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Neutrosophic Positive Implicative $\mathcal{N}$-Ideals in BCK-Algebras

Young Bae Jun [1], Florentin Smarandache [2], Seok-Zun Song [3,*] and Madad Khan [4]

1 Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com
2 Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA; fsmarandache@gmail.com
3 Department of Mathematics, Jeju National University, Jeju 63243, Korea
4 Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad 45550, Pakistan; madadmath@yahoo.com
* Correspondence: szsong@jejunu.ac.kr

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Abstract: The notion of a neutrosophic positive implicative $\mathcal{N}$-ideal in BCK-algebras is introduced, and several properties are investigated. Relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic positive implicative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic positive implicative $\mathcal{N}$-ideal are considered. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal are provided. An extension property of a neutrosophic positive implicative $\mathcal{N}$-ideal based on the negative indeterminacy membership function is discussed.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-ideal; neutrosophic positive implicative $\mathcal{N}$-ideal

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1. Introduction

There are many real-life problems which are beyond a single expert. It is because of the need to involve a wide domain of knowledge. As a generalization of the intuitionistic fuzzy set, paraconsistent set and intuitionistic set, the neutrosophic logic and set is introduced by F. Smarandache [1] and it is a useful tool to deal with uncertainty in several social and natural aspects. Neutrosophy provides a foundation for a whole family of new mathematical theories with the generalization of both classical and fuzzy counterparts. In a neutrosophic set, an element has three associated defining functions such as truth membership function (T), indeterminate membership function (I) and false membership function (F) defined on a universe of discourse X. These three functions are independent completely. The neutrosophic set has vast applications in various fields (see [2–6]).

In order to provide mathematical tool for dealing with negative information, Y. B. Jun, K. J. Lee and S. Z. Song [7] introduced the notion of negative-valued function, and constructed $\mathcal{N}$-structures. M. Khan, S. Anis, F. Smarandache and Y. B. Jun [8] introduced the notion of neutrosophic $\mathcal{N}$-structures, and it is applied to semigroups (see [8]) and BCK/BCI-algebras (see [9]). S. Z. Song, F. Smarandache and Y. B. Jun [10] studied a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras. As well-known, BCK-algebras originated from two different ways: one of them is based on set theory, and another is from classical and non-classical propositional calculi (see [11]). The bounded commutative BCK-algebras are precisely MV-algebras. For MV-algebras, see [12]. The background of this study is displayed in the second section. In the third section, we introduce the notion of a neutrosophic positive implicative $\mathcal{N}$-ideal in BCK-algebras, and investigate several properties. We discuss relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic positive implicative $\mathcal{N}$-ideal, and provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal. We consider characterizations of a neutrosophic positive implicative $\mathcal{N}$-ideal. We establish an extension property of a neutrosophic positive implicative $\mathcal{N}$-ideal based on the negative indeterminacy membership function. Conclusions are provided in the final section.

2. Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation "$*$" and a special element "0" in which the following conditions are satisfied:

(I) $((x * y) * (x * z)) * (z * y) = 0,$
(II) $(x * (x * y)) * y = 0,$
(III) $x * x = 0,$
(IV) $x * y = y * x = 0 \ \Rightarrow \ x = y$

for all $x, y, z \in X$. By a BCK-algebra, we mean a BCI-algebra X satisfying the condition

$$(\forall x \in X)(0 * x = 0).$$

A partial ordering $\preceq$ on X is defined by

$$(\forall x, y \in X)\,(x \preceq y \ \Rightarrow \ x * y = 0).$$

Every BCK/BCI-algebra X verifies the following properties.

$$(\forall x \in X)\ (x * 0 = x), \tag{1}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y). \tag{2}$$

Let I be a subset of a BCK/BCI-algebra. Then I is called an *ideal* of X if it satisfies the following conditions.

$$0 \in I, \tag{3}$$
$$(\forall x, y \in X)\,(x * y \in I,\ y \in I \ \Rightarrow \ x \in I)\,. \tag{4}$$

Let I be a subset of a BCK-algebra. Then I is called a *positive implicative ideal* of X if the Condition (3) holds and the following assertion is valid.

$$(\forall x, y, z \in X)\,((x * y) * z \in I,\ y * z \in I \ \Rightarrow \ x * z \in I)\,. \tag{5}$$

Any positive implicative ideal is an ideal, but the converse is not true (see [13]).

