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Asymptotic and Pseudoholomorphic Solutions of Singularly Perturbed Differential and Integral Equations in the Lomov's Regularization Method

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Abstract: We consider a singularly perturbed integral equation with weakly and rapidly varying kernels. The work is a continuation of the studies carried out previously, but these were focused solely on rapidly changing kernels. A generalization for the case of two kernels, one of which is weakly, and the other rapidly varying, has not previously been carried out. The aim of this study is to investigate the effects introduced into the asymptotics of the solution of the problem by a weakly varying integral kernel. In the second part of the work, the problem of constructing exact (more precise, pseudo-analytic) solutions of singularly perturbed problems is considered on the basis of the method of holomorphic regularization developed by one of the authors of this paper. The power series obtained with the help of this method for the solutions of singularly perturbed problems (in contrast to the asymptotic series constructed in the first part of this paper) converge in the usual sense.

Keywords: singularly perturbed; integral equations; regularization of the integral; weakly and rapidly changing kernel; holomorphic integrals; family of homomorphisms; asymptotic and pseudoholomorphic solutions

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1. Introduction

In the first part of this work, we consider a singularly perturbed equation in which integral operators contain both weakly and rapidly changing kernels. The problem of constructing a regularized asymptotic solution for this problem, uniformly applicable over the entire time interval under consideration, was previously solved but only for rapidly varying kernels (see, for example References [1–4]). A generalization for the case of two kernels, one of which is weakly, and the other rapidly varying, has not previously been carried out. The aim of the present study is to investigate the effects introduced into the asymptotics of the solution by a weakly varying kernel. Notice that this problem was not considered from the point of view of other methods of asymptotic integration (for example, using the methods of References [5–7]).

The second part of our paper is devoted to the construction of approximate solutions of singularly perturbed problems using the method of holomorphic regularization [8,9]. The analysis of asymptotic methods for solving singularly perturbed problems shows that the solutions of such problems depend in two ways on a small parameter: regularly and singularly. This dependence is especially vividly demonstrated by the method of regularization of Lomov. Moreover, regularized series representing solutions of singularly perturbed problems can converge in the usual sense. In this connection, it became necessary to study a special class of functions—pseudoholomorphic functions. This very important part of the complex analysis is designed to substantiate the main provisions of the so-called analytic theory of singular perturbations. On the other hand, the relevance of the theory is also supported by the fact that pseudoholomorphic functions, in contrast to holomorphic functions, are determined when the conditions of the implicit function theorem are violated.

The concept of a pseudoanalytic (pseudoholomorphic) function and the associated concept of an essentially singular manifold are of a general mathematical nature, although they arose in the framework of the regularization method for singular perturbations. First of all, they reflect the new concept of a pseudoholomorphic solution of singularly perturbed problems, i.e., such a solution, which is representable in the form of a series converging in the usual (but not asymptotic) sense in powers of a small parameter. We must also take into account the fact that the modern mathematical theory of the boundary layer [1], along with the Vasilyeva–Butuzov–Nefedov boundary-function method [5] and the method of barrier functions [10], widely uses the notion of a pseudoholomorphic solution. The importance of considering singularly perturbed problems from the standpoint of the method of pseudoholomorphic solutions is illustrated by applications (see, for example, References [11,12]).

2. An Equivalent Integro-Differential System and Its Regularization

We consider the singularly perturbed equation

$$\varepsilon y\left(t,\varepsilon\right)=\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta}K_2(t,s)y(s,\varepsilon)ds+\int_0^t K_1(t,s)y(s,\varepsilon)ds+h(t), t\in[0,T]. \tag{1}$$

Differentiating Equation (1) with respect to t, will have

$$\varepsilon^2\left(\frac{dy(t,\varepsilon)}{dt}\right)=\int\limits_0^t\left(\mu\left(t\right)e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}K_2\left(t,s\right)y\left(s,\varepsilon\right)+\varepsilon\cdot e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}\left(\frac{\partial}{\partial t}K_2\left(t,s\right)\right)y\left(s,\varepsilon\right)\right)\,ds+$$
$$+\varepsilon\cdot K_2\left(t,t\right)y\left(t,\varepsilon\right)+\varepsilon\cdot\int\limits_0^t\left(\frac{\partial}{\partial t}K_1\left(t,s\right)\right)y\left(s,\varepsilon\right)\,ds+\varepsilon\cdot K_1\left(t,t\right)y\left(t,\varepsilon\right)+\varepsilon\cdot\frac{d}{dt}h\left(t\right),$$

or

$$\begin{aligned}&\varepsilon^2\frac{dy}{dt}=\left(K_1\left(t,t\right)+K_2\left(t,t\right)\right)\varepsilon y+\mu\left(t\right)z+\\&+\int\limits_0^t e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}\frac{\partial}{\partial t}\,K_2\left(t,s\right)\varepsilon y\left(s,\varepsilon\right)\,ds+\int\limits_0^t\frac{\partial}{\partial t}\,K_1\left(t,s\right)\varepsilon y\left(s,\varepsilon\right)\,ds+\varepsilon\cdot\frac{d}{dt}h\left(t\right),\end{aligned} \tag{2}$$

where $z\left(t,\varepsilon\right)=\int\limits_0^t e^{\frac{1}{\varepsilon}\int\limits_s^t\mu(\theta)d\theta}K_2\left(t,s\right)y\left(s,\varepsilon\right)ds$. By differentiating this function with respect to t, we also obtain

$$\varepsilon\frac{dz}{dt}=\mu\left(t\right)\cdot z+\int\limits_0^t\left(\varepsilon\cdot e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}\left(\frac{\partial}{\partial t}K_2\left(t,s\right)\right)y\left(s,\varepsilon\right)\right)\,ds+\varepsilon\cdot K_2\left(t,t\right)y. \tag{3}$$

Finally, denoting by $\varepsilon y=v$, rewriting Equations (2) and (3) in the form

$$\begin{aligned}&\varepsilon\frac{dv}{dt}=\left(K_1\left(t,t\right)+K_2\left(t,t\right)\right)v+\mu\left(t\right)z+\\&+\int\limits_0^t e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}\frac{\partial}{\partial t}\,K_2\left(t,s\right)v\left(s,\varepsilon\right)\,ds+\int\limits_0^t\frac{\partial}{\partial t}\,K_1\left(t,s\right)v\left(s,\varepsilon\right)\,ds+\varepsilon\cdot\dot{h}\left(t\right),\\&\varepsilon\frac{dz}{dt}=\mu\left(t\right)\cdot z+\int\limits_0^t\left(e^{\frac{\int_s^t\mu(\theta)\,d\theta}{\varepsilon}}\left(\frac{\partial}{\partial t}K_2\left(t,s\right)\right)v\left(s,\varepsilon\right)\right)\,ds+K_2\left(t,t\right)v.\end{aligned}$$

We have obtained an integro-differential system of equations

$$\varepsilon\begin{pmatrix}\frac{dv}{dt}\\ \frac{dz}{dt}\end{pmatrix}=\begin{pmatrix}K_1(t,t)+K_2(t,t) & \mu(t)\\ K_2(t,t) & \mu(t)\end{pmatrix}\begin{pmatrix}v\\ z\end{pmatrix}+$$
$$+\int_0^t e^{\frac{\int_s^t \mu(\theta)\,d\theta}{\varepsilon}}\begin{pmatrix}\frac{\partial}{\partial t}K_2(t,s) & 0\\ \frac{\partial}{\partial t}K_2(t,s) & 0\end{pmatrix}\begin{pmatrix}v(s,\varepsilon)\\ z(s,\varepsilon)\end{pmatrix}ds+\int_0^t\begin{pmatrix}\frac{\partial}{\partial t}K_1(t,s) & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v(s,\varepsilon)\\ z(s,\varepsilon)\end{pmatrix}ds+$$
$$+\varepsilon\begin{pmatrix}\dot h(t)\\ 0\end{pmatrix},\ \begin{pmatrix}v(0,\varepsilon)\\ z(0,\varepsilon)\end{pmatrix}=\begin{pmatrix}h(0)\\ 0\end{pmatrix},$$

or

$$\begin{gathered}\varepsilon\frac{dw}{dt}=A(t)w+\int_0^t B(t,s)\,w(s,\varepsilon)ds+\\ +\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\mu(\theta)d\theta}G(t,s)\,w(s,\varepsilon)ds+\varepsilon H(t),w(0,\varepsilon)=w^0\equiv\begin{pmatrix}h(0)\\ 0\end{pmatrix},\end{gathered}\tag{4}$$

where $w=\{v,z\}$, matrixes $A(t)$, $A_1(t)$, $B(t,s)$, $G(t,s)$, and the vector function $H(t)$ have the form

$$A(t)=\begin{pmatrix}K_1(t,t)+K_2(t,t) & \mu(t)\\ K_2(t,t) & \mu(t)\end{pmatrix},\ B(t,s)=\begin{pmatrix}\frac{\partial K_1(t,s)}{\partial t} & 0\\ 0 & 0\end{pmatrix},$$
$$G(t,s)=\begin{pmatrix}\frac{\partial K_2(t,s)}{\partial t} & 0\\ \frac{\partial K_2(t,s)}{\partial t} & 0\end{pmatrix},\ H(t)=\begin{pmatrix}\dot h(t)\\ 0\end{pmatrix},\ w^0\equiv\begin{pmatrix}h(0)\\ 0\end{pmatrix}.$$

The roots of the characteristic equation of matrix $A(t)$:

$$\lambda^2-(\mu(t)+K_1(t,t)+K_2(t,t))\,\lambda+\mu(t)\,K_1(t,t)=0$$

form the spectrum $\sigma(A(t))=\{\lambda_1(t),\lambda_2(t)\}$ of the matrix $A(t)$. We assume that the following conditions hold:

1) $h(t),\mu(t)\in C^\infty([0,T],\mathbb{C})$, $K_j(t,s)\in C^\infty(0\le s\le t\le T,\mathbb{C}), j=1,2;$
2) $\mu(t)\neq 0, \operatorname{Re}\mu(t)\le 0$, $\lambda_j(t)\neq 0$, $\operatorname{Re}\lambda_j(t)$ $\forall t\in[0,T]$, $j=1,2.$

We denote by $\lambda_3(t)\equiv\mu(t)$ and (according to the method [13] of Lomov) we introduce regularizing variables

$$\tau_j=\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta\equiv\frac{\psi_j(t)}{\varepsilon},\ j=1,2,3.\tag{5}$$

For the extension $\tilde w=\{v(t,\tau,\varepsilon),z(t,\tau,\varepsilon)\}$, we get the following system:

$$\begin{gathered}\frac{\partial\tilde w}{\partial t}+\sum_{j=1}^3\lambda_j(t)\frac{\partial\tilde w}{\partial\tau_j}-A(t)\tilde w-\int_0^t B(t,s)\,\tilde w(s,\tfrac{\psi(s)}{\varepsilon},\varepsilon)ds-\\ -\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_3(\theta)d\theta}G(t,s)\,\tilde w(s,\tfrac{\psi(s)}{\varepsilon},\varepsilon)ds=\varepsilon H(t),\tilde w(t,\tau,\varepsilon)|_{t=0,\tau=0}=w^0,\end{gathered}\tag{6}$$

where $\tau=(\tau_1,\tau_2,\tau_3)$, $\psi=(\psi_1,\psi_2,\psi_3)$. However, Equation (6) cannot be considered completely regularized, since the integral operator

$$J\tilde w=\int_0^t B(t,s)\,\tilde w(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds+\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_3(\theta)d\theta}G(t,s)\,\tilde w(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$$

has not been regularized. To regularize the operator $J\tilde w$, we introduce a class $M_\varepsilon=U|_{\tau=\frac{\psi(t)}{\varepsilon}}$, asymptotically invariant with respect to the operator J (see Reference [13], p. 62). In this case, we take as the space U the vector-valued functions representable by the sums of the form

$$w(t,\tau)=\sum_{j=1}^3 w_j(t)e^{\tau_j}+w_0(t),w_j(t)\in C([0,T],\mathbb{C}^2),j=\overline{0,3}.\tag{7}$$

