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Article

Yang–Baxter Equations, Computational Methods and Applications

Florin F. Nichita

Simion Stoilow Institute of Mathematics of the Romanian Academy,
21 Calea Grivitei Street, 010702 Bucharest, Romania; E-Mail: florin.nichita@imar.ro;
Tel.: +40-21-319-6506; Fax: +40-21-319-6505

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Abstract: Computational methods are an important tool for solving the Yang–Baxter equations (in small dimensions), for classifying (unifying) structures and for solving related problems. This paper is an account of some of the latest developments on the Yang–Baxter equation, its set-theoretical version and its applications. We construct new set-theoretical solutions for the Yang–Baxter equation. Unification theories and other results are proposed or proven.

Keywords: Yang–Baxter equation; computational methods; universal gate; non-associative structures; associative algebras; Jordan algebras; Lie algebras

MSC classifications: 6T10; 16T25; 17B01; 17B60; 17C05; 17C50; 17D99; 65D20; 65L09; 68R99; 97M10; 97M80; 97N50; 97N80; 97P20; 97R20

1. Introduction

The current paper is an extension of [1], a paper based on a presentation at the INASE (Institute for Natural Sciences and Engineering) conference in Barcelona. Our interaction with the participants and some earlier proceedings of INASE influenced the development of it.

Computational methods were an important tool for solving the Yang–Baxter equation and Yang–Baxter system, in dimension two, in the papers [2,3], where the authors used the Grobner basis.

The discovery of the Yang–Baxter Equation ([4]) in theoretical physics and statistical mechanics (see [5–7]) has led to many applications in these fields and in quantum groups, quantum computing, knot theory, braided categories, the analysis of integrable systems, quantum mechanics, *etc*. (see [8]). The interest in this equation is growing, as new properties of it are found and its solutions are not classified yet (see also [9,10]).

One of the striking properties of this equation is its unifying feature (see, for example, [11–13]). Another unification of non-associative structures was recently obtained using the so-called UJLA structures ([1,14,15]), which could be seen as structures that comprise the information encapsulated in associative algebras, Lie algebras and Jordan algebras. These unifications have similarities with the properties of quantum computers, whose quantum gates can execute many operations at the same time (a classical gate executes just one operation at a time). Several Jordan structures have applications in quantum group theory, and exceptional Jordan algebras have played an important role in recent fundamental physical theories, namely in the theory of super-strings (see [16]).

The quantum computer can be used to solve large computational problems from number theory and optimization. An example is Shor's algorithm, a quantum algorithm that determines quickly and effectively the prime factors of a big number. With enough qubii, such a computer could use Shor's algorithm to break the algorithms' encryption used today.

The organization of our paper is as follows. In the next section, we give some preliminaries on the Yang–Baxter equation, and we explain its importance for constructing quantum gates and obtaining link invariants. Section 3 deals with the set-theoretical Yang–Baxter equation. New solutions for it are presented. Section 4 deals with transcendental numbers, computational methods and some applications. In Section 5, we discuss algorithms and interpretations of the Yang–Baxter equation in computer science. Section 6 is about unification theories for non-associative algebras and their connections with the previous sections. A conclusions section ends our paper.

2. Yang–Baxter Equations

The Yang–Baxter equation first appeared in theoretical physics, in a paper by the Nobel laureate C.N. Yang and in statistical mechanics in R.J. Baxter's work. It has applications in many areas of physics, informatics and mathematics. Many scientists have used computer calculations or the axioms of various algebraic structures in order to solve this equation, but the full classification of its solutions remains an open problem (see [1,8,10,17–22]).

In this paper, tensor products are defined over the field k.

