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Article

On Elliptic and Hyperbolic Modular Functions and the Corresponding Gudermann Peeta Functions

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Abstract: In this article, we move back almost 200 years to Christoph Gudermann, the great expert on elliptic functions, who successfully put the twelve Jacobi functions in a didactic setting. We prove the second hyperbolic series expansions for elliptic functions again, and express generalizations of many of Gudermann's formulas in Carlson's modern notation. The transformations between squares of elliptic functions can be expressed as general Möbius transformations, and a conjecture of twelve formulas, extending a Gudermannian formula, is presented. In the second part of the paper, we consider the corresponding formulas for hyperbolic modular functions, and show that these Möbius transformations can be used to prove integral formulas for the inverses of hyperbolic modular functions, which are in fact Schwarz-Christoffel transformations. Finally, we present the simplest formulas for the Gudermann Peeta functions, variations of the Jacobi theta functions. 2010 Mathematics Subject Classification: Primary 33E05; Secondary 33D15.

Keywords: hyperbolic series expansion; Carlson's modern notation; hyperbolic modular function; Möbius transformation; Schwarz-Christoffel transformation; Peeta function

1. Introduction

The elliptic integrals were first classified by Euler and Legendre, and then Gauss, Jacobi and Abel started to study their inverses, the elliptic functions. Starting in the 1830s, Gudermann published a series of papers in German and Latin, with the aim of presenting these functions in a didactic way, and to introduce a short notation for them. This notation, with a small modification, has survided until the present day. Jacobi, in 1829, had found quickly converging Fourier series expansions for most of the twelve elliptic functions, which have been put in q-hypergeometric form in the authors article [1]. As Gudermann [2] showed, there are second series expansions for the twelve elliptic functions, starting from the imaginary period, which are not so quickly converging for all values of the variables; these expansions were also found, without proof, by Glaisher [3]. Since these hyperbolic expansions are virtually unknown today, we prove them again, and also put them into q-hypergeometric form in section two. There are many series expansions for squareroots of rational functions of elliptic functions; as a bonus we also prove some of these. However, before this, we introduce the q-hypergeometric notation in this first section, this can also be found in the book [4].

In section three, we generalize many of Gudermanns formulas to the very general Carlson [5] notation, where many formulas can be put into one single equation by using a clever code, and the symmetry of these functions. This notation has been known for many years, but was only recently published; by coincidence, the author saw it when he was asked to review this article by Carlson. In particular, a formula with squareroots, stated without proof by Gudermann, is generalized to a conjecture of two formulas with squareroots, or twelve elliptic function formulas, which generalize four formulas with squareroots for trigonometric and hyperbolic functions. Gudermann was the first to point out the close relationship between trigonometric and hyperbolic functions. We also state four Möbius transformations in Carlson's notation, and generalize Gudermanns formulas for artanh.

In section four, we come to the hyperbolic modular functions, which have not yet appeared in the English literature; the function $\mathcal{SN}(u)$ is the inverse of an hyperbolic integral, which is formed by changing two minuses to plus in the elliptic integral of the first kind. We calculate the poles, periods, Möbius transformations for squares, and special values of the hyperbolic modular functions. Finally, we compute several addition formulas using a short notation for these functions.

In section five, we consider the Peeta functions, which are theta functions with imaginary function value. We show that the hyperbolic modular functions can be expressed as quotients of Peeta functions, and that the four Peeta functions are solutions of a certain heat equation with the variable q as parameter.

Before presenting the q-series formulas in the next section, we present the necessary definitions.

An elliptic integral is given by

$$F(z) = \int_0^z \frac{dx}{\sqrt[2]{(1-x^2)(1-(kx)^2)}} \tag{1}$$

where $0 < k < 1$.

Abel and Jacobi, inspired by Gauss, discovered that inverting $F(z)$ gave the doubly periodic elliptic function

$$F^{-1}(\omega) = \mathrm{sn}(\omega) \tag{2}$$

In connection with elliptic functions k always denotes the modulus.

Definition 1. Let $\delta > 0$ be an arbitrarily small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The power function is defined by

$$q^a \equiv e^{a \log(q)} \tag{3}$$

Definition 2. The q-factorials and the tilde operator are defined by

$$\langle a; q\rangle_n \equiv \begin{cases} 1, & n = 0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n = 1, 2, \dots \end{cases} \tag{4}$$

$$\langle \widetilde{a}; q\rangle_n \equiv \prod_{m=0}^{n-1}(1+q^{a+m}) \tag{5}$$

Definition 3. The q-hypergeometric series is defined by

$${}_2\phi_1(\widehat{a}, \widehat{b}; \widehat{c}|q; z) \equiv \sum_{n=0}^{\infty} \frac{\langle \widehat{a}; q\rangle_n \langle \widehat{b}; q\rangle_n}{\langle 1; q\rangle_n \langle \widehat{c}; q\rangle_n} z^n \tag{6}$$

where

$$\widehat{a} \equiv a \vee \widetilde{a} \tag{7}$$

It is assumed that the denominator contains no zero factors.

By $\sqrt{z}$ we mean the branch $|z|^{\frac{1}{2}}\exp(i\frac{1}{2}\arg z)$. Everywhere we have $y \equiv \frac{\pi u}{2K'}$. To maintain a symmetrical form, we put according to Jacobi and Glaisher $q' \equiv e^{-\pi\frac{K}{K'}}$. The following lemma will be used in the proofs.

Lemma 1.1. *A Fourier series for the logarithmic potential [6, p. 76].*

$$\log(1 - 2q^{2k}\cos(2x) + q^{4k}) = -\sum_{n=1}^{\infty} \frac{2q^{2kn}}{n}\cos(2nx) \tag{8}$$

2. Hyperbolic series expansions

The following series were published for the first time by Gudermann in [2, p. 106], see also [7].

Theorem 2.1.

$$\frac{\mathrm{cn}u\,\mathrm{dn}u}{\mathrm{sn}u} = \frac{\pi}{2K'}\left[\frac{2}{\sinh(2y)} - 8\sum_{m=1}^{\infty}\frac{q'^{4m-2}}{1-q'^{4m-2}}\sinh((4m-2)y)\right] \tag{9}$$

$$\frac{\mathrm{sn}u}{\mathrm{cn}u\,\mathrm{dn}u} = \frac{4\pi}{K'k'^2}\sum_{m=1}^{\infty}\frac{q'^{2m-1}}{1-q'^{4m-2}}\sinh((4m-2)y) \tag{10}$$

$$\frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u} = \frac{\pi}{2K'}\left(\tanh y + 4\sum_{m=1}^{\infty}\frac{q'^m}{1+(-q')^m}\sinh(2my)\right) \tag{11}$$

$$\frac{\mathrm{cn}u}{\mathrm{sn}u\,\mathrm{dn}u} = \frac{\pi}{2K'}\left(\coth y + 4\sum_{m=1}^{\infty}\frac{(-q')^m}{1+(-q')^m}\sinh(2my)\right) \tag{12}$$

$$\frac{\mathrm{sn}u\,\mathrm{cn}u}{\mathrm{dn}u} = \frac{\pi}{2k^2K'}\left[\tanh\ y + 4\sum_{m=1}^{\infty}\frac{(-q')^m}{1+q'^m}\sinh(2my)\right] \tag{13}$$

$$\frac{\mathrm{dn}u}{\mathrm{sn}u\,\mathrm{cn}u} = \frac{\pi}{2K'}\left[\coth\ y + 4\sum_{m=1}^{\infty}\frac{q'^m}{1+q'^m}\sinh(2my)\right] \tag{14}$$

Proof. We only prove Equation (9), the other formulas are proved similarly.

$$\begin{aligned}
&\log \mathrm{sn} u = \log\left(\frac{2q^{\frac{1}{4}}\sin x}{k^{\frac{1}{2}}}\right)\\
&+\sum_{n=1}^{\infty}\log(1-2q^{2n}\cos 2x+q^{4n})-\log(1-2q^{2n-1}\cos 2x+q^{4n-2})\\
&\overset{\text{by}(8)}{=}\log\left(\frac{2q^{\frac{1}{4}}\sin x}{k^{\frac{1}{2}}}\right)+\sum_{m,n=1}^{\infty}\frac{2\cos 2mx(q^{m(2n-1)}-q^{2mn})}{m}\\
&=\log\left(\frac{2q^{\frac{1}{4}}\sin x}{k^{\frac{1}{2}}}\right)+\sum_{m=1}^{\infty}\frac{2q^{m}\cos 2mx}{m(1+q^{m})}
\end{aligned} \tag{15}$$

The derivative with respect to u finally gives (9).

