamond_{SW(\lambda)} v$ for all $u, v \in [0, 1]$. Additively, consider $u_0 \in \mathcal{F}$ such that $\vartheta(u_0) \neq 0$. Let $D \subset \mathcal{F}$ such that for every $u \in S(u_0) \backslash D$, there exists $u' \neq u$ such that $\vartheta(u')\mathcal{L}(u, u', \alpha) \geq \vartheta(u)$ for all $\alpha > 0$. Then $S(u_0) \cap D \neq \emptyset$.*

Proof. Consider $\mathcal{F}$ endowed with $\diamond_{SW(\lambda)}$, where $\lambda > -1$. Define $\mathcal{L}'(u, v, \alpha) = \log_{\lambda+2}[(\lambda + 1)\mathcal{L}(u, v, \alpha) + 1] = \log_{\tau+1}(\tau\mathcal{L}(u, v, \alpha) + 1)$ and $\vartheta'(u) = \log_{\tau+1} \vartheta(u)$, $(\tau = \lambda + 1)$, for all $u, v \in \mathcal{F}$ and $\alpha > 0$. Also, we can easily know that ϑ' is a nontrivial u.s.c. and upper bounded function from $\mathcal{F}$ into $\mathbb{R}^+$. We claim that $(\mathcal{F}, \mathcal{L}', \diamond_L)$ is a complete fuzzy b-metric space. To demonstrate that, we only prove the triangle inequality, that is, for all $u, v, w \in \mathcal{F}$ and $\alpha, \beta > 0$,

$$\mathcal{L}'(u, v, \alpha + \beta) \geq \mathcal{L}'(u, w, \frac{\alpha}{s}) \diamond_L \mathcal{L}'(w, v, \frac{\beta}{s}) = \mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}'\left(w, v, \frac{\beta}{s}\right) - 1.$$

Firstly, by the definition of $\mathcal{L}'$, we can deduce that

$$\begin{aligned}
&\mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}'\left(w, v, \frac{\beta}{s}\right) \\
&= \log_{\tau+1}\left(\tau\mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + 1\right) + \log_{\tau+1}\left(\tau\mathcal{L}'\left(w, v, \frac{\beta}{s}\right) + 1\right) \\
&= \log_{\tau+1}\left(\tau\left[\tau\mathcal{L}'\left(u, w, \frac{\alpha}{s}\right)\mathcal{L}'\left(w, v, \frac{\beta}{s}\right) + \mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}'\left(w, v, \frac{\beta}{s}\right)\right] + 1\right).
\end{aligned}$$

From Lemma 5, we have

$$\tau\mathcal{L}'\left(u, w, \frac{\alpha}{s}\right)\mathcal{L}'\left(w, v, \frac{\beta}{s}\right) + \mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}'\left(w, v, \frac{\beta}{s}\right) \leq (\tau + 1)\mathcal{L}(u, v, \alpha + \beta) + 1.$$

Next, we can infer that

$$\begin{aligned}
&\mathcal{L}'(u, v, \alpha + \beta) + 1 \\
&= \log_{\tau+1}(\tau\mathcal{L}(u, v, \alpha + \beta) + 1) + 1 \\
&= \log_{\tau+1}[(\tau + 1)(\tau\mathcal{L}(u, v, \alpha + \beta) + 1)] \\
&= \log_{\tau+1}[\tau[(\tau + 1)\mathcal{L}(u, v, \alpha + \beta) + 1] + 1] \\
&\geq \log_{\tau+1}\left[\tau\left[\tau\mathcal{L}'\left(u, w, \frac{\alpha}{s}\right)\mathcal{L}'\left(w, v, \frac{\beta}{s}\right) + \mathcal{L}'\left(u, w, \frac{\alpha}{s}\right) + \mathcal{L}'\left(w, v, \frac{\beta}{s}\right)\right] + 1\right].
\end{aligned}$$