We must show that the image $Jw(t,\tau)$ of the functions of the form of Equation (7) can be represented in the form of a series

$$Jw(t,\tau) = \sum_{k=0}^{\infty} \varepsilon^k (\sum_{j=1}^{3} w_1^{(k)}(t) e^{\tau_j} + w_0^{(0)}(t))|_{\tau=\frac{\psi(t)}{\varepsilon}},$$

converging asymptotically to Jw (as $\varepsilon \to +0$) and that this convergence is uniform with respect to $t \in [0,T]$. Substituting Equation (7) into $Jw(t,\tau)$, we obtain

$$\begin{aligned}
& Jw\,(t,\tau) = \int_0^t B\,(t,s) \left(\sum_{j=1}^3 w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} + w_0(s)\right) ds + \\
& + \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} G\,(t,s) \left(\sum_{j=1}^3 w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} + w_0(s)\right) ds \equiv \\
& \equiv \int_0^t B\,(t,s)\, w_0(s) ds + \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} G\,(t,s)\, w_0(s) ds + \\
& + \sum_{j=1}^3 \int_0^t B\,(t,s)\, w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds + \\
& + \sum_{j=1}^3 \int_0^t G\,(t,s)\, w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta + \frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} ds \equiv \\
& \equiv \int_0^t B\,(t,s)\, w_0(s) ds + e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t G\,(t,s)\, w_3(s) ds + \\
& + \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} G\,(t,s)\, w_0(s) ds + \sum_{j=1}^3 \int_0^t B\,(t,s)\, w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds + \\
& + \sum_{k=1}^2 \int_0^t G\,(t,s)\, w_k(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_k(\theta)d\theta + \frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} ds.
\end{aligned} \tag{7a}$$

Applying the operation of integration by parts, we find that

$$\begin{aligned}
& \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} G\,(t,s)\, w_0(s) ds = -\varepsilon \int_0^t \frac{G(t,s)w_0(s)}{\lambda_3(s)} de^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} = \\
& = \varepsilon \left[\frac{G(t,0)w_0(0)}{\lambda_3(0)} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} - \frac{G(t,t)w_0(t)}{\lambda_3(t)}\right] + \\
& + \varepsilon \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} \frac{\partial}{\partial s}\left(\frac{G(t,s)w_0(s)}{\lambda_3(s)}\right) ds = \\
& = \sum_{m=0}^\infty \varepsilon^{m+1} [\left(I_3^m\left(G\,(t,s)\, w_0\,(s)\right)\right)_{s=0} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} - \left(I_3^m\left(G\,(t,s)\, w_0\,(s)\right)\right)_{s=t}];
\end{aligned}$$

$$\begin{aligned}
& \int_0^t B\,(t,s)\, w_j(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds = \varepsilon \int_0^t \frac{B(t,s)w_j(s)}{\lambda_j(s)} de^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} = \\
& = \varepsilon \left[\frac{B(t,t)w_j(t)}{\lambda_j(t)} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \frac{B(t,0)w_j(0)}{\lambda_j(0)}\right] - \\
& - \int_0^t \frac{\partial}{\partial s}\left(\frac{B(t,s)w_j(s)}{\lambda_j(s)}\right) e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds = \\
& = \sum_{m=0}^\infty (-1)^m \varepsilon^{m+1} [\left(I_j^m\left(B\,(t,s)\, w_j\,(s)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \left(I_j^m\left(B\,(t,s)\, w_j\,(s)\right)\right)_{s=0}];
\end{aligned}$$

$$\begin{aligned}
& \int_0^t G\,(t,s)\, w_k(s) e^{\frac{1}{\varepsilon}\int_0^s \lambda_k(\theta)d\theta + \frac{1}{\varepsilon}\int_s^t \lambda_3(\theta)d\theta} ds = e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t e^{\frac{1}{\varepsilon}\int_0^s [\lambda_k(\theta)-\lambda_3(\theta)]d\theta} G\,(t,s)\, w_k(s) ds = \\
& = \varepsilon e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t \frac{G(t,s)w_k(s)}{\lambda_k(s)-\lambda_3(s)} de^{\frac{1}{\varepsilon}\int_0^s [\lambda_k(\theta)-\lambda_3(\theta)]d\theta} = \\
& = \varepsilon e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \{[\int_0^t \frac{G(t,t)w_k(t)}{\lambda_k(t)-\lambda_3(t)} e^{\frac{1}{\varepsilon}\int_0^t [\lambda_k(\theta)-\lambda_3(\theta)]d\theta} - \frac{G(t,0)w_k(0)}{\lambda_k(0)-\lambda_3(0)}] - \\
& - \int_0^t e^{\frac{1}{\varepsilon}\int_0^s [\lambda_k(\theta)-\lambda_3(\theta)]d\theta} \frac{\partial}{\partial s}\left(\frac{G(t,s)w_k(s)}{\lambda_k(s)-\lambda_3(s)}\right) ds\} = \\
& = \varepsilon [\int_0^t \frac{G(t,t)w_k(t)}{\lambda_k(t)-\lambda_3(t)} e^{\frac{1}{\varepsilon}\int_0^t \lambda_k(\theta)d\theta} - \frac{G(t,0)w_k(0)}{\lambda_k(0)-\lambda_3(0)} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta}] - \\
& - \varepsilon e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} \int_0^t e^{\frac{1}{\varepsilon}\int_0^s (\lambda_k(\theta)-\lambda_3(\theta))d\theta} \frac{\partial}{\partial s}\left(\frac{G(t,s)w_k(s)}{\lambda_k(s)-\lambda_3(s)}\right) ds = \\
& = \sum_{m=0}^\infty (-1)^m \varepsilon^{m+1} [\left(I_{k3}^m\left(G\,(t,s)\, w_k(s)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_k(\theta)d\theta} - \\
& - \left(I_{k3}^m\left(G\,(t,s)\, w_k(s)\right)\right)_{s=0} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta}],
\end{aligned}$$

where operators are introduced:

$$\begin{aligned}
& I_j^0 = \frac{1}{\lambda_j(s)},\, I_j^m = \frac{1}{\lambda_j(s)}\frac{\partial}{\partial s} I_j^{m-1},\, m \geq 1,\, j = 1,2,3; \\
& I_{k3}^0 = \frac{1}{\lambda_k(s)-\lambda_3(s)},\, I_{k3}^m = \frac{1}{\lambda_k(s)-\lambda_3(s)}\frac{\partial}{\partial s} I_{k3}^{m-1},\, m \geq 1,\, k = 1,2.
\end{aligned} \tag{8}$$

Consequently, for the operator $Jw\left(t,\tau\right)$ there is a decomposition

$$
\begin{array}{l}
Jw\left(t,\tau\right) \equiv \int_0^t B\left(t,s\right) w_0(s)ds + e^{\frac{1}{\varepsilon}\int_0^s \lambda_3(\theta)d\theta} \int_0^t G\left(t,s\right) w_3(s)ds + \\
+\sum_{m=0}^{\infty} \varepsilon^{m+1}\left[\left(I_3^m\left(G\left(t,s\right) w_0\left(s\right)\right)\right)_{s=0} e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta} - \left(I_3^m\left(G\left(t,s\right) w_0\left(s\right)\right)\right)_{s=t}\right] \\
+\sum_{m=0}^{\infty} \left(-1\right)^m \varepsilon^{m+1}\sum_{j=1}^{3}\left[\left(I_j^m\left(B\left(t,s\right) w_j\left(s\right)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \right. \\
\left. -\left(I_j^m\left(B\left(t,s\right) w_j\left(s\right)\right)\right)_{s=0}\right] + \\
+\sum_{m=0}^{\infty} \left(-1\right)^m \varepsilon^{m+1}\sum_{k=1}^{2}\left[\left(I_{k3}^m\left(G\left(t,s\right) w_k(s)\right)\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_k(\theta)d\theta} - \right. \\
\left. -\left(I_{k3}^m\left(G\left(t,s\right) w_k(s)\right) e^{\frac{1}{\varepsilon}\int_0^t \lambda_3(\theta)d\theta}\right)_{s=0}\right].
\end{array} \tag{9}
$$

It is not hard to show (see Reference [14]) that the series on the right-hand side of Equation (9) converges to $Jw(t,\varepsilon)$ (as $\varepsilon \to +0$) uniformly with respect to $t \in [0,T]$. We introduce operators of order (on ε) $R_\nu : U \to U$:

$$
\begin{array}{l}
R_0 w\left(t,\tau\right) \equiv \int_0^t B\left(t,s\right) w_0(s)ds + e^{\tau_3}\int_0^t G\left(t,s\right) w_3(s)ds, \\
R_1 w\left(t,\tau\right) = \frac{G(t,0)w_0(0)}{\lambda_3(0)} e^{\tau_3} - \frac{G(t,t)w_0(t)}{\lambda_3(t)} + \\
+\sum\limits_{j=1}^{3}\left[\frac{B(t,t)w_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{B(t,0)w_j(0)}{\lambda_j(0)}\right] + \\
+\sum\limits_{k=1}^{2}\left[\frac{(G(t,t)w_k(t))}{\lambda_k(t)-\lambda_3(t)} e^{\tau_k} - \frac{(G(t,0)w_k(0))}{\lambda_k(0)-\lambda_3(0)} e^{\tau_3}\right],
\end{array} \tag{10}
$$

$$
\begin{array}{l}
R_{m+1} w\left(t,\tau\right) = \left[\left(I_3^m\left(G\left(t,s\right) w_0\left(s\right)\right)\right)_{s=0} e^{\tau_3} - \left(I_3^m\left(G\left(t,s\right) w_0\left(s\right)\right)\right)_{s=t}\right] + \\
+(-1)^m \sum\limits_{j=1}^{3}\left[\left(I_j^m\left(B\left(t,s\right) w_j\left(s\right)\right)\right)_{s=t} e^{\tau_j} - \left(I_j^m\left(B\left(t,s\right) w_j\left(s\right)\right)\right)_{s=0}\right] + \\
+(-1)^m \sum\limits_{k=1}^{2}\left[\left(I_{k3}^m\left(G\left(t,s\right) w_k(s)\right)\right)_{s=t} e^{\tau_k} - \left(I_{k3}^m\left(G\left(t,s\right) w_k(s)\right) e^{\tau_3}\right)_{s=0}\right], \\
m \geq 1, \tau = \frac{\psi(t)}{\varepsilon}.
\end{array}
$$