For V a k-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$ and by $I : V \to V$ the identity map of the space V; for $R : V \otimes V \to V \otimes V$ a k-linear map, let $R^{12} = R \otimes I,\ R^{23} = I \otimes R,\ R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 2.1. *A Yang–Baxter operator is k-linear map $R : V \otimes V \to V \otimes V$, which is invertible, and it satisfies the braid condition (sometimes called the Yang–Baxter equation):*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1}$$

If R satisfies Equation (1), then both $R \circ \tau$ and $\tau \circ R$ satisfy the quantum Yang–Baxter equation (QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{2}$$

Therefore, Equations (1) and (2) are equivalent.

For A to be a (unitary) associative k-algebra and $\alpha, \beta, \gamma \in k$, the authors of [18] defined the k-linear map $R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A$,

$$a \otimes b \mapsto \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b \tag{3}$$

which is a Yang–Baxter operator if and only if one of the following cases holds:
(i) $\alpha = \gamma \neq 0, \quad \beta \neq 0$; (ii) $\beta = \gamma \neq 0, \quad \alpha \neq 0$; (iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.

An interesting property of Equation (3) can be visualized in knot theory, where the link invariant associated with $R^A_{\alpha,\beta,\gamma}$ is the Alexander polynomial (*cf.* [23,24]).

For $(L, [,])$ a Lie super-algebra over k, $z \in Z(L) = \{z \in L : [z, x] = 0 \ \forall x \in L\}$, $|z| = 0$ and $\alpha \in k$, the authors of the papers [25] and [22] defined the following Yang–Baxter operator: ϕ^L_α : $L \otimes L \longrightarrow L \otimes L$,

$$x \otimes y \mapsto \alpha[x, y] \otimes z + (-1)^{|x||y|} y \otimes x \tag{4}$$

This construction could lead to interesting "bozonization" constructions, a technique often used in constructing (super) quantum groups. For example, the FRT (Faddeev-Reshetikhin-Takhtajan) algebras associated with ϕ^L_α for a Lie algebra and a Lie super-algebra might be related via such a construction.

Remark 2.2. *In dimension two,* $R^A_{\alpha,\beta,\alpha}$ *gives a universal quantum gate (see [1]):*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

which, according to [26], is related to the CNOT (controlled NOT) gate:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6}$$

Remark 2.3. *The author of the paper [27] obtains the abstract controlled-not by a composition of a comonoid and a monoid. That formula could be related to the well-known Yang–Baxter operator* $a \otimes b \mapsto \sum a_1 \otimes a_2 b$ *and leads us to the open problem of comparing this operator to Equation (3).*

Remark 2.4. *The matrix Equation (5) can be interpreted as a sum of the Yang–Baxter operators* I *and* $-I$*, using the techniques of [28].*

3. The Set-Theoretical Yang–Baxter Equation

Definition 3.1. *For an arbitrary set* X*, the map* $S : X \times X \to X \times X$*, is a solution for the set-theoretical Yang–Baxter equation if:*

$$S^{12} \circ S^{13} \circ S^{23} = S^{23} \circ S^{13} \circ S^{12} \tag{7}$$

(Here $S^{12} = S \times I, \ S^{23} = I \times S$*, etc.)*

There are many examples of solutions for Equation (7): from "brace" structures, from relations on sets, *etc.*, and they are related to other interesting structures (see, for example, [21,29–34]). However, this equation is not completely solved yet.

Next, we present some explicit solutions for Equation (7); we extend some constructions from [1], and then, we give new constructions of solutions for Equation (7).

