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Article

Golden Ratio and a Ramanujan-Type Integral

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Abstract: In this paper, we give a pedagogical introduction to several beautiful formulas discovered by Ramanujan. Using these results, we evaluate a Ramanujan-type integral formula. The result can be expressed in terms of the Golden Ratio.

Keywords: golden ratio; Ramanujan integral; Rogers–Ramanujan continued fraction

1. Introduction

The celebrated Rogers–Ramanujan continued fraction is defined as follows. For $|q| < 1$,

$$R(q) := \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \ddots}}} \tag{1}$$

For excellent introductions, see [1,2]. For a recent introduction, see [3], especially Chapter 11.

Ramanujan discovered the following remarkable integral formula regarding $R(q)$, which is recorded in his "lost" notebook:

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left(-\frac{1}{5}\int_q^1 \frac{(1-t)^5(1-t^2)^5(1-t^3)^5\cdots}{(1-t^5)(1-t^{10})(1-t^{15})\cdots}\frac{dt}{t}\right) \tag{2}$$

This was first proved by Andrews [4]. See also Section 14.4 of [5] and Chapter 15 of [3]. This article is a pedagogical introduction to this remarkable identity. As a corollary, we will derive the following integral identity:

$$\ln\left(\sqrt{4\phi+3}-\phi^2\right) = -\frac{1}{5}\int_{e^{-2\pi}}^1 \frac{(1-t)^5(1-t^2)^5(1-t^3)^5\cdots}{(1-t^5)(1-t^{10})(1-t^{15})\cdots}\frac{dt}{t} \tag{3}$$

Here, $\phi = (1+\sqrt{5})/2$ is the Golden Ratio. The integrand in this equation is the same integrand that appears in Equation (2).

2. Proof of Equation (3)

We will break up this task into several steps (*BIG STEPS 1, 2* and *3* below).

BIG STEP 1: A differential equation for $R(q)$

First we recall the differential version of Equation (2):

$$5q\frac{d}{dq}\ln R(q) = \prod_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{5n})} \tag{4}$$

This will allows us to find a formula analogous to Equation (2) in the next step.

In order to make our discussion self-contained and motivated, we will give some details on one of the known proofs, following the work of Dobbie [6] (as it illustrates many nice tricks in dealing with q-series).

Equation (4) is not easy to prove. One may ask, "How do we take the derivative of the *continued fraction* $R(q)$ on the left-hand side of it?" What comes to rescue us is the following remarkable identity due to Rogers and Ramanujan (who discovered it independently):

$$R(q) = q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})} \tag{5}$$

Before we move on, we note that the infinite products on right-hand side of Equation (5) are related to the first and the second Rogers–Ramanujan identities; e.g., see [7]; see also Part II of [3], where readers will find three different proofs of these famous identities.

By using Equation (5), we can rewrite Equation (4) as

$$1-5\left(\sum_{n=1}^{\infty}\frac{(5n-1)q^{5n-1}}{1-q^{5n-1}}+\frac{(5n-4)q^{5n-4}}{1-q^{5n-4}}-\frac{(5n-2)q^{5n-2}}{1-q^{5n-2}}-\frac{(5n-3)q^{5n-3}}{1-q^{5n-3}}\right)=\prod_{n=1}^{\infty}\frac{(1-q^n)^5}{(1-q^{5n})} \tag{6}$$

To prove the last identity, we will turn to yet another identity:

$$\begin{aligned}\frac{x}{(1-x)^2}-\frac{z}{(1-z)^2}+\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}(x^j+x^{-j}-z^j-z^{-j}) &= \frac{(x-z)(1-xz)}{(1-x)^2(1-z)^2}\times\\ \times\prod_{n=1}^{\infty}\frac{(1-xzq^n)(1-x^{-1}z^{-1}q^n)(1-x^{-1}zq^n)(1-xz^{-1}q^n)(1-q^n)^4}{(1-xq^n)^2(1-x^{-1}q^n)^2(1-zq^n)^2(1-z^{-1}q^n)^2}\end{aligned} \tag{7}$$

Now it seems that we are making things worse: it looks like we just traded one difficult identity for another identity that is more difficult! But it turns out, to one's surprise, Equation (7) is more manageable than Equation (6). The reason is that there are more variables (x and z) in Equation (7). These variables allow us to explore the symmetries of the identity, which are essential to its proof.

Before we prove Equation (7), let us indicate how it implies Equation (6). This is be done by setting $x = e^{2\pi i/5}$ and $z = x^2$. We will let readers fill in the details (see Dobbie's original paper, or Section 15.2 of [3]).

Let us prove Equation (7). We will follow closely the discussion in Section 15.2 of [3].