□

Theorem 2.2. *Hyperbolic series for* $\sqrt{\frac{1\pm t}{1\mp t}}$, $t\in\{\mathrm{cd}u, \mathrm{cn}u, \mathrm{dn}u\}$ *[2, p. 107]. The comodulus* k' *is small.*

$$k'\sqrt{\frac{1+\mathrm{cd}u}{1-\mathrm{cd}u}}=\frac{\pi}{K'}\left[\frac{1}{\sinh\ y}-4\sum_{m=1}^{\infty}\frac{q'^{4m-2}}{1-q'^{2m-1}}\sinh((4m-2)y)\right] \tag{16}$$

$$k'\sqrt{\frac{1-\mathrm{cd}u}{1+\mathrm{cd}u}}=\frac{4\pi}{K'}\sum_{m=1}^{\infty}\frac{q'^{2m-1}}{1-q'^{4m-2}}\sinh((2m-1)y) \tag{17}$$

$$\sqrt{\frac{1-\mathrm{cn}u}{1+\mathrm{cn}u}}=\frac{\pi}{2K'}\left(\tanh\tfrac{y}{2}+4\sum_{m=1}^{\infty}\frac{q'^{m}}{1+(-q')^{m}}\sinh(my)\right) \tag{18}$$

$$\sqrt{\frac{1+\mathrm{cn}u}{1-\mathrm{cn}u}}=\frac{\pi}{2K'}\left(\coth\tfrac{y}{2}+4\sum_{m=1}^{\infty}\frac{(-q')^{m}}{1+(-q')^{m}}\sinh(my)\right) \tag{19}$$

$$k\sqrt{\frac{1-\mathrm{dn}u}{1+\mathrm{dn}u}}=\frac{\pi}{2K'}\left[\tanh\ \tfrac{y}{2}+4\sum_{m=1}^{\infty}\frac{(-q')^{m}}{1+q'^{m}}\sinh(my)\right] \tag{20}$$

$$k\sqrt{\frac{1+\mathrm{dn}u}{1-\mathrm{dn}u}}=\frac{\pi}{2K'}\left[\coth\ \tfrac{y}{2}+4\sum_{m=1}^{\infty}\frac{q'^{m}}{1+q'^{m}}\sinh(my)\right] \tag{21}$$

Proof. All formulas are proved with the help of the previous theorem. We first observe that

$$k'\frac{\mathrm{sn}(\frac{u}{2})}{\mathrm{cn}(\frac{u}{2})\ \mathrm{dn}(\frac{u}{2})}=\sqrt{\frac{1-\mathrm{cd}u}{1+\mathrm{cd}u}} \tag{22}$$

$$\frac{\mathrm{sn}(\frac{u}{2})\mathrm{dn}(\frac{u}{2})}{\mathrm{cn}(\frac{u}{2})}=\sqrt{\frac{1-\mathrm{cn}u}{1+\mathrm{cn}u}} \tag{23}$$

$$k\frac{\mathrm{sn}(\frac{u}{2})\mathrm{cn}(\frac{u}{2})}{\mathrm{dn}(\frac{u}{2})}=\sqrt{\frac{1-\mathrm{dn}u}{1+\mathrm{dn}u}} \tag{24}$$

The Formulas (16) and (17) follow from Formula (22), the Formulas (18) and (19) follow from Formula (23) and finally, Formulas (20) and (21) follow from Formula (24). □

Theorem 2.3. *The following 12 series, found by Gudermann [8] and Glaisher [3, S. 18], define the second series expansions of the corresponding elliptic functions.*

$$\text{[8, p. 366 (6), p. 367 (3)] } \mathrm{sn}u = \frac{\pi}{2K'k}\left[\tanh y + 4\sum_{m=1}^{\infty}\frac{(-1)^m q'^{2m}}{1+q'^{2m}}\cosh(2my)\right] \tag{25}$$

$$\text{[8, p. 366 (4), p. 368 (11)] } \mathrm{cn}u = \frac{\pi}{2K'k}\left[\frac{1}{\cosh y} + 4\sum_{m=1}^{\infty}\frac{(-1)^m q'^{2m-1}}{1+q'^{2m-1}}\cosh((2m-1)y)\right] \tag{26}$$

$$\text{[8, p. 366 (5), p. 368 (19)] } \mathrm{dn}u = \frac{\pi}{2K'}\left[\frac{1}{\cosh y} - 4\sum_{m=1}^{\infty}\frac{(-1)^m q'^{2m-1}}{1-q'^{2m-1}}\sinh((4m-2)y)\right] \tag{27}$$

$$\text{[8, p. 366 (3), p. 367 (7)] } \mathrm{ns}u = \frac{\pi}{2K'}\left[\coth y + 4\sum_{m=1}^{\infty}\frac{q'^{2m}}{1+q'^{2m}}\sinh(2my)\right] \tag{28}$$

$$\text{[8, p. 366 (7), p. 368 (15)] } \mathrm{nc}u = \frac{2\pi}{K'k'}\sum_{m=1}^{\infty}\frac{q'^{\frac{2m-1}{2}}}{1+q'^{2m-1}}\sinh((2m-1)y) \tag{29}$$

$$\text{[8, p. 367 (11), p. 368 (20)] } \mathrm{nd}u = \frac{2\pi}{K'k'}\sum_{m=1}^{\infty}\frac{(-1)^{m+1} q'^{\frac{2m-1}{2}}}{1-q'^{2m-1}}\cosh((2m-1)y) \tag{30}$$

$$\text{[8, p. 366 (8), p. 368 (23)] } \mathrm{sc}u = \frac{2\pi}{K'k'}\sum_{m=1}^{\infty}\frac{q'^{\frac{2m-1}{2}}}{1-q'^{2m-1}}\sinh((2m-1)y) \tag{31}$$

$$\text{[8, p. 368 (24)] } \mathrm{cs}u = \frac{\pi}{2K'}\left[\frac{1}{\mathrm{sinh} y} - 4\sum_{m=1}^{\infty}\frac{q'^{2m-1}}{1-q'^{2m-1}}\sinh((2m-1)y)\right] \tag{32}$$

$$\text{[8, p. 367 (10), p. 368 (12)] } \mathrm{sd}u = \frac{2\pi}{K'kk'}\sum_{m=1}^{\infty}\frac{(-1)^{n-1} q'^{\frac{2m-1}{2}}}{1+q'^{2m-1}}\sinh((2m-1)y) \tag{33}$$

$$\text{[8, p. 365 (1), p. 368 (16)] } \mathrm{ds}u = \frac{\pi}{2K'}\left[\frac{1}{\mathrm{sinh} y} + 4\sum_{m=1}^{\infty}\frac{q'^{2m-1}}{1+q'^{2m-1}}\sinh((2m-1)y)\right] \tag{34}$$

$$[8, \text{(p. 367 (4)), p. 367 (12)]}\ \mathrm{cd}u = \frac{\pi}{2K'k}\left[1+4\sum_{m=1}^{\infty}\frac{(-q')^m}{1+q'^{2m}}\sinh(2my)\right] \tag{35}$$

$$[8, \text{p. 366 (9), p. 367 (8)]}\ \mathrm{dc}u = \frac{\pi}{2K'}\left[1+4\sum_{m=1}^{\infty}\frac{q'^m}{1+q'^{2m}}\cosh(2my)\right] \tag{36}$$

Proof. By addition and subtraction of the Formulas (76) and (77) we obtain Formulas (37)–(40):

$$\mathrm{ds}u+\mathrm{cs}u = \frac{\pi}{2K'}\left[\frac{2}{\sinh\ y} - 8\sum_{m=1}^{\infty}\frac{q'^{4m-2}}{1-q'^{4m-2}}\sinh((2m-1)y)\right] \tag{37}$$

$$\mathrm{ds}u-\mathrm{cs}u = \frac{4\pi}{K'}\sum_{m=1}^{\infty}\frac{q'^{2m-1}}{1-q'^{4m-2}}\sinh((2m-1)y) \tag{38}$$

$$\mathrm{ns}u+\mathrm{cs}u = \frac{\pi}{2K'}\left(\tanh\tfrac{y}{2} + 4\sum_{m=1}^{\infty}\frac{q'^m}{1+(-q')^m}\sinh(my)\right) \tag{39}$$

$$\mathrm{ns}u-\mathrm{cs}u = \frac{\pi}{2K'}\left(\coth\tfrac{y}{2} + 4\sum_{m=1}^{\infty}\frac{(-q')^m}{1+(-q')^m}\sinh(my)\right) \tag{40}$$

New additions and subtractions give Formulas (34), (32) and (28). The substitution $u \mapsto u+iK'$ gives Formulas (26), (27) and (25). □