We consider a three-dimensional Euclidean space and a point $P(a,b,c)$ of it.
The symmetry of the point $P(a,b,c)$ about the origin is defined as follows:
$S_O(a,b,c) = (-a,\ -b,\ -c)$.
The symmetries of the point $P(a,b,c)$ about the axes $OX,\ OY,\ OZ$ are defined as follows:
$S_{OX}(a,b,c) = (a,\ -b,\ -c)$,
$S_{OY}(a,b,c) = (\ -a,\ b,\ -c)$,
$S_{OZ}(a,b,c) = (\ -a,\ -b,\ c)$.
The symmetries of the point $P(a,b,c)$ about the planes $XOY,\ XOZ,\ YOZ$ are defined as follows:
$S_{XOY}(a,b,c) = (a,\ b,\ -c)$,
$S_{XOZ}(a,b,c) = (a,\ -b,\ c)$,
$S_{YOZ}(a,b,c) = (\ -a,\ b,\ c)$.
The above symmetries with the identity map form a group:
$\{I, S_{OX}, S_{OY}, S_{OZ}, S_{XOY}, S_{XOZ}, S_{YOZ}, S_O\}$,
which contains a subgroup isomorphic with Klein's group: $\{I, S_{OX}, S_{OY}, S_{OZ}\}$.
One could check the following instances of the Yang–Baxter equation:
$S_{XOY} \circ S_{XOZ} \circ S_{YOZ} = S_{YOZ} \circ S_{XOZ} \circ S_{XOY}$,
$S_{OX} \circ S_{OY} \circ S_{OZ} = S_{OZ} \circ S_{OY} \circ S_{OX}$.

Theorem 3.2. *The following is a two-parameter family of solutions for the set-theoretical braid condition:*

$S : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}, \quad (x,y) \mapsto (y^{\alpha},\ x^{\beta} y^{1-\alpha\beta}) \quad \forall \alpha, \beta \in \mathbb{N}^*$

Proof. We observe that the map $(x,y) \mapsto (x^m y^n, x^p y^q)$ is a solution for the set-theoretical braid condition if and only if the following relations hold:

$mnq = 0,\ mpq = 0,\ mq^2 = m^2 q,\ m^2 + mnp = m,\ q^2 + npq = q.$

We leave the analysis of this system of non-linear equations as a computational problem, and we pick just the above solution of this system.

Another approach to study this theorem might be by using the properties of generalized means from [35]. □

Theorem 3.3. *The following is a two-parameter family of solutions for the set-theoretical braid condition:*

$R : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}, \quad (z,w) \mapsto (\alpha w,\ \beta z + (1-\alpha\beta) w) \quad \forall \alpha, \beta \in \mathbb{C}$

Proof. One way to prove this theorem is to follow the steps of the above proof. □

Remark 3.4. *Another way to prove Theorem 3.3 is to relate it to Theorem 3.2. Thus, in some cases, the exponential function,* $f : \mathbb{C} \to \mathbb{C},\ z \mapsto e^z$*, is a morphism of the above solutions for (7):* $(f \times f) \circ R = S \circ (f \times f)$ *.*

Further approaches could be by using computational methods, and extended results for Dieudonne modules (see [36]) are expected.

Other examples of solutions for Equation (7) will be given in the Section 5, and they will be related to informatics.

4. Transcendental Numbers and Applications

The following identity, containing the transcendental numbers e and π, is well known (see more details about transcendental numbers in [37]):

$$e^{i\pi} + 1 = 0. \tag{8}$$

Let J be the following matrix (for $\alpha \in \mathbb{R}^*$):

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\alpha} i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ \alpha i & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

then, similar to Equation (8), the following formula holds:

$$e^{\pi\ J} + I_2 = 0_4 \qquad J,\ I_2, 0_4 \in \mathcal{M}_4(\mathbb{C})$$

For $x \in \mathbb{C}$, let $R(x) = \cos x I_2 + \sin x J = e^{xJ} : (\mathbb{C} \times \mathbb{C})^{\otimes 2} \to (\mathbb{C} \times \mathbb{C})^{\otimes 2}$. It satisfies the colored Yang–Baxter equation:

$$R^{12}(x) \circ R^{23}(x+y) \circ R^{12}(y) \ = \ R^{23}(y) \circ R^{12}(x+y) \circ R^{23}(x) \qquad \forall\, x, y \in \mathbb{C}\ . \tag{10}$$

Furthermore, $R(x) = e^{xJ}$ is a solution for the following differential matrix equation:

$$Y' = JY \tag{11}$$

which is related, for example, to [38]. The combination of the properties Equations (10) and (11), has applications in computing the Hamiltonian of many body systems in physics.