Lemma 1 ([13]). *A subset I of a BCK-algebra X is a positive implicative ideal of X if and only if I is an ideal of X which satisfies the following condition.*

$$(\forall x, y \in X)\,((x * y) * y \in I \ \Rightarrow \ x * y \in I)\,. \tag{6}$$

We refer the reader to the books [13,14] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} := \inf\{a_i \mid i \in \Lambda\}.$$

We denote the collection of functions from a set X to $[-1,0]$ by $\mathcal{F}(X,[-1,0])$. An element of $\mathcal{F}(X,[-1,0])$ is called a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on X). An ordered pair (X,f) of X and an $\mathcal{N}$-function f on X is called an $\mathcal{N}$*-structure* (see [7]).

A *neutrosophic* $\mathcal{N}$*-structure* over a nonempty universe of discourse X (see [8]) is defined to be the structure

$$X_{\mathbf{N}} := \left\{ \frac{x}{(T_N(x),I_N(x),F_N(x))} \mid x \in X \right\} \tag{7}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

For the sake of simplicity, we will use the notation $X_{\mathbf{N}}$ or $X_{\mathbf{N}} := \frac{X}{(T_N,I_N,F_N)}$ instead of the neutrosophic $\mathcal{N}$-structure in (7).

Recall that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the following condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)\,.$$

3. Neutrosophic Positive Implicative $\mathcal{N}$-ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 1 ([9])**.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X. Then $X_{\mathbf{N}}$ is called a neutrosophic $\mathcal{N}$-ideal of X if the following condition holds.*

$$(\forall x,y \in X)\begin{pmatrix} T_N(0) \leq T_N(x) \leq \bigvee\{T_N(x*y),T_N(y)\} \\ I_N(0) \geq I_N(x) \geq \bigwedge\{I_N(x*y),I_N(y)\} \\ F_N(0) \leq F_N(x) \leq \bigvee\{F_N(x*y),F_N(y)\} \end{pmatrix}. \tag{8}$$

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic positive implicative $\mathcal{N}$-ideal of X if the following assertions are valid.*

$$(\forall x \in X)\,(T_N(0) \leq T_N(x),\ I_N(0) \geq I_N(x),\ F_N(0) \leq F_N(x))\,, \tag{9}$$

$$(\forall x,y,z \in X)\begin{pmatrix} T_N(x*z) \leq \bigvee\{T_N((x*y)*z),T_N(y*z)\} \\ I_N(x*z) \geq \bigwedge\{I_N((x*y)*z),I_N(y*z)\} \\ F_N(x*z) \leq \bigvee\{F_N((x*y)*z),F_N(y*z)\} \end{pmatrix}. \tag{10}$$

Example 1. *Let $X = \{0,1,2,3,4\}$ be a BCK-algebra with the Cayley table in Table 1.*

Table 1. Cayley table for the binary operation "$*$".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Let

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.9,-0.2,-0.7)}, \frac{1}{(-0.7,-0.6,-0.7)}, \frac{2}{(-0.5,-0.7,-0.6)}, \frac{3}{(-0.1,-0.4,-0.4)}, \frac{4}{(-0.3,-0.8,-0.2)} \right\}$$

be a neutrosophic $\mathcal{N}$-structure over X. Then $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.

If we take $z = 0$ in (10) and use (1), then we have the following theorem.

Theorem 1. *Every neutrosophic positive implicative $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-ideal.*

The following example shows that the converse of Theorem 1 does not holds.

Example 2. *Let $X = \{0, a, b, c\}$ be a BCK-algebra with the Cayley table in Table 2.*

Table 2. Cayley table for the binary operation "$*$".