Theorem 2.4. *According to Heine, these 12 series can be written as follows.*

$$\mathrm{sn}u = \frac{\pi}{2K'k}\left[\tanh y - i\mathrm{Im}\left(-2+2\,{}_2\phi_1(1,\widetilde{0};\widetilde{1}|q'^2;-q'^2e^{-2y})\right)\right] \tag{41}$$

$$\mathrm{cn}u = \frac{\pi}{2K'k}\left[\frac{1}{\cosh y} - \mathrm{Re}\left[e^{-y}\frac{4q'}{1+q'}\,{}_2\phi_1(1,\widetilde{\tfrac{1}{2}};\widetilde{\tfrac{3}{2}}|q'^2;-q'^2e^{-2y})\right]\right] \tag{42}$$

$$\mathrm{dn}u = \frac{\pi}{2K'}\left[\frac{1}{\cosh y} + \mathrm{Re}\left(\frac{4q'e^{-y}}{1-q'}\,{}_2\phi_1(1,\tfrac{1}{2};\tfrac{3}{2}|-q'^2;-q'^2e^{-2y})\right)\right] \tag{43}$$

$$\mathrm{ns}u = \frac{\pi}{2K'}\left[\frac{1}{\tanh y} - i\mathrm{Im}\left(-2+2\,{}_2\phi_1(1,\widetilde{0};\widetilde{1}|q'^2;q'^2e^{-2y})\right)\right] \tag{44}$$

$$\mathrm{nc}u = \frac{2\pi}{K'k'}\mathrm{Re}\left[\frac{q'^{\frac{1}{2}}e^{-y}}{1+q'}\,{}_2\phi_1(1,\widetilde{\tfrac{1}{2}};\widetilde{\tfrac{3}{2}}|q'^2;q'e^{-2y})\right] \tag{45}$$

$$\mathrm{nd}u = \frac{2\pi}{K'k'}\mathrm{Re}\left[e^{-y}\frac{q'^{\frac{1}{2}}}{1-q'}\,{}_2\phi_1(1,\tfrac{1}{2};\tfrac{3}{2}|q'^2;-q'e^{-2y})\right] \tag{46}$$

$$\mathrm{sc}u = \frac{-2\pi i}{K'k'}\mathrm{Im}\left[\frac{q'^{\frac{1}{2}}e^{-y}}{1-q'}\,{}_2\phi_1(1,\tfrac{1}{2};\tfrac{3}{2}|q'^2;q'e^{-2y})\right] \tag{47}$$

$$\mathrm{cs}u = \frac{\pi}{2K'}\left[\frac{1}{\mathrm{sinh}y} - i\,\mathrm{Im}\left[e^{-y}\frac{4q'}{1-q'}\,{}_2\phi_1(1,\tfrac{1}{2};\tfrac{3}{2}|q'^2;q'^2e^{-2y})\right]\right] \tag{48}$$

$$\mathrm{sd}u = \frac{-2\pi i}{K'kk'}\mathrm{Im}\left[e^{-y}\frac{q'^{\frac{1}{2}}}{1+q'}\,{}_2\phi_1(1,\widetilde{\tfrac{1}{2}};\widetilde{\tfrac{3}{2}}|q'^2;-q'e^{-2y})\right] \tag{49}$$

$$\mathrm{ds}u = \frac{\pi k'}{2K'k}\left[\frac{1}{\mathrm{sinh}y} - i\,\mathrm{Im}\left[e^{-y}\frac{4q'}{1+q'}\,{}_2\phi_1(1,\widetilde{\tfrac{1}{2}};\widetilde{\tfrac{3}{2}}|q'^2;q'^2e^{-2y})\right]\right] \tag{50}$$

$$\mathrm{cd}u = \frac{\pi}{2K'k}\left[-1 + 2\,\mathrm{Re}\,{}_2\phi_1(1,\widetilde{0};\widetilde{1}|q'^2;-q'e^{-2y})\right] \tag{51}$$

$$\mathrm{dc}u = \frac{\pi}{2K'}\left[-1 + 2\,\mathrm{Re}\,{}_2\phi_1(1,\widetilde{0};\widetilde{1}|q'^2;q'e^{-2y})\right] \tag{52}$$

3. Some New Elliptic Function Formulas in Carlsons Notation

Bille Carlson (1924–2013) [5] managed to simplify the great number of elliptic function formulas into a series of very general formulas. First put

$$\{\text{p,q,r}\} \equiv \{\text{c,d,n}\} \tag{53}$$

and use Glaisher's abbreviations for Jacobis elliptic functions. Thus q is not a q-analogue in this section.

Furthermore, we put

$$\triangle(\text{ p,q}) \equiv \text{ps}^2 - \text{qs}^2,\ \text{p,q} \in \{\text{c,d,n}\} \tag{54}$$

which implies that

$$\begin{aligned}&\triangle(\text{n,c}) = -\triangle(\text{c,n}) = 1,\ \triangle(\text{n,d}) = -\triangle(\text{d,n}) = k^2\\&\triangle(\text{d,c}) = -\triangle(\text{c,d}) = k'^2\end{aligned} \tag{55}$$

The default function values are u, k. All formulas apply for $u \in \Sigma$ (Riemann sphere). It is well-known that

$$\begin{aligned}&\lim_{k\to 0^+}\mathrm{sn}x = \lim_{k\to 0^+}\mathrm{sd}x = \mathrm{sin}x,\ \lim_{k\to 0^+}\mathrm{cn}x = \cos x,\ \lim_{k\to 0^+}\mathrm{dn}x = 1\\&\lim_{k\to 0^+}\mathrm{sc}x = \tan x,\ \lim_{k\to 1^-}\mathrm{sn}x = \tanh x,\ \lim_{k\to 1^-}\mathrm{cn}x = \lim_{k\to 1^-}\mathrm{dn}x = \frac{1}{\cosh x}\\&\lim_{k\to 1^-}\mathrm{sc}x = \lim_{k\to 1^-}\mathrm{sd}x = \mathrm{sinh}x\end{aligned} \tag{56}$$

All formulas in this section lie between these two limits, *i.e.*, for all limits in k, we get known trigonometric and hyperbolic function (or trivial) formulas. We first give one of Carlsons results; all other formulas are presumably new.

Theorem 3.1. *Addition formulas [5, p. 248]. Put* $\mathrm{ps}_i \equiv \mathrm{ps}(u_i, k),\ i = 1, 2$, *and similar notation for the other functions. Then*

$$\mathrm{ps}(u_1+u_2,k) = \frac{\mathrm{ps}_1\mathrm{qs}_2\mathrm{rs}_2 - \mathrm{ps}_2\mathrm{qs}_1\mathrm{rs}_1}{\mathrm{ps}_2^2 - \mathrm{ps}_1^2} = \frac{\mathrm{ps}_1^2\mathrm{ps}_2^2 - \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})}{\mathrm{ps}_1\mathrm{qs}_2\mathrm{rs}_2 + \mathrm{ps}_2\mathrm{qs}_1\mathrm{rs}_1} \tag{57}$$

$$\mathrm{sp}(u_1+u_2,k) = \frac{\mathrm{sp}_1^2 - \mathrm{sp}_2^2}{\mathrm{sp}_1\mathrm{qp}_2\mathrm{rp}_2 - \mathrm{sp}_2\mathrm{qp}_1\mathrm{rp}_1} = \frac{\mathrm{sp}_1\mathrm{qp}_2\mathrm{rp}_2 + \mathrm{sp}_2\mathrm{qp}_1\mathrm{rp}_1}{1 - \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathrm{sp}_2^2} \tag{58}$$

$$\mathrm{pq}(u_1+u_2,k) = \frac{\mathrm{ps}_1\mathrm{qs}_2\mathrm{rs}_2 - \mathrm{ps}_2\mathrm{qs}_1\mathrm{rs}_1}{\mathrm{qs}_1\mathrm{ps}_2\mathrm{rs}_2 - \mathrm{qs}_2\mathrm{ps}_1\mathrm{rs}_1} = \frac{\mathrm{ps}_1\mathrm{qs}_1\ \mathrm{ps}_2\mathrm{qs}_2 + \triangle(\mathrm{p,q})\mathrm{rs}_1\mathrm{rs}_2}{\mathrm{qs}_1^2\mathrm{qs}_2^2 + \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})} \tag{59}$$

$$\mathrm{pq}(u_1+u_2,k) = \frac{\mathrm{pq}_1\mathrm{sq}_1\mathrm{rq}_2 - \mathrm{pq}_2\mathrm{sq}_2\mathrm{rq}_1}{\mathrm{pq}_2\mathrm{sq}_1\mathrm{rq}_2 - \mathrm{pq}_1\mathrm{sq}_2\mathrm{rq}_1} = \frac{\mathrm{pq}_1\mathrm{pq}_2 + \triangle(\mathrm{p,q})\mathrm{sq}_1\mathrm{rq}_1\mathrm{sq}_2\mathrm{rq}_2}{1 + \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})\mathrm{sq}_1^2\mathrm{sq}_2^2} \tag{60}$$

Remark 1. A special case of Formula (58) was first given by Gudermann 1838 [9, p. 21:19]. Two special cases of Formula (60) were first given by Gudermann 1838 [9, p. 18:4, p. 21:18].