Then, the image $Jw(t,\tau)$ can be written in the form

$$
Jw(t,\tau) = R_0 w(t,\tau) + \sum_{m=0}^{\infty} \varepsilon^{m+1} R_{m+1} w(t,\tau), \tag{11}
$$

where $\tau = \frac{\psi(t)}{\varepsilon}$. We now extend the operator J on the series of the form

$$
\tilde{w}(t,\tau,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k w_k(t,\tau) \tag{12}
$$

with coefficients $w_k(t,\tau) \in U, k \geq 0$. The formal extension $\tilde{J}$ of the operator J on the series of the form of Equation (12) is called the operator

$$
\tilde{J}\tilde{w}(t,\tau,\varepsilon) \overset{def}{=} \sum_{\nu=0}^{\infty} \varepsilon^{\nu} \sum_{s=0}^{\nu} R_{\nu-s} w_s(t,\tau). \tag{13}
$$

In spite of the fact that the extension in Equation (13) of the operator J is defined formally, it is quite possible to use it (see Theorem 3 below) in constructing an asymptotic solution of a finite order in ε. Now, it is easy to write out the regularized (with respect to Equation (1)) problem:

$$
\frac{\partial \tilde{w}}{\partial t} + \sum_{j=1}^{3} \lambda_j\left(t\right) \frac{\partial \tilde{w}}{\partial \tau_j} - A(t)\tilde{w} - \tilde{J}\tilde{w} = \varepsilon H(t), \tilde{w}(t,\tau,\varepsilon)|_{t=0,\tau=0} = w^0. \tag{14}
$$

3. The Solvability of Iterative Problems and the Asymptotic Convergence of Formal Solutions to the Exact Ones

Substituting the series of Equation (12) into Equation (14) and equating the coefficients for the same powers of ε, we obtain the following iteration problems:

$$L_0 w_0(t,\tau) \equiv \sum_{j=1}^{3} \lambda_j(t) \frac{\partial w_0}{\partial \tau_j} - A(t) w_0 - R_0 w_0 = 0, w_0(0,0) = w^0; \tag{15a}$$

$$L_0 w_1(t,\tau) = -\frac{\partial w_0}{\partial t} + R_1 w_0 + H(t), w_1(0,0) = 0; \tag{15b}$$

$$L_0 w_2(t,\tau) = -\frac{\partial w_1}{\partial t} + R_1 w_1 + R_2 w_0;\ w_2(0,0) = 0; \tag{15c}$$

$$\cdots$$

$$\begin{gathered} L_0 w_k(t,\tau) = -\tfrac{\partial w_{k-1}}{\partial t} + R_1(t) w_{k-1} + R_2 w_{k-2} + \\ + \ldots + R_k w_0, w_k(0,0) = 0, k \geq 1, \end{gathered} \tag{15d}$$

where $R_0 w(t,\tau) \equiv R_0 \left(\sum_{j=1}^{3} w_1(t) e^{\tau_j} + w_0(t) \right) = \int_0^t B(t,s) w_0(s) ds + e^{\tau_3} \int_0^t G(t,s) w_3(s) ds.$

Turning to the formulation of theorems on the normal and unique solvability of the iterative problems of Equations (15a)–(15d), we denote by

$$\varphi_j(t) \equiv \begin{pmatrix} \varphi_j^1(t) \\ \varphi_j^2(t) \end{pmatrix} = \begin{pmatrix} \lambda_j(t) - \mu(t) \\ K_2(t,t) \end{pmatrix}, j = 1,2,$$

the eigenvectors of the matrix $A(t)$. As the eigenvectors $\chi_j(t)$ of the matrix $A^*(t)$ we take the columns of the matrix $\left(\Phi^{-1}(t)\right)^* \equiv (\chi_1(t), \chi_2(t))$, where $\Phi(t) = (\varphi_1(t), \varphi_2(t))$ is the matrix whose columns are the eigenvectors of the matrix $A(t)$. Therefore, if $\varphi_j(t)$ is $\lambda_j(t)$-eigenvector of the matrix $A(t)$, then $\chi_j(t)$ is an $\bar{\lambda}_j(t)$-eigenvector of the matrix $A^*(t)$, and the systems $\{\varphi_j(t)\}$ and $\{\chi_k(t)\}$ are biorthonormal (see Reference [14], pp. 81–83), that is,

$$(\varphi_j(t), \chi_k(t)) \equiv \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k \end{cases} \quad (j,k = 1,2).$$

Each of the iterative systems of Equation (15d) has the form

$$L_0 w(t,\tau) \equiv \sum_{j=1}^{3} \lambda_j(t) \frac{\partial w}{\partial \tau_j} - A(t) w - R_0 w = P(t,\tau), \tag{16}$$

where $P(t,\tau) = \sum_{j=1}^{3} P_j(t) e^{\tau_j} + P_0(t) \in U$. We prove the following assertion.

Theorem 1. *Suppose that the conditions (1) –(2) are satisfied and $P(t,\tau) \in U$. Then, the system of Equation (16) is solvable in the space U if and only if*

$$(P_j(t), \chi_j(t)) \equiv 0\, \forall t \in [0,T],\ j = 1,2. \tag{17}$$

Proof. We will determine the solution of the system of Equation (16) as the sum of Equation (7). Substituting Equation (7) into Equation (16) and equating separately the coefficients of e^{τ_j} and the free terms, we have

$$(\lambda_k(t) I - A(t)) w_k(t) = P_k(t),\ k = 1,2, \tag{18a}$$

$$(\lambda_3(t) I - A(t)) w_3(t) - \int_0^t G(t,s) w_3(s) ds = P_3(t), \tag{18b}$$

$$-A(t)\,w_0(t) - \int_0^t B(t,s)\,w_0(s)\,ds = P_0(t)\,. \tag{18c}$$

For the systems of Equation (18a) to be solvable in space $C^\infty\left([0,T],\mathbb{C}^2\right)$, it is necessary and sufficient that the identities of Equation (17) hold (see, for example, Reference [14], p. 84). Moreover, these systems have a solution in the form of vector functions

$$w_k(t) = \alpha_k(t)\,\varphi_k(t) + \sum_{s=1,s\neq k}^{2} \frac{(P_k(t),\chi_s(t))}{\lambda_k(t)-\lambda_s(t)}\varphi_s(t)\,,\; k=1,2,$$

where $\alpha_k(t) \in C^\infty\left([0,T],\mathbb{C}^1\right)$ are arbitrary functions. Since $\lambda_3(t) \notin \sigma(A(t))$ and $0 \notin \sigma(A(t))$, the systems of Equations (18b) and (18c) can also be rewritten in the form

$$\begin{array}{l} w_3(t) - \int_0^t (\lambda_3(t)\,I - A(t))^{-1}\,G(t,s)\,w_3(s)\,ds = (\lambda_3(t)\,I - A(t))^{-1}\,P_3(t)\,, \\ w_0(t) + \int_0^t A^{-1}(t)\,B(t,s)\,w_0(s)\,ds = -A^{-1}(t)\,P_0(t)\,. \end{array} \tag{19}$$

These Volterra integral systems have kernels belonging to the class $C^\infty\left([0,T],\mathbb{C}^{2\times 2}\right)$, so they have unique solutions in the space $C^\infty\left([0,T],\mathbb{C}^2\right)$. The theorem is proved. □

Remark 1. *It follows from the proof of Theorem 1 that if the conditions of Equation (17) are satisfied, then the system of Equation (17) has the following solution in the space U :*

$$\begin{array}{l} w(t,\tau) = \sum_{k=1}^2 \left[\alpha_k(t)\,\varphi_k(t) + \sum_{s=1,s\neq k}^2 p_{ks}(t)\,\varphi_s(t)\right] e^{\tau_k} + w_3(t)\,e^{\tau_3} + w_0(t)\,, \\ \left(p_{ks}(t) \equiv \frac{(P_k(t),\chi_s(t))}{\lambda_k(t)-\lambda_s(t)},\, k,s=1,2\right), \end{array} \tag{20}$$

where $\alpha_k(t) \in C^\infty\left([0,T],\mathbb{C}^1\right)$ are arbitrary functions, and vector-valued functions $w_3(t)\,, w_0(t)$ are solutions of the integral systems of Equation (19).

We now consider the system of Equation (16) under additional conditions

$$\begin{array}{l} w(0,0) = w^*, \\ < -\frac{\partial w}{\partial t} + R_1 w + Q(t,\tau), \chi_j(t)e^{\tau_j} >\equiv 0,\; j=1,2, \end{array} \tag{21}$$

where $Q(t,\tau) = \sum_{j=1}^3 Q_j(t)e^{\tau_j} + Q_0(t)$ are known functions of class $U, w^* \in \mathbb{C}^2$ is a known constant vector, the operator R_1 is defined by the equality of Equation (10), and by the $<,>$ we denote the inner product (for each $t \in [0,T]$) in space U :

$$< p(t,u), q(t,u) >\equiv< \sum_{j=1}^3 p_j(t)e^{\tau_j} + p_0(t), \sum_{j=1}^3 q_j(t)e^{\tau_j} + q_0(t) > \overset{def}{=} \sum_{k=0}^3 (p_k(t), q_k(t)),$$

where $(\,,)$ is an ordinary inner product in $\mathbb{C}^2$. The following assertion holds true.

Theorem 2. *Suppose that the conditions (1)–(2) hold and the vector function $P(t,\tau) \in U$ satisfies the conditions of Equation (17). Then, the system of Equation (16) under additional conditions of Equation (21) is uniquely solvable in U.*

Proof. Since the conditions of Equation (17) are satisfied, the system of Equation (16) has a solution for Equation (20) in the space U, where $\alpha_j(t)$ are arbitrary functions for now. Subordinating Equation (18) to the initial condition $w(0,0) = w^*$, we obtain the equality

$$\alpha_1(0)\,\varphi_1(0) + p_{12}(0)\,\varphi_2(0) + \alpha_2(0)\,\varphi_2(0) + p_{21}(0)\,\varphi_1(0) = w^*,$$

where $w^* = w_* - w_3(0) - w_0(0)$. Multiplying both sides of this equation scalarly in turn by $\chi_1(0)$ and $\chi_2(0)$, taking into account the biorthonormality of the eigenvector systems $\{\varphi_j(t)\}, \{\chi_k(t)\}$, we have

$$\alpha_1(0) = (w^*, \chi_1(0)) - p_{21}(0), \alpha_2(0) = (w^*, \chi_2(0)) - p_{12}(0). \tag{22}$$

We now calculate the expression $-\frac{\partial w}{\partial t} + R_1 w + Q(t,\tau)$. Taking into account Equation (21) and the form of the operator $R_1 w(t,\tau)$, we have (here and everywhere below, a fatty dot denotes differentiation with respect to t.)