$*$	**0**	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let

$$X_{\mathbf{N}} = \left\{ \frac{0}{(t_0, i_2, f_0)}, \frac{a}{(t_1, i_1, f_2)}, \frac{b}{(t_1, i_1, f_2)}, \frac{c}{(t_2, i_0, f_1)} \right\}$$

be a a neutrosophic $\mathcal{N}$-structure over X where $t_0 < t_1 < t_2$, $i_0 < i_1 < i_2$ and $f_0 < f_1 < f_2$ in $[-1, 0]$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. But it is not a neutrosophic positive implicative $\mathcal{N}$-ideal of X since

$$T_N(b * a) = T_N(a) = t_1 \not\leq t_0 = \bigvee\{T_N((b * a) * a), T_N(a * a)\},$$
$$I_N(b * a) = I_N(a) = i_1 \not\geq i_2 = \bigwedge\{I_N((b * a) * a), I_N(a * a)\},$$

or

$$F_N(b * a) = F_N(a) = f_2 \not\leq f_0 = \bigvee\{F_N((b * a) * a), F_N(a * a)\}.$$

Given a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$, we define the following sets.

$$T_N^{\alpha} := \{x \in X \mid T_N(x) \leq \alpha\},$$
$$I_N^{\beta} := \{x \in X \mid I_N(x) \geq \beta\},$$
$$F_N^{\gamma} := \{x \in X \mid F_N(x) \leq \gamma\}.$$

Then we say that the set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is the (α, β, γ)-*level set* of $X_{\mathbf{N}}$ (see [9]). Obviously, we have

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}.$$

Theorem 2. *If $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X, then T_N^{α}, I_N^{β} and F_N^{γ} are positive implicative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ whenever they are nonempty.*

Proof. Assume that T_N^{α}, I_N^{β} and F_N^{γ} are nonempty for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $x \in T_N^{\alpha}$, $y \in I_N^{\beta}$ and $z \in F_N^{\gamma}$ for some $x, y, z \in X$. Thus $T_N(0) \leq T_N(x) \leq \alpha$, $I_N(0) \geq$

$I_N(y) \geq \beta$, and $F_N(0) \leq F_N(z) \leq \gamma$, that is, $0 \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Let $(x*y)*z \in T_N^\alpha$ and $y*z \in T_N^\alpha$. Then $T_N((x*y)*z) \leq \alpha$ and $T_N(y*z) \leq \alpha$, which imply that

$$T_N(x*z) \leq \bigvee\{T_N((x*y)*z), T_N(y*z)\} \leq \alpha,$$

that is, $x*z \in T_N^\alpha$. If $(a*b)*c \in I_N^\beta$ and $b*c \in I_N^\beta$, then $I_N((a*b)*c) \geq \beta$ and $I_N(b*c) \geq \beta$. Thus

$$I_N(a*c) \geq \bigwedge\{I_N((a*b)*c), I_N(b*c)\} \geq \beta,$$

and so $a*c \in I_N^\beta$. Finally, suppose that $(u*v)*w \in F_N^\gamma$ and $v*w \in F_N^\gamma$. Then $F_N((u*v)*w) \leq \gamma$ and $F_N(v*w) \leq \gamma$. Thus

$$F_N(u*w) \leq \bigvee\{F_N((u*v)*w), F_N(v*w)\} \leq \gamma,$$

that is, $u*w \in F_N^\gamma$. Therefore T_N^α, I_N^β and F_N^γ are positive implicative ideals of X. □

Corollary 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a positive implicative ideal of X.*

Proof. Straightforward. □

The following example illustrates Theorem 2.

Example 3. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the Cayley table in Table 3.*

Table 3. Cayley table for the binary operation "$*$".

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Let

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8,-0.3,-0.7)}, \frac{1}{(-0.7,-0.6,-0.4)}, \frac{2}{(-0.4,-0.4,-0.5)}, \frac{3}{(-0.3,-0.5,-0.6)}, \frac{4}{(-0.2,-0.9,-0.1)} \right\}$$

be a neutrosophic $\mathcal{N}$-structure over X. Routine calculations show that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then

$$T_N^\alpha = \begin{cases} \emptyset & \text{if } \alpha \in [-1, -0.8), \\ \{0\} & \text{if } \alpha \in [-0.8, -0.7), \\ \{0, 1\} & \text{if } \alpha \in [-0.7, -0.4), \\ \{0, 1, 2\} & \text{if } \alpha \in [-0.4, -0.3), \\ \{0, 1, 2, 3\} & \text{if } \alpha \in [-0.3, -0.2), \\ X & \text{if } \alpha \in [-0.2, 0], \end{cases}$$