Put $\mathrm{ps}_i \equiv \mathrm{ps}(u_i, k),\ i = 1, 2$, and similar notation for the other functions.

Theorem 3.2. *Formulas for elliptic functions corresponding to product formulas for trigonometric functions.*

$$\begin{aligned} \mathrm{ps}(u_1+u_2,k) + \mathrm{ps}(u_1-u_2,k) &= \frac{2\mathrm{ps}_1\mathrm{qs}_2\mathrm{rs}_2}{\mathrm{ps}_2^2 - \mathrm{ps}_1^2} \\ \mathrm{ps}(u_1-u_2,k) - \mathrm{ps}(u_1+u_2,k) &= \frac{2\mathrm{ps}_2\mathrm{qs}_1\mathrm{rs}_1}{\mathrm{ps}_2^2 - \mathrm{ps}_1^2} \end{aligned} \tag{61}$$

Proof. Use Formulas (57). □

A special case of Formula (61) was first given by Gudermann 1838 [10, 2: p. 151].

Theorem 3.3. *Formulas for elliptic functions corresponding to product formulas for trigonometric functions.*

$$\begin{aligned} \mathrm{sp}(u_1+u_2,k) + \mathrm{sp}(u_1-u_2,k) &= \frac{2\mathrm{sp}_1\mathrm{qp}_2\mathrm{rp}_2}{1 - \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathrm{sp}_2^2} \\ \mathrm{sp}(u_1+u_2,k) - \mathrm{sp}(u_1-u_2,k) &= \frac{2\mathrm{sp}_2\mathrm{qp}_1\mathrm{rp}_1}{1 - \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathrm{sp}_2^2} \end{aligned} \tag{62}$$

Proof. Use Formulas (58). □

Special cases of Formula (62) were first given by Legendre [11, p. 4] 1828 , Jacobi 1829 [12, p. 191], Laurent [13, p. 95] and by Gudermann 1838 [10, 1: p. 151, 7: p. 152].

Theorem 3.4. *Formulas for elliptic functions corresponding to product formulas for trigonometric functions.*

$$\begin{aligned} \mathrm{pq}(u_1+u_2,k) + \mathrm{pq}(u_1-u_2,k) &= \frac{2\mathrm{pq}_1\mathrm{pq}_2}{1 + \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})\mathrm{sq}_1^2\mathrm{sq}_2^2} \\ \mathrm{pq}(u_1+u_2,k) - \mathrm{pq}(u_1-u_2,k) &= \frac{2\triangle(\mathrm{p,q})\mathrm{sq}_1\mathrm{rq}_1\mathrm{sq}_2\mathrm{rq}_2}{1 + \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})\mathrm{sq}_1^2\mathrm{sq}_2^2} \end{aligned} \tag{63}$$

Proof. Use formula (60). □

Special cases of Formula (63) were first given by Gudermann 1838 [10, 3: p. 151, 4,5: p. 152].

Theorem 3.5.

$$\mathrm{sp}(u_1+u_2,k)\mathrm{sp}(u_1-u_2,k)=\frac{\mathrm{sp}_1^2-\mathrm{sp}_2^2}{1-\triangle(\mathrm{p,q})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathrm{sp}_2^2} \tag{64}$$

$$\mathrm{pq}(u_1+u_2,k)\mathrm{pq}(u_1-u_2,k)=\frac{1+\triangle(\mathrm{q,p})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathrm{sp}_2^2}{1+\triangle(\mathrm{p,q})\triangle(\mathrm{q,r})\mathrm{sq}_1^2\mathrm{sq}_2^2} \tag{65}$$

Proof. Use Formulas (61), (62), and (63). □

Special cases of Formulas (64) and (65) were first given by Jacobi 1829 [12, p. 209] and by Gudermann 1838 [10, p. 153].

Theorem 3.6. *Put $f_i = f(2u_i), i = 1, 2$. Then we have*

$$\mathrm{pq}(u_1+u_2,k)\mathrm{pq}(u_1-u_2,k)=\frac{\mathrm{pq}_1\mathrm{rq}_2+\mathrm{rq}_1\mathrm{pq}_2}{\mathrm{rq}_1+\mathrm{rq}_2} \tag{66}$$

$$\frac{\mathrm{pq}\,(u_1-u_2,k)}{\mathrm{pq}(u_1+u_2,k)}=\frac{\mathrm{sq}_1+\mathrm{sq}_2}{\mathrm{sq}_1\mathrm{pq}_2+\mathrm{sq}_2\mathrm{pq}_1}=\frac{\mathrm{sq}_1\mathrm{pq}_2+\mathrm{sq}_2\mathrm{pq}_1}{\mathrm{sq}_1+\mathrm{sq}_2} \tag{67}$$

Special cases of Formula (66) were given by Gudermann 1838 [10, (4), (7) p. 156]. Special cases of Formula (67) were given by Gudermann 1838 [10, (13), (15) p. 158].

Theorem 3.7. *For $p = n$, Formula (68) holds unaltered. In the two other cases we only use one of the two factors* qn, rn *in either numerator or denominator. This gives the six formulas*

$$\frac{\mathrm{sp}(u_1-u_2,k)}{\mathrm{sp}(u_1+u_2,k)}=\frac{\mathrm{sn}(2u_1)\mathrm{qn}(2u_2)-\mathrm{sn}(2u_2)\mathrm{qn}(2u_1)}{\mathrm{sn}(2u_1)\mathrm{rn}(2u_2)+\mathrm{sn}(2u_2)\mathrm{rn}(2u_1)} \tag{68}$$

Special case of Formula (68) were given by Gudermann 1838 [10, (8), (9), (11) p. 157].

Theorem 3.8. *Bisection*

$$\mathrm{sp}^2(\frac{u}{2},k)=\frac{1}{\triangle(\mathrm{p,q})}\frac{1-\mathrm{qp}}{1+\mathrm{rp}} \tag{69}$$

Special cases of the following formulas were given by Gudermann 1838 [10, p. 148 f].

Theorem 3.9. *Addition formulas [5, p. 248]. Put $\mathrm{ps}_i \equiv \mathrm{ps}(u_i,k),\ i = 1, 2$, and similar notation for the other functions.*

$$1+\mathrm{pq}(u_1\pm u_2,k)=\frac{(\mathrm{pq}_1+\mathrm{pq}_2)(\mathrm{sr}_1\mp\mathrm{sr}_2)}{\mathrm{sr}_1\mathrm{pq}_2\mp\mathrm{sr}_2\mathrm{pq}_1} \tag{70}$$

$$1-\mathrm{pq}(u_1\pm u_2,k)=\frac{(\mathrm{pq}_1-\mathrm{pq}_2)(\mathrm{sr}_1\pm\mathrm{sr}_2)}{\pm\mathrm{sr}_2\mathrm{pq}_1-\mathrm{sr}_1\mathrm{pq}_2} \tag{71}$$

Half of the following conjecture was given in [2, p. 109]. We have the well-known formulas

$$\frac{1}{2}\sqrt{\frac{1+\cos x}{1-\cos x}}+\frac{1}{2}\sqrt{\frac{1-\cos x}{1+\cos x}}=\frac{1}{\sin x} \tag{72}$$

$$\frac{1}{2}\sqrt{\frac{1+\cos x}{1-\cos x}}-\frac{1}{2}\sqrt{\frac{1-\cos x}{1+\cos x}}=\cot x \tag{73}$$

$$\frac{1}{2}\sqrt{\frac{1+\cosh x}{\cosh x-1}}-\frac{1}{2}\sqrt{\frac{\cosh x-1}{1+\cosh x}}=\frac{1}{\sinh x} \tag{74}$$

$$\frac{1}{2}\sqrt{\frac{1+\cosh x}{\cosh x-1}}+\frac{1}{2}\sqrt{\frac{\cosh x-1}{1+\cosh x}}=\coth x \tag{75}$$

Conjecture 3.10. We have the twelve formulas

$$\frac{\sqrt{\triangle(\mathrm{p,q})}}{2}\sqrt{\frac{\mathrm{pq}+1}{\mathrm{pq}-1}}+\frac{\sqrt{\triangle(\mathrm{p,q})}}{2}\sqrt{\frac{\mathrm{pq}-1}{\mathrm{pq}+1}}=f \tag{76}$$

$$\frac{\sqrt{\triangle(\mathrm{p,q})}}{2}\sqrt{\frac{\mathrm{pq}+1}{\mathrm{pq}-1}}-\frac{\sqrt{\triangle(\mathrm{p,q})}}{2}\sqrt{\frac{\mathrm{pq}-1}{\mathrm{pq}+1}}=g \tag{77}$$

where $f,g\in\{\mathrm{qs},\mathrm{ps}\}$. We choose the right hand side that has the correct limits for $\lim_{k\to 0^+}$ and $\lim_{k\to 1^-}$ in Formulas (72)–(75). Ten of the formulas have one limit among these four formulas, and the remaining two (with $\mathrm{nc}u$) have two limits.