$$\begin{aligned}
&-\tfrac{\partial w}{\partial t} + R_1 w + Q(t,\tau) = -\textstyle\sum_{k=1}^{2}\left(\alpha_k(t)\varphi_k(t) + p_{ks}(t)\varphi_s(t)\right)^{\bullet} e^{\tau_k} - \\
&-\dot{w}_3(t)e^{\tau_3} - \dot{w}_0(t) + \tfrac{G(t,0)w_0(0)}{\lambda_3(0)}e^{\tau_3} - \tfrac{G(t,t)w_0(t)}{\lambda_3(t)} + \\
&+\textstyle\sum_{j=1}^{3}\left[\tfrac{B(t,t)w_j(t)}{\lambda_j(t)}e^{\tau_j} - \tfrac{B(t,0)w_j(0)}{\lambda_j(0)}\right] + \\
&+\textstyle\sum_{k=1}^{2}\left[\tfrac{G(t,t)w_k(t)}{\lambda_k(t)-\lambda_3(t)}e^{\tau_k} - \tfrac{G(t,0)w_k(0)}{\lambda_k(0)-\lambda_3(0)}e^{\tau_3}\right] + \textstyle\sum_{j=1}^{3} Q_j(t)e^{\tau_j} + Q_0(t).
\end{aligned}$$

When writing the conditions of Equation (21) in this expression, it is necessary to preserve only terms containing exponentials e^{τ_1} and e^{τ_2}, that is, Equation (21) is equivalent to the conditions

$$\begin{aligned}
&< -\sum_{k=1}^{2}\left(\alpha_k(t)\varphi_k(t) + \sum_{s=1,s\neq k}^{2} p_{ks}(t)\varphi_s(t)\right)^{\bullet} e^{\tau_k} + \\
&+\sum_{k=1}^{2}\left(\tfrac{B(t,t)}{\lambda_k(t)} + \tfrac{G(t,t)}{\lambda_k(t)-\lambda_3(t)}\right)\left(\alpha_k(t)\varphi_k(t) + \sum_{s=1,s\neq k}^{2} p_{ks}(t)\varphi_s(t)\right)e^{\tau_k} + \\
&+\sum_{k=1}^{2} Q_j(t)e^{\tau_k}, \chi_j(t)e^{\tau_j} > \equiv 0, j = 1,2,
\end{aligned}$$

or

$$\begin{aligned}
&\left(-(\alpha_1(t)\varphi_1(t) + p_{12}(t)\varphi_2(t))^{\bullet} + \left(\tfrac{B(t,t)}{\lambda_1(t)} + \tfrac{G(t,t)}{\lambda_1(t)-\lambda_3(t)}\right)(\alpha_1(t)\varphi_1(t) + p_{12}(t)\varphi_2(t)) + \right.\\
&\left.+Q_1(t), \chi_1(t)\right) \equiv 0, \\
&\left(-(\alpha_2(t)\varphi_2(t) + p_{21}(t)\varphi_1(t))^{\bullet} + \left(\tfrac{B(t,t)}{\lambda_2(t)} + \tfrac{G(t,t)}{\lambda_2(t)-\lambda_3(t)}\right)(\alpha_2(t)\varphi_2(t) + p_{21}(t)\varphi_1(t)) + \right.\\
&\left.+Q_2(t), \chi_2(t)\right) \equiv 0.
\end{aligned}$$

Performing inner multiplication here, we obtain differential equations

$$\dot{\alpha}_1(t) + \left(\dot{\varphi}_1(t) - \left(\frac{B(t,t)}{\lambda_1(t)} + \frac{G(t,t)}{\lambda_1(t)-\lambda_3(t)}\right)\varphi_1(t), \chi_1(t)\right)\alpha_1(t) = g_1(t),$$

$$\dot{\alpha}_2(t) + \left(\dot{\varphi}_2(t) - \left(\frac{B(t,t)}{\lambda_2(t)} + \frac{G(t,t)}{\lambda_2(t)-\lambda_3(t)}\right)\varphi_2(t), \chi_2(t)\right)\alpha_2(t) = g_2(t),$$

where $g_j(t)$ are known scalar functions, $j = 1,2$. Adding the initial conditions of Equation (22) to these equations, we find uniquely the functions $\alpha_j(t)$ in the solution of Equation (20) of the system of Equation (16), and therefore, we construct a solution of this system in the space U in a unique way. The theorem is proved. □

Applying Theorems 1 and 2 to iterative problems, we uniquely determine their solutions in space U and construct the series of Equation (12). As in Reference [2], we prove the following assertion.

Theorem 3. *Assume that the conditions (1)–(2) are satisfied for the system of Equation (2). Then, for* $\varepsilon \in (0, \varepsilon_0]$ *(*$\varepsilon_0 > 0$ *is sufficiently small) the system of Equation (2) has a unique solution* $w(t,\varepsilon) \in C^1([0,T], \mathbb{C}^2)$*; and here we have the estimate*

$$||w(t,\varepsilon) - w_{\varepsilon N}(t)||_{C[0,T]} \leq c_N \varepsilon^{N+1},\ N = 0,1,2,...,$$

where $w_{\varepsilon N}(t)$ is the restriction (for $\tau = \frac{\psi(t)}{\varepsilon}$) N-partial sum of the series of Equation (12) (with coefficients $w_k(t,\tau) \in U$, satisfying the iterative problems of Equation (15d)), the constant $c_N > 0$ does not depend on ε at $\varepsilon \in (0, \varepsilon_0]$.

Since $y(t,\varepsilon) = \frac{1}{\varepsilon} v(t,\varepsilon)$, the series

$$\frac{1}{\varepsilon}\sum_{k=0}^{\infty} v_k\left(t, \frac{\psi(t)}{\varepsilon}\right) \equiv \frac{1}{\varepsilon} v_0\left(t, \frac{\psi(t)}{\varepsilon}\right) + v_1\left(t, \frac{\psi(t)}{\varepsilon}\right) + \varepsilon v_2\left(t, \frac{\psi(t)}{\varepsilon}\right) + \dots$$

is an asymptotic solution (for $\varepsilon \to +0$) of the original problem of Equation (1), that is, the estimate

$$||y(t,\varepsilon) - \sum_{k=-1}^{N} \varepsilon^k v_{k+1}\left(t, \frac{\psi(t)}{\varepsilon}\right)||_{C[0,T]} \le C_N \varepsilon^{N+1},\ N = -1, 0, 1, \dots, \tag{23}$$

is correct, where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

Conclusion 1. The influence of the weakly varying integral kernel $K_0(t,s)$ on the asymptotic of the solution of the problem of Equation (1) consists of two factors: Firstly, the kernel $K_0(t,s)$ participates in the formation of the matrix $A(t)$ and its eigenvectors and eigenvalues, secondly, it participates in the construction of the limit operator L_0, which leads to an additional integral system $w_0(t) + \int_0^t A^{-1}(t) B(t,s) w_0(s)\,ds = -A^{-1}(t) P_0(t)$ in the solvability of conditions Equation (17) of iterative problems.

4. The Limit Transition in the Problem of Equation (1). Solving the Initialization Problem

It follows from Equation (23) that the exact solution of the problem of Equation (1) is represented in the form

$$\begin{array}{l} y(t,\varepsilon) = \frac{1}{\varepsilon} v_0\left(t, \frac{\psi(t)}{\varepsilon}\right) + v_1\left(t, \frac{\psi(t)}{\varepsilon}\right) + \varepsilon F(t,\varepsilon), \\ \|F(t,\varepsilon)\|_{\mathbb{C}^n} \le \bar{F} = \text{const}\,(\forall (t,\varepsilon) \in [0,T] \times (0,\varepsilon_0]), \end{array} \tag{24}$$

therefore, in order to study the passage to the limit (for $\varepsilon \to +0$) in the solution of the problem of Equation (1), it is necessary to find the solutions of the two iteration problems of Equation (15d) ($k = 0, 1$) under the conditions of Equation (18) for the solvability of the third problems of Equation (15c). We start with the problem of Equation (15a):

$$\begin{array}{l} L_0 w_0(t,\tau) \equiv \sum\limits_{j=1}^{3} \lambda_j(t) \frac{\partial w_0}{\partial \tau_j} - A(t) w_0 - R_0 w_0 = 0, w_0(0,0) = w^0 \\ \left(R_0 w(t,\tau) = \int_0^t B(t,s) w_0(s) ds + e^{\tau_3} \int_0^t G(t,s) w_3(s) ds\right). \end{array} \tag{15a}$$

Since the right-hand side of the system of Equation (15a) $P^{(0)}(t,\tau) = \sum_{j=1}^{3} P_j^{(0)}(t) e^{\tau_j} + P_0^{(0)}(t)$ is identically zero, it has (according to Theorem 1) a solution

$$w_0(t,\tau) = \sum_{k=1}^{2} \alpha_k^{(0)}(t) \varphi_k(t) e^{\tau_k} + w_3^{(0)}(t) e^{\tau_3} + w_0^{(0)}(t),$$

where the vector functions $w_3^{(0)}(t), w_0^{(0)}(t)$ satisfy the equations

$$\begin{array}{l} w_3^{(0)}(t) - \int_0^t (\lambda_3(t) I - A(t))^{-1} G(t,s) w_3^{(0)}(s)\,ds = 0, \\ w_0^{(0)}(t) + \int_0^t A^{-1}(t) B(t,s) w_0(s)\,ds = 0. \end{array}$$

These equations are homogeneous, and therefore, they have the unique solutions $w_3^{(0)}(t) = w_0^{(0)}(t) \equiv 0$, and the solution of the system of Equation (15a) is written in the form

$$w_0(t,\tau) = \sum_{k=1}^{2} \alpha_k^{(0)}(t)\,\varphi_k(t)\,e^{\tau_k}. \tag{25}$$

Let $\chi_k(t) = \{\chi_k^1(t), \chi_k^2(t)\}$, $k = 1,2$. Subordinating Equation (24) to the initial condition $w_0(0,0) = w^0$, we find the values

$$\alpha_k^{(0)}(0) = \left(w^0, \chi_k(0)\right) = h(0)\,\bar{\chi}_k^1(0),\ k = 1,2. \tag{26}$$

For the final computation of the functions $\alpha_k^{(0)}(t)$, we pass to the next iteration problem

$$L_0 w_1(t,\tau) = -\sum_{k=1}^{2}\left(\alpha_k^{(0)}(t)\,\varphi_k(t)\right)^{\bullet} e^{\tau_k} + R_1 w_0 + H(t),\, w_1(0,0) = 0, \tag{15b}$$

where

$$\begin{aligned} &R_1 w_0 = R_1\left(\textstyle\sum_{k=1}^{2}\alpha_k^{(0)}(t)\,\varphi_k(t)\,e^{\tau_k}\right) = \\ &= \textstyle\sum_{j=1}^{3}\left[\frac{B(t,t)\alpha_j^{(0)}(t)\varphi_k(t)}{\lambda_j(t)}e^{\tau_j} - \frac{B(t,0)\alpha_j^{(0)}(0)\varphi_j(0)}{\lambda_j(0)}\right] + \\ &+ \textstyle\sum_{k=1}^{2}\left[\frac{G(t,t)\alpha_k^{(0)}(t)\varphi_k(t)}{\lambda_k(t)-\lambda_3(t)}e^{\tau_k} - \frac{G(t,0)\alpha_k^{(0)}(0)\varphi_k(0)}{\lambda_k(0)-\lambda_3(0)}e^{\tau_3}\right]. \end{aligned}$$