$$I_N^\beta = \begin{cases} \emptyset & \text{if } \beta \in (-0.3, 0], \\ \{0\} & \text{if } \beta \in (-0.4, -0.3], \\ \{0,2\} & \text{if } \beta \in (-0.5, -0.4], \\ \{0,2,3\} & \text{if } \beta \in (-0.6, -0.5], \\ \{0,1,2,3\} & \text{if } \beta \in (-0.9, -0.6], \\ X & \text{if } \beta \in [-1, -0.9], \end{cases}$$

and

$$F_N^\gamma = \begin{cases} \emptyset & \text{if } \gamma \in [-1, -0.7), \\ \{0\} & \text{if } \gamma \in [-0.7, -0.6), \\ \{0,3\} & \text{if } \gamma \in [-0.6, -0.5), \\ \{0,2,3\} & \text{if } \gamma \in [-0.5, -0.4), \\ \{0,1,2,3\} & \text{if } \gamma \in [-0.4, -0.1), \\ X & \text{if } \gamma \in [-0.1, 0], \end{cases}$$

which are positive implicative ideals of X.

Lemma 2 ([9])**.** *Every neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the following assertions:*

$$(x, y \in X)\,(x \preceq y \;\Rightarrow\; T_N(x) \le T_N(y), I_N(x) \ge I_N(y), F_N(x) \le F_N(y))\,. \tag{11}$$

We discuss conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal.

Theorem 3. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \begin{pmatrix} T_N(x*y) \le T_N((x*y)*y), \\ I_N(x*y) \ge I_N((x*y)*y), \\ F_N(x*y) \le F_N((x*y)*y) \end{pmatrix}. \tag{12}$$

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. If z is replaced by y in (10), then

$$\begin{aligned} T_N(x*y) &\le \bigvee\{T_N((x*y)*y), T_N(y*y)\} \\ &= \bigvee\{T_N((x*y)*y), T_N(0)\} = T_N((x*y)*y), \end{aligned}$$

$$\begin{aligned} I_N(x*y) &\ge \bigwedge\{I_N((x*y)*y), I_N(y*y)\} \\ &= \bigwedge\{I_N((x*y)*y), I_N(0)\} = I_N((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} F_N(x*y) &\le \bigvee\{F_N((x*y)*y), F_N(y*y)\} \\ &= \bigvee\{F_N((x*y)*y), F_N(0)\} = F_N((x*y)*y) \end{aligned}$$

by (III) and (9).

Conversely, let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X satisfying (12). Since

$$((x*z)*z)*(y*z) \preceq (x*z)*y = (x*y)*z$$

for all $x, y, z \in X$, we have

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N(((x*z)*z)*(y*z)) \leq T_N((x*y)*z), \\ I_N(((x*z)*z)*(y*z)) \geq I_N((x*y)*z), \\ F_N(((x*z)*z)*(y*z)) \leq F_N((x*y)*z) \end{array} \right).$$

by Lemma 2. It follows from (8) and (12) that

$$\begin{aligned} T_N(x*z) &\leq T_N((x*z)*z) \\ &\leq \bigvee\{T_N(((x*z)*z)*(y*z)), T_N(y*z)\} \\ &\leq \bigvee\{T_N((x*y)*z), T_N(y*z)\}, \end{aligned}$$

$$\begin{aligned} I_N(x*z) &\geq I_N((x*z)*z) \\ &\geq \bigwedge\{I_N(((x*z)*z)*(y*z)), I_N(y*z)\} \\ &\geq \bigwedge\{I_N((x*y)*z), I_N(y*z)\}, \end{aligned}$$

and

$$\begin{aligned} F_N(x*z) &\leq F_N((x*z)*z) \\ &\leq \bigvee\{F_N(((x*z)*z)*(y*z)), F_N(y*z)\} \\ &\leq \bigvee\{F_N((x*y)*z), F_N(y*z)\}. \end{aligned}$$

Therefore $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. $\square$

Lemma 3 ([9])**.** *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x*y \preceq z \Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right). \tag{13}$$

Lemma 4. *If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition (13), then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Proof. Since $0*x \preceq x$ for all $x \in X$, we have $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$ for all $x \in X$ by (13). Note that $x*(x*y) \preceq y$ for all $x, y \in X$. It follows from (13) that $T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\}$, $I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\}$, and $F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\}$ for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. $\square$

Theorem 4. *For any neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X, the following assertions are equivalent.*

(1) *$X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.*
(2) *$X_{\mathbf{N}}$ satisfies the following condition.*

$$((x*y)*y)*a \preceq b \Rightarrow \begin{cases} T_N(x*y) \leq \bigvee\{T_N(a), T_N(b)\}, \\ I_N(x*y) \geq \bigwedge\{I_N(a), I_N(b)\}, \\ F_N(x*y) \leq \bigvee\{F_N(a), F_N(b)\}, \end{cases} \tag{14}$$

for all $x, y, a, b \in X$.

