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Letter

Doily as Subgeometry of a Set of Nonunimodular Free Cyclic Submodules

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Abstract: In this paper, it is shown that there exists a particular associative ring with unity of order 16 such that the relations between non-unimodular free cyclic submodules of its two-dimensional free left module can be expressed in terms of the structure of the generalized quadrangle of order two. Such a doily-centered geometric structure is surmised to be of relevance for quantum information.

Keywords: associative ring with unity; free cyclic submodules; generalized quadrangle

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Let R be a finite associative ring with unity (1), and R^2 its free left module. The set $R(a,b) = \{(\alpha a, \alpha b)|(a,b) \in R^2, \alpha \in R\}$ is called a cyclic submodule of R^2. $R(a,b)$ is called *free* if the mapping $\alpha \mapsto (\alpha a, \alpha b)$ is injective. A pair/vector $(a,b) \in R^2$ is called *unimodular* over R if there exist $c, d \in R$ such that $ac + bd = 1$. It is well-known (see, for example, [1]) that if (a,b) is unimodular, then $R(a,b)$ is free. A great majority of finite rings have the property that all their free $R(a,b)$'s are generated by unimodular vectors. Here we shall consider a specific ring where this is not true—that is, a ring that also features free $R(a,b)$'s containing no unimodular vector. In what follows, we shall call such free cyclic submodules "non-unimodular". Our ring R is a non-commutative one of order 16, defined as follows:

$$R = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \mid a, b, c, d \in GF(2) \right\}.$$

Labeling the 16 matrices as follows:

$$0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 1 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 2 \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 3 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$4 \equiv \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 5 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 6 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 7 \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$8 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 9 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 10 \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 11 \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$12 \equiv \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ 13 \equiv \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 14 \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ 15 \equiv \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we see that 0 is the additive identity, 1 is the multiplicative identity, and the only invertible elements are 1, 2, 4, and 7. The ring features two (two-sided) maximal ideals, namely:

$$I_l = \{0, 3, 5, 6, 8, 11, 13, 14\}$$

and

$$I_r = \{0, 3, 5, 6, 9, 10, 12, 15\},$$

and its Jacobson radical reads:

$$J = \{0, 3, 5, 6\}.$$

From the above-given matrix representation of R, we find that R^2 contains nine distinct free cyclic submodules generated by non-unimodular vectors, which are listed in Table 1.

Table 1. The explicit form of the nine non-unimodular free cyclic submodules.

α	$R(3,8)$	$R(5,8)$	$R(6,8)$	$R(8,11)$	$R(8,13)$	$R(8,14)$	$R(8,6)$	$R(8,5)$	$R(8,3)$
0	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
1	$(3,8)$	$(5,8)$	$(6,8)$	$(8,11)$	$(8,13)$	$(8,14)$	$(8,6)$	$(8,5)$	$(8,3)$
2	$(3,11)$	$(5,11)$	$(6,11)$	$(11,8)$	$(11,14)$	$(11,13)$	$(11,6)$	$(11,5)$	$(11,3)$
3	$(0,3)$	$(0,3)$	$(0,3)$	$(3,3)$	$(3,3)$	$(3,3)$	$(3,0)$	$(3,0)$	$(3,0)$
4	$(3,13)$	$(5,13)$	$(6,13)$	$(13,14)$	$(13,8)$	$(13,11)$	$(13,6)$	$(13,5)$	$(13,3)$
5	$(0,5)$	$(0,5)$	$(0,5)$	$(5,5)$	$(5,5)$	$(5,5)$	$(5,0)$	$(5,0)$	$(5,0)$
6	$(0,6)$	$(0,6)$	$(0,6)$	$(6,6)$	$(6,6)$	$(6,6)$	$(6,0)$	$(6,0)$	$(6,0)$
7	$(3,14)$	$(5,14)$	$(6,14)$	$(14,13)$	$(14,11)$	$(14,8)$	$(14,6)$	$(14,5)$	$(14,3)$
8	$(0,8)$	$(0,8)$	$(0,8)$	$(8,8)$	$(8,8)$	$(8,8)$	$(8,0)$	$(8,0)$	$(8,0)$
9	$(3,0)$	$(5,0)$	$(6,0)$	$(0,3)$	$(0,5)$	$(0,6)$	$(0,6)$	$(0,5)$	$(0,3)$
10	$(3,3)$	$(5,3)$	$(6,3)$	$(3,0)$	$(3,6)$	$(3,5)$	$(3,6)$	$(3,5)$	$(3,3)$
11	$(0,11)$	$(0,11)$	$(0,11)$	$(11,11)$	$(11,11)$	$(11,11)$	$(11,0)$	$(11,0)$	$(11,0)$
12	$(3,5)$	$(5,5)$	$(6,5)$	$(5,6)$	$(5,0)$	$(5,3)$	$(5,6)$	$(5,5)$	$(5,3)$
13	$(0,13)$	$(0,13)$	$(0,13)$	$(13,13)$	$(13,13)$	$(13,13)$	$(13,0)$	$(13,0)$	$(13,0)$
14	$(0,14)$	$(0,14)$	$(0,14)$	$(14,14)$	$(14,14)$	$(14,14)$	$(14,0)$	$(14,0)$	$(14,0)$
15	$(3,6)$	$(5,6)$	$(6,6)$	$(6,5)$	$(6,3)$	$(6,0)$	$(6,6)$	$(6,5)$	$(6,3)$