To be able to compute the inverses of the elliptic functions, we must first prove a number of Möbius transformations between squares of elliptic functions which govern their transformations. Most of these can be summarized in the formulas

$$\mathrm{pq}^2=\frac{\triangle(\mathrm{p,r})+\mathrm{rs}^2}{\triangle(\mathrm{q,r})+\mathrm{rs}^2} \tag{78}$$

$$\mathrm{sr}^2=\frac{\mathrm{pq}^2-1}{\triangle(\mathrm{p,r})-\triangle(\mathrm{q,r})\mathrm{pq}^2} \tag{79}$$

$$\mathrm{sp}^2=\frac{1-\mathrm{pq}^2}{\triangle(\mathrm{q,p})\mathrm{pq}^2} \tag{80}$$

$$\mathrm{pq}^2=\frac{\triangle(\mathrm{p,r})\mathrm{qr}^2+\triangle(\mathrm{q,p})}{\triangle(\mathrm{q,r})\mathrm{qr}^2} \tag{81}$$

Formula (81) is its own inverse. With these formulas we can prove integral formulas for the twelve inverse elliptic functions like in [14, p. 102] and [15, 22.15]. The integral formulas for the inverses of the elliptic functions are Schwarz-Christoffel mappings from the periodic rectangle of each elliptic function. The Formulas (78) to (81) do not map each periodic rectangle to the next, but if we take all rectangles in the vicinity of the origin and agree to start each equation solving with the prerequisite $\Re(z)>0$, the formulas give correct values on the Riemann sphere.

We conclude with a few formulas with the function $\mathrm{artanh}(x)$, which again generalize Gudermanns results.

Theorem 3.11. *We have the eighteen formulas*

$$\log\sqrt{\frac{1+\mathrm{pq}(u_1\pm u_2,k)}{1-\mathrm{pq}(u_1\pm u_2,k)}}=\mathrm{artanh}\left(\frac{\mathrm{pq}_2}{\mathrm{pq}_1}\right)\pm\mathrm{artanh}\left(\frac{\mathrm{sr}_2}{\mathrm{sr}_1}\right) \tag{82}$$

$$\log\sqrt{\frac{1+\mathrm{pq}(u_1-u_2,k)}{1+\mathrm{pq}(u_1+u_2,k)}}=\mathrm{artanh}\left(\frac{\mathrm{sr}_2}{\mathrm{sr}_1}\right)-\mathrm{artanh}\left(\frac{\mathrm{sr}_2\mathrm{pq}_1}{\mathrm{sr}_1\mathrm{pq}_2}\right) \tag{83}$$

$$\log\sqrt{\frac{1-\mathrm{pq}(u_1+u_2,k)}{1-\mathrm{pq}(u_1-u_2,k)}}=\mathrm{artanh}\left(\frac{\mathrm{sr}_2}{\mathrm{sr}_1}\right)+\mathrm{artanh}\left(\frac{\mathrm{sr}_2\mathrm{pq}_1}{\mathrm{sr}_1\mathrm{pq}_2}\right) \tag{84}$$

Proof. Use Formulas (70) and (71). □

Special cases were given by Gudermann 1838 [10, p. 150].

Theorem 3.12. *Assume that $p \neq n$ and $q = n$. Then we have the two formulas*

$$\log\sqrt{\frac{\mathrm{sp}(a+b)}{\mathrm{sp}(a-b)}} = \frac{1}{2}\mathrm{artanh}\left(\frac{\mathrm{sn}(2b)}{\mathrm{sn}(2a)}\right) + \frac{1}{2}\mathrm{artanh}\left(\frac{\mathrm{sr}(2b)}{\mathrm{sr}(2a)}\right) \tag{85}$$

Proof. Use Formula (68). □

Special cases of Formula (85) were given by Gudermann 1838 [10, p. 157].

Theorem 3.13. *We have the six formulas*

$$\log\sqrt{\frac{\mathrm{pq}(a+b)}{\mathrm{pq}(a-b)}} = \frac{1}{2}\mathrm{artanh}\left(\frac{\mathrm{sq}(2b)\mathrm{pq}(2a)}{\mathrm{sq}(2a)\mathrm{pq}(2b)}\right) - \frac{1}{2}\mathrm{artanh}\left(\frac{\mathrm{sq}(2b)}{\mathrm{sq}(2a)}\right) \tag{86}$$

Proof. Use Formula (67). □

Special cases of Formula (86) were given by Gudermann 1838 [10, 14, 16 p. 158].

4. Hyperbolic Modular Functions

We will again consider two inverse functions.

Definition 4. Let $0 < k < 1$ and consider the following hyperbolic integral.

$$u \equiv F(x) \equiv \int_0^x \frac{dt}{\sqrt{1+t^2}\sqrt{1+k^2t^2}} \tag{87}$$

Now put $x = \mathcal{SN}(u) \equiv F^{-1}(u)$, the hyperbolic modular sine for the module k. Then we further define

$$\mathcal{CN}(u) \equiv \sqrt{1+x^2} \tag{88}$$

the hyperbolic modular cosine for the module k

$$\mathcal{SC}(u) \equiv \frac{x}{\sqrt{1+x^2}} \tag{89}$$

the hyperbolic modular tangent for the module k

$$\mathcal{DN}(u) \equiv \sqrt{1+k^2x^2} \tag{90}$$

the hyperbolic difference for the module k.

Definition 5. Just like for the elliptic functions, we use the Glaischer notation as follows:

$$\mathcal{NS}u \equiv \frac{1}{\mathcal{SN}u},\ \mathcal{NC}u \equiv \frac{1}{\mathcal{CN}u},\ \mathcal{ND}u \equiv \frac{1}{\mathcal{DN}u},\ \mathcal{CD}u \equiv \frac{\mathcal{CN}u}{\mathcal{DN}u},\ \textit{etc.} \tag{91}$$

We find that

$$\begin{aligned}
&\lim_{k\to 0^+} \mathcal{SN}x = \lim_{k\to 0^+} \mathcal{SD}x = \sinh x,\ \lim_{k\to 0^+} \mathcal{SN}x = \cosh x,\ \lim_{k\to 0^+} \mathcal{DN}x = 1 \\
&\lim_{k\to 0^+} \mathcal{SC}x = \tanh x \lim_{k\to 1^-} \mathcal{SN}x = \tan x \\
&\lim_{k\to 1^-} \mathcal{CN}x = \lim_{k\to 1^-} \mathcal{DN}x = \frac{1}{\cos x},\ \lim_{k\to 1^-} \mathcal{SC}x = \lim_{k\to 1^-} \mathcal{SD}x = \sin x
\end{aligned} \tag{92}$$

Definition 6. We put

$$\{\mathcal{P}, \mathcal{Q}, \mathcal{R}\} \equiv \{\mathcal{C}, \mathcal{D}, \mathcal{N}\} \tag{93}$$

Furthermore, we put

$$\triangle(\mathcal{P}, \mathcal{Q}) \equiv \mathcal{PS}^2 - \mathcal{QS}^2 \tag{94}$$

Definition 7. The Gudermannian function $l(x)$ relates the circular functions and hyperbolic functions without using complex numbers. It is given by

$$\begin{aligned}
l(x) &\equiv \int_0^x \frac{dt}{\cosh t} = \arcsin(\tanh x) = \arctan(\sinh x) \\
&= 2\arctan\left[\tanh\left(\tfrac{1}{2}x\right)\right] = 2\arctan(e^x) - \tfrac{1}{2}\pi
\end{aligned} \tag{95}$$

The inverse function or the Mercator function is given by

$$\begin{aligned}
\mathcal{L}(x) &\equiv \int_0^x \frac{dt}{\cos t} = \log\frac{1+\sin x}{\cos x} = \log\sqrt{\frac{1+\sin x}{1-\sin x}} \\
&= \log[\tan x + \sec x] = \log\left[\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}x\right)\right] \\
&= \operatorname{artanh}(\sin x) = \operatorname{arsinh}(\tan x)
\end{aligned} \tag{96}$$

The function $\mathcal{L}(x)$ is the inverse of $l(x)$. Legendre calculated tables for this function.