Proof. Suppose that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Theorem 1. Let $x, y, a, b \in X$ be such that $((x * y) * y) * a \preceq b$. Then

$$T_N(x * y) \leq T_N(((x * y) * y)) \leq \bigvee\{T_N(a), T_N(b)\},$$
$$I_N(x * y) \geq I_N(((x * y) * y)) \geq \bigwedge\{I_N(a), I_N(b)\},$$
$$F_N(x * y) \leq F_N(((x * y) * y)) \leq \bigvee\{F_N(a), F_N(b)\}$$

by Theorem 3 and Lemma 3.

Conversely, let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X that satisfies (14). Let $x, a, b \in X$ be such that $x * a \preceq b$. Then $((x * 0) * 0) * a \preceq b$, and so

$$T_N(x) = T_N(x * 0) \leq \bigvee\{T_N(a), T_N(b)\},$$
$$I_N(x) = I_N(x * 0) \geq \bigwedge\{I_N(a), I_N(b)\},$$
$$F_N(x) = F_N(x * y) \leq \bigvee\{F_N(a), F_N(b)\}.$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Lemma 4. Since $((x * y) * y) * ((x * y) * y) \preceq 0$, it follows from (14) and (9) that

$$T_N(x * y) \leq \bigvee\{T_N((x * y) * y), T_N(0)\} = T_N((x * y) * y),$$
$$I_N(x * y) \geq \bigwedge\{I_N((x * y) * y), I_N(0)\} = I_N((x * y) * y),$$
$$F_N(x * y) \leq \bigvee\{F_N((x * y) * y), F_N(0)\} = F_N((x * y) * y),$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X by Theorem 3. □

Lemma 5 ([9])**.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are positive implicative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.*

Proof. If T_N^{α}, I_N^{β} and F_N^{γ} are positive implicative ideals of X, then T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X. Thus $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Lemma 5. Let $x, y \in X$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ such that $T_N((x * y) * y) = \alpha$, $I_N((x * y) * y) = \beta$ and $F_N((x * y) * y) = \gamma$. Then $(x * y) * y \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Since $T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$ is a positive implicative ideal of X, it follows from Lemma 1 that $x * y \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Hence

$$T_N(x * y) \leq \alpha = T_N((x * y) * y),$$
$$I_N(x * y) \geq \beta = I_N((x * y) * y),$$
$$F_N(x * y) \leq \gamma = F_N((x * y) * y).$$

Therefore $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X by Theorem 3. □

Lemma 6 ([9]). *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ satisfies the condition (12) if and only if it satisfies the following condition.*

$$(\forall x, y, z \in X) \begin{pmatrix} T_N((x*z)*(y*z)) \leq T_N((x*y)*z), \\ I_N((x*z)*(y*z)) \geq I_N((x*y)*z), \\ F_N((x*z)*(y*z)) \leq F_N((x*y)*z) \end{pmatrix}. \tag{15}$$

Corollary 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X if and only if $X_{\mathbf{N}}$ satisfies (15).*

Proof. It follows from Theorem 3 and Lemma 6. □

Theorem 6. *For any neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X, the following assertions are equivalent.*

(1) *$X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.*
(2) *$X_{\mathbf{N}}$ satisfies the following condition.*

$$((x*y)*z)*a \preceq b \;\Rightarrow \begin{cases} T_N((x*z)*(y*z)) \leq \bigvee\{T_N(a), T_N(b)\}, \\ I_N((x*z)*(y*z)) \geq \bigwedge\{I_N(a), I_N(b)\}, \\ F_N((x*z)*(y*z)) \leq \bigvee\{F_N(a), F_N(b)\}, \end{cases} \tag{16}$$

for all $x, y, z, a, b \in X$.