Hence, $\mathcal{L}'(u,v,\alpha+\beta)+1 \geq \mathcal{L}'(u,w,\frac{\alpha}{s}) + \mathcal{L}'\left(w,v,\frac{\beta}{s}\right)$, $(\mathcal{F},\mathcal{L}',\diamond_L)$ is a fuzzy b-metric space. By the definition of $\mathcal{L}'$, it is easy to see that if $\{u_n\}$ is a Cauchy sequence in $\left(\mathcal{F},\mathcal{L},\diamond_{SW(\lambda)}\right)$ then it is a Cauchy in $(\mathcal{F},\mathcal{L}',\diamond_L)$, so $(\mathcal{F},\mathcal{L}',\diamond_L)$ is a complete fuzzy b-metric space. From the assumption of Corollary 2, we know that for every $u \in S(u_0)\backslash D$, there exists $u' \neq u$ such that $\vartheta(u')\mathcal{L}(u,u',\alpha) \geq \vartheta(u)$ for all $\alpha > 0$. Thus, we have

$$\begin{aligned}
\mathcal{L}'(u,u',\alpha) &= \log_{\tau+1}(\tau\mathcal{L}(u,u',\alpha)+1) \\
&\geq \log_{\tau+1}[(\tau+1)\mathcal{L}(u,u',\alpha)] \\
&\geq \log_{\tau+1}\mathcal{L}(u,u',\alpha)+1 \\
&\geq \log_{\tau+1}\vartheta(u) - \log_{\tau+1}\vartheta(u')+1 \\
&= \vartheta'(u) - \vartheta'(u')+1.
\end{aligned}$$

Therefore, $\mathcal{L}'$ and ϑ' satisfied the conditions in Corollary 1, and we can obtain that $S(u_0) \cap D \neq \emptyset$. □

Remark 2. *According to Corollary 1 and 2, if the minimum norm $\wedge$ is used instead of $\diamond$, the conclusions still hold true. This indicates that under specific conditions, the t-norm does not need to satisfy the Archimedean condition.*

4. Application

Building upon the Oettli–Théra theorem, we establish EVP, which is significant in optimization problems.

Theorem 3 (EVP)**.** *Let $(\mathcal{F},\mathcal{L},\diamond)$ be a complete Fb-MS and let $\vartheta : \mathcal{F} \to [0,1]$ be a non-trivial and upper semicontinuous mapping. Assume that $\diamond$ is a continuous t-norm and satisfies the Archimedean condition. Consider $u_0 \in \mathcal{F}$ such that $\vartheta(u_0) \neq 0$. Then there exists $\bar{u} \in S(u_0)$ and $\alpha_0 > 0$ such that for any $u \in \mathcal{F}\backslash\{\bar{u}\}$,*

$$\vartheta(u) \diamond \mathcal{L}(\bar{u},u,\alpha_0) < \vartheta(\bar{u}).$$

Proof. For all $u \in \mathcal{F}$, define a set-valued mapping $\mathcal{H} : \mathcal{F} \to 2^{\mathcal{F}}$ as follows,

$$\mathcal{H}(u) := \{v \in \mathcal{F} | \vartheta(v) \diamond \mathcal{L}(u,v,\alpha) \geq \vartheta(u), v \neq u,\ \forall \alpha > 0\}.$$

Choose $D := \{u \in \mathcal{F} : \mathcal{H}(u) = \emptyset\}$. If $\hat{u} \notin D$, that is, $\mathcal{H}(\hat{u}) \neq \emptyset$, then, there exists $v \in \mathcal{F}$ with $v \neq \hat{u}$ such that

$$\vartheta(v) \diamond \mathcal{L}(\hat{u},v,\alpha) \geq \vartheta(\hat{u}),\ \forall \alpha > 0.$$

Hence, for all $\hat{u} \in S(u_0)$, the conditions of Theorem 2 are satisfied. Consequently, there exists $\bar{u} \in D$, such that $\mathcal{H}(\bar{u}) = \emptyset$, that is, for all $u \neq \bar{u}$, there exists $\alpha_0 > 0$ such that $\vartheta(u) \diamond \mathcal{L}(u,\bar{u},\alpha_0) < \vartheta(\bar{u})$. □

Remark 3. *According to Theorem 3, we can generalize Theorem 3.9 in [16] from fuzzy metric spaces to fuzzy b-metric spaces without adding any conditions. Also, we can derive Theorem 2.2 in [2].*

Due to the universality of Fb-MS, we further apply the Oettli–Théra theorem to fixed point problems and minimization problems, thereby extending its scope.

Theorem 4 (CFPT)**.** *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and let $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ be a mapping. Suppose that $\diamond$ is a continuous and Archimedean t-norm, and $\vartheta : \mathcal{F} \to [0,1]$ is a non-trivial upper semicontinuous function. Assume that for all $u \in \mathcal{F}$ and $\alpha > 0$*

$$\vartheta(\mathcal{T}u) \diamond \mathcal{L}(\mathcal{T}u, u, \alpha) \geq \vartheta(u)$$

holds. Then $\mathcal{T}$ has a fixed point.