Since $\mathcal{L}\varphi > \varphi$ it follows that [9, p. 27]

$$\sinh u < \mathcal{SN}u < \sinh \mathcal{L}u \tag{97}$$

We also find that

$$\cosh u < \mathcal{CN}u < \cosh \mathcal{L}u,\ \tanh u < \mathcal{SC}u < \tanh \mathcal{L}u \tag{98}$$

The hyperbolic elliptic functions can also be transformed to the hyperbolic potential functions by putting

$$\mathcal{SN}u = \sinh\psi,\ \mathcal{CN}u = \cosh\psi \text{ and } \mathcal{SC}u = \tanh\psi \tag{99}$$

The arc ψ is is called the *hyperbolic amplitude* of the argument u for the module kt; or $\psi = \mathcal{A}\mathrm{m}\ u$, and vice versa $u = \mathcal{A}\mathrm{rg}\ \mathcal{A}\mathrm{m}\ (\psi)$.

Next we have [9, p. 27]

$$\mathrm{D}\mathcal{SN}u = \mathcal{CN}u\mathcal{DN}u,\ \mathrm{D}\mathcal{CN}u = \mathcal{SN}u\mathcal{DN}u \tag{100}$$

$$\mathrm{D}\mathcal{SC}u = \frac{\mathcal{DN}u}{\mathcal{CN}^2u},\ \mathrm{D}\mathcal{DN}u = k^2\mathcal{SN}u\mathcal{CN}u \tag{101}$$

Below is a list of the inverse of the four hyperbolic modular functions:

When $t = \mathcal{SN}u$, so $u = \mathcal{A}\mathrm{rg}\mathcal{SN}t$;
When $t = \mathcal{CN}u$, so $u = \mathcal{A}\mathrm{rg}\mathcal{CN}t$;
When $t = \mathcal{SC}u$, so $u = \mathcal{A}\mathrm{rg}\mathcal{SC}t$;
When $t = \mathcal{DN}u$, so $u = \mathcal{A}\mathrm{rg}\mathcal{DN}t$;

We have

$$\mathrm{D}\mathcal{A}\mathrm{m}u = \mathcal{DN}u \tag{102}$$

Formula (87) is equivalent to

$$\mathcal{A}\mathrm{rg}\mathcal{A}\mathrm{m}(t) = \int_0^{\mathrm{arsinh}t} \frac{d\psi}{\sqrt{1 + k^2 \mathrm{sinh}^2(\psi)}} \tag{103}$$

The poles and periods are shown in the following table:

half period	poles iK	poles $K' + iK$	poles K'	poles 0	periods
iK	$\mathcal{SC}$	$\mathcal{ND}$	$\mathcal{DN}$	$\mathcal{CS}$	$2iK, 4K' + 4iK, 4K'$
$K' + iK$	$\mathcal{NC}$	$\mathcal{SD}$	$\mathcal{CN}$	$\mathcal{DS}$	$4iK, 2K' + 2iK, 4K'$
K'	$\mathcal{DC}$	$\mathcal{CD}$	$\mathcal{SN}$	$\mathcal{NS}$	$4iK, 4K' + 4iK, 2K'$

We have the following special values for the twelve hyperbolic modular functions:

u	$\mathcal{SN}$	$\mathcal{CN}$	$\mathcal{DN}$	$\mathcal{SC}$	$\mathcal{SD}$	$\mathcal{ND}$	$\mathcal{CD}$	$\mathcal{CS}$	$\mathcal{DS}$	$\mathcal{NC}$	$\mathcal{DC}$	$\mathcal{NS}$
0	0	1	1	0	0	1	1	∞	∞	1	1	∞
iK	i	0	k'	∞	$\frac{i}{k'}$	$\frac{1}{k'}$	0	0	$-ik'$	∞	∞	$-i$
K'	∞	∞	∞	1	$\frac{1}{k}$	0	$\frac{1}{k}$	1	k	0	k	0
$2K'$	0	-1	-1	0	0	-1	1	∞	∞	-1	1	∞
$2K' + 2iK$	0	-1	-1	0	0	-1	-1	∞	∞	-1	-1	∞
$2iK$	0	-1	1	0	0	1	-1	∞	∞	-1	-1	∞
$K' + iK$	$\frac{i}{k}$	$\frac{ik'}{k}$	0	$\frac{1}{k'}$	∞	∞	∞	k'	0	$\frac{-ik}{k'}$	0	$-ik$

Just like the trigonometric and hyperbolic potential functions can be transformed to each other by multiplication by i, the trigonometric and hyperbolic modular functions can also be mapped to each other with utter avoidance of imaginary forms [9, p. 31]. These transformations look like this (we use the Glaisher notation and $'$ means the module k'):

$$\begin{cases} \mathcal{SN}ui = i\mathrm{sn}u, \ \mathcal{CN}ui = \mathrm{cn}u \\ \mathcal{SC}ui = i\mathrm{sc}u, \ \mathcal{DN}ui = \mathrm{dn}u \end{cases} \tag{104}$$

$$\begin{cases} \mathrm{sn}ui = i\mathcal{SN}u, \ \mathrm{cn}ui = \mathcal{CN}u \\ \mathrm{sc}ui = i\mathcal{SC}u, \ \mathrm{dn}ui = \mathcal{DN}u \end{cases} \tag{105}$$

Then we have by the Jacobi imaginary transformation [12, p.85], compare with [16, p.596: 17.4.41].

$$\begin{cases} \mathcal{SN}u = \mathrm{sc}'u \\ \mathcal{CN}u = \mathrm{nc}'u \\ \mathcal{SC}u = \mathrm{sn}'u \\ \mathcal{DN}u = \mathrm{dc}'u \end{cases} \tag{106}$$

This implies

$$\mathcal{A}\mathrm{m}ui = i\mathrm{am}u,\ \mathrm{am}ui = i\mathcal{A}\mathrm{m}u \tag{107}$$

To be able to compute the inverses of the hyperbolic modular functions, we must first prove a large number $<\ 132$ of Möbius transformations between squares of hyperbolic modular functions which govern their transformations. Most of these can be summarized in the formulas

$$\mathcal{PQ}^2 = \frac{\triangle(\mathcal{R},\mathcal{P}) + \mathcal{RS}^2}{\triangle(\mathcal{R},\mathcal{Q}) + \mathcal{RS}^2} \tag{108}$$

$$\mathcal{SR}^2 = \frac{\mathcal{PQ}^2 - 1}{\triangle(\mathcal{R},\mathcal{P}) - \triangle(\mathcal{R},\mathcal{Q})\mathcal{PQ}^2} \tag{109}$$

$$\mathcal{SP}^2 = \frac{1 - \mathcal{PQ}^2}{\triangle(\mathcal{P},\mathcal{Q})\mathcal{PQ}^2} \tag{110}$$

$$\mathcal{PQ}^2 = \frac{\triangle(\mathcal{P},\mathcal{R})\mathcal{QR}^2 + \triangle(\mathcal{Q},\mathcal{P})}{\triangle(\mathcal{Q},\mathcal{R})\mathcal{QR}^2} \tag{111}$$

Formulas (108) and (109) are inverse to each other. Formula (111) is its own inverse. It is the same as the previous Formula (81).

We can now easily prove the following integral formulas:

$$\mathcal{SN}^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1+t^2}\sqrt{1+k^2t^2}}, 0 < x \leq \infty \tag{112}$$

$$\mathcal{CN}^{-1}(x) = \int_1^x \frac{dt}{\sqrt{t^2-1}\sqrt{k'^2+k^2t^2}},\ \infty \geq x > 1 \tag{113}$$

$$\mathcal{SC}^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k'^2t^2}}, 0 < x \leq 1 \tag{114}$$

$$\mathcal{DN}^{-1}(x) = \int_1^x \frac{dt}{\sqrt{t^2-1}\sqrt{t^2-k'^2}}, \infty \geq x > 1 \tag{115}$$

$$\mathcal{SD}^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1+k'^2t^2}\sqrt{1-k^2t^2}}, \tfrac{1}{k} \geq x > 0 \tag{116}$$

$$\mathcal{DC}^{-1}(x) = \int_x^1 \frac{dt}{\sqrt{t^2-k^2}\sqrt{1-t^2}}, k \leq x < 1 \tag{117}$$

$$\mathcal{DS}^{-1}(x) = \int_x^\infty \frac{dt}{\sqrt{k'^2+t^2}\sqrt{t^2-k^2}}, x \geq k > 0 \tag{118}$$

$$\mathcal{CD}^{-1}(x) = \int_1^x \frac{dt}{\sqrt{t^2-1}\sqrt{1-k^2t^2}}, \tfrac{1}{k} \leq x < 1 \tag{119}$$

$$\mathcal{CS}^{-1}(x) = \int_x^1 \frac{dt}{\sqrt{1-t^2}\sqrt{k'^2-t^2}}, -\infty \leq x < 1 \tag{120}$$

$$\mathcal{NS}^{-1}(x) = \int_x^\infty \frac{dt}{\sqrt{1+t^2}\sqrt{k^2+t^2}}, \ \infty > x \geq 0 \tag{121}$$

$$\mathcal{NC}^{-1}(x) = \int_x^1 \frac{dt}{\sqrt{1-t^2}\sqrt{k^2+k'^2t^2}}, \ 0 \leq x < 1 \tag{122}$$

$$\mathcal{ND}^{-1}(x) = \int_x^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k'^2t^2}}, \ 0 \leq x < 1 \tag{123}$$

Formulas (112)–(115) can be found in [9, p. 28].