Keeping, as in Theorem 2, only the terms containing exponentials e^{τ_1} and e^{τ_2}, we write down conditions of Equation (17) in the form (see Equation (26)):

$$\begin{aligned} &\dot{\alpha}_k^{(0)}(t) = \left(\frac{B(t,t)\,\varphi_k(t)}{\lambda_k(t)} + \frac{G(t,t)\,\varphi_k(t)}{\lambda_k(t)-\lambda_3(t)} - \dot{\varphi}_k(t), \chi_k(t)\right)\alpha_k^{(0)}(t), \\ &\alpha_k^{(0)}(0) = h(0)\,\bar{\chi}_k^1(0), k = 1,2, \end{aligned}$$

from which we find that

$$\alpha_k^{(0)}(t) = h(0)\,\bar{\chi}_k^1(0)\,e^{\int_0^t q_k(\theta)d\theta},\ k = 1,2, \tag{27}$$

where it is denoted: $q_k(t) \equiv \left(\frac{B(t,t)\varphi_k(t)}{\lambda_k(t)} + \frac{G(t,t)\varphi_k(t)}{\lambda_k(t)-\lambda_3(t)} - \dot{\varphi}_k(t), \chi_k(t)\right), k = 1,2$. Thus, the solution of the problem of Equation (15a) is found in the form of Equation (25), where the functions $\alpha_k^{(0)}(t)$ are Equation (27). Similarly, we can find the solution of the problem of Equation (15b). However, having in mind to solve the initialization problem in the future, we must put $v_0\left(t, \frac{\psi(t)}{\varepsilon}\right) \equiv 0$ in Equation (24). This identity holds if and only if $\alpha_k^{(0)}(t) \equiv 0\ \ (k = 1,2) \Leftrightarrow\ h(0) = 0$ (remember that $v_0\left(t, \frac{\psi(t)}{\varepsilon}\right) = \sum_{k=1}^{2}\alpha_k^{(0)}(t)\varphi_k^1(t)\,e^{\frac{\psi_k(t)}{\varepsilon}},\ \ \varphi_j^1(t) = \lambda_j(t) - \mu(t),\ j = 1,2$ and see Equation (27)), we will therefore carry out further calculations for $h(0) = 0$. In this case, $w_0(t,\tau) \equiv 0, R_1 w_0 \equiv 0$, and the problem of Equation (15b) takes the form

$$L_0 w_1(t,\tau) \equiv \sum_{j=1}^{3}\lambda_j(t)\frac{\partial w_1}{\partial \tau_j} - A(t)w_1 - R_0 w_1 = H(t), w_1(0,0) = 0.$$

Since here $P^{(1)}(t,\tau) = H(t)\ \ \left(P_j^{(1)}(t) \equiv 0,\ j = 1,2,3, P_0^{(1)}(t) = H(t)\right)$, in formula

$$w_1(t,\tau) = \sum_{k=1}^{2}\left[\alpha_k^{(1)}(t)\,\varphi_k(t) + \sum_{s=1,s\neq k}^{2} p_{ks}^{(1)}(t)\,\varphi_s(t)\right]e^{\tau_k} + w_3^{(1)}(t)\,e^{\tau_3} + w_0^{(1)}(t)$$

for the solution of the problem of Equation (15b) functions $p_{ks}^{(1)}(t) \equiv 0$ $(k,s=1,2)$, functions $w_3^{(1)}(t)$ and $w_0^{(1)}(t)$ are solutions of the integral equations

$$
\begin{array}{l}
w_3^{(1)}(t) - \int_0^t (\lambda_3(t)\, I - A(t))^{-1}\, G(t,s)\, w_3^{(1)}(s)\, ds = 0 \Leftrightarrow w_3^{(1)}(t) \equiv 0, \\
w_0^{(1)}(t) + \int_0^t A^{-1}(t)\, B(t,s)\, w_0^{(1)}(s)\, ds = -A^{-1}(t)\, H(t),
\end{array}
\tag{28}
$$

therefore, the solution of the problem will be as follows:

$$
w_1(t,\tau) = \sum_{k=1}^{2} \alpha_k^{(1)}(t)\, \varphi_k(t)\, e^{\tau_k} + w_0^{(1)}(t), \tag{29}
$$

where $\alpha_k^{(1)}(t)$, for the time being, are arbitrary functions, $k=1,2$, and the vector-valued function $w_0^{(1)}(t)$ is a solution of the system of Equation (28). Subordinating Equation (29) to the initial condition $w_1(0,0)=0$, we obtain

$$
\begin{array}{l}
\sum_{k=1}^{2} \alpha_k^{(1)}(0)\, \varphi_k(0) = -w_0^{(1)}(0) \equiv A^{-1}(0)\, H(0) \Rightarrow \\
\alpha_k^{(1)}(0) = \left(A^{-1}(0)\, H(0), \chi_k(0)\right) = \left(H(0), A^{-1}(0)\, \chi_k(0)\right) = \\
= \left(H(0), \bar{\lambda}_k(0)\, \chi_k(0)\right) = \lambda_k(0)\left(H(0), \chi_k(0)\right),
\end{array}
$$

i.e.,

$$
\left\{
\begin{array}{l}
\alpha_1^{(1)}(0) = \lambda_1(0)\, \dot{h}(0)\, \bar{\chi}_1^1(0), \\
\alpha_2^{(1)}(0) = \lambda_2(0)\, \dot{h}(0)\, \bar{\chi}_2^1(0).
\end{array}
\right.
\tag{30}
$$

For the final calculation of the solution of Equation (29) of the problem of Equation (15b), let us pass to the following problem (note that $w_0 \equiv 0$):

$$
L_0 w_2(t,\tau) = -\frac{\partial w_1}{\partial t} + R_1 w_1, \quad w_2(0,0) = 0. \tag{15c}
$$

Substituting here the function of Equation (29), we obtain the system

$$
\begin{array}{l}
L_0 w_2(t,\tau) = -\sum_{k=1}^{2} \left(\alpha_k^{(1)}(t)\, \varphi_k(t)\right)^{\bullet} e^{\tau_k} + \\
+ \frac{G(t,0) w_0^{(1)}(0)}{\lambda_3(0)} e^{\tau_3} - \frac{G(t,t) w_0^{(1)}(t)}{\lambda_3(t)} + \\
+ \sum_{j=1}^{3} \left[\frac{B(t,t)\alpha_j^{(1)}(t)\varphi_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{B(t,0)\alpha_j^{(1)}(0)\varphi_j(0)}{\lambda_j(0)} \right] + \\
+ \sum_{k=1}^{2} \left[\frac{G(t,t)\alpha_k^{(1)}(t)\varphi_k(t)}{\lambda_k(t)-\lambda_3(t)} e^{\tau_k} - \frac{G(t,0)\alpha_k^{(1)}(0)\varphi_k(0)}{\lambda_k(0)-\lambda_3(0)} e^{\tau_3} \right] = 0.
\end{array}
$$

Keeping here, as in Theorem 2, only terms containing exponentials e^{τ_1} and e^{τ_2}, we write the conditions of Equation (17) for the solvability of this system in the form

$$
\begin{array}{l}
\dot{\alpha}_k^{(1)}(t) = \left(\frac{B(t,t)\varphi_k(t)}{\lambda_k(t)} + \frac{G(t,t)\varphi_k(t)}{\lambda_k(t)-\lambda_3(t)} - \dot{\varphi}_k(t), \chi_k(t) \right) \alpha_k^{(1)}(t), \\
\alpha_1^{(1)}(0) = \lambda_1(0)\, \dot{h}(0)\, \bar{\chi}_1^1(0), \alpha_2^{(1)}(0) = \lambda_2(0)\, \dot{h}(0)\, \bar{\chi}_2^1(0),
\end{array}
$$

from which we uniquely find the functions $\alpha_k^{(1)}(t)$:

$$
\alpha_k^{(1)}(t) = \lambda_k(0)\, \dot{h}(0)\, \bar{\chi}_k^1(0)\, e^{\int_0^t q_k(\theta) d\theta}, \; k = 1,2,
$$

and therefore, we uniquely construct the solution of Equation (29) of the problem of Equation (15b). In this case, the equality holds (remember that $w_0(t,\tau) \equiv 0$)

$$w(t,\varepsilon)=\varepsilon\left(\sum_{k=1}^{2}\lambda_k(0)\,\dot h(0)\,\bar\chi_k^1(0)\,e^{\int_0^t q_k(\theta)d\theta}\varphi_k(t)\,e^{\frac{\psi_k(t)}{\varepsilon}}+w_0^{(1)}(t)\right)+\varepsilon^2F_1(t,\varepsilon)\Rightarrow$$

$$\Rightarrow y(t,\varepsilon)=\sum_{k=1}^{2}\lambda_k(0)\,\dot h(0)\,\bar\chi_k^1(0)\,e^{\int_0^t q_k(\theta)d\theta}\varphi_k^1(t)\,e^{\frac{\psi_k(t)}{\varepsilon}}+v_0^{(1)}(t)+\varepsilon f_1(t,\varepsilon),\tag{31}$$

where $w_0^{(1)}(t)=\left\{v_0^{(1)}(t),z_0^{(1)}(t)\right\}$ is the solution of the integral system

$$\begin{cases}v_0^{(1)}(t)+\int\limits_0^t\dfrac{\left(\frac{\partial}{\partial t}K_1(t,s)\right)v_0^{(1)}(s)}{K_1(t,t)}\,ds=-\dfrac{\frac{d}{dt}h(t)}{K_1(t,t)},\\ z_0^{(1)}(t)-\left(\int\limits_0^t\dfrac{K_2(t,t)\left(\frac{\partial}{\partial t}K_1(t,s)\right)v_0^{(1)}(s)}{\mu(t)K_1(t,t)}\,ds\right)=\dfrac{K_2(t,t)\left(\frac{d}{dt}h(t)\right)}{\mu(t)K_1(t,t)}.\end{cases}\tag{32a}$$

It follows from Equation (31) that when $\operatorname{Re}\lambda_k(t)<0$ $(\forall t\in[0,T],\,k=1,2)$ there is a passage to the limit

$$\left\|y(t,\varepsilon)-v_0^{(1)}(t)\right\|_{C[\delta,T]}\to 0\ \ (\varepsilon\to+0),$$

where $\delta\in(0,T)$ is an arbitrary fixed constant, and $w_0^{(1)}(t)=\left\{v_0^{(1)}(t),z_0^{(1)}(t)\right\}$. However, in our case, there can be purely imaginary eigenvalues $(\operatorname{Re}\lambda_k(t)\equiv 0)$, so the indicated limit transition does not hold. The following problem is posed: to find a class $\Sigma=\{h(t),K_1(t,s),K_2(t,s)\}$ of initial data of Equation (1) for which the passage to the limit

$$\left\|y(t,\varepsilon)-v_0^{(1)}(t)\right\|_{C[0,T]}\to 0\ \ (\varepsilon\to+0),\tag{$*$}$$

takes place on the whole segment $[0,T]$, including the boundary layer zone. This task is called *the initialization problem.* . It is clear from Equation (31) that the limit transition $(*)$ occurs if and only if $\dot h(0)=0$, therefore, the following result follows from Equation (32a).

Theorem 4. *Suppose that the conditions (1)–(2) are satisfied. Then, the passage to the limit $(*)$ holds if and only if $h(0)=\dot h(0)=0$ (here, $v_0^{(1)}(t)$ is the solution of the first equation of the system of Equation (32a)).*

Conclusion 2. Thus, the initialization class Σ has the form $\Sigma=\left\{h(t):h(0)=\dot h(0)=0\right\}$. Here, the kernels $K_j(t,s)$ can be arbitrary, provided that conditions (1)–(2) are satisfied.