Proof. Suppose that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Theorem 1. Let $x, y, z, a, b \in X$ be such that $((x*y)*z)*a \preceq b$. Using Corollary 2 and Lemma 3, we have

$$\begin{aligned} T_N((x*z)*(y*z)) &\leq T_N(((x*y)*z)) \leq \bigvee\{T_N(a), T_N(b)\}, \\ I_N((x*z)*(y*z)) &\geq I_N(((x*y)*z)) \geq \bigwedge\{I_N(a), I_N(b)\}, \\ F_N((x*z)*(y*z)) &\leq F_N(((x*y)*z)) \leq \bigvee\{F_N(a), F_N(b)\} \end{aligned}$$

for all $x, y, z, a, b \in X$.

Conversely, let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X that satisfies (16). Let $x, y, a, b \in X$ be such that $((x*y)*y)*a \preceq b$. Then

$$\begin{aligned} T_N(x*y) &= T_N((x*y)*(y*y)) \leq \bigvee\{T_N(a), T_N(b)\}, \\ I_N(x*y) &= I_N((x*y)*(y*y)) \geq \bigwedge\{I_N(a), I_N(b)\}, \\ F_N(x*y) &= F_N((x*y)*(y*y)) \leq \bigvee\{F_N(a), F_N(b)\} \end{aligned}$$

by (III), (1) and (16). It follows from Theorem 4 that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. □

Theorem 7. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X. Then $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X if and only if $X_{\mathbf{N}}$ satisfies (9) and*

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x*y) \leq \bigvee\{T_N(((x*y)*y)*z), T_N(z)\}, \\ I_N(x*y) \geq \bigwedge\{I_N(((x*y)*y)*z), I_N(z)\}, \\ F_N(x*y) \leq \bigvee\{F_N(((x*y)*y)*z), F_N(z)\} \end{pmatrix}. \tag{17}$$

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Theorem 1, and so the condition (9) is valid. Using (8), (III), (1), (2) and (15), we have

$$\begin{aligned} T_N(x*y) &\leq \bigvee\{T_N((x*y)*z), T_N(z)\} \\ &= \bigvee\{T_N(((x*z)*y)*(y*y)), T_N(z)\} \\ &\leq \bigvee\{T_N(((x*z)*y)*y), T_N(z)\} \\ &= \bigvee\{T_N(((x*y)*y)*z), T_N(z)\}, \end{aligned}$$

$$\begin{aligned} I_N(x*y) &\geq \bigwedge\{I_N((x*y)*z), I_N(z)\} \\ &= \bigwedge\{I_N(((x*z)*y)*(y*y)), I_N(z)\} \\ &\geq \bigwedge\{I_N(((x*z)*y)*y), I_N(z)\} \\ &= \bigwedge\{I_N(((x*y)*y)*z), I_N(z)\}, \end{aligned}$$

and

$$\begin{aligned} F_N(x*y) &\leq \bigvee\{F_N((x*y)*z), F_N(z)\} \\ &= \bigvee\{F_N(((x*z)*y)*(y*y)), F_N(z)\} \\ &\leq \bigvee\{F_N(((x*z)*y)*y), F_N(z)\} \\ &= \bigvee\{F_N(((x*y)*y)*z), F_N(z)\} \end{aligned}$$

for all $x, y, z \in X$. Therefore (17) is valid.

Conversely, if $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-structure over X satisfying two Conditions (9) and (17), then

$$\begin{aligned} &T_N(x) = T_N(x*0) \leq \bigvee\{T_N(((x*0)*0)*z), T_N(z)\} = \bigvee\{T_N(x*z), T_N(z)\}, \\ &I_N(x) = I_N(x*0) \geq \bigwedge\{I_N(((x*0)*0)*z), I_N(z)\} = \bigwedge\{I_N(x*z), I_N(z)\}, \\ &F_N(x) = F_N(x*0) \leq \bigvee\{F_N(((x*0)*0)*z), F_N(z)\} = \bigvee\{F_N(x*z), F_N(z)\} \end{aligned}$$

for all $x, z \in X$. Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. Now, if we take $z = 0$ in (17) and use (1), then

$$\begin{aligned} T_N(x*y) &\leq \bigvee\{T_N(((x*y)*y)*0), T_N(0)\} \\ &= \bigvee\{T_N((x*y)*y), T_N(0)\} = T_N((x*y)*y), \end{aligned}$$

$$\begin{aligned} I_N(x*y) &\geq \bigwedge\{I_N(((x*y)*y)*0), I_N(0)\} \\ &= \bigwedge\{I_N((x*y)*y), I_N(0)\} = I_N((x*y)*y), \end{aligned}$$

and

$$\begin{aligned} F_N(x*y) &\leq \bigvee\{F_N(((x*y)*y)*0), F_N(0)\} \\ &= \bigvee\{F_N((x*y)*y), F_N(0)\} = F_N((x*y)*y) \end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3 that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. $\square$