The addition formulas for hyperbolic modular functions are

Theorem 4.1. *Put* $\mathcal{PS}_i \equiv \mathcal{PS}(u_i, k), \ i = 1, 2,$ *and similar notation for the other functions.*

$$\mathcal{PS}(u_1+u_2,k) = \frac{\mathcal{PS}_1\mathcal{QS}_2\mathcal{RS}_2 - \mathcal{PS}_2\mathcal{QS}_1\mathcal{RS}_1}{\mathcal{PS}_2^2 - \mathcal{PS}_1^2} = \frac{\triangle(\mathcal{P},\mathcal{Q})\triangle(\mathcal{P},\mathcal{R}) - \mathcal{PS}_1^2\mathcal{PS}_2^2}{\mathcal{PS}_1\mathcal{QS}_2\mathcal{RS}_2 + \mathcal{PS}_2\mathcal{QS}_1\mathcal{RS}_1} \tag{124}$$

$$\begin{aligned}\mathcal{SP}(u_1+u_2,k) &= \frac{\mathcal{SP}_1^2 - \mathcal{SP}_2^2}{\mathcal{SP}_1\mathcal{QP}_2\mathcal{RP}_2 - \mathcal{SP}_2\mathcal{QP}_1\mathcal{RP}_1} \\ &= \frac{\mathcal{SP}_1\mathcal{QP}_2\mathcal{RP}_2 + \mathcal{SP}_2\mathcal{QP}_1\mathcal{RP}_1}{1 - \triangle(\mathcal{P},\mathcal{Q})\triangle(\mathcal{P},\mathcal{R})\mathcal{SP}_1^2\mathcal{SP}_2^2}\end{aligned} \tag{125}$$

$$\begin{aligned}\mathcal{PQ}(u_1+u_2,k) &= \frac{\mathcal{PS}_1\mathcal{QS}_2\mathcal{RS}_2 - \mathcal{PS}_2\mathcal{QS}_1\mathcal{RS}_1}{\mathcal{QS}_1\mathcal{PS}\mathrm{ps}_2\mathcal{RS}_2 - \mathcal{QS}_2\mathcal{PS}_1\mathcal{RS}_1} \\ &= \frac{\mathcal{PS}_1\mathcal{QS}_1\mathcal{PS}_2\mathcal{QS}_2 - \triangle(\mathcal{P},\mathcal{Q})\mathcal{RS}_1\mathcal{RS}_2}{\mathcal{QS}_1^2\mathcal{QS}_2^2 + \triangle(\mathcal{P},\mathcal{Q})\triangle(\mathcal{Q},\mathcal{R})}\end{aligned} \tag{126}$$

$$\begin{aligned}\mathcal{PQ}(u_1+u_2,k) &= \frac{\mathcal{PQ}_1\mathcal{SQ}_1\mathcal{RQ}_2 - \mathcal{PQ}_2\mathcal{SQ}_2\mathcal{RQ}_1}{\mathcal{PQ}_2\mathcal{SQ}_1\mathcal{RQ}_2 - \mathcal{PQ}_1\mathcal{SQ}_2\mathcal{RQ}_1} \\ &= \frac{\mathcal{PQ}_1\mathcal{PQ}_2 - \triangle(\mathcal{P},\mathcal{Q})\mathcal{SQ}_1\mathcal{RQ}_1\mathcal{SQ}_2\mathcal{RQ}_2}{1 + \triangle(\mathcal{P},\mathcal{Q})\triangle(\mathcal{Q},\mathcal{R})\mathcal{SQ}_1^2\mathcal{SQ}_2^2}\end{aligned} \tag{127}$$

Proof. Use Formula (105). □

Special cases of Formulas (124) to (127) were given in [17]. Formulas (124) and (125) are inverse to each other.

Theorem 4.2. *Addition formulas for complex arguments. Put* $\mathrm{ps}_1 \equiv \mathrm{ps}(u_1,k), \ \mathcal{PS} \equiv \mathcal{PS}(u_2,k)$, *and similar notation for the other functions.*

$$\mathrm{ps}(u_1+iu_2,k) = \frac{\mathrm{ps}_1\mathcal{QSRS} - i\mathcal{PS}\mathrm{qs}_1\mathrm{rs}_1}{\mathcal{PS}^2 + \mathrm{ps}_1^2} = \frac{\mathcal{PS}^2\mathrm{ps}_2^2 + \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})}{\mathrm{ps}_1\mathcal{QSRS} + i\mathcal{PS}\mathrm{qs}_1\mathrm{rs}_1} \tag{128}$$

$$\mathrm{sp}(u_1+iu_2,k) = \frac{\mathcal{SP}^2 + \mathrm{sp}_1^2}{\mathrm{sp}_1\mathcal{QPRP} - i\mathcal{SP}\mathrm{qp}_1\mathrm{rp}_1} = \frac{\mathrm{sp}_1\mathcal{QPRP} + i\mathcal{SP}\mathrm{qp}_1\mathrm{rp}_1}{1 + \triangle(\mathrm{p,q})\triangle(\mathrm{p,r})\mathrm{sp}_1^2\mathcal{SP}^2} \tag{129}$$

$$\mathrm{pq}(u_1+iu_2,k) = \frac{\mathrm{ps}_1\mathcal{QSRS} - i\mathcal{PS}\mathrm{qs}_1\mathrm{rs}_1}{\mathrm{qs}_1\mathcal{PSRS} - i\mathcal{QS}\mathrm{ps}_1\mathrm{rs}_1} = \frac{\mathrm{ps}_1\mathrm{qs}_1\mathcal{PSQS} + i\triangle(\mathrm{p,q})\mathrm{rs}_1\mathcal{RS}}{\mathrm{qs}_1^2\mathcal{QS}^2 - \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})} \tag{130}$$

$$\mathrm{pq}(u_1+iu_2,k) = \frac{\mathrm{pq}_1\mathrm{sq}_1\mathcal{RQ} - i\mathcal{PQSQ}\mathrm{rq}_1}{\mathcal{PQRQ}\mathrm{sq}_1 - i\mathrm{pq}_1\mathcal{SQ}\mathrm{rq}_1} = \frac{\mathrm{pq}_1\mathcal{PQ} + i\triangle(\mathrm{p,q})\mathrm{sq}_1\mathrm{rq}_1\mathcal{SQRQ}}{1 - \triangle(\mathrm{p,q})\triangle(\mathrm{q,r})\mathrm{sq}_1^2\mathcal{SQ}^2} \tag{131}$$

Proof. Use Formula (105).
□

Special cases of Formulas (128) to (131) were first given by Gudermann 1838 [9, p. 33].

5. The Peeta Functions

The Peeta functions were first introduced by Gudermann [18, p. 79], to be able to express the hyperbolic modular functions by theta functions in a manner similar to the elliptic functions.

Eagle [7, p. 81] has also spoken of the great importance of these functions.

The four Peeta functions are defined as follows:

Definition 8.

$$\psi_1(z,q) \equiv -i\theta_1(iz,q); \psi_2(z,q) \equiv \theta_2(iz,q); \psi_3(z,q) \equiv \theta_3(iz,q); \psi_4(z,q) \equiv \theta_4(iz,q) \tag{132}$$

This is equivalent to

$$\begin{aligned} \psi_1(z,q) &\equiv 2\sum_{n=0}^{\infty}(-1)^n \mathrm{QE}((n+\frac{1}{2})^2)\sinh(2n+1)z \\ \psi_2(z,q) &\equiv 2\sum_{n=0}^{\infty}\mathrm{QE}((n+\frac{1}{2})^2)\cosh(2n+1)z \\ \psi_3(z,q) &\equiv 1+2\sum_{n=1}^{\infty}\mathrm{QE}(n^2)\cosh(2nz) \\ \psi_4(z,q) &\equiv 1+2\sum_{n=1}^{\infty}(-1)^n\mathrm{QE}(n^2)\cosh(2nz) \end{aligned} \tag{133}$$

where $\mathrm{QE}(x) \equiv q^x$, $q \equiv \exp(\pi i t), t \in \mathfrak{U}$.