Example 1. *Consider the equation*

$$\varepsilon y(t,\varepsilon)=\int\limits_0^t e^{-\frac{1}{\varepsilon}(t-s)}\left(e^{-s}-1\right)y(s,\varepsilon)\,ds+\int\limits_0^t\left(-2e^{-s}y(s,\varepsilon)\right)ds+t^2.\tag{32b}$$

Here, $h(t)=t^2$, $\mu(t)=-1$, $K_1(t,s)=-2e^{-s}$, $K_2(t,s)=e^{-s}-1$. *The characteristic equation of the matrix* $A(t)=\begin{bmatrix}-e^{-t}-1 & -1\\ e^{-t}-1 & -1\end{bmatrix}$ *has two roots* $\lambda_1(t)=-2$, $\lambda_2(t)=-e^{-t}$. *Using the algorithm developed above, we find that*

$$v_0^{(1)}(t)+\int\limits_0^t\frac{\left(\frac{\partial}{\partial t}K_1(t,s)\right)v_0^{(1)}(s)}{K_1(t,t)}\,ds=-\frac{\frac{d}{dt}h(t)}{K_1(t,t)}\Leftrightarrow v_0^{(1)}(t)=te^t.$$

Since $\dot{h}(0) = 0$, the main term of the asymptotic of the solution of our Equation (32b) coincides with $v_0^{(1)}(t)$ (see Equation (31)). By Theorem 4, there is a passage to the limit:

$$\left\| y(t,\varepsilon) - te^t \right\|_{C[0,T]} \to 0 \,(\varepsilon \to +0).$$

We note that the function $v_0^{(1)}(t) = te^t$ is a solution of the integral equation $\int\limits_0^t \left(-2e^{-s}y(s)\right) ds + t^2 = 0$, which is degenerative with respect to Equation (1). If only $h(0) = 0$, but $\dot{h}(0) \neq 0$, then from Equation (31), we would have obtained that

$$y(t,\varepsilon) = v_1\left(t, \frac{\psi(t)}{\varepsilon}\right) + \varepsilon F(t,\varepsilon),$$

and the function $v_1\left(t, \frac{\psi(t)}{\varepsilon}\right)$ contains exponents $e^{\frac{-2t}{\varepsilon}}$ and $e^{\frac{1}{\varepsilon}\left(e^{-t}-1\right)}$, which prevent uniform convergence of the solution $y(t,\varepsilon)$ on the whole interval $[0,T]$ to the limit function. In this case, uniform convergence will occur only outside the boundary layer $[\delta, T]$ $(\delta \in (0,T))$.

The analysis of asymptotic methods for solving singularly perturbed problems shows that the solutions of such problems depend in two ways on a small parameter: regularly and singularly. This dependence is especially vividly demonstrated by the method of regularization of Lomov. Moreover, regularized series representing solutions of singularly perturbed problems can converge in the usual sense. In this connection, it became necessary to study a special class of functions—pseudoholomorphic functions. This very important part of the complex analysis is designed to substantiate the main provisions of the so-called analytic theory of singular perturbations. On the other hand, the relevance of the theory is also dictated by the fact that pseudoholomorphic functions, in contrast to holomorphic functions, are determined when the conditions of the implicit function theorem are violated. The concept of a pseudoanalytic (pseudoholomorphic) function and the associated concept of an essentially singular manifold are of a general mathematical nature, although they arose in the framework of the regularization method for singular perturbations. First of all, they reflect the new concept of a pseudoholomorphic solution of singularly perturbed problems, i.e., such a solution, which is representable in the form of a series converging in the usual (but not asymptotic) sense in powers of a small parameter. We must also take into account the fact that the modern mathematical theory of the boundary layer [13], along with the Vasilyeva–Butuzov–Nefedov's boundary-function method [5,6], widely uses the concept of a pseudoholomorphic solution. The following sections of our work are devoted to the construction of exactly such solutions [15].

5. Pseudoholomorphic Functions in the Theory of Singular Perturbations. Basic Concepts and Statements

We consider the set of functions $F(z,w,\varepsilon)$, where $w = (w_1, \ldots, w_k)$, $F = (F_1, \ldots, F_k)$, holomorphic in a polydisc $D = D_{z_0} \times D_{w_0} \times D_0$, in which

$$D_{z_0} = \{z : |z - z_0| < R_0\}, D_{w_0} = \{w : |w_j - w_{0,j}| < R_j, \ j = \overline{1,k}\}, D_0 = \{\varepsilon : |\varepsilon| < \varepsilon_0\}.$$

Definition 1. *A function $w(z,\varepsilon)$, defined implicitly by the equation*

$$F(z,w,\varepsilon) = 0, \tag{33}$$

is said to be pseudoholomorphic at a point of $\varepsilon = 0$ of rank r, if the following conditions are satisfied:

1^0. $F(z_0, w_0, 0) = 0$;

2^0. $\partial_{w_j} F_i \Big|_{\varepsilon=0} = 0 \ \forall (z,w) \in D_{z_0} \times D_{w_0}$, $i = \overline{1,r}$, $j = \overline{1,k}$;

3^0. $\det \|f_{ij}\| \neq 0 \ \forall (z,w) \in D_{z_0} \times D_{w_0}$, *where* $f_{ij} = \partial^2_{\varepsilon w_j} F_i \Big|_{\varepsilon=0}$, $i = \overline{1,r}$, $j = \overline{1,k}$; $f_{ij} = \partial_{w_j} F_i \Big|_{\varepsilon=0}$, $i = \overline{r+1,k}$, $j = \overline{1,k}$.

4^0. *$w(z,\varepsilon)$ is unbounded in any sufficiently small neighborhood of a point $\varepsilon = 0$ and there exists a set $E_0 \subset D_0$, for which the point $\varepsilon = 0$ is a limit point and such that it is bounded on a set $T_{z_0} \times E_0$, where T_{z_0} is a compact that belongs D_{z_0} and contains a point z_0.*

From definition , it follows that

$$\begin{aligned} F_i(z,w,\varepsilon) &= \varphi_i(z) - \varepsilon U_{i,1}(z,w) - \ldots - \varepsilon^n U_{i,n}(z,w) - \ldots, \quad i = \overline{1,r}; \\ F_i(z,w,\varepsilon) &= U_{i,0}(z,w) + \varepsilon U_{i,1}(z,w) + \ldots + \varepsilon^n U_{i,n}(z,w) + \ldots, \quad i = \overline{r+1,k}, \end{aligned} \tag{34}$$

and these series converge uniformly on any compact set $D_{z_0} \times D_{w_0}$ in some neighborhood of the point $\varepsilon = 0$ (depending on the compact).

We compose the following system of equations:

$$\begin{cases} U_{1,1}(z,w) = \varphi_1(z)/\varepsilon, \\ \qquad \ldots \\ U_{r,1}(z,w) = \varphi_r(z)/\varepsilon, \\ U_{r+1,0}(z,w) = 0, \\ \qquad \ldots \\ U_{k,0}(z,w) = 0, \end{cases} \tag{35}$$

which will be used in the future. We shall call Equation (35) *the main system*.

Suppose that the entire functions $\Psi_1, \ldots, \Psi_r$ of one variable with the asymptotic values $a_1, \ldots, a_r$ are such that the sets $\omega_i = \{q_i : \ \ q_i = \Psi_i(\varphi_i(z)/\varepsilon)\} \subset \mathbb{C}_{q_i}$ are bounded if $z \in T_{z_0}$ and $\varepsilon \in E_0$, where T_{z_0} and E_{z_0} are sets satisfying the condition 4^0 of the Definitions 1. We also assume that the points a_i close these sets: $\bar{\omega}_i = \omega_i \cup \{a_i\}$, $i = \overline{1,r}$. We introduce the notations: $\Psi = (\Psi_1, \ldots, \Psi_r)$, $\varphi = (\varphi_1, \ldots, \varphi_r)$, $a = (a_1, \ldots, a_r)$.

Definition 2. *The set $\Omega(\Psi, \varphi, T_{z_0}, E_0) = \omega_1 \times \ldots \times \omega_r \subset \mathbb{C}_{q_1} \times \ldots \times \mathbb{C}_{q_2}$ is called an essentially singular manifold, generated by the functions Ψ and φ on the set $T_{z_0} \times E_0$; we call the set $\bar{\Omega}(\Psi, \varphi, T_{z_0}, E_0) = \bar{\omega}_1 \times \ \ldots \times \ \bar{\omega}_r$ an extended essentially singular manifold.*

Let us formulate sufficient conditions for the existence of a pseudoholomorphic function. For this, along with the system of Equation (35), we consider the system

$$\begin{cases} U_{1,1}(z,w) = q_1, \\ \qquad \ldots \\ U_{r,1}(z,w) = q_r, \\ U_{r+1,0}(z,w) = 0, \\ \qquad \ldots \\ U_{k,0}(z,w) = 0. \end{cases} \tag{36}$$

Theorem 5. *If a functionw $= W_0(z,q)$ that is a solution of the system of Equation (36) is holomorphic on a compact $\bar{Q} = T_{z_0} \times \bar{\Omega}(\Psi, \varphi, T_{z_0}, E_0)$ and maps it to a polydisk D_{w_0}, then the function $w(z,\varepsilon)$, implicitly defined by Equation (33), is pseudoholomorphic at the point $\varepsilon = 0$.*

Proof. We represent the vector of Equation (33) in the form of a system as follows:

$$\begin{cases} U_{1,1}(z,w) + \varepsilon U_{1,2}(z,w) + \ldots + \varepsilon^n U_{1,n+1}(z,w) + \ldots = \varphi_1(z)/\varepsilon, \\ \quad\ldots \\ U_{r,1}(z,w) + \varepsilon U_{r,2}(z,w) + \ldots + \varepsilon^n U_{r,n+1}(z,w) + \ldots = \varphi_r(z)/\varepsilon, \\ U_{r+1,0}(z,w) + \varepsilon U_{r+1,1}(z,w) + \ldots + \varepsilon^n U_{r+1,n}(z,w) + \ldots = 0, \\ \quad\ldots \\ U_{k,0}(z,w) + \varepsilon U_{k,1}(z,w) + \ldots + \varepsilon^n U_{k,n}(z,w) + \ldots = 0, \end{cases} \tag{37}$$

and calculate the values of the functions $\Psi_1, \ldots, \Psi_r$ from the left and right parts of the first r equations:

$$\Psi_1(U_{1,1}(z,w) + \varepsilon U_{1,2}(z,w) + \ldots + \varepsilon^n U_{1,n+1}(z,w) + \ldots) = \Psi_1(\varphi_1(z)/\varepsilon),$$
$$\ldots$$
$$\Psi_r(U_{r,1}(z,w) + \varepsilon U_{r,2}(z,w) + \ldots + \varepsilon^n U_{r,n+1}(z,w) + \ldots) = \Psi_r(\varphi_r(z)/\varepsilon),$$

and then in the left-hand sides of these equations we distinguish the main terms:

$$\begin{gathered} \Psi_1(U_{1,1}(z,w)) + \varepsilon V_1(z,w,\varepsilon) = \Psi_1(\varphi_1(z)/\varepsilon), \\ \ldots \\ \Psi_r(U_{r,1}(z,w)) + \varepsilon V_r(z,w,\varepsilon) = \Psi_r(\varphi_r(z)/\varepsilon). \end{gathered} \tag{38}$$