Proof. From Theorem 3, we know that there exists $\bar{u} \in \mathcal{F}$ and $\alpha_0 > 0$ such that for all $u \neq \bar{u}$,

$$\vartheta(u) \diamond \mathcal{L}(u, \bar{u}, \alpha_0) < \vartheta(\bar{u}).$$

We claim that $\bar{u} = \mathcal{T}\bar{u}$, otherwise, $\bar{u} \neq \mathcal{T}(\bar{u})$. By the conditions of Theorem 4, we have

$$\vartheta(\mathcal{T}\bar{u}) \diamond \mathcal{L}(\mathcal{T}\bar{u}, \bar{u}, \alpha) \geq \vartheta(\bar{u}) \tag{5}$$

for all $\alpha > 0$. Moreover, from Theorem 3, we know that $\vartheta(\mathcal{T}\bar{u}) \diamond \mathcal{L}(\mathcal{T}\bar{u}, \bar{u}, \alpha_0) < \vartheta(\bar{u})$, which contradicts (5). Hence, $\bar{u} = \mathcal{T}\bar{u}$. □

Similarly, we can give the version of set-valued Caristi-Kirk's fixed point theorem.

Theorem 5 (CKFPT)**.** *Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and $\vartheta : \mathcal{F} \to [0,1]$ be a non-trivial and upper semicontinuous function. Let $\mathcal{T}$ be a set-valued mapping from $\mathcal{F}$ into $2^{\mathcal{F}}$. Suppose that for all $u \in \mathcal{F}$, there exists $v \in \mathcal{T}(u)$ such that*

$$\vartheta(v) \diamond \mathcal{L}(u, v, \alpha) \geq \vartheta(u), \ \forall \alpha > 0.$$

Then there exists $\bar{u} \in \mathcal{F}$ such that $\mathcal{T}(\bar{u}) = \{\bar{u}\}$.

Remark 4. *Similar to Corollaries 1 and 2, if we replace $\diamond$ with $u \diamond v \geq u \diamond_L v$ or $u \diamond v \geq u \diamond_{SW(\lambda)} v$ in Theorem 4, we can derive ([16] Corollaries 3.3, 3.4).*

Next, we establish TMT in fuzzy b-metric spaces, which can solve the minimization problem.

Theorem 6. *(TMT) Let $(\mathcal{F}, \mathcal{L}, \diamond)$ be a complete Fb-MS and let $\vartheta(u) : \mathcal{F} \to [0,1]$ be a non-trivial and upper semicontinuous function. Assume that $\diamond$ is a continuous t-norm and satisfies the Archimedean condition. Suppose that for each $\hat{u} \in \mathcal{F}$ with $\vartheta(\hat{u}) < \sup_{u \in \mathcal{F}} \vartheta(u)$, there exists $u' \neq \hat{u}$ such that*

$$\vartheta(u') \diamond \mathcal{L}(\hat{u}, u', \alpha) \geq \vartheta(\hat{u}), \ \forall \alpha > 0.$$

Then there exists $\bar{u} \in \mathcal{F}$ such that $\vartheta(\bar{u}) = \sup_{u \in \mathcal{F}} \vartheta(u)$.

Proof. First, we suppose that for all $\hat{u} \in \mathcal{F}$, $\vartheta(\hat{u}) < \sup_{u \in \mathcal{F}} \vartheta(u)$. By the assumptions of Theorem 6, we know that there exists $u' \neq \hat{u}$ such that

$$\vartheta(u') \diamond \mathcal{L}(\hat{u}, u', \alpha) \geq \vartheta(\hat{u}), \ \forall \alpha > 0.$$

Next, for all $\hat{u} \in \mathcal{F}$, we define $\mathcal{T}(\hat{u}) = u'$, hence, $\mathcal{T}$ satisfies the conditions of Theorem 4. Thus, we obtain that $\mathcal{T}$ has a fixed point, i.e., there exists $\tilde{u} \in \mathcal{F}$ such that $\tilde{u} = \mathcal{T}(\tilde{u})$, which contradicts the definition of $\mathcal{T}$. Therefore, there exists $\bar{u} \in \mathcal{F}$ such that $\vartheta(\bar{u}) = \sup_{u \in \mathcal{F}} \vartheta(u)$. □