Summarizing the above results, we have a characterization of a neutrosophic positive implicative $\mathcal{N}$-ideal.

Theorem 8. *For a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X, the following assertions are equivalent.*

(1) $X_{\mathbf{N}}$ *is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.*
(2) $X_{\mathbf{N}}$ *is a neutrosophic $\mathcal{N}$-ideal of X satisfying the condition (12).*
(3) $X_{\mathbf{N}}$ *is a neutrosophic $\mathcal{N}$-ideal of X satisfying the condition (15).*
(4) $X_{\mathbf{N}}$ *satisfies two conditions (9) and (17).*
(5) $X_{\mathbf{N}}$ *satisfies the condition (14).*
(6) $X_{\mathbf{N}}$ *satisfies the condition (3).*

For any fixed numbers $\xi_T, \xi_F \in [-1, 0)$, $\xi_I \in (-1, 0]$ and a nonempty subset G of X, a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}^G$ over X is defined to be the structure

$$X_{\mathbf{N}}^G := \frac{X}{(T_N^G, I_N^G, F_N^G)} = \left\{ \frac{x}{(T_N^G(x), I_N^G(x), F_N^G(x))} \mid x \in X \right\} \tag{18}$$

where T_N^G, I_N^G and F_N^G are $\mathcal{N}$-functions on X which are given as follows:

$$T_N^G : X \to [-1, 0],\ x \mapsto \begin{cases} \xi_T & \text{if } x \in G, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_N^G : X \to [-1, 0],\ x \mapsto \begin{cases} \xi_I & \text{if } x \in G, \\ -1 & \text{otherwise,} \end{cases}$$

and

$$F_N^G : X \to [-1, 0],\ x \mapsto \begin{cases} \xi_F & \text{if } x \in G, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 9. *Given a nonempty subset G of X, a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}^G$ over X is a neutrosophic positive implicative $\mathcal{N}$-ideal of X if and only if G is a positive implicative ideal of X.*

Proof. Assume that G is a positive implicative ideal of X. Since $0 \in G$, it follows that $T_N^G(0) = \xi_T \leq T_N^G(x)$, $I_N^G(0) = \xi_I \geq I_N^G(x)$, and $F_N^G(0) = \xi_F \leq F_N^G(x)$ for all $x \in X$. For any $x, y, z \in X$, we consider four cases:

Case 1. $(x * y) * z \in G$ and $y * z \in G$,
Case 2. $(x * y) * z \in G$ and $y * z \notin G$,
Case 3. $(x * y) * z \notin G$ and $y * z \in G$,
Case 4. $(x * y) * z \notin G$ and $y * z \notin G$.

Case 1 implies that $x * z \in G$, and thus

$$\begin{aligned} T_N^G(x * z) &= T_N^G((x * y) * z) = T_N^G(y * z) = \xi_T, \\ I_N^G(x * z) &= I_N^G((x * y) * z) = I_N^G(y * z) = \xi_I, \\ F_N^G(x * z) &= F_N^G((x * y) * z) = F_N^G(y * z) = \xi_F. \end{aligned}$$

Hence

$$\begin{aligned} T_N^G(x * z) &\leq \bigvee\{T_N^G((x * y) * z), T_N^G(y * z)\}, \\ I_N^G(x * z) &\geq \bigwedge\{I_N^G((x * y) * z), I_N^G(y * z)\}, \\ F_N^G(x * z) &\leq \bigvee\{F_N^G((x * y) * z), F_N^G(y * z)\}. \end{aligned}$$