Using the notations introduced earlier, we rewrite the system of Equation (36):

$$\begin{cases} \Psi_1(U_{1,1}(z,w)) + \varepsilon V_1(z,w,\varepsilon) = q_1, \\ \quad\ldots \\ \Psi_r(U_{r,1}(z,w)) + \varepsilon V_r(z,w,\varepsilon) = q_r, \\ U_{r+1,0}(z,w) = 0, \\ \quad\ldots \\ U_{k,0}(z,w) = 0. \end{cases} \tag{39}$$

When $\varepsilon = 0$, the system of Equation (39) has a solution $w = W_0(z,q)$, holomorphic on a set $\bar{Q}$, that which maps to a compact, belonging to D_{w_0} , and therefore, in accordance with the implicit function theorem, in some neighborhood σ_{zq} of each point $(z,q) \in \bar{Q}$ this system has a solution w that is holomorphic at the point $\varepsilon = 0 : W(z,q,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n W_n(z,q)$. From the covering $\{\sigma_{zq}\}$ of a compact set $\bar{Q}$, we choose a finite subcover, then the function $W(z,q,\varepsilon)$ will be holomorphic uniformly on $\bar{Q}$ in a neighborhood $|\varepsilon| < \varepsilon_1$, where ε_1 is the smallest number of the corresponding finite subcoverings. The boundedness of the function $w(z,\varepsilon) = W(z, \Psi_1(\varphi_1(z)/\varepsilon), \ldots, \Psi_r(\varphi_r(z)/\varepsilon), \varepsilon)$ for $\varepsilon \to 0$ ($\varepsilon \in E_0$) follows from the fact that the point (z,ε) belongs to an extended essentially singular manifold $\bar{\Omega}(\Psi, \varphi, T_{z_0}, E_0)$. The theorem is proved. □

Remark 2. *It follows from Theorem 5 that a pseudoholomorphic function decomposes into a power series with coefficients that depend in a singular way on ε:*

$$W(z,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n W_n(z, \Psi_1(\varphi_1(z)/\varepsilon), \ldots, \Psi_r(\varphi_r(z)/\varepsilon)) \tag{40}$$

and this series converges for $|\varepsilon| < \varepsilon_1$ ($\varepsilon \in E_0$) *uniformly on* T_{z_0}.

6. *-Pseudoholomorphic Functions

In applications, for example, in the mathematical theory of the boundary layer [3], we have to impose less restrictive conditions on pseudomorphic functions.

Definition 3. *A *-transformation of a function* $F(z,w,\varepsilon) = (F_1, ..., F_k)$*, defined by the equalities of Equation (34), is a vector-valued function of* $(k+3)$ *variables:*

$$F_*(z,w,\varepsilon,\varepsilon_*) = (F_{*1}(z,w,\varepsilon,\varepsilon_*), \ldots, F_{*k}(z,w,\varepsilon,\varepsilon_*)),$$

where the components with numbers $i = \overline{1,r}$ *have the form*

$$F_{*i}(z,w,\varepsilon,\varepsilon_*) = \varphi_i(z) - \varepsilon_* U_{i,1}(z,w) - \ldots - \varepsilon_*\varepsilon^{n-1}U_{1,n}(z,w) - \ldots,$$

(that is, they are obtained from $F_i(z,w,\varepsilon)$ *by replacing* ε^n *by* $\varepsilon_*\varepsilon^{n-1}, n = 1, 2, \ldots$*), and when* $i = \overline{r+1,k}$ *they remain unchanged:* $F_{*i}(z,w,\varepsilon,\varepsilon_*) \equiv F_i(z,w,\varepsilon)$*.*

Obviously, the function $F_*(z,w,\varepsilon,\varepsilon_*)$ is holomorphic in a polydisc $D \times D_{0*}$, where $D_{0*} = \{\varepsilon_* : |\varepsilon_*| < \varepsilon_0\}$, and the equation $F_*(z,w,\varepsilon,\varepsilon_*) = 0$ implicitly defines a function$w = w_*(z,\varepsilon,\varepsilon_*)$ for which the equality $w(z,\varepsilon) = w_*(z,\varepsilon,\varepsilon_*)$ holds true.

Definition 4. *A function* $w(z,\varepsilon)$ *is said to be *-pseudoholomorphic, if the function* $w_*(z,\varepsilon,\varepsilon_*)$ *is holomorphic with respect to the second variable at the point* $\varepsilon = 0$ *uniformly with respect to*$z \in T_{z_0}$ *for each fixed* $\varepsilon_* \in E_0$*.*

Theorem 6. *If a function* $W_0(z,q)$ *is holomorphic on a set* $Q = T_{z_0} \times \Omega(\Psi, \varphi, T_{z_0}, E_0)$ *and maps it to a polydisk* D_{w_0}*, then the function* $w(z,\varepsilon)$ *is * -pseudoholomorphic at a point* $\varepsilon = 0$*.*

Proof. We fix $\varepsilon_* \in E_0$, then choose arbitrarily $z \in T_{z_0}$, and let $q_* = \Psi(\varphi(z)/\varepsilon_*)$. It is clear that for the system

$$\begin{cases} \Psi_1(U_{1,1}(z,w)) + \varepsilon V_1(z,w,\varepsilon) = q_{1*}, \\ \qquad \ldots \\ \Psi_r(U_{r,1}(z,w)) + \varepsilon V_r(z,w,\varepsilon) = q_{r*}, \\ F_{r+1}(z,w,\varepsilon) = 0, \\ \qquad \ldots \\ F_k(z,w,\varepsilon) = 0 \end{cases} \tag{41}$$

the conditions of the implicit function theorem are satisfied, and since the set of all such q_* compacts (for a fixed $\varepsilon_*$and $z \in T_{z_0}$), the proof is completed in the same way as in the previous theorem. □

Corollary 1. *Thus, the solution of the system of Equation (41) can be represented in the form of a series in powers of* ε *:*

$$w(z,\varepsilon,\varepsilon_*) = \sum_{n=0}^{\infty} \varepsilon^n W_n(z, \Psi_1(\varphi_1(z)/\varepsilon_*), \ldots, \Psi_r(\varphi_r(z)/\varepsilon_*)) \tag{42}$$

which converges uniformly on T_{z_0} *at* $|\varepsilon| < \varepsilon_1$*, where* $\varepsilon_1 > 0$ *and depends on* ε_**. In addition, from the proof of Theorem 6, it follows that if* $\varepsilon_* = \varepsilon$ *(*ε *is fixed and belongs to the circle of convergence of this series), then uniform convergence will be observed even on a narrower set* $T_{z_0*} \subset T_{z_0}$ *(*$z_0 \in T_{z_{0*}}$*).*

The main question that arises in connection with the notion of *-pseudoholomorphy is the following: when can a *-pseudoholomorphic function be extended to the whole compact T_{z_0}? The answer to this question will be given in the scalar case, i.e., when $n = r = 1$. Note that in this case

$$F(z,w,\varepsilon) = \varphi(z) - \varepsilon U_1(z,w) - \ldots - \varepsilon^n U_n(z,w) - \ldots \tag{43}$$

and $\partial_w U_1(z,w) \neq 0$ in the in bidisk $D_{z_0} \times D_{w_0}$.

Furthermore, we assume that the condition (R) is fulfilled: all the functions participating in the analysis take real values, when their arguments are real.

Let $\mathcal{A}(D_{z_0})$ and $\mathcal{A}(D_{z_0} \times D_{w_0})$, where $D_{w_0} = \{w : \ |w - w_0| \ < R\}$ be the algebras of holomorphic functions, respectively, in the domains D_{z_0} and $D_{z_0} \times D_{w_0}$. In connection with the condition (R), we will assume that z_0 and w_0 are real.

Theorem 7. *If $\{H_\varepsilon\}$ is a holomorphic at the point $\varepsilon = 0$ family of homomorphisms of an algebra $\mathcal{A}(D_{z_0})$ into an algebra $\mathcal{A}(D_{z_0} \times D_{w_0})$ such that $H_0 = I$ and the functions $\varphi(z)$, $\mathcal{F}(z, w, \varepsilon) \equiv H_\varepsilon[\varphi(z)]$ satisfy the condition (R), and the conditions of Theorem 6 on the compact set $T_{z_0} = [z_0,\ z_0 + \Delta] \subset D_{z_0}$ hold true, then the function $w(z, \varepsilon)$, implicitly defined by the equation $\mathcal{F}(z, w, \varepsilon) = 0$, admits a pseudoholomorphic extension to T_{z_0}.*

We preface the proof of Theorem 7 with the following lemma.

Lemma 1. *The mappings $H_\varepsilon : \mathcal{A}(D_{z_0}) \to \mathcal{A}(D_{z_0} \times D_{w_0})$ for each sufficiently small ε satisfy the commutation relation*

$$H_\varepsilon[\varphi(z)] = \varphi(H_\varepsilon[z]). \tag{44}$$

Proof. Indeed, since $\varphi(z) \in \mathcal{A}_{z_0}$, then $\varphi(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$, and, therefore,

$$H_\varepsilon\left[\sum_{k=0}^{\infty} c_k (z - z_0)^k\right] = \sum_{k=0}^{\infty} c_k H_\varepsilon[(z - z_0)^k] = \sum_{k=0}^{\infty} c_k (H_\varepsilon[z - z_0])^k =$$

$$= \sum_{k=0}^{\infty} c_k (H_\varepsilon[z] - H_\varepsilon[z_0])^k = \sum_{k=0}^{\infty} c_k (H_\varepsilon[z] - z_0)^k = \varphi(H_\varepsilon[z]),$$

thus, Equation (44) is proved. □

Proof of Theorem 7. We differentiate Equation (12) with respect to z and w:

$$\partial_z H_\varepsilon[\varphi(z)] = \varphi'(H_\varepsilon[z]) \partial_z H_\varepsilon[z],$$
$$\partial_w H_\varepsilon[\varphi(z)] = \varphi'(H_\varepsilon[z]) \partial_w H_\varepsilon[z],$$

from which, it follows that

$$\varepsilon \mathcal{F}_z + f(z, w, \varepsilon) \mathcal{F}_w = 0, \tag{45}$$

where $f(z, w, \varepsilon) = -\varepsilon \partial_z H_\varepsilon[z] / \partial_w H_\varepsilon[z]$ is a holomorphic function at the point $\varepsilon = 0$, which differs from zero in the domain $D_{z_0} \times D_{w_0}$ for a sufficiently small ε. Equation (45) is the equation of integrals of the differential equation

$$\varepsilon \frac{dw}{dz} = f(z, w, \varepsilon), \tag{46}$$

and we seek its solution in the form of a series in powers of ε , assuming the operator ∂_z to be a subordinate operator $f\partial_w$. We have [8], for an arbitrary function $\varphi(z) \in \mathcal{A}_{z_0}$, that

$$\begin{gathered} \mathcal{F}(z, w, \varepsilon) \equiv H_\varepsilon[\varphi(z)] = \\ = \varphi(z) - \varepsilon \int_{w_0}^{w} \frac{\varphi'(z) dw_1}{f(z, w_1, \varepsilon)} + \varepsilon^2 \int_{w_0}^{w} \left(\frac{\partial}{\partial z} \int_{w_0}^{w_1} \frac{\varphi'(z) dw_2}{f(z, w_2, \varepsilon)}\right) \frac{dw_1}{f(z, w_1, \varepsilon)} - \dots . \end{gathered} \tag{47}$$