Computational methods could be employed for finding matrices J with these properties in higher dimensions.

The presentation [39] was related to the following formula with transcendental numbers: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$. Thus, the experimental results presented at that time were related to the Gauss bell function.

Next, we solve an open problem proposed in [1]. This theorem, which is related to the transcendental numbers e and π, was solved, using computational methods, thanks to an observation of Dr. Mihai Cipu.

Theorem 4.1. $\sum_1^n \frac{1}{k^2} < \frac{2}{3}\left(\frac{n+1}{n}\right)^n \quad \forall n \in \mathbb{N}^*$.

Proof. We evaluate the first three values for the above expressions:

n=1	1	1.(3)
n=2	1.25	1.5
n=3	1.36(1)	1.58...

The inequality is true in these cases, and we will use the following inequality: $\sum_1^n \frac{1}{k^2} < \frac{\pi^2}{6} < 1.645 \quad \forall n \geq 4$, in order to finish our proof. The above inequality is a consequence of the Basel problem: $\sum_1^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$. (A new proof for this identity was recently given in [40].) Note also that the sequence in the right-hand side is increasing. The last step of the proof is shown below.

n= 4	1.4236(1)	1.6276...
n= 5	...	1.65(8)
∞	1.644934067...	1.812187886...

□

Other recent problems relating e and π are listed below. Numerical and experimental results are very important for studying them.

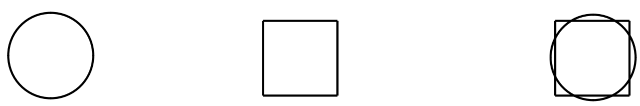
$$| e^{1-z} + e^{\bar{z}} | > \pi \qquad \forall z \in \mathbb{C} ,$$

$$\int_a^b e^{-x^2} dx \; < \; \frac{e^e}{\pi}\left(\frac{1}{e^{\pi a}} - \frac{1}{e^{\pi b}}\right) , \;\; \forall\, a, b \in \mathbb{R}, \; a \leq b,$$

$$x^2 + e > \pi x \quad \forall x \in \mathbb{R};$$

the last inequality holds because $\Delta = \pi^2 - 4e = -1,003522913... < 0$. We conjecture that $4e - \pi^2 = 1,003522913...$ is a transcendental number. Theorem 4.1 and numerical results could give a partial answer for this problem.

The geometrical interpretation of the formula $\pi^2 < 4e$ could be stated as: "The length of the circle with diameter π is almost equal (and less) to the perimeter of a square with edges of length e".



The area of the above circle is greater than the area of the square, because $\pi^3 > 4e^2$.

Open problems: For an arbitrary convex closed curve, we consider the largest diameter (D). (It can be found by considering the center of mass of a body, which corresponds to the domain inside the given curve.)

(i) The equation:

$$x^2 - \frac{L}{2}x + A = 0 \tag{12}$$

and its implications are not completely understood. For example, if the given curve is an ellipse, solving this equation in terms of the semi-axes of the ellipse is an unsolved problem.

(ii) We conjecture that the following system of equations is an inverse of Equation (12). We consider two functions with second order derivatives, such that:

$$f : [0, D] \to \mathbb{R},\ f \geq 0,\ f'' \leq 0,\quad g : [0, D] \to \mathbb{R},\ g \leq 0,\ g'' \geq 0,$$
$$\int_0^D \sqrt{1 + (f'(x))^2} + \sqrt{1 + (g(x)')^2}\, dx \ = L\ ,\quad \int_0^D f(x) - g(x)\, dx \ = A\ .$$

Remark 4.2. *Graphics for arbitrary closed convex curves related to the above open problems could be represented using graphing calculators and computers. Thus, some numerical (experimental) results can be obtained. This direction seems to be a challenging one for computer scientists, and it has applications for representations similar to those from [41].*

Remark 4.3. *The equation $x^i = i^x$ for $x \in \mathbb{R}^*_+$, has no solutions (see [1]).*

At this moment, we do not have convincing numerical/experimental results for solving the following generalization of the above equation: $z^i = i^z \quad z \in \mathbb{C}^*$.