We shall show that the way in which these free cyclic submodules are interwoven is intricately related to the structure of the generalized quadrangle of order two, the doily. To this end, we employed a duad-syntheme model of the latter (see, for example, [2]). Take a six-element set $S = \{1, 2, 3, 4, 5, 6\}$. Let us call a two-element subset of S a duad, and a set of three duads forming a partition of S a syntheme. Then, the point-line incidence structure whose points are 15 duads and whose lines are 15 synthemes, with incidence being containment, is isomorphic to the doily. The structure of the doily is illustrated in Figure 1 (left). Here, the points of the doily are represented by circles, labeled by duads, and its lines are represented by nine straight segments, three concentric circles, and three arcs of circles; one can readily check that each line corresponds to a syntheme. Next, we employ the following bijection between the 15 duads and 15 nontrivial vectors $(a, b) \in R^2$, where $a, b \in J$ and $(a, b) \neq (0, 0)$:

$$\{1,2\} \leftrightarrow (3,3),\ \ \{1,3\} \leftrightarrow (5,3),\ \ \{1,4\} \leftrightarrow (0,6),\ \ \{1,5\} \leftrightarrow (3,6),\ \ \{1,6\} \leftrightarrow (5,0),$$
$$\{2,3\} \leftrightarrow (6,0),\ \ \{2,4\} \leftrightarrow (3,5),\ \ \{2,5\} \leftrightarrow (0,5),\ \ \{2,6\} \leftrightarrow (6,3),\ \ \{3,4\} \leftrightarrow (5,5),$$
$$\{3,5\} \leftrightarrow (6,5),\ \ \{3,6\} \leftrightarrow (0,3),\ \ \{4,5\} \leftrightarrow (3,0),\ \ \{4,6\} \leftrightarrow (5,6),\ \ \{5,6\} \leftrightarrow (6,6).$$

We thus get a new labeling of the points of the doily in terms of these particular non-unimodular vectors of R^2, as illustrated in Figure 1 (right). From a comparison of the latter figure with Table 1, it follows that each submodule shares with the doily seven vectors, forming three concurrent lines—as depicted in Figure 2. From this figure, one can easily discern that six lines of the doily have a different status than the other nine, as each of them belongs to three submodules. Given the fact that each point of concurrence belongs to two such lines, the nine points and the six lines are found to form inside the doily a point-line incidence structure isomorphic to the generalized quadrangle of type GQ(2,1) (see [2]).

Figure 1. A pictorial representation of the doily, with its points labeled by duads (**left**) and non-unimodular vectors (**right**); both a duad $\{a, b\}$ and a vector (a, b) are abbreviated to ab.