By uniqueness, the solution of the equation $\mathcal{F}(z, w, \varepsilon) = 0$ is the solution $\tilde{w}_1(z, \varepsilon)$ of the Cauchy problem for the differential Equation (46) with the initial condition $\tilde{w}_1(z_0, \varepsilon) = w_0$, which, in accordance with Theorem 7, is a *-pseudoholomorphic function in a neighborhood $|\varepsilon| \ < \varepsilon_1$ (see Corollary 1) and is defined on some interval $[z_0,\ z_0 + \Delta_1] \subset [z_0,\ z_0 + \Delta]$ (recall that Equation (46) is considered in the real domain). We will assume that the small parameter in Equation (46) satisfies the inequality $0 < \varepsilon < \varepsilon_1$, $\varphi'(z) < 0 \ \forall z \in [z_0,\ z_0 + \Delta]$. We show how in the real case we can find Δ_1. Thus, the series

$$\tilde{W}_1(z, \varepsilon, \varepsilon_*) = \sum_{n=0}^{\infty} \varepsilon^n W_n(z, \Psi(\varphi(z)/\varepsilon_*)), \tag{48}$$

where Ψ is an entire function, that satisfies Theorem 6, in the scalar case, converges uniformly on the interval $T_{z_0} = [z_0,\ z_0+\Delta]$ (ε_* and ε are fixed!). Suppose also (without loss of generality) that an essentially singular manifold is a half-open interval $(p,\ \Psi(0))$, where p is the asymptotic value of the function Ψ, and hence the set $Q = T_{z_0} \times \Omega(\Psi, \varphi, T_{z_0}, E_0)$ is a rectangle. If $\varepsilon > \varepsilon_*$, then $w_1(z,\varepsilon) = \tilde{W}_1(z,\varepsilon,\varepsilon)$ it is defined on the entire segment T_{z_0} (ie $\Delta_1 = \Delta$), because the graph of the function $q = \varphi(z)$ completely belongs to Q. If $\varepsilon < \varepsilon_*$, then $w_1(z,\varepsilon) = \tilde{W}_1(z,\varepsilon,\varepsilon)$, where $z \in [z_0,\ z_0+\Delta_1]$ and Δ_1 is found from the equation $\varphi(z_0+\Delta_1)/\varepsilon = \varphi(z_0+\Delta_1)/\varepsilon_*$. We now consider the Cauchy problem

$$\begin{aligned} &\varepsilon \tfrac{dw}{dz} = f(z,w,\varepsilon), \\ &w(z_1,\varepsilon) = v_1, \end{aligned} \tag{49}$$

where $z_1 = z_0 + \Delta_1$, $v_1 = \tilde{w}_1(z_1,\varepsilon)$. The general integral of this equation can be represented in the form

$$\int_{v_1}^{w} \frac{\varphi'(z)dw_1}{f(z,w_1,\varepsilon)} - \varepsilon \int_{v_1}^{w} \left(\frac{\partial}{\partial z} \int_{v_1}^{w_1} \frac{\varphi'(z)dw_2}{f(z,w_2,\varepsilon)} \right) \frac{dw_1}{f(z,w_1,\varepsilon)} + \ldots = \frac{\varphi(z)-\varphi(z_1)}{\varepsilon}. \tag{50}$$

The solution $\tilde{w}_2(z,\varepsilon)$, obtained from it, is defined on the interval $[z_1,\ z_2]$, where $z_2 = z_1 + \Delta_2$ and Δ_2 is determined from the equation $\frac{\varphi(z_1+\Delta_2)-\varphi(z_1)}{\varepsilon} = \frac{\varphi(z_0+\Delta)}{\varepsilon_*}$. If $|\varphi'(z)| \le l\ \forall z \in T_{z_0}$, then in accordance with the Lagrange theorem we have

$$\Delta_2 \ge \frac{\varepsilon \varphi(z_0+\Delta)}{\varepsilon_* l}. \tag{51}$$

Then, Equation (46) is considered with the initial condition $w(z_2,\varepsilon) = v_2$, when $v_2 = \tilde{w}_2(z_2,\varepsilon)$. A general integral analogous to Equation (50) is constructed, and so on. Since the estimate of Equation (51) is constant on an interval T_{z_0}, then in a finite number of steps the solution will be constructed on it. The Theorem is proved.

We give two examples of constructing pseudoholomorphic solutions in the real domain.

Example 2. *We consider the Cauchy problem for the scalar equation* $(n = r = 1)$

$$\begin{cases} \varepsilon y' = f(t,y), & t \in [t_0, T], \\ y(t_0,\varepsilon) = y_0. \end{cases} \tag{52}$$

We assume that the function $f(t,y)$ admits a holomorphic extension to the bidisk $D_{t_0} \times D_{y_0}$, where $D_{t_0} = \{z : |z - t_0| < R_0,\ R_0 > T\}$, $D_{y_0} = \{w : |w - y_0| < R\}$, and is not equal to zero there. Then, the general integral has the form:

$$\begin{aligned} &\varphi(t) - \varepsilon\varphi'(t) \textstyle\int_{y_0}^{y} \frac{dy_1}{f(t,y_1)} + \varepsilon \int_{y_0}^{y} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_1} \frac{\varphi'(t)dy_2}{f(t,y_2)} \right) \frac{dy_1}{f(t,y_1)} - \\ &-\varepsilon^2 \textstyle\int_{y_0}^{y} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_1} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_2} \frac{\varphi'(t)dy_3}{f(t,y_3)} \right) \frac{dy_2}{f(t,y_2)} \right) \frac{dy_1}{f(t,y_1)} + \ldots = 0. \end{aligned}$$

*Hence, we obtain a *-pseudoholomorphic solution*

$$y(t,\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n Y_n \left(t, \frac{\varphi(z)}{\varepsilon} \right), \tag{53}$$

where $\varphi(t) \in \mathcal{A}_{t_0}$ and such that the conditions of Theorem 6 are satisfied.

We write out the formulas for the first terms of the series of Equation (53):

$$Y_1 = -\left.\frac{V_1}{V_2}\right|_{y=Y_0(t,\varphi(t)/\varepsilon)}, \quad Y_2 = -\left.\frac{V_{11}V_2^2 - 2V_{12}V_1V_2 + V_{22}V_1^2}{2V_2^2}\right|_{y=Y_0(t,\varphi(t)/\varepsilon)},$$

where

$$V_1 = -\int_{y_0}^{y} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_1} \frac{\varphi'(t) dy_2}{f(t, y_2)} \right) \frac{dy_1}{f(t, y_1)};$$

$$V_2 = \frac{\varphi'(t)}{f(t, y)};$$

$$V_{11} = 2\int_{y_0}^{y} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_1} \left(\frac{\partial}{\partial t} \int_{y_0}^{y_2} \frac{\varphi'(t) dy_3}{f(t, y_3)} \right) \frac{dy_2}{f(t, y_2)} \right) \frac{dy_1}{f(t, y_1)};$$

$$V_{12} = -\frac{1}{f(t, y)} \left(\frac{\partial}{\partial t} \int_{y_0}^{y} \frac{\varphi'(t) dy_1}{f(t, y_1)} \right);$$

$$V_{22} = -\frac{\varphi'(t) f'_y(t, y)}{f^2(t, y)}.$$

We recall that $y = Y_0(t, \varphi(t)/\varepsilon)$ is the bounded solution (for $\varepsilon \to +0$) of the equation

$$\varphi'(t) \int_{y_0}^{y} \frac{dy_1}{f(t, y_1)} = \frac{\varphi(t)}{\varepsilon}.$$

In particular, if $f(t, y) = y^2 - e^{2t}$, $t_0 = 0$, $y_0 = 0$, then

$$y(t, \varepsilon) = e^t th \frac{1 - e^t}{\varepsilon} + \frac{\varepsilon}{2} th^2 \frac{1 - e^t}{\varepsilon} + \dots .$$

Example 3. *Consider the Cauchy problem for Tikhonov's system [7] (here, $n = 2$, $r = 1$)*

$$\begin{cases} y' = g(t, y, v), \\ \varepsilon v' = f(t, y, v), \quad t \in [t_0, T], \end{cases} \\ y(t_0, \varepsilon) = y_0, \quad v(t_0, \varepsilon) = v_0. \tag{54}$$

Denote by $\bar{y}(t)$ the solution of the limit problem $f(t, y, y') = 0$, $y(t_0) = 0$, and by $L = \partial_t + g\partial_y$ – the first-order linear partial differential operator. Then,

$$\begin{cases} \varphi(t) - \varepsilon \int_{v_0}^{v} \frac{\varphi'(t) dv_1}{f(t,y,v_1)} - \varepsilon^2 \int_{v_0}^{v} \left(L \int_{v_0}^{v_1} \frac{\varphi'(t) dv_2}{f(t,y,v_2)} \right) \frac{dv_1}{f(t,y,v_1)} - \dots = 0, \\ y - \bar{y}(t) - \varepsilon \int_{v_0}^{v} \frac{L(y - \bar{y}(t)) dv_1}{f(t,y,v_1)} + \varepsilon^2 \int_{v_0}^{v} \left(L \int_{v_0}^{v_1} \frac{L(y - \bar{y}(t)) dv_2}{f(t,y,v_2)} \right) \frac{dv_1}{f(t,y,v_1)} - \dots = 0 \end{cases}$$

*are independent first integrals of the system of Equation (54). Hence, we obtain a *-pseudoholomorphic solution of this system:*

$$y(t, \varepsilon) = \bar{y}(t) + \varepsilon \int_{v_0}^{v} \frac{L(y - \bar{y}(t)) dv_1}{f(t, y, v_1)} \Bigg|_{\substack{y = \bar{y}(t) \\ v = V_0(t, \varphi(t)/\varepsilon)}} + \dots;$$

$$v(t, \varepsilon) = V_0(t, \varphi(t)/\varepsilon) + \varepsilon \frac{f(t,y,v)}{\varphi'(t)} \left[\int_{v_0}^{v} \left(L \int_{v_0}^{v_1} \frac{\varphi'(t) dv_2}{f(t,y,v_2)} \right) \frac{dv_1}{f(t,y,v_1)} - \right. \\ \left. - \int_{v_0}^{v} \frac{L(y - \bar{y}(t)) dv_1}{f(t,y,v_1)} \cdot \frac{\partial}{\partial y} \int_{v_0}^{v} \frac{\varphi'(t) dv_1}{f(t,y,v_1)} \right] \Bigg|_{\substack{y = \bar{y}(t) \\ v = V_0(t, \varphi(t)/\varepsilon)}} + \dots .$$

Here, $v = V_0(t, \varphi(t)/\varepsilon)$ is the bounded solution (for $\varepsilon \to +0$) of the equation

$$\varphi'(t) \int_{v_0}^{v} \frac{dv_1}{f(t, \bar{y}(t), v_1)} = \frac{\varphi(t)}{\varepsilon}.$$

Conclusion 3. The algorithms developed in this paper allow one to theoretically substantiate two main approaches in the general theory of singular perturbations: an approach related to approximate (asymptotic) solutions, and an approach related to pseudoholomorphic (exact) solutions of such problems.

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