Another definition is

$$\begin{aligned} \psi_1(z,q) &= 2q^{\frac{1}{4}}\sinh(x)\prod_{k=1}^{\infty}(1-q^{2k})(1-2q^{2k}\cosh(2z)+q^{4k}) \\ \psi_2(z,q) &= 2q^{\frac{1}{4}}\cosh(z)\prod_{k=1}^{\infty}(1-q^{2k})(1+2q^{2k}\cosh(2z)+q^{4k}) \\ \psi_3(z,q) &= \prod_{k=1}^{\infty}(1-q^{2k})(1+2q^{2k-1}\cosh(2z)+q^{4k-2}) \\ \psi_4(z,q) &= \prod_{k=1}^{\infty}(1-q^{2k})(1-2q^{2k-1}\cosh(2z)+q^{4k-2}) \end{aligned} \tag{134}$$

This implies [18, p. 86]

$$
\begin{aligned}
\log \psi_1(z,q) &= f_1(q) + \log\sinh(z) - \sum_{m=1}^{\infty} \frac{2q^{2m}}{m(1-q^{2m})}\cosh(2mz) \\
\log \psi_2(z,q) &= f_2(q) + \log\cosh(z) + \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}q^{2m}}{m(1-q^{2m})}\cosh(2mz) \\
\log \psi_3(z,q) &= f_3(q) + \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}q^{m}}{m(1-q^{2m})}\cosh(2mz) \\
\log \psi_4(z,q) &= f_4(q) - \sum_{m=1}^{\infty} \frac{2q^{m}}{m(1-q^{2m})}\cosh(2mz)
\end{aligned}
\tag{135}
$$

The hyperbolic modular functions can be expressed in terms of Peeta functions as follows [18, p. 79]:

Theorem 5.1.

$$
\begin{aligned}
\mathcal{SN}(u) &= k^{-\frac{1}{2}}\frac{\psi_1(z,q)}{\psi_4(z,q)} \\
\mathcal{CN}(u) &= \left(\frac{k'}{k}\right)^{\frac{1}{2}}\frac{\psi_2(z,q)}{\psi_4(z,q)} \\
\mathcal{DN}(u) &= k'^{\frac{1}{2}}\frac{\psi_3(z,q)}{\psi_4(z,q)}
\end{aligned}
\tag{136}
$$

The Peeta functions have the following periodic properties:

$y=$	z	$z+\frac{1}{2}\log q$	$z+\log q$	$z+\frac{\pi i}{2}$	$z+\pi i$
$\psi_1(y)$	$\psi_1(z)$	$q^{-\frac{1}{4}}e^z\psi_4(z)$	$-q^{-1}e^{2z}\psi_1(z)$	$-i\psi_2(z)$	$-\psi_1(z)$
$\psi_2(y)$	$\psi_2(z)$	$q^{-\frac{1}{4}}e^z\psi_3(z)$	$q^{-1}e^{2z}\psi_2(z)$	$-i\psi_1(z)$	$-\psi_2(z)$
$\psi_3(y)$	$\psi_3(z)$	$q^{-\frac{1}{4}}e^z\psi_2(z)$	$q^{-1}e^{2z}\psi_3(z)$	$\psi_4(z)$	$\psi_3(z)$
$\psi_4(y)$	$\psi_4(z)$	$-q^{-\frac{1}{4}}e^z\psi_1(z)$	$-q^{-1}e^{2z}\psi_4(z)$	$\psi_3(z)$	$\psi_4(z)$

Theorem 5.2. *The Peeta functions have the following zeros:*

$$
\begin{aligned}
&\psi_1(m\pi i + n\log q, q) = 0 \\
&\psi_2(\tfrac{\pi i}{2} + m\pi i + n\log q, q) = 0 \\
&\psi_3(\tfrac{\pi i}{2} + \tfrac{1}{2}\log q + m\pi i + n\log q, q) = 0 \\
&\psi_4(\tfrac{1}{2}\log q + m\pi i + n\log q, q) = 0
\end{aligned}
\tag{137}
$$

where $m, n \in \mathbb{Z}$.

Theorem 5.3. *The four Peeta functions satisfy the following heat equation*

$$
\frac{\partial^2 \psi_i(z,q)}{\partial z^2} = 4q\frac{\partial \psi_i(z,q)}{\partial q},\ i = 1,\ldots,4
\tag{138}
$$

Corollary 5.4. *Generalization of heat equation to n variables. Put*

$$\triangle_N \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2},\ u(x_1, \ldots, x_N, q) \equiv \prod_{i=1}^{N} \psi_{k(i)}(x_i, q),\ k(i) \in \{1, \ldots, 4\} \tag{139}$$

Then

$$\triangle_N u = 4q \frac{\partial u}{\partial q} \tag{140}$$

Proof. Use [19, p. 390]. □

By scaling in q, we can transform Formula (140) to other heat equations.

Example 1. Heat transfer in friction-free (non-viscous) fluid flow.

$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial u}{\partial t} \tag{141}$$

Several formulas for theta functions, like logarithmic derivative, immediately transfer to Peeta functions.

Conflicts of Interest

The author declare no conflict of interest.

References

1. Ernst, T. Die Jacobi-Gudermann-Glaisherschen elliptischen Funktionen nach Heine. *Hadronic. J.* **2010**, *33*, 273–302.
2. Gudermann, C. Theorie der Modular-Functionen and der Modular-Integrale: Reihen for die Logarithmen der Modular-Functionen, welche nach den Cosinus der Vielfachen von ηu and $\eta' u$ fortschreiten. *J. Reine Angew. Math.* **1840**, *20*, 103–167.
3. Glaisher, J.W.L. On the series which represent the twelve elliptic and the four Zeta functions. *Mess. Math.* **1888**, *18*, 1–84.
4. Ernst, T. *A Comprehensive Treatment of q-Calculus*; Springer: Basel, Switzerland, 2012.
5. Carlson, B.C. Symmetry in c, d, n of Jacobian elliptic functions. *J. Math. Anal. Appl.* **2004**, *299*, 242–253.
6. Enneper, A. *Elliptische Funktionen. Theorie and Geschichte. Zweite Auflage. Neu bearbeitet und herausgegeben von Felix Müller*; L. Nebert, 1890.
7. Eagle, A. *The Elliptic Functions as They Should Be: An Account, with Applications, of the Functions in a New Canonical Form*; Galloway and Porter, Ltd.: Cambridge, England, UK, 1958.
8. Gudermann, C. *Theorie der Modular-Functionen und der Modular-Integrale*; G. Reimer Publisher: Berlin, Germany, 1844.
9. Gudermann, C. Theorie der Modular-Functionen and der Modular-Integrale: Allgemeiner Character der Potenzial-Funktionen. *J. Reine Angew. Math.* **1838**, *18*, 1–54.
10. Gudermann, C. Theorie der Modular-Functionen and der Modular-Integrale: Die Modular-Functionen von $\frac{iK'}{2}$ and $\frac{K \pm iK'}{2}$. *J. Reine Angew. Math.* **1838**, *18*, 142–175.

11. Legendre, A.M. *Traité des fonctions elliptiques et des intégrales eulériennes*; Gauthier-Villars: Paris, France, 1828.
12. Jacobi, C.G.J. *Fundamenta Nova*; Königsberg, 1829.
13. Laurent, H. *Théorie élémentaire des fonctions elliptiques*; Gauthier-Villars: Paris, France, 1880.
14. Neville, E. *Elliptic Functions: A Primer*; Pergamon Press: Oxford, UK, 1971.
15. Olver, F.W.J. *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010.
16. Abramowitz, M.; Stegun, I. *Handbook of Mathematical Funktionen with Formulas, Graphs, and Mathematical Tables*; U.S. Department of Commerce: Washington, DC, USA, 1970.
17. Gudermann, C. Theorie der Modular-Functionen and der Modular-Integrale: Ausdruck von u and el$u$ durch am$u$ in Reihen, welche nach Potenzen des Moduls k fortschreiten. *J. Reine Angew. Math.* **1839**, *19*, 45–83.
18. Gudermann, C. Theorie der Modular-Functionen und der Modular-Integrale: Die Modular-Functionen, dargestellt als Producte unendlich vieler Factoren. *J. Reine Angew. Math.* **1840**, *20*, 62–87.
19. Widder, D.V. Series expansions of solutions of the heat equation in n dimensions. *Ann. Mat. Pura Appl.* **1961**, *55*, 389–409.