5. The Yang–Baxter Equations in Informatics

The Yang–Baxter equation represents some kind of compatibility condition in logic.

More explicitly, let us consider three logical sentences $p,\ q,\ r$, and let us suppose that if all of them are true, then the conclusion A could be drawn, and if $p,\ q,\ r$ are all false, then the conclusion C can be drawn.

Modeling this situation by the map R, defined by $(p,\ q)\ \mapsto (p' = p \vee q,\ q' = p \wedge q)$, helps us to comprise our analysis: we can apply R to the pair $(p,\ q)$, then to $(q',\ r)$ and, finally, to $(p',\ q'')$.

The Yang–Baxter equation guarantees that the order in which we start this analysis is not important; more explicitly, in this case, it states that $(\ (p')',\ q''',\ r')\ =\ (\ p',\ q''',\ (r')')$. In other words, the map $(p,\ q)\ \mapsto (p \vee q,\ p \wedge q)$ is a solution for Equation (7).

The sorting of numbers (see, for example, [1,42]) is an important problem in informatics, and the Yang–Baxter equation is related to it. The following "bubble sort" algorithm is related to the right-hand side of Equation (1).

```
int m, aux;
m=L;
while (m)
{
for (int i=1; i≤L-1; i++)
if (s[L-i] ≥ s[L+1-i])
{
aux = s[L+1-i];
s[L+1-i] = s[L-i];
s[L-i] = aux;
}
m - -;
```

}
The maipart of another sorting algorithm, related to the left-hand side of (2), is given below (see [1]).

```
    if (s[i]≥s[j])
{
aux=s[i];
s[i]=s[j];
s[j]=aux;
}
```

Ordering three numbers is related to a common solution of Equations (1) and (2): $R(a,b) = (min(a,b), max(a,b))$. In a similar manner, one can find the greatest common divisor and the least common multiple of three numbers, using another common solution of Equations (1) and (2): $R'(a,b) = (gcd(a,b), lcm(a,b))$.

Since R and R' can be extended to braidings in certain monoidal categories, we obtain interpretations for the cases when we deal with more numbers. The "divide et impera" algorithm for finding the maximum/minimum (or the greatest common divisor/least common multiple) of a sequence of numbers could be related to Yang–Baxter systems and to the gluing procedure from [28].

6. Non-Associative Algebras and Their Unifications

The main non-associative structures are Lie algebras and Jordan algebras. Arguable less known, Jordan algebras have applications in physics, differential geometry, ring geometries, quantum groups, analysis, biology, *etc*. (see [16,43,44]).

Formulas (3) and (4) lead to the unification of associative algebras and Lie (super-)algebras in the framework of Yang–Baxter structures (see [45,46]). For the invertible elements of a Jordan algebra, one can associate a symmetric space and, after that, a Yang–Baxter operator. Thus, the Yang–Baxter equation can be thought of as a unifying equation.

Another unification for these structures will be presented below.

Definition 6.1. *For the vector space V, let* $\eta : V \otimes V \to V, \;\; \eta(a \otimes b) = ab$, *be a linear map that satisfies:*

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \tag{13}$$

$$(a^2b)a = a^2(ba), \; (ab)a^2 = a(ba^2), \; (ba^2)a = (ba)a^2, \; a^2(ab) = a(a^2b) \tag{14}$$

$\forall\, a, b, c \in V$.