- *Step I: Rewriting the sum side of Equation (7)*

Our goal is to show that the left-hand side of Equation (7) is the same as

$$\sum_{n=-\infty}^{\infty} \frac{xq^n}{(1-xq^n)^2} - \frac{zq^n}{(1-zq^n)^2} \tag{8}$$

Indeed, let us consider the sum involving x in Equation (8). We break it up according to $n = 0$, $n > 0$ and $n < 0$:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{xq^n}{(1-xq^n)^2} &= \frac{x}{(1-x)^2} + \sum_{n=1}^{\infty} \frac{xq^n}{(1-xq^n)^2} + \frac{xq^{-n}}{(1-xq^{-n})^2} \\
&= \frac{x}{(1-x)^2} + \sum_{n=1}^{\infty} \frac{xq^n}{(1-xq^n)^2} + \frac{x^{-1}q^n}{(1-x^{-1}q^n)^2} \\
&= \frac{x}{(1-x)^2} + \sum_{j,n=1}^{\infty} jx^j q^{jn} + jx^{-j}q^{jn} \\
&= \frac{x}{(1-x)^2} + \sum_{j=1}^{\infty} \frac{jx^j q^j}{1-q^j} + \frac{jx^{-j}q^j}{1-q^j}
\end{aligned}$$

This is the same as the sum involving x on the left-hand side of Equation (7). Doing the same with the sum involving z in Equation (8) gives the rest of the left-hand side of Equation (7).

- *Step II: Identifying the poles of the infinite product in Equation (7)*

Let us denote by $F(x, z)$ the right-hand side of Equation (7). First we treat $F(x, z)$ as a function of x. At the same time, we will treat z as a parameter that is not an integral power of q. We claim that $F(x, z)$ has poles of order two at

$$x = q^n$$

with $n \in \mathbb{Z}$.

Indeed, the denominator $(1 - x)^2$ in $F(x, z)$ implies $x = 1$ is a pole of order 2. Similarly, the denominator $(1 - x^{\pm 1}q^j)^2$ implies $x = q^{\pm j}$ is a pole of order 2. This proves our claim.

Next we want to find the partial fraction expansion of $F(x, z)$. To this end, we need to determine the symmetries of $F(x, z)$.

- *Step III: Exploring the symmetries of $F(x, z)$*

Readers can easily verify the following:

$$\begin{aligned}
F(x,z) &= -F(z,x) && (9) \\
F(x,z) &= F(qx,z) && (10) \\
&= F(q^{-1}x,z) && (11)
\end{aligned}$$

- *Step IV: Finding the partial fraction expansion of $F(x, z)$*

Our goal is to prove that

$$F(x,z) = \sum_{n\in\mathbb{Z}} \frac{xq^n}{(1-xq^n)^2} + H(x,z) := G(x) + H(x,z) \tag{12}$$

Here $H(x, z)$ is a "remainder" term that has a Laurent expansion in x. Note that, by comparing with *Step I* above, Equation (12) implies that we have "half" of Equation (7).

To prove Equation (12), we start with the observation that *Step II* implies

$$F(x,z) = \frac{a_0(x)}{(1-x)^2} + \sum_{n=1}^{\infty} \frac{a_n(x)}{(1-xq^n)^2} + \frac{a_{-n}(x)}{(1-x^{-1}q^n)^2} + H(x,z) \tag{13}$$

We need to determine a_m.

First we show that $a_0(x) = x$. The part of $F(x,z)$ that contributes to $a_0(x)$ comes solely from the overall prefactor (note that the infinite product becomes 1 as $x \to 1$). Regarding this prefactor, we note that

$$\frac{(x-z)(1-xz)}{(1-x)^2(1-z)^2} = \frac{x}{(1-x)^2} - \frac{z}{(1-z)^2}$$

This implies the principal part at $x = 1$ is

$$\frac{x}{(1-x)^2}$$

and therefore $a_0(x) = x$.

For $n > 0$, we have

$$a_{\pm n}(x) = x^{\pm 1} q^n \tag{14}$$

Indeed, $a_n = xq^n$ follows from the fact that $a_0(x) = x$ and Equation (10), and $a_{-n}(x) = x^{-1}q^n$ follows from $a_0(x) = x$ and Equation (11).

By folding the sum in x in Equation (13), we obtain Equation (12).

- *Step V: Determining* $H(x,z)$

This is the final step: let us show that

$$H(x,z) = -G(z) \tag{15}$$

We recall that both H and G were defined in Equation (12). Previously, $H(x,z)$ represented what cannot be determined by understanding the pole structure of $F(x,z)$. What Equation (15) says is that, $H(x,z)$ can indeed be written as something known (*i.e.*, G)—but there is a catch: the argument of G on the right-hand side of Equation (15) is z, not x. In fact, the same equation tells us the $H(x,z)$ is independent of x. Let us turn to the proof.