Remark 5. *It is worth noting that we can derive Theorems 3.1 and 3.13 in [16] from Theorems 4 and 6.*

Theorem 7 (Equivalence)**.** *Theorems 2–4 and 6 are equivalent.*

Proof. Since

$$\text{Theorem } 2 \Rightarrow \text{Theorem } 3 \Rightarrow \text{Theorem } 4 \Rightarrow \text{Theorem } 6,$$

next, we only need to verify Theorem 6 ⇒ Theorem 2. Suppose that $S(u_0) \cap D = \emptyset$. Then for every $u \in S(u_0)$, there exists $u' \neq u$ such that

$$\vartheta(u') \diamond \mathcal{L}(u, u', \alpha) \geq \vartheta(u), \ \forall \alpha > 0. \tag{6}$$

Notably, inequality (6) states that $u' \in S(u)$, and so, $u' \in S(u_0)$. Hence, for all $u \in S(u_0)$, the conditions of Theorem 6 are satisfied, and we can obtain that there exists $\bar{u} \in S(u_0)$ such that $\vartheta(\bar{u}) = \sup_{u \in S(u_0)} \vartheta(u)$. Furthermore, for each $u \in S(u_0)$ with $u \neq \bar{u}$, we have $\vartheta(u) < \vartheta(\bar{u})$ and $\mathcal{L}(u, \bar{u}, \alpha) < 1$ for all $\alpha > 0$. This implies that

$$\vartheta(u) \diamond \mathcal{L}(u, \bar{u}, \alpha) < \vartheta(\bar{u}) \diamond 1 = \vartheta(\bar{u}), \ \forall \alpha > 0,$$

a contradiction with (6). Therefore, $S(u_0) \cap D \neq \emptyset$. □

5. Conclusions

We propose some new properties of fuzzy b-metrics and provide corresponding lemmas to demonstrate the existence of the right limit of fuzzy b-metrics when α tends to 0^+. This effectively removes the restriction on the triangle inequality coefficient in fuzzy b-metric spaces. By utilizing these lemmas, we establish the Oettli–Théra theorem for the first time in KM-type fuzzy b-metric spaces, which is an extension of the results in existing fuzzy metric spaces, fundamentally expanding the application scope of the Oettli–Théra theorem.

The extended Oettli–Théra theorem can also derive EVP, CFPT, and TMT, greatly enriching the research on optimization problems, fixed point problems, and minimization problems in KM-type fuzzy b-metric spaces. Notably, these conclusions generalize and enhance the relevant results in references [1–3,16,17,19]. Furthermore, the results also demonstrate the utility of the methods proposed in this paper, providing a research tool for further exploration in this field.

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References

1. Oettli, W.; Théra, M. Equivalents of Ekeland's principle. *Bull. Aust. Math. Soc.* **1993**, *48*, 385–392. [CrossRef]
2. Ekeland, I. On the variational principle. *J. Math. Anal. Appl.* **1974**, *47*, 324–353. [CrossRef]
3. Takahashi, W. Existence theorems generalizing fixed point theorems for multivalued mappings. *J. Fixed Point Theory Appl.* **1991**, *252*, 397–406.
4. Blaszkiewicz, P.; Cmiel, H.; Linzi, A.; Szewczyk, P. Caristi-Kirk and Oettli-Théra ball spaces and applications. *J. Fixed Point Theory Appl.* **2019**, *21*, 98. [CrossRef]
5. Zeng, J.; Li, S.J. An Ekeland's variational principle for set-valued mappings with applications. *J. Comput. Appl. Math.* **2009**, *230*, 477–484. [CrossRef]