If Case 2 is valid, then $T_N^G(y*z)=0$, $I_N^G(y*z)=-1$ and $F_N^G(y*z)=0$. Thus

$$\begin{aligned} T_N^G(x*z) &\leq 0 = \bigvee\{T_N^G((x*y)*z), T_N^G(y*z)\}, \\ I_N^G(x*z) &\geq -1 = \bigwedge\{I_N^G((x*y)*z), I_N^G(y*z)\}, \\ F_N^G(x*z) &\leq 0 = \bigvee\{F_N^G((x*y)*z), F_N^G(y*z)\}. \end{aligned}$$

For the Case 3, it is similar to the Case 2.
For the Case 4, it is clear that

$$\begin{aligned} T_N^G(x*z) &\leq \bigvee\{T_N^G((x*y)*z), T_N^G(y*z)\}, \\ I_N^G(x*z) &\geq \bigwedge\{I_N^G((x*y)*z), I_N^G(y*z)\}, \\ F_N^G(x*z) &\leq \bigvee\{F_N^G((x*y)*z), F_N^G(y*z)\}. \end{aligned}$$

Therefore $X_{\mathbf{N}}^G$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X.

Conversely, suppose that $X_{\mathbf{N}}^G$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then $(T_N^G)^{\frac{\xi_T}{2}}=G$, $(I_N^G)^{\frac{\xi_I}{2}}=G$ and $(F_N^G)^{\frac{\xi_F}{2}}=G$ are positive implicative ideals of X by Theorem 2. □

We consider an extension property of a neutrosophic positive implicative $\mathcal{N}$-ideal based on the negative indeterminacy membership function.

Lemma 7 ([13])**.** *Let A and B be ideals of X such that $A\subseteq B$. If A is a positive implicative ideal of X, then so is B.*

Theorem 10. *Let*

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{\frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x\in X\right\}$$

and

$$X_{\mathbf{M}} := \frac{X}{(T_M, I_M, F_M)} = \left\{\frac{x}{(T_M(x), I_M(x), F_M(x))} \mid x\in X\right\}$$

be neutrosophic $\mathcal{N}$-ideals of X such that $X_{\mathbf{N}}(=,\leq,=)X_{\mathbf{M}}$, that is, $T_N(x)=T_M(x)$, $I_N(x)\leq I_M(x)$ and $F_N(x)=F_M(x)$ for all $x\in X$. If $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X, then so is $X_{\mathbf{M}}$.

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X. Then T_N^α, I_N^β and F_N^γ are positive implicative ideals of X for all $\alpha,\beta,\gamma\in[-1,0]$ by Theorem 2. The condition $X_{\mathbf{N}}(=,\leq,=)X_{\mathbf{M}}$ implies that $T_N^{\xi_T}=T_M^{\xi_T}$, $I_N^{\xi_I}\subseteq I_M^{\xi_I}$ and $F_N^{\xi_F}=F_M^{\xi_F}$. It follows from Lemma 7 that T_M^α, I_M^β and F_M^γ are positive implicative ideals of X for all $\alpha,\beta,\gamma\in[-1,0]$. Therefore $X_{\mathbf{M}}$ is a neutrosophic positive implicative $\mathcal{N}$-ideal of X by Theorem 5. □

4. Conclusions

The aim of this paper is to study neutrosophic $\mathcal{N}$-structure of positive implicative ideal in BCK-algebras, and to provide a mathematical tool for dealing with several informations containing uncertainty, for example, decision making problem, medical diagnosis, graph theory, pattern recognition, etc. As a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set, F. Smarandache have developed neutrosophic set (NS) in [1,15]. In this manuscript, we have discussed the notion of a neutrosophic positive implicative $\mathcal{N}$-ideal in BCK-algebras, and investigated several properties. We have considered relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic positive implicative $\mathcal{N}$-ideal. We have provided conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal, and considered characterizations of a neutrosophic positive implicative $\mathcal{N}$-ideal. We have established an extension property of a neutrosophic positive implicative $\mathcal{N}$-ideal based on the negative indeterminacy membership function.

Various sources of uncertainty can be a challenge to make a reliable decision. Based on the results in this paper, our future research will be focused to solve real-life problems under the opinions of experts in a neutrosophic set environment, for example, decision making problem, medical diagnosis etc. The future works also may use the study neutrosophic set theory on several related algebraic structures, BL-algebras, MTL-algebras, R_0-algebras, MV-algebras, EQ-algebras and lattice implication algebras etc.

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