Then, (V, η) *is called a "UJLA structure".*

Remark 6.2. *The UJLA structures unify Jordan algebras, Lie algebras and (non-unital) associative algebras; results for UJLA structures could be "decoded" in the properties of Jordan algebras, Lie algebras or (non-unital) associative algebras. This unification resembles the properties of quantum computers.*

Remark 6.3. *The UJLA structures unify in a natural manner the above-mentioned non-associative structures. Thus, if* $(A,\ \theta)$*, where* $\theta : A \otimes A \to A,\ \ \theta(a \otimes b) = ab$*, is a (non-unital) associative algebra, then we define* $(A,\ \theta')$*, where* $\theta'(a \otimes b) = \alpha ab\ +\ \beta ba$*, for* $\alpha,\ \beta \in k$*.*

If $\alpha = \frac{1}{2}$ *and* $\beta = \frac{1}{2}$*, then* $(A,\ \theta')$ *is a Jordan algebra.*

If $\alpha = 1$ *and* $\beta = -1$*, then* $(A,\ \theta')$ *is a Lie algebra.*

If $\alpha = 0$ *and* $\beta = 1$*, then* $(A,\ \theta')$ *is the opposite algebra of* $(A,\ \theta)$*, and if* $\alpha = 1$ *and* $\beta = 0$*, then* $(A,\ \theta')$ *is the algebra* $(A,\ \theta)$*.*

If we put no restrictions on α *and* β*, then* $(V,\ \theta)$ *is a UJLA structure.*

Theorem 6.4. *For* V *a* k*-space,* $f : V \to k$ *a* k*-map,* $\alpha,\ \beta \in k$*, and* $e \in V$*, such that* $f(e) = 1$*, the following structures can be associated:*

(i) (V, M, e)*, a unital associative algebra, where* $M(v \otimes w) = f(v)w + vf(w) - f(v)f(w)e$*;*

(ii) $(V,\ [,])$*, a Lie algebra, where* $[v,\ w] = f(v)w - vf(w)$*;*

(iii) $(V,\ \mu)$*, a Jordan algebra, where* $\mu(v \otimes w) = f(v)w + vf(w)$*;*

(iv) $(V, M_{\alpha,\beta})$*, a UJLA structure, where* $M_{\alpha,\beta}(v \otimes w) = \alpha f(v)w + \beta vf(w)$*.*

Proof. (i) The proof is direct. We denote by " $\cdot$ " the operation M, in order to simplify our presentation. We observe that " $\cdot$ " is commutative. We first prove that e is the unity of our algebra: $x \cdot e = f(x)e + f(e)x - f(e)f(x)e = x = e \cdot x$.

Next, we prove the associativity of " $\cdot$ ": $(x \cdot y) \cdot z = f(x)f(y)z + f(x)f(z)y - f(x)f(y)f(z)e + f(x)f(y)z + xf(y)f(z) - f(x)f(y)f(z)e - f(x)f(y)z$;

$x \cdot (y \cdot z) = f(x)f(y)z + f(x)yf(z) - f(x)f(y)f(z)e + xf(y)f(z) - f(x)f(y)f(z)e$.

It follows that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(ii) In this case

$(x \cdot y) \cdot z = f(x)f(z)y - f(y)f(z)x$;

$(y \cdot z) \cdot x = f(y)f(x)z - f(z)f(x)y$;

$(z \cdot x) \cdot y = f(z)f(y)x - f(x)f(y)z$.

The Jacobi identity is verified.