Since $H(x,z)$ is a Laurent expansion in x (cf. the sentence right after Equation (12)), we can write it as

$$H(x,z) = \sum_{n\in\mathbb{Z}} b_n(z) x^n \tag{16}$$

To determine b_n, we need to know the symmetry of $H(x,z)$.

First we note that $G(x)$ satisfies

$$G(x) = G(xq)$$

This can be easily verified and we will omit the detail of its proof. This, with Equation (10), implies

$$H(x,z) = H(xq,z) \tag{17}$$

Indeed,

$$\begin{aligned} H(x,z) &= F(x,z) - G(x) \\ &= F(xq,z) - G(xq) \\ &= H(xq,z) \end{aligned}$$

Equation (17) implies that only $b_0(z)$ survives in the expansion in Equation (16):

$$H(x,z) = b_0(z)$$

This, with Equation (12), implies

$$F(x,z) = G(x) + b_0(z) \tag{18}$$

By Equations (9) and (18), we have

$$G(x) + b_0(z) = -G(z) - b_0(x)$$

Rearranging this equation gives

$$G(x) + b_0(x) = -(G(z) + b_0(z)) \tag{19}$$

Since the left-hand side of this equation depends on x and the right-hand side on z, we conclude that both sides equal a constant independent of either x or z. If we call this constant $-\lambda$, we have

$$b_0(z) = \lambda - G(z) \tag{20}$$

Equations (18) and (20) imply

$$F(x,z) = G(x) - G(z) + \lambda \tag{21}$$

We want to show that $\lambda = 0$. To this end, we note that Equations (9) and (21) imply

$$0 = F(x,z) + F(z,z) = 2\lambda$$

This shows that $\lambda = 0$, and, with Equation (21), we arrive at our final conclusion,

$$F(x,z) = G(x) - G(z)$$

which is Equation (7); cf. the definition of G in Equation (12) and *Step I*.

BIG STEP 2: Integrating Equation (4)

Integrating the differential equation (4) gives

$$R(q) = A\ \exp\left(-\frac{1}{5}\int_q^{q^*}\prod_{n=1}^{\infty}\frac{(1-t^n)^5}{(1-t^{5n})}\cdot\frac{dt}{t}\right) \tag{22}$$

where $A = R(q^*)$.

Let us set $q^* = 1$. As $q \to 1^-$, we have

$$R(q) \to \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ddots}}} = \frac{1}{\phi}$$

Hence $A = 1/\phi$. This gives Equation (2).

There is another choice for q^* (hence A). Ramanujan discovered the following remarkable result

$$R(e^{-2\pi}) = \sqrt{2+\phi} - \phi \tag{23}$$

This is one of striking formulas that convinced Hardy that Ramanujan's result is deep. On Equation (23) (and several other results), Hardy said [7]:

> I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.

With this understood, we let $q^* = e^{-2\pi}$ in Equation (22), and we have (with Equation (23)),

$$R(q) = (\sqrt{2+\phi} - \phi)\ \exp\left(-\frac{1}{5}\int_q^{e^{-2\pi}} \prod_{n=1}^{\infty} \frac{(1-t^n)^5}{(1-t^{5n})}\cdot\frac{dt}{t}\right) \tag{24}$$

Before we turn to the last BIG STEP, let us pause and sketch a proof of Equation (23). We will follow Watson [8]. He believed that this is how Ramanujan derived this striking result. For another proof, see Section 12.1 of [3].

The key of this proof is to start with

$$\prod_{n=1}^{\infty}\left(\frac{1-q^{n/5}}{1-q^{5n}}\right) = \frac{\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/10}}{\sum_{n\in\mathbb{Z}}(-1)^n q^{5n(3n+1)/2}} \tag{25}$$

To obtain the second equality, we have used Euler's pentagonal theorem (e.g., see [1–3,9–11]): For $|q| < 1$,

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(3j+1)/2}$$

Looking carefully at the sum in the numerator reveals that it can be written as

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n+1)/10} &= \sum_{m\in\mathbb{Z}} a_m q^m + q^{2/5}\sum_{m\in\mathbb{Z}} b_m q^m + q^{1/5}\sum_{m\in\mathbb{Z}} c_m q^m \\ &:= \widetilde{J_1} + q^{2/5}\,\widetilde{J_2} + q^{1/5}\,\widetilde{J_3} \end{aligned} \tag{26}$$

Here $\widetilde{J_i}$ are integral power series of q.

In fact, we have more: careful analysis reveals that $\widetilde{J_3}$ is actually the denominator of Equation (25):

$$-\widetilde{J_3} = \sum_{n\in\mathbb{Z}}(-1)^n q^{5n(3n+1)/2} \tag{27}$$

This is remarkable!