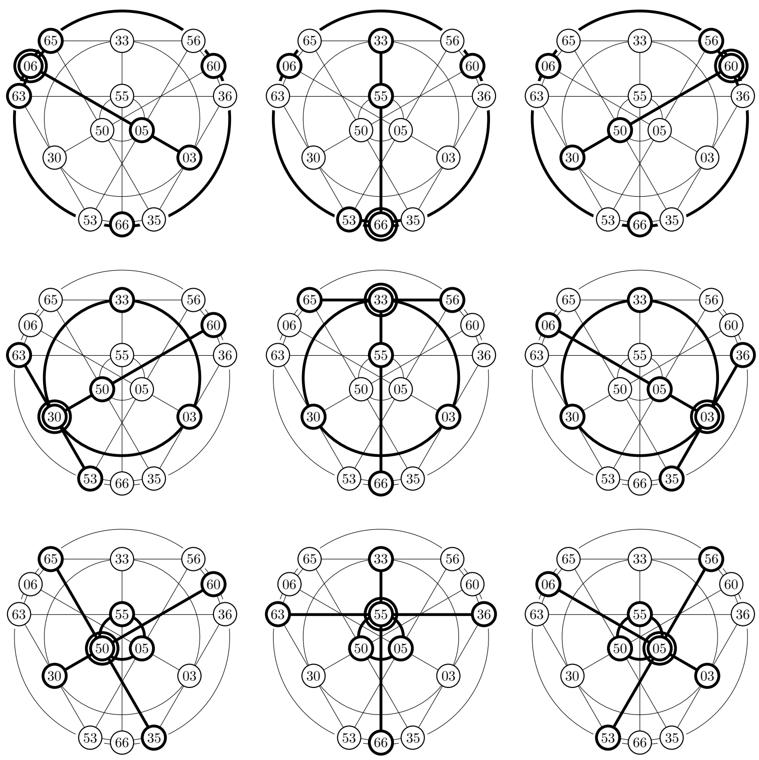

Figure 2. A graphical illustration of "Jacobson traces" of individual non-unimodular free cyclic submodules in the core doily; consecutively from top left to bottom right, they are shown the traces of $R(6,8)$, $R(8,14)$, $R(8,6)$, $R(8,3)$, $R(8,11)$, $R(3,8)$, $R(8,5)$, $R(8,13)$, and $R(5,8)$. In each subfigure, the corresponding concurrent lines are shown in boldface, the point of concurrence being encircled.

A complete view of the relation between individual submodules is outlined in Figure 3. The pronounced automorphism of order three of the figure stems from the fact that the submodules form three disjoint triples according to the number of shared vectors. One further observes that all vectors lying on our submodules acquire values from the ideal I_l. It can readily be verified that a completely analogous geometric structure is obtained if we take the free *right* module, in which case the corresponding vectors have entries from the ideal I_r. Obviously, the two structures share the same doily, as its points are labeled by vectors from J^2. At this point it is well worth recalling the existence of a similar geometrical structure in the case of the smallest ring of ternions and its three-dimensional free left (and also right) module [3]. There, the associated geometry, referred to as the "Fano-snowflake", has its center isomorphic to the Fano plane, a generalized triangle of order two (see also [4] for generalization to an arbitrary ring of ternions).

Figure 3. Visualisation of the full geometric structure formed by the nine nonunimodular free cyclic submodules, with the doily lying in its center. For each submodule there are shown all of its vectors except for the trivial one. For six submodules pairs of identical symbols are employed to identify the corresponding points of concurrence; the remaining three cases are obtained by swapping the figure with respect to the vertical axis passing through the center. The distinguished GQ(2,1) of the doily is shown in bold.

The occurrence of the doily in this remarkable nonunimodular ring-theoretic setting is quite intriguing. We checked, case by case, all non-commutative associative rings with unity up to order 31 inclusive and found, up to the isomorphism, no other ring having a similar property. However, there does exist a ring of order 32 which is a promising candidate in this respect. This particular ring features 21 non-unimodular free cyclic submodules whose mutual relations seem to be expressible in terms of the structure of the split Cayley hexagon of order two; the latter is, like the doily, a generalized polygon and a distinguished subgeometry of the symplectic polar space of type $W(5,2)$ at that. We, therefore, surmise that the "phenomenon" described above is an example of a more general mathematical rule linking ring geometry and polar spaces (or generalized polygons).

This doily-based setting is also of considerable interest in view of possible physical applications. For example, among the finite geometric concepts relevant for the theory of quantum information, the doily—though in various disguises—has been recognized to play the foremost role. Firstly, being isomorphic to the symplectic polar space of type $W(3,2)$, it underlies the commutation relations between the elements of the two-qubit Pauli group [5] and provides us with the simplest settings for observable proofs of quantum contextuality. Secondly, being isomorphic to a non-singular quadric of type $\mathcal{Q}(4,2)$, it also lies in the heart of a remarkable magic three-qubit Veldkamp line of form theories of gravity and its four-qubit extensions [6]. Finally, being a subquadrangle of a generalized quadrangle of type GQ$(2,4)$, it enters in an essential way certain black-hole entropy formulas and the so-called black-hole/qubit correspondence [7].

It is, in particular, the first aspect of the above-mentioned three ones that our non-unimodular doily has a particular link to. For if we label its 15 points by the 15 elements (identity disregarded) of the two-qubit Pauli group in such a way that each of its 15 lines features three pairwise commuting elements, then each of the ten labeled copies of GQ(2,1) (also known in the physics community as Mermin "magic" squares) contained in the doily furnishes an observable-based proof of quantum contextuality (see, e.g., Section 2 and Figure 3 of [6] for a particularly illustrative description of this quantum phenomenon). It is quite appealing to realize that not only does our non-unimodular geometry exhibit the doily, but it also *selects* in the latter a particular copy of GQ(2,1) (see Figure 3). Moreover, it has recently been shown [8] that this contextual character of GQ(2,1) is preserved also for doilies embedded in arbitrary symplectic polar spaces of type $W(2N-1,2)$ and, thus, labeled by elements of the arbitrary N-qubit Pauli group, $N \geq 2$.

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