We leave Cases (iii) and (iv) to be proven by the reader. □

Remark 6.5. *The above theorem produces new examples of non-associative structures, and it finds common information encapsulated in these non-associative structures.*

Theorem 6.6. *Let* (V, η) *be a UJLA structure, and* $\alpha,\ \beta \in k$*. Then,* $(V, \eta'),\ \ \eta'(a \otimes b) = \alpha ab + \beta ba$ *is a UJLA structure.*

Remark 6.7. *The classification of UJLA structures is an open problem, and it is more difficult than the problem of classifying associative algebras (which is an open problem for higher dimensions).*

Theorem 6.8. *Let* V *be a vector space over the field* k*, and* $p, q \in k$*. For* $f, g : V \to V$*, we define* $M(f \otimes g) = f * g = f *_{p,q} g = pf \circ g + qg \circ f : V \to V$*. Then:*

(i) $(End_k(V),\ *_{p,q})$ *is a UJLA structure* $\forall p, q \in k$*.*

(ii) For $\phi : End_k(V) \to End_k(V \otimes V)$ *a morphism of UJLA structures (i.e.,* $\phi(f * g) = \phi(f) * \phi(g)$ *),* $W = \{f : V \to V | f \circ M = M \circ \phi(f)\}$ *is a sub-UJLA structure of the structure defined in (i). In other words,* $f * g \in W,\ \forall f, g \in W$*.*

7. Conclusions and Further Implications

Surveying topics from abstract algebra to computational methods and from computer science to number theory, the current paper relates these subjects by unifying theories and the celebrated Yang–Baxter equation. We present new results (Theorems 3.2, 3.3, 4.1, 6.4, 6.6 and 6.8), and we propose several open problems and new interpretations.

Unifying the main non-associative structures, the UJLA structures are structures that resemble the properties of quantum computers. Quantum computers could help in solving hard problems in number theory and optimization theory, because they have large computational power. For example, such a computer could use Shor's algorithm to break the algorithms' encryption.

From some solutions of the Yang–Baxter equation, one could construct abstract universal gates from quantum computing. We explained how this equation is related to computer programming; we studied problems about transcendental numbers; and we used computational methods in order to solve problems related to these topics.

Related to Equation (10), there is a long-standing open problem. The following system of equations, obtained in [10] and extended in [9], is not completely classified:

$$\begin{aligned}&(\beta(v,w)-\gamma(v,w))(\alpha(u,v)\beta(u,w)-\alpha(u,w)\beta(u,v))\\&\qquad+(\alpha(u,v)-\gamma(u,v))(\alpha(v,w)\beta(u,w)-\alpha(u,w)\beta(v,w))=0\end{aligned} \tag{15}$$

$$\begin{aligned}&\beta(v,w)(\beta(u,v)-\gamma(u,v))(\alpha(u,w)-\gamma(u,w))\\&\qquad+(\alpha(v,w)-\gamma(v,w))(\beta(u,w)\gamma(u,v)-\beta(u,v)\gamma(u,w))=0\end{aligned} \tag{16}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\alpha(u,w)-\gamma(u,w))+\alpha(v,w)\gamma(u,w)(\gamma(u,v)-\alpha(u,v))\\&\qquad+\gamma(v,w)(\alpha(u,v)\gamma(u,w)-\alpha(u,w)\gamma(u,v))=0\end{aligned} \tag{17}$$

$$\begin{aligned}&\alpha(u,v)\beta(v,w)(\beta(u,w)-\gamma(u,w))+\beta(v,w)\gamma(u,w)(\gamma(u,v)-\beta(u,v))\\&\qquad+\gamma(v,w)(\beta(u,v)\gamma(u,w)-\beta(u,w)\gamma(u,v))=0\end{aligned} \tag{18}$$

$$\begin{aligned}&\alpha(u,v)(\alpha(v,w)-\gamma(v,w))(\beta(u,w)-\gamma(u,w))\\&\qquad+(\beta(u,v)-\gamma(u,v))(\alpha(u,w)\gamma(v,w)-\alpha(v,w)\gamma(u,w))=0\end{aligned} \tag{19}$$

From the transdisciplinary (see [47,48]) point of view and attempting to relate art and science, Equation (12) could be called the "cubism equation". In the same manner, the inverse system could be related to Art Nouveau (for example, recall the architecture of Casa Mila by Gaudi).

Conflicts of Interest

The author declares no conflict of interest.

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