Equations (25)–(27) imply that

$$q^{-1/5} \prod_{n=1}^{\infty} \left(\frac{1 - q^{n/5}}{1 - q^{5n}} \right) = q^{-1/5} J_1 - 1 + q^{1/5} J_2 \tag{28}$$

where we have defined $J_i := -\widetilde{J_i} \,/\, \widetilde{J_3}$ for $i = 1, 2$.

The next step is truly miraculous: one can show that

$$J_1(q)\, J_2(q) = -1$$

and

$$J_1(q) = \frac{q^{1/5}}{R(q)}$$

The proof of these identities are long and we refer readers to Watson's paper [8] (or see Section 12.2 of [3]).

With Equation (28), these identities imply

$$q^{-1/5} \prod_{n=1}^{\infty} \left(\frac{1 - q^{n/5}}{1 - q^{5n}} \right) = \frac{1}{R(q)} - 1 - R(q) \tag{29}$$

which is a key step in proving Equation (23). As an aside, we note that Equation (29) can be used to derive Ramanujuan's "Most Beautiful Identity": for $|q| < 1$,

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^5}{(1 - q^n)^6}$$

Here $p(n)$ is the partition function defined by $\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$. For an excellent introduction to this amazing identity, see a recent paper by Hirschhorn [12] (and stay tuned for his forthcoming book). To understand how Equation (29) implies this identity, see Chapter 16 of [3].

At $q = e^{-2\pi}$, the left-hand side of Equation (29) can be shown to reduce to $\sqrt{5}$. This is the consequence of a famous identity satisfied by the eta function $\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$, where $q = e^{2\pi i z}$ and Im $z > 0$:

$$\eta\left(-\frac{1}{z} \right) = \sqrt{\frac{z}{i}}\, \eta(z) \tag{30}$$

(see, e.g., Apostol's book [13] for a proof; see also Appendix A of [3], which sketches three different proofs of this important result). In other words, we have

$$\frac{1}{R(e^{-2\pi})} - 1 - R(e^{-2\pi}) = \sqrt{5}$$

Solving this equation gives the desired result $R(e^{-2\pi}) = \sqrt{2 + \phi} - \phi$.

BIG STEP 3: The punch line

This is actually a small step! By equating Equation (2) and Equation (24), we have

$$\frac{1}{\phi} \exp\left(-\frac{1}{5} \int_q^1 \prod_{n=1}^{\infty} \frac{(1 - t^n)^5}{(1 - t^{5n})} \cdot \frac{dt}{t} \right) = (\sqrt{2 + \phi} - \phi) \exp\left(-\frac{1}{5} \int_q^{e^{-2\pi}} \prod_{n=1}^{\infty} \frac{(1 - t^n)^5}{(1 - t^{5n})} \cdot \frac{dt}{t} \right)$$

Rearranging this equation gives

$$\phi\,(\sqrt{2+\phi}-\phi)=\exp\left(-\frac{1}{5}\int_{e^{-2\pi}}^{1}\prod_{n=1}^{\infty}\frac{(1-t^n)^5}{(1-t^{5n})}\cdot\frac{dt}{t}\right)$$

which is equivalent to our key result Equation (3). □

3. Final Remarks

Recently the present author, inspired by Equation (5), discovered a Wallis-type formula for the Golden Ratio (see [14]):

$$\frac{1}{\phi}=\frac{1\times6\times11\times16\times\cdots4\times9\times14\times19\times\cdots}{2\times7\times12\times17\times\cdots3\times8\times13\times18\times\cdots}\tag{31}$$

To see this *heuristically*: let $q\to1$ in Equation (5) and apply the L'Hôpital's rule. See [14] for a simple but rigorous proof. (One of the reviewers kindly showed us a proof of it. It turns out, it is essentially same as that in [14]!) Note that one may regard Equation (5) as a q-deformation of Equation (31).

This formula further implies (again, see [14] for details):

$$\phi^3=9+\frac{1}{6}\mathbf{K}_{n=1}^{\infty}\left(\frac{-(5n+1)(5n+2)(5n+3)(5n+4)}{10(5(n+1)^2+1)}\right)\tag{32}$$

where we have used the $\mathbf{K}$ notation for continued fractions (where $\mathbf{K}$ is for *Kettenbrüche* in German):

$$\mathbf{K}_{n=1}^{\infty}\left(\frac{B_n}{A_n}\right)=\cfrac{B_1}{A_1+\cfrac{B_2}{A_2+\cfrac{B_3}{\ddots}}}.$$

Here is an open question: find a q-deformation of Equation (32).

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