6. Eshghinezhad, S.; Fakhar, M. Vectorial form of Ekeland-Type variational principle in locally convex spaces and its applications. *Fixed Point Theory Appl.* **2010**, *1*, 276294. [CrossRef]
7. Ansari, Q.H. Vectorial form of Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory. *J. Math. Anal. Appl.* **2007**, *334*, 561–575. [CrossRef]
8. Almezel, S.; Ansari, Q.H.; Khamsl, M.A. *Topics in Fixed Point Theory*; Springer: Berlin, Germany, 2013.
9. Gutiérrez, C.; Kassay, G.; Novo, V.; Ródenas-Pedregosa, J.L. Ekeland variational principles in vector equilibrium problems. *SIAM J. Optim.* **2017**, *27*, 2405–2425. [CrossRef]
10. Qiu, J.H.; Soubeyranc, A.; He, F. Equilibrium set-valued variational principles and the lower boundedness condition with application to psychology. *Optimization* **2024**, *73*, 673–705. [CrossRef]
11. Georgiev, V.; Ilchev, A.; Zlatanov, B. A variational principle, fixed points and coupled fixed points on $\mathbb{P}$ sets. *J Fixed Point Theory Appl.* **2024**, *26*, 1–16. [CrossRef]
12. Al-Homidan, S.; Ansari, Q.H.; Kassay, G. Vectorial form of Ekeland variational principle with applications to vector equilibrium problems. *Optimization* **2020**, *69*, 415–436. [CrossRef]
13. Zhang, C.L.; Huang, N.J. On Ekeland's variational principle for interval-valued functions with applications. *Fuzzy Sets Syst.* **2021**, *436*, 152–174. [CrossRef]
14. Aydi, H.; Karapinar, E.; Vetro, C. On Ekeland's variational principle in partial metric space. *Appl. Math. Inf. Sci.* **2015**, *9*, 257–262. [CrossRef]
15. Qiu, J.H. Set-valued Ekeland variational principles in fuzzy metric spaces. *Fuzzy Sets Syst.* **2014**, *245*, 43–62. [CrossRef]
16. Abbasi, N.; Golshan, H.M. Caristi's fixed point theorem and its equivalences in fuzzy metric spaces. *Kybernetika* **2016**, *52*, 929–942. [CrossRef]
17. Wu, J.R.; Xiao, T. Caristi's fixed point theorem, Ekeland's variational principle and Takahashi's maximization theorem in fuzzy quasi-metric spaces. *Topol Appl.* **2021**, *302*, 107801. [CrossRef]
18. Bota, M.; Molnar, A.; Varga, C. On Ekeland's variational principle in *b*-metric spaces. *Fixed Point Theory* **2011**, *12*, 21–28.
19. Kramosil, I.; Michalek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
20. Abbas, M.; Ali, B.; Romaguera, S. Multivalued caristi's type mappings in fuzzy metric spaces and a characterization of fuzzy metric completeness. *Filomat* **2015**, *29*, 1217–1222. [CrossRef]
21. Garcia, J.G.; Rodriguez-Lopez, J.; Romaguera, S. On fuzzy uniformities induced by a fuzzy metric space. *Fuzzy Sets Syst.* **2018**, *330*, 52–78. [CrossRef]
22. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
23. Gregori, V.; Minana, J.; Morillas, S. Some questions in fuzzy metric spaces. *Fuzzy Sets Syst.* **2012**, *204*, 71–85. [CrossRef]
24. Zhu, J.; Zhong, C.K.; Wang, G.P. Vector-valued variational principle in fuzzy metric space and its applications. *Fuzzy Sets Syst.* **2001**, *119*, 343–354. [CrossRef]
25. Xiao, J.Z.; Zhu, X.H.; Jin, X. Fixed point theorems for nonlinear contractions in Kaleva-Seikkala's type fuzzy metric spaces. *Fuzzy Sets Syst.* **2012**, *200*, 65–83. [CrossRef]
26. Czerwik, S. Contraction mappings in *b*-metric spaces. *Acta Math. Univ. Ostrav.* **1993**, *1*, 5–11.
27. Lu, N.; He, F.; Du, W.S. On the best areas for Kannan system and Chatterjea system in *b*-metric spaces. *Optimization* **2021**, *70*, 973–986. [CrossRef]
28. Sedghi, S.; Shobe, N. Common fixed point theorems in *b*-fuzzy metric spaces. *Nonlinear Funct. Anal. Appl.* **2013**, *17*, 349–359.
29. Romaguera, S. Concerning Fuzzy b-Metric Spaces. *Mathematics* **2023**, *11*, 4625. [CrossRef]
30. Nadaban, S. Fuzzy b-Metric Spaces. *Int. J. Comput. Commun.* **2016**, *11*, 273–281. [CrossRef]
31. Raki, D.; Mukheimer, A.; Doenovi, T.; Mitrović, Z.D.; Radenović, S. On some new fixed point results in fuzzy b-metric spaces. *J. Inequal Appl.* **2020**, *2020*, 99. [CrossRef]
32. Batul, S.; Mehmood, F.; Hussain, A.; Sagheer, D.; Aydi, H.; Mukheimer, A. Multivalued contraction maps on fuzzy b-metric spaces and an application. *AIMS Math.* **2022**, *7*, 5925–5942. [